Constructive justification of impredicative definitions in type theory

The relation between types, sets and propositions is a fundamental topic in logic and foundations of mathematics. The discovery of Curry-Howard Isomorphism or Correspondence is without any doubt a milestone of mathematical logic. Curry-Howard states a link between logical systems and computation: one can think of a proposition as a type and of a proof of that proposition as a program. A type is just the specification of a program and in view of this correspondence logical operations have a nice operational side. Martin-Löf Type Theory (MLTT) ([3],[4]) is arguably the formal system in which this correspondence is exploited in the more systematic way. In MLTT sets enter into this correspondence too: one can read "p:A'', i.e. p is a program or term of type A, as p is an element of the set A. The point to be stressed is that sets are not to be understood as the *arbitrary* sets of ZF, but they are inductively generated: there is a rule that states what is an element of a given set and when two elements are equal. Having rules forming sets fulfills the idea of sets as objects built up from below, a *predicative* constraint. The result is an intuitionistic and predicative system.

There are some quite good reasons to consider MLTT as the *basic* foundational framework for constructive mathematics. The strict adhesion to Curry-Howard Isomorphism with the consequent identification of types, sets and propositions guarantees that the theory has a natural computational interpretation. The presence of meaning explanations gives a rigorous, even though informal, justification of any type construction. Other constructive theories such as CZF(*Constructive Zermelo-Fraenkel)* can be interpreted in MLTT. Furthermore, the theory is predicative: it contains inductive types that are built up from below and increase logical strength, without assuming any impredicative principle. This makes sense if one considers the traditional idea that impredicative definitions are troublesome for those who retain that mathematical entities are constructions. However, the first system of type theory proposed by Martin-Löf contained a type of all types and quantification over types, but Girard showed that it was inconsistent by an ingenious type-theoretical formulation of Burali-Forti paradox ([1]). Martin-Löf recast the theory by keeping the identification between types, sets and propositions and introducing a universe of small types that cannot contain itself. The result was a rejection of impredicativity. The point of Girard's paradox is that "full'' Curry-Howard and impredicativity are not compatible. Martin-Löf's choice is a clear manifestation of the constructive concerns about impredicative principles, together with the conviction that Curry-Howard is indisputably a main feature of any constructive system that aims to be computationally effective.

The trouble with impredicative definitions is the fact that they are viciously circular: an entity is defined by quantifying over the set to which it belongs. Those who have a constructive view of mathematics (in a broad sense) are at odds with the justification of such kind of principles. The idea is that the second order universal quantifier " for all X...'' (suppose that X is ranging over sets) runs over all the possible instances of X and then we have to consider also the circular case when X is the set we are defining. It is quite clear that such a reading of second order quantification is scarcely justifiable from a constructive point of view: then banning impredicative definitions is reasonable. But that is not the only available reading: treating it as the privileged interpretation is a classical view, not compatible with the constructive assumptions behind theories like MLTT.

My thesis is that we can show that impredicative definitions are constructively justified and we can accept even the stronger claim that their acceptance is more natural in a constructive framework than a rigid adhesion to Curry-Howard Isomorphism.

The argument goes as follows: a detailed analysis of some results concerning Girard's System F ([1]) (second order or polymorphic lambda calculus, with lambda-abstraction over types that is the same as second order universal quantification) is carried out, by focusing on the way the circular case is handled in the normalization proof ([5]). The key remark is that quantification over types is purely schematic and a term of the universal type has a computational behavior that is not influenced by the type taken as input. In short, the steps of the computation are completely determined by a generic instance, there is no need of running over all the individual instances ([2]). The harmful circularity is strictly related to a classical understanding of the quantification.

At the same time, we can observe that the understanding of Curry-Howard underlying MLTT is not so natural. The correspondence works out well when we are looking at the types that correspond to logical constants, but when one takes into account inductive types, like natural numbers, the identification with propositions is a bit of a stretch. This naive remark is made rigorous by considering the new research program known as Homotopy Type Theory (HoTT)([6]). Informally, types are seen as spaces and terms as points. The point to be stressed is that this interpretation is naturally induced by the most peculiar rules of MLTT without assuming any additional principle, namely those concerning identity types. The result is a hierarchy of types, where only the lower levels are identified with propositions and sets. In short, not all types are propositions or sets, while the converse still holds. This suggests a revision of Curry-Howard. HoTT has at least two interesting features from a foundational point of view:
1) Classical logic works fine for the lower levels of the hierarchy, even if the theory is constructive.
2) It is a system closed to the mathematical practice: we can prove things about some areas of pure mathematics directly inside the type-theoretical framework.

In the end, I will discuss briefly as these two perspectives, justification of impredicativity coming from System F and revision of Curry-Howard coming from HoTT, can be joined together. The idea is to add an impredicative *universe* to HoTT, in which quantification over types is allowed without any restriction. The conclusion is that we can retain impredicative definitions in a constructive framework, vindicating a quite natural aspect of mathematics and restricting Curry-Howard. As additional advantage, the debate between classical and constructive foundations is revised in a more pluralistic sense, shedding a new light on an old dispute.

References

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