The Hard Lefschetz Theorem, some history and recent progress

Hélène Esnault, Freie Universität Berlin

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Acknowledgements

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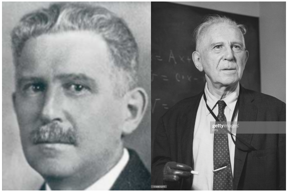

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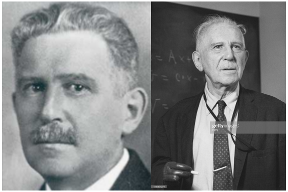

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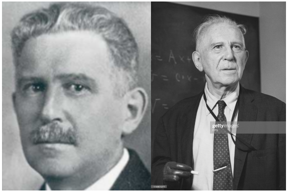

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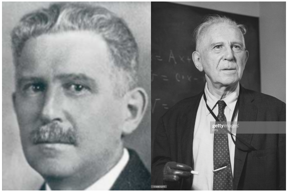

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He lost his both hands and a forearm in an industrial accident in 1907 and moved towards mathematics. He became a US citizen in 1912.

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Probably under the influence of the Institute for Advanced Study (which was then physically located at the department of Mathematics of Princeton U.), Princeton U. was an exception. Lefschetz became a Princeton professor in 1924.

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During WWII, many US mathematicians participated to the Manhattan project. I seems Lefschetz did not, he 'undertook applied mathematical work directed at the war effort' (Sylvia Nasar).

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His contribution to mathematics in Mexico is overwhelming and is still today in the heart of many Mexican mathematicians.

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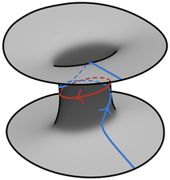
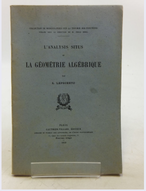


If you google 'vanishing cycles' for a nice picture, you'll get tons of cute ...stolen bicycles!

He studied the topology of hyperplane sections of algebraic varieties, yielding the weak Lefschetz theorems. He defined the Lefschetz pencils, their monodromy, studied the Picard-Lefschetz formulae on vanishing cycles.

Lefschetz theory II



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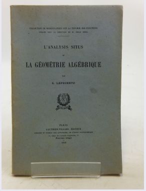

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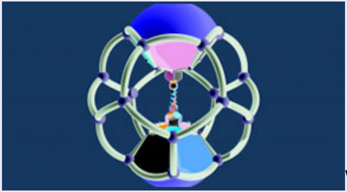
Wikipedia French: *The name 'Analysis Situs' had been given by Leibniz to a research he led during his whole life on a symbolism which would be to geometry what the algebraic symbolism is to numbers.*

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We owe Lefschetz the 'semantic' we use.

Quotations on Lefschetz

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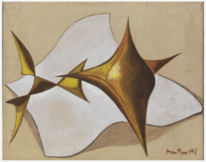
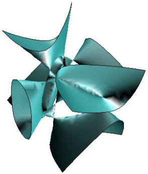
Sylvia Nasar, A Beautiful Mind, 1998, confirmed by Phillip Griffiths.

Some pictures: projective varieties



Man Ray, *Mathematical Objects*, 1934-35. Gelatin silver print. Steven Mendelson, Pittsburgh, Pennsylvania



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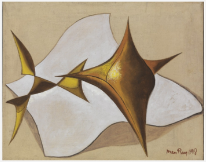

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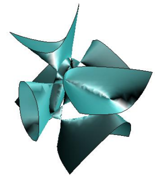

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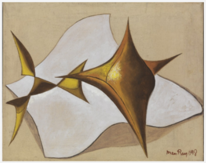

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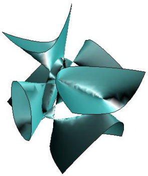

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The first approximation here is that even if the set of zeroes of polynomial equations is empty, the mere fact that we have equations entitles us to talk on varieties. The second approximation is that we have in addition to characterize the smoothness notion, which essentially means (over fields like the complex numbers) that if we caress the geometric object with our fingers, it never pricks.

Hard Lefschetz in complex geometry

Theorem (Formulated by Lefschetz 1924, proved by Hodge 1941)

Let X be a complex projective manifold of dimension d, and η be the Chern class in Betti cohomology $H^2(X, \mathbb{Z})$ of an ample line bundle. Then for all $i \in \mathbb{N}$, the cup-product map

$$\cup \eta^i : H^{d-i}(X, \mathbb{Q}) \to H^{d+i}(X, \mathbb{Q})$$

is an isomorphism.

Hard Lefschetz in complex geometry II

It is a *structural theorem* which enables one to write cohomology as a direct sum of 'primitive cohomologies' (called Lefschetz decomposition). It has *immense consequences*.

Hard Lefschetz in complex geometry: Hodge theory

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In particular, Hard Lefschetz holds more generally for Kähler manifolds, so compact complex manifolds endowed with a Kähler form. Its class $\eta \in H^2(X, \mathbb{R})$ is then the one in the theorem.

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Hard Lefschetz is one central piece of *Hodge theory* which controls cohomology of compact Kähler manifolds, e.g. of smooth complex projective manifolds.

Hard Lefschetz in complex geometry: non-abelian Hodge theory (Simpson)

In 1992 Simpson defined the notion of *non-abelian Hodge theory* on smooth complex projective varieties.

Hard Lefschetz in complex geometry: non-abelian Hodge theory (Simpson)II

Let $\rho : \pi_1^{\mathrm{top}}(X, x) \to GL_r(\mathbb{C})$ be a representation of the topological fundamental group.

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It is a topological bundle on X, defined as the quotient of the product $X_x \times \mathbb{C}^r$ by the diagonal action of $\pi_1^{\mathrm{top}}(X, x)$. Here $X_x \to X$ is the universal cover associated to the choice of x, $\pi_1^{\mathrm{top}}(X, x)$ acts on X_x as the group of deck transformations and on $\mathbb{C}^r$ via ρ.

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Then *non-abelian* refers to the replacement of X by $(X, \mathcal{V})$.

Hard Lefschetz in complex geometry: non-abelian Hodge theory (Simpson) III

Under the assumption that ρ is semi-simple, Simpson showed the existence of a *harmonic metric* on $\mathcal{V}$ made a $\mathcal{C}^\infty$ bundle. This enabled him to develop a Hodge theory with this metric and notably to prove the non-abelian Hard Lefschetz.

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Theorem (Simpson 1992)

Let X be a compact Kähler manifold of dimension d, $\mathcal{V}$ be a semi-simple local system of rank r, and η be the Chern class in Betti cohomology $H^2(X, \mathbb{C})$ of a Kähler class. Then for all $i \in \mathbb{N}$, the cup-product map

$$\cup \eta^i : H^{d-i}(X, \mathcal{V}) \to H^{d+i}(X, \mathcal{V})$$

is an isomorphism.

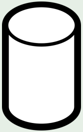
Examples

Example (Finite monodromy)

Of course, if $\mathcal{V}$ has finite monodromy, we can replace X by a finite topological cover $Y \to X$ on which $\mathcal{V}$ becomes a trivial local system, apply Hard Lefschetz on Y, and descend the information by a trace argument. However, finite monodromy local systems are *not often.*

Examples II

Example (Rank 1)

e.g. if $r = 1$, we understand well *their moduli space*: it contains a complex torus as a finite index subgroup. A complex torus is like on the picture: a circle (on the top) cross the (strictly positive, here from top to bottom to infinity) line. 

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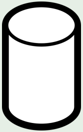
e.g. if $r = 1$, we understand well *their moduli space*: it contains a complex torus as a finite index subgroup. A complex torus is like on the picture: a circle (on the top) cross the (strictly positive, here from top to bottom to infinity) line. 

Example

The points which correspond to torsion (i.e. with finite monodromy) rank 1 local systems are all located on one horizontal section. So they *are not dense in the real topology*. It makes it difficult to deduce Simpson's theorem from the standard Hard Lefschetz. We remark however already here for later discussion that they are *Zariski dense*.

The Riemann hypothesis

The Riemann hypothesis (1859) predicts that for $s \in \mathbb{C}$, the Riemann zeta function

$$\xi(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots,$$

which converges for $\mathrm{Re}(s) > 1$, has all but its trivial zeroes $-2, -4, \dots$ located on the vertical line $\mathrm{Re}(s) = \frac{1}{2}$.

The Weil conjectures

Weil (1948) (see e.g. a Lecture I have at the Bibliothèque Nationale de France, which is linked on my webpage) defined the zeta function of a smooth projective variety X defined over a finite field $\mathbb{F}_q$, by *counting the number of $\mathbb{F}_{q^n}$-rational points* of X as n goes to infinity.

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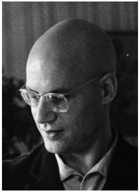
He predicted the existence of a cohomology theory $H^i(X)$ on which the Frobenius F acts, which would express the zeta function as the trace of the Frobenius acting on cohomology.

The Grothendieck school in relation to the Weil conjectures



ℓ-adic étale cohomology theory has been developed by the Grothendieck school in the 60s, 70s. Dwork (1960), independently of the Grothendieck school, and then Grothendieck (1965) proved the necessary properties of the zeta function.

Grothendieck-Lefschetz trace formula

In summary, for X_0 a smooth projective variety of dimension d defined over the finite field $\mathbb{F}_q$, X its base change to an algebraic closure $\overline{\mathbb{F}}_q$, and F the 'geometric' Frobenius acting on ℓ-adic cohomology $H^i(X)$, one has the Grothendieck-Lefschetz trace formula

$$|X_0(\mathbb{F}_q)| = \mathrm{Trace}(F|H^\bullet(X)) = \sum_{i=0}^{2d} (-1)^i \mathrm{Trace}(F|H^i(X))$$

The Weil conjectures II: Deligne

Weil, in analogy to the Riemann hypothesis, predicted that *read via complex embeddings, the absolute value of the eigenvalues of the Frobenius acting on* H^i *depends only on* i *and are different for different* i.

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In 1973 Deligne proves the Weil conjectures. In 1980 he publishes a more conceptual proof.

The Weil conjectures and Hard Lefschetz (Deligne)

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Theorem (Deligne 1980)

Let X_0 be a smooth projective variety of dimension d defined over a finite field $\mathbb{F}_q$, let X be its base change to an algebraic closure $\overline{\mathbb{F}}_p$. Let η be the Chern class in ℓ-adic cohomology $H^2(X, \mathbb{Q}_\ell)$ of an ample line bundle. Then for all $i \in \mathbb{N}$, the cup-product map

$$\cup \eta^i : H^{d-i}(X, \mathbb{Q}_\ell) \to H^{d+i}(X, \mathbb{Q}_\ell)$$

is an isomorphism.

The Weil conjectures and Hard Lefschetz (Deligne) II

What replaces Harmonic Theory in Hodge theory on the complex side is Deligne's theory of *weights* for varieties defined over a finite field.

The Weil conjectures and Hard Lefschetz (Deligne) II

What replaces Harmonic Theory in Hodge theory on the complex side is Deligne's theory of *weights* for varieties defined over a finite field.

It enables him to conclude that the intersection pairing restricted to the space of Lefschetz' vanishing cycles is non-degenerated.

Examples

Example (Finite monodromy)

Of course as we saw on the complex side, we could replace $\mathbb{Q}_\ell$ coefficients by any ℓ-adic local system with finite monodromy $\mathcal{V}$. Such a $\mathcal{V}$ is necessarily defined over the base change of X_0 to a finite extension of $\mathbb{F}_q$. We would apply Deligne's Hard Lefschetz on the pull-back of $\mathcal{V}$ to a finite étale cover $Y \to X$ which kills the monodromy, and then descend via a trace argument.

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Example (Rank 1)

For example in rank $r = 1$. However, *most* ℓ-adic local systems are *not* definable over the base change of X_0 to a finite extension of $\mathbb{F}_q$. The aim of the last part of the lecture is to explain that we can make more precise this *most*. This leads to a proof of a new case of Hard Lefschetz for which we do *not have Deligne's theory of weights* at disposal.

Zariski density of torsion points.

Theorem (E-Kerz 2019)

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$.

- *There is a noetherian space $\mathcal{G}$ over $\overline{\mathbb{Q}}_\ell$, such that its points $\mathcal{G}(\overline{\mathbb{Q}}_\ell)$ form a subgroup of discrete index in the group of rank 1 ℓ-adic local systems.*
- *With respect to the Zariski topology stemming from this noetherianity, the locus $S_\circ$ of points which violate Hard Lefschetz is Zariski constructible.*
- *The Zariski closure of $S_\circ$, if not empty, necessarily is the Zariski closure of its torsion points.*

Hard Lefschetz in rank 1 is positive characteristic.

Corollary (E-Kerz 2019)

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$, let $\mathcal{V}$ be a rank 1 local system. Let η be the Chern class in ℓ-adic cohomology $H^2(X, \mathbb{Q}_\ell)$ of an ample line bundle. Then for all $i \in \mathbb{N}$, the cup-product map

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Logic of proof: Drinfeld

The logic of proof, as exposed, relies on Drinfeld's work: Indeed, if not empty, $S_\circ$ must contain torsion points, which violates Deligne's Hard Lefschetz.

Proof

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Thank you for your attention.