Preface

Emil Artin was born on March 3, 1898 in Vienna. His father was an art dealer, and his mother was an opera singer. After his father died, his mother married again, and lived in Reichenberg, Bohemia, where Artin obtained his "Reifeprüfung" in 1916. After studying one semester at the University in Vienna, he was drafted and served in an infantry regiment until the end of the war. In January 1919 he continued his studies at the University of Leipzig. He studied there with Herglotz, towards whom he kept a heartfelt appreciation throughout his life. Herglotz was the only person whom Artin recognized as having been his "teacher." Artin got his PhD in 1921, spent one year at the University of Göttingen, and then went to Hamburg University. He became Privatdozent in 1923, Ausserordentlicher Professor in 1925, and Ordentlicher Professor in 1926 at the age of 28.

He married Natalie Jasny in 1929. They had three children, Karin, Michael, both born in Hamburg, and Thomas, born later in America, to which he emigrated in 1937. He spent one year at the University of Notre Dame, then was at Indiana University in Bloomington from 1938 to 1946, at which time he moved to Princeton, where he stayed from 1946 to 1958. He returned to Hamburg in 1958, and remained there until his death, of a heart attack, on December 20, 1962.

This volume includes all of Artin's papers.

It is not our intention to discuss Artin's mathematical works, but we thought it might be worth while to mention briefly some of his conjectures, not all of which were published.

The first one was, in effect, the Riemann hypothesis in function fields. In his thesis, Artin discussed hyperelliptic fields over finite constant fields as analogues of quadratic number fields, and pointed out that the analogue of the classical Riemann hypothesis seemed to be true for them. The proof was eventually given by Hasse for fields of genus 1 (Int. Congress, Oslo, 1936) and by Weil in the general case (*Comptes Rendus*, 1941).

A little later, he defined the non-abelian L-series, and conjectured their integrality, as well as a Riemann hypothesis for them. Both of these conjectures are still unproved, but it is interesting to note that Weil's methods allowed him to prove them in the function field case, and showed the close connection between the two (exhibiting them as two aspects of a more general phenomenon, the positive definiteness of the trace in the ring of correspondences).

Artin's guess that the *L*-series were meromorphic, and that this property should follow from a theorem on group characters was proved correct by Brauer (*Annals of Math*. 1947).

To prove that his general *L*-series coincided with the classical *L*-series in the case of abelian extensions, Artin was led to conjecture the reciprocity law. He succeeded in proving it four years later. Furthermore, he observed that, by means of the reciprocity law, Hilbert's famous conjecture that ideals become principal in the maximal unramified abelian extension could be reduced to a purely group theoretical statement involving the transfer. This statement, now known as the principal ideal theorem, was proved by Furtwangler, and a new way of looking at the transfer, suggested by Artin, enabled Iyanaga to give a much shorter proof soon after.

Artin's concern with the decomposition laws for primes in algebraic number fields led him to a conjecture in elementary number theory, as follows. Let a be a fixed integer $\neq 0, \pm 1$. In order that a be a primitive root for a prime p not dividing a, it is obviously necessary and sufficient that for no prime q the conditions

$$(*) \qquad p \equiv 1 \pmod{q} \quad \text{and} \quad a^{(p-1)/q} \equiv 1 \pmod{p}$$

be simultaneously satisfied. For given q, let M_q be the set of primes p satisfying the conditions $(*)$. Then M_q is just the set of primes p which split completely in the splitting field K_q of the polynomial $x^q - a$, and consequently has density $1/k_q$, where k_q is the degree of K_q over the field of rational numbers. Thinking the conditions $(*)$ for different q's to be independent, Artin conjectured that the set of primes p for which a is primitive root has density

$$\prod_q \left(1 - \frac{1}{k_q}\right).$$

In case a is square free, we have $k_q = q(q-1)$ for all q. Computations by Lehmer a few years ago showed a discrepancy in some cases. When Artin learned of this, he realized that the conditions (*) are not independent. For example, if $a = \left(\frac{-1}{q_0}\right) q_0$ for some odd prime q_0, then $K_2 \subset K_{q_0}$ (omitting the index a for simplicity). Hence $M_2 \supset M_{q_0}$ and the factor $\left(1 - \frac{1}{k_{q_0}}\right)$ should be deleted from the above product to get the correct density.

In the course of a conversation, he mentioned to us that one has to make the "obvious" modification of the above product by a rational factor to take into account the dependence of the fields K_q. Unfortunately we did not go into this matter in detail with him, but it seems to us that the necessary modification is the following. For each square-free integer $m > 0$, let

$$K_m = \prod_{q|m} K_q$$

be the compositum of the fields K_q for primes q dividing m, and let k_m be the absolute degree of K_m. Then the set of primes p such that the conditions (*) are satisfied for no q dividing m has density

$$\rho(m) = \sum_{d|m} \frac{\mu(d)}{k_d},$$

where μ is the Moebius function, and the sum runs over the positive divisors d of m. Hence the conjecture should be that the set of primes p for which a is a primitive root has density

$$\rho = \lim_{m} \left(\sum_{d|m} \frac{\mu(d)}{k_d} \right)$$

where the limit is taken over all square-free m, ordered by divisibility.

To get a more explicit expression for ρ, one proves that if m is odd the fields K_q for $q|m$ are completely linearly disjoint. Consequently, for odd m, we have

$$k_m = \prod_{q|m} k_q,$$

and k_{2m} is equal to k_m or to $2k_m$, according as $\sqrt{a}$ is or is not contained in the field of m-th roots of unity. Let $a = a_0 b^2$ with a_0 square free. Then the condition for $\sqrt{a}$ to be contained in the field of m-th roots of unity for odd square-free m is that a_0 divide m and be congruent to 1 (mod 4). Putting all this together one finds that the conjectured density ρ of the set of primes p for which a is a primitive root is

$$\rho = A \cdot \prod_{q} \left(1 - \frac{1}{k_q} \right),$$

with

$$A = \begin{cases} 1 & \text{if } a_0 \not\equiv 1 \pmod 4, \\ 1 - \dfrac{\mu(a_0)}{\prod_{q|a_0} (k_q - 1)} & \text{if } a_0 \equiv 1 \pmod 4, \end{cases}$$

where a_0 is the square-free part of a. Here k_q is the absolute degree of the splitting field of the polynomial $x^q - a$, and consequently $k_q = q(q-1)$ unless a is of the form $\pm c^n$ for some integer c and some integer $n > 1$.

It is interesting to note that the analogue of Artin's conjecture on primitive roots in function fields over finite fields has been proved by students of Hasse, using the Riemann hypothesis in function fields (cf. Hasse's discussion, *Annales Academiae Scientiarum Fennicae*, Helsinki 1952).

We now pass to an entirely different kind of question. In 1935, Tsen proved that there do not exist non-trivial division algebras of finite rank over a function field in one variable, over an algebraically closed constant field. Analysing Tsen's proof, Artin was led to make the following definition. A field K is quasi-algebraically closed (QAC) if every form (homogeneous polynomial) of degree d in n variables with coefficients in K has a non-trivial zero in K provided $n > d$. He then observed that the method used by Tsen could be used to prove that a function field as above is QAC. In view of Wedderburn's theorem, he then conjectured that finite fields are QAC. This was proved almost immediately by Chevalley (*Hamburg Abh.* 1936). (It seems that Dickson had made the same conjecture a number of years earlier.) By class field theory, it was known that the field Ω obtained from the rational numbers by adjoining all roots of unity has no non-trivial division algebra over it. In view of this, and by analogy with function fields over finite fields, Artin also conjectured that the field Ω is QAC. Concerning number fields themselves, he suggested that an analogous statement should be true provided $n > d^2$, and provided the number field is totally imaginary. He also made the corresponding local conjectures. None of the global conjectures is proved, and the fact that p-adic fields satisfy the "$n > d^2$" property is also unproved.

Finally, let us mention that Artin was always interested in "geometric algebra", and in the theory of finite groups. In this field, he conjectured that a simple group of order g divisible by a prime number $p > g^{1/3}$ is of known type. This was proved by Brauer and Reynolds (*Annals* 1958).

Artin loved teaching at all levels. Even though occupying research professorships, he never failed to give, regularly, courses in elementary Calculus. His lectures and seminars were reknowned for their perfection and excitement. They contributed much towards spreading his point of view in algebra, for which van der Waerden's text, derived from lectures by Artin and Emmy Noether, has been the fundamental reference for the past 30 years.

They also inspired his students, towards whom his generosity and affection were unsurpassed.

S. Lang

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