Differential equations: Qualitative theory

System of differential equations: $\dot{x} = f(t, x)$ or $\dot{x} = f(x)$

Basic questions

1. Do there exist equilibrium solutions $x(t) = a$?
2. Let $x(t), \tilde{x}(t)$ be two solutions with $\tilde{x}(0)$ close to $x(0)$.
   Will $\tilde{x}(t)$ remain close to $x(t)$ for all future time, or will $\tilde{x}(t)$ diverge from $x(t)$ as t approaches infinity ?
   Stability
3. What happens to solutions $x(t)$ as t approaches infinity?
   Do all solutions approach equilibrium values? If not, do they at least approach a periodic solution?

Stability of solutions

- The solution $x(t)$ of $\dot{x} = f(x)$ is *stable* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ we have $||x(t) - \tilde{x}(t)|| < \varepsilon$, for all $t > 0$.
- The solution $x(t)$ of $\dot{x} = f(x)$ is *unstable* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists a solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ but $||x(t) - \tilde{x}(t)|| \geq \varepsilon$, for some $t > 0$.
- $||a|| = ||a||_\infty = \max\{|a_1|, \dots, |a_n|\}$ denotes the (maximum) *norm* of the vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$.

Stability of linear systems

- Consider a linear system
$$\dot{x} = Ax, \text{ with } A \in \mathbb{R}^{n \times n} \qquad (*)$$
- **Theorem** (cf. Braun, Differential equations, Chapt. 4.2)
  - Every solution $x(t)$ of (*) is stable if all eigenvalues of A have negative real part < 0.
  - Every solution $x(t)$ of (*) is unstable if at least one eigenvalue of A has positive real part > 0.
  - If all eigenvalues of A have real part ≤ 0 and $\lambda_1, \dots, \lambda_s$ have zero real part, let k_j be the multiplicity of λ_j.
    Every solution $x(t)$ of (*) is stable if A has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j, j = 1, \dots, s$. Otherwise, every solution $x(t)$ of (*) is unstable.

Reminder: Linear approximation

- One dimensional case
  - $f : I \to \mathbb{R}$ differentiable, $I \subseteq \mathbb{R}$ interval, $a \in I$
  - Linear approximation
$$f(x) = f(a) + f'(a) \cdot (x - a) + \text{higher order terms}$$
  - Taylor expansion
$$\begin{aligned} f(x) &= f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2}(x - a)^2 + \dots \\ &\quad + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots \end{aligned}$$

- n-dimensional case
  - $f : D \to \mathbb{R}^n$ differentiable, $a \in D \subseteq \mathbb{R}^n$
  - Jacobi matrix

$$\frac{\partial f}{\partial x}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \dots & \frac{\partial f_n}{\partial x_n}(a) \end{pmatrix}$$

  - Linear approximation

$$f(x) = f(a) + \frac{\partial f}{\partial x}(a)(x - a) + \text{higher order terms}$$

Linearisation around a critical point

- $x = a$ critical point of $\dot{x} = f(x)$
- Linearisation

$$\dot{x} = \frac{\partial f}{\partial x}(a)(x - a) + \text{higher order terms}$$

- Study linear equation with constant coefficients

$$\dot{z} = \frac{\partial f}{\partial z}(a)(z - a)$$

- Shift point a to the origin by putting $y = z - a$ and $A = \frac{\partial f}{\partial x}(a)$.
- Linearised system

$$\dot{y} = Ay$$

Asymptotic stability

- Consider the autonomous system $\dot{x} = f(x)$.
- A solution $x(t)$ is *asymptotically stable* if it is stable and if every solution $\tilde{x}(t)$ which starts sufficiently close to $x(t)$ must approach $x(t)$ as t approaches infinity.

Stability of equilibrium solutions

Theorem (cf. Braun, Differential equations, Chapt. 4.3)

- Consider the system $\dot{x} = f(x)$ and suppose f has continuous second-order partial derivatives.
- Let $x(t) = a$ be an equilibrium solution and $A = \frac{\partial f}{\partial x}(a)$.
- The equilibrium solution $x(t) = a$ is asymptotically stable if all the eigenvalues of A have negative real part.
- The equilibrium solution $x(t) = a$ is unstable if at least one eigenvalue of A has positive real part.
- The stability of the equilibrium solution $x(t) = a$ cannot be determined from the stability of the equilibrium solution $y(t) = 0$ of the linear system $\dot{y} = Ay$ if all the eigenvalues of A have negative real part ≤ 0 and at least one eigenvalue of A has zero real part.