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ON A RATIONALITY QUESTION IN THE GROTHENDIECK RING OF VARIETIES

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Abstract. We discuss elementary rationality questions in the Grothendieck ring of varieties for the quotient of a finite dimensional vector space over a characteristic 0 field by a finite group.

1. Introduction

Let k be a field. One defines the *Grothendieck group of varieties* $K_0(\mathrm{Var}_k)$ *over* k [8, Definition 2.1] to be the free abelian group generated by k-schemes modulo the subgroup spanned the scissor relations

$$[X] = [X \setminus Z] + [Z]$$

where $Z \subset X$ is a closed subscheme. The product

$$[X \times_k Y] = [X] \cdot [Y]$$

for two k-schemes makes it a commutative ring, with unit $1 = [\mathrm{Spec}\, k]$. As the underlying topological space of the complement $X \setminus X_{\mathrm{red}}$ is empty, $[X] = [X_{\mathrm{red}}]$. This justifies the terminology "varieties" rather than "schemes".

In characteristic 0, first examples of 0-divisors in this ring were shown to exist by Poonen [9]. He constructed two abelian varieties A, B over $\mathbb{Q}$ such that

$$0 = ([A] - [B]) \cdot ([A] + [B]) \in K_0(\mathrm{Var}_{\mathbb{Q}})$$

but with

$$[A \otimes_{\mathbb{Q}} k] \neq [B \otimes_{\mathbb{Q}} k] \in K_0(\mathrm{Var}_k)$$

for all field extensions $\mathbb{Q} \hookrightarrow k$. The main tool to distinguish those two classes relies ultimately on a deep insight in the structure of birational morphisms, gathered in the *Weak Factorization Theorem* [1]. It implies both the presentation of $K_0(\mathrm{Var}_k)$ as the free group generated by smooth projective varieties modulo the blow up relation [2] and the isomorphism $K_0(\mathrm{Var}_k)/\langle \mathbb{L} \rangle \xrightarrow{\cong} \mathbb{Z}[SB]$ [5]. Here $\mathbb{L}$ is the class of the affine line $\mathbb{A}^1$ over k, $\langle \mathbb{L} \rangle$ is the ideal spanned by it, $\mathbb{Z}[SB]$ is the free abelian group on stably birational classes of projective smooth k-varieties, endowed with

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the ring structure stemming from the product of varieties over k. So there are no relations in $\mathbb{Z}[SB]$ and this allows to recognize certain classes. Of course this does not help in understanding $\mathbb{L}$, and the question whether or not $\mathbb{L}$ is a 0-divisor remains open.

Later Kollár [4] used $\mathbb{Z}[SB]$ to distinguish in characteristic 0 the $K_0(\mathrm{Var}_k)$-classes of non-trivial Severi-Brauer varieties from trivial ones. Rökaeus [10] and Nicaise [8], using in addition specialization of $K_0(\mathrm{Var}_k)$ from k to finite fields, studied 0-divisors which are classes of 0-dimensional varieties, in particular those of the form $\mathrm{Spec}\, K$ for a non-trivial field extension of a number field k. This indicates that one can not expect "descent". For two k-varieties X and Y the equality

$$[X \times_k \mathrm{Spec}\, K] = [Y \times_k \mathrm{Spec}\, K] \in K_0(\mathrm{Var}_K)$$

implies

$$[X] \cdot [\mathrm{Spec}\, K] = [Y] \cdot [\mathrm{Spec}\, K] \in K_0(\mathrm{Var}_k).$$

However, the relation $[X] \cdot [\mathrm{Spec}\, K] = [Y] \cdot [\mathrm{Spec}\, K] \in K_0(\mathrm{Var}_k)$ does not imply the equality $[X] = [Y] \in K_0(\mathrm{Var}_k)$.

For applications of the Grothendieck ring, it is of importance to understand the class of quotients $[X/G]$ where X is a variety and G is a finite group acting on it. In [6, Lemma 5.1], Looijenga shows that if k is an algebraically closed field of characteristic 0, and if G is a finite abelian group acting linearly on a finite dimensional k-vector space V, then

$$[V/G] = \mathbb{L}^{\dim_k V} \in K_0(\mathrm{Var}_k). \tag{1.1}$$

In fact the formula (1.1), as well as its proof, remain valid if k is any field of characteristic 0 containing the $|G|$-th roots of 1. However the condition that G be abelian is essential, as shown by Ekedahl. Indeed, [3, Proposition 3.1, ii)] together with [3, Corollary 5.2] show that for $G \subset GL(V)$, $V \cong \mathbb{C}^n$ as in Saltman's example [11], the class of $\lim_{m\to\infty}[V^m/G]/\mathbb{L}^{nm}$ in the completion $\widehat{K_0(\mathrm{Var}_{\mathbb{C}})}$ of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ by the dimension filtration, is not equal to 1. This implies in particular that for m large enough, $\mathbb{L}^{nm} \neq [V^m/G] \in K_0(\mathrm{Var}_{\mathbb{C}})$.

In this note, we discuss possible simple generalizations of Looijenga's formula in various ways. Our first result is the following.

Lemma 1.1. *Let G be a finite abelian group with quotient $G \to \Gamma$. Let k be a field of characteristic 0 and let $K \supset k$ be an abelian Galois extension with Galois group Γ. Assume, that the Galois action of Γ on K lifts to a k-linear action of G on a finite dimensional K-vector space V. If, for $N = \exp(G)$, all N-th roots of 1 lie in k, then (1.1) holds, i.e.*

$$[V/G] = \mathbb{L}^{\dim_K V} \in K_0(\mathrm{Var}_k).$$

The condition that k contains the N-th roots of 1 is really necessary. In particular, if one allows the group G to act non-trivially on the ground field, the equation (1.1) is not compatible with descent to smaller ground fields.

Example 1.2. *Assume $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-1})$, $V = K \otimes_{\mathbb{Q}} \mathbb{Q}^2$, and let G be the subgroup of the group of $\mathbb{Q}$-linear automorphisms of V spanned by*

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where the chosen basis of K as a 2-dimensional vector space over $\mathbb{Q}$ is $(1, \sqrt{-1})$. The group G is cyclic of order 4 and

$$\mathbb{L}^2 \neq [V/G] \in K_0(\mathrm{Var}_{\mathbb{Q}}).$$

If $G \subset GL_k(V)$ is a finite group acting linearly on a finite dimensional vector space V over a characteristic 0 field k, then G acts semi-simply. So as a G-representation, $V = \bigoplus_i V_i \otimes T_i$, where V_i is an irreducible representation with $\mathrm{Hom}_G(V_i, V_j) = \delta_{ij} \cdot k$, and T_i is the trivial representation of dimension m_i equal to the multiplicity of V_i in V. If G is commutative of exponent N and if the N-th roots of 1 lie in k, then $d_i = \dim_k V_i = 1$. Since V_i/G is normal and one dimensional, it is smooth. So the starting point of Looijenga's proof of (1.1) is the simple observation that there is a k-isomorphism $V_i/G \cong V_i$ of k-varieties. The proof of (1.1) then proceeds by stratifying V.

For $d_i \geq 2$, the quotient V_i/G might be singular, thus it can not be isomorphic to V_i, not even over a field extension. Nevertheless, one can show that the formula (1.1) remains true for irreducible two dimensional representations, or after stratifying, whenever all the d_i are 1 or 2 and G is a prime power order cyclic group.

Proposition 1.3. *Let k be a field of characteristic 0 and let V be a finite dimensional k-vector space. Let $G \to GL_k(V)$ be a linear representation of a finite abelian group.*

1) *If $\dim_k V \leq 2$, then (1.1) holds true.*
2) *If G is cyclic of prime power order, and if each irreducible subrepresentation V_i has $\dim(V_i) \leq 2$, then (1.1) holds true.*

The main reason for the restriction to $\dim(V_i) \leq 2$ is that in this case $\mathbb{P}(V_i) \cong \mathbb{P}^1_k$ and hence $\mathbb{P}(V_i)/G \cong \mathbb{P}^1_k$ as well. If V is an irreducible representation of dimension $d \geq 3$ a similar statement fails, and we were unable to prove the equation (1.1).

2. Proof of Lemma 1.1

By assumption $G \subset GL_k(V)$ lifts the action of the quotient Γ on K, hence writing

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\varrho} \Gamma \longrightarrow 1,$$

one has $\sigma(\lambda \cdot v) = \gamma(\lambda) \cdot \sigma(v)$, for all $\sigma \in G$, $\gamma = \varrho(\sigma)$, for all $\lambda \in K$ and for all $v \in V$. In particular H is a subgroup of $GL_K(V)$. This defines the fiber square

$$\begin{tikzcd} V/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ V/G \arrow[r] & \operatorname{Spec} k. \end{tikzcd} \tag{2.1}$$

By the rationality assumption, $\mu_N(k) \cong_k \mathbb{Z}/N$, for $N = \exp(G)$, and hence the characters of G are k-rational. So writing $\hat{H}$ for the character group of H and $V_\chi(H)$ for the eigenspace with respect to the character χ of H, one has a fortiori the K-eigenspace decomposition

$$V = \bigoplus_{\chi \in \hat{H}} V_\chi(H).$$

Since G is commutative the subspace $V_\chi(H)$ of V is G-invariant.

Now on the geometric side, one proceeds as in Looijenga's Bourbaki lecture [6, Lemma 5.1]. Write

$$V = \prod_{\chi \in \hat{H}} V_\chi(H)$$

for the product as K-schemes. For $\{0\} = \operatorname{Spec} K$ one sets $V_\chi^\times = V_\chi(H) \setminus \{0\}$ and defines the stratification

$$V = \bigsqcup_{I \subset \hat{H}} V_I, \quad \text{with} \quad V_I = \prod_{\chi \in I} V_\chi^\times. \tag{2.2}$$

The product in (2.2) is defined over K. The $\mathbb{G}_m$-fibration $V_\chi^\times \to \mathbb{P}(V_\chi(H))$ is the structure map of the geometric line bundle $\mathcal{O}_{\mathbb{P}(V_\chi(H))}(-1)$, restricted to the complement of the zero-section. It is defined over K and G-equivariant. The subgroup H acts trivially on $\mathbb{P}(V_\chi(H))$ and by multiplication with χ on the geometric fibres of $V_\chi^\times \to \mathbb{P}(V_\chi(H))$.

So for $I \subset \hat{H}$ given, the K-morphism

$$V_I \to \prod_{\chi \in I} \mathbb{P}(V_\chi(H))$$

is a G-equivariant fibration, locally trivial for the Zariski topology. The fibres are isomorphic to $\mathbb{G}_m^{\#I} \cong \prod_{\chi \in I} \mathbb{G}_{m,\chi}$, with $\mathbb{G}_{m,\chi} \cong \mathbb{G}_m$, hence

$$[V_I] = [\mathbb{G}_m^{\#I}] \cdot \prod_{\chi \in I} [\mathbb{P}_K^{r_\chi}] \quad \text{in} \quad K_0(\mathrm{Var}_K). \tag{2.3}$$

The action of H is trivial on $\prod_{\chi \in I} \mathbb{P}(V_\chi(H))$ and on the factor $\mathbb{G}_{m,\chi}$ of $\mathbb{G}_m^{\#I}$ the group H acts by multiplication with χ. One obtains an induced K-morphism

$$V_I/H \to \prod_{\chi \in I} \mathbb{P}(V_\chi(H))$$

which is still a Zariski locally trivial fibration with fibre

$$\mathbb{G}_m^{\#I} \cong \big(\prod_{\chi\in I} \mathbb{G}_{m,\chi}\big)/H. \tag{2.4}$$

The G-action respects the decomposition $V_I = \prod_{\chi\in I} V_\chi(H)$ and on $\mathbb{P}(V_\chi(H))$, it factors through Γ, that is one has a splitting of $\mathrm{Aut}(\mathbb{P}(V_\chi(H))/k) \to \mathrm{Aut}(K/k) = \Gamma$. This implies

$$\big(\prod_{\chi\in I} \mathbb{P}(V_\chi(H))\big)/G = \big(\prod_{\chi\in I} \mathbb{P}(V_\chi(H))\big)/\Gamma$$

as well as

$$(\mathbb{P}(V_\chi(H))/\Gamma) \otimes_k K = \mathbb{P}(V_\chi(H)).$$

From this one deduces

$$\begin{aligned}\big(\prod_{\chi\in I} \mathbb{P}(V_\chi(H))\big)/\Gamma = \big(\prod_{K,\chi\in I} \big((\mathbb{P}(V_\chi(H))/\Gamma) \otimes_k K\big)\big)/\Gamma = \\ \big(\big(\prod_{k,\chi\in I} (\mathbb{P}(V_\chi(H))/\Gamma)\big) \otimes_k K\big)/\Gamma = \prod_{k,\chi\in I} (\mathbb{P}(V_\chi(H))/\Gamma).\end{aligned}$$

Here we underline by the lower indices $_{K,k}$ where we took the fiber products. Remark that $\mathbb{P}(V_\chi(H))/\Gamma$ is a k-form of $\mathbb{P}_k^{r_\chi}$ for $r_\chi = \dim_K V_\chi(H) - 1$.

The fiber square (2.1) is the composite of two fibre squares

$$\begin{tikzcd} V_I/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{\chi\in I} \mathbb{P}(V_\chi(H)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ V_I/G \arrow[r] & \prod_{\chi\in I} \big(\mathbb{P}(V_\chi(H))/\Gamma\big) \arrow[r] & \operatorname{Spec} k. \end{tikzcd} \tag{2.5}$$

Claim 2.1. The k-form $\mathbb{P}(V_\chi(H))/\Gamma$ of $\mathbb{P}_k^{r_\chi}$ is split, the k-morphism

$$V_I/G \to \prod_{\chi\in I} \mathbb{P}(V_\chi(H))/\Gamma$$

is a $\mathbb{G}_m^{\#I}$-fibration, locally trivial for the Zariski topology, and hence

$$[V_I/G] = [\mathbb{G}_m^{\#I}] \cdot \prod_{\chi\in I} [\mathbb{P}_k^{r_\chi}] \quad \text{in} \quad K_0(\mathrm{Var}_k).$$

Proof. By assumption k contains the N-th roots of 1 for $N = \exp(G)$ and hence the characters $\chi \in \hat{H}$ are defined over k.

Then $V_\chi(H)$, regarded as a k-vector space, has a G-eigenvector v. The line $\langle v\rangle_K$ defines a point $c \in \mathbb{P}(V_\chi(H))(K)$. Since the action of G on $K(c) = K$ factors through the Galois action of Γ on $K(c)$, the image of c lies in $(\mathbb{P}(V_\chi(H))/G)(k)$.

In addition, in (2.4) the action of H on $\prod_{\chi\in I} \mathbb{G}_{m,\chi}$ is given by multiplication with χ, hence it is defined over k. Then $\big[\prod_{\chi\in I} \mathbb{G}_{m,\chi}\big]/H$ is obtained by base extension from a k-variety, isomorphic to $\mathbb{G}_m^{\#I}$.

Using that the left hand side of (2.5) is a fibre product and that

$$V_I/H \to \prod_{\chi \in I} \mathbb{P}(V_\chi(H))$$

is Zariski locally trivial with fibre $\big[\prod_{\chi\in I} \mathbb{G}_{m,\chi}\big]/H$ this implies the second assertion in Claim 2.1. □

By (2.2) and (2.3) $[V/G] = \sum_{I\subset \hat{H}} \Big([\mathbb{G}_m^{\#I}] \cdot \prod_{\chi\in I} [\mathbb{P}_k^{r_\chi}]\Big)$.

This decomposition just depends on the dimensions $r_\chi+1$ of the subspaces $V_\chi(H)$. So if W_χ denotes any k-vectorspace of this dimension and $W = \bigoplus_{\chi\in\hat{H}} W_\chi$, one finds in $K_0(\mathrm{Var}_k)$

$$\mathbb{L}^{\dim_K V} = \mathbb{L}^{\dim_k W} = \sum_{I\subset\hat{H}} \prod_{\chi\in I} [W_\chi^\times] = \sum_{I\subset \hat{H}} \Big([\mathbb{G}_m^{\#I}] \cdot \prod_{\chi\in I}[\mathbb{P}_k^{r_\chi}]\Big) = [V/G].$$

This finishes the proof of Lemma 1.1. □

3. Verification of the properties in Example 1.2

In the standard basis e_1, e_2 of $\mathbb{Q}^2$ and the basis $(1, \sqrt{-1})$ of $K/\mathbb{Q}$, we write

$$\sigma : (x_1 + \sqrt{-1}y_1)e_1 + (x_2 + \sqrt{-1}y_2)e_2 \mapsto (-x_1 + \sqrt{-1}y_1)e_2 + (x_2 - \sqrt{-1}y_2)e_1.$$

As σ is $\mathbb{Q}$-linear, it leaves the origin of V invariant, thus acts on $V^\times = V \setminus \{0\}$. One has $\sigma^2 = -\mathrm{Id}$ and this defines the extension

$$0 \longrightarrow H := \langle\sigma^2\rangle \longrightarrow G \longrightarrow \Gamma := \langle\gamma\rangle \longrightarrow 0$$

with $\quad \Gamma = \langle\gamma\rangle \cong \mathbb{Z}/2 = \mathrm{Aut}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}), \quad$ and $\quad \gamma(\sqrt{-1}) = -\sqrt{-1}$.
Thus one has the fiber square

$$\begin{tikzcd} V/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ V/G \arrow[r] & \operatorname{Spec} \mathbb{Q} \end{tikzcd}$$

The $\mathbb{G}_m$-bundle $V^\times \to \mathbb{P}^1_K$ is compatible with the G-action. The subgroup H acts trivially on $\mathbb{P}^1_K$ while σ acts via

$$\bar{\sigma} : (x_1 + \sqrt{-1}y_1 : x_2 + \sqrt{-1}y_2) \mapsto (x_2 - \sqrt{-1}y_2 : -x_1 + \sqrt{-1}y_1).$$

This yields the fiber squares

$$\begin{tikzcd} V^\times/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^1_K \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d] \\ V^\times/G \arrow[r] & \mathbb{P}^1_K/G \arrow[r] & \operatorname{Spec} \mathbb{Q} \end{tikzcd}$$

Claim 3.1. $\mathbb{P}^1_K/G$ is a genus 0 curve over $\mathbb{Q}$ without a rational point.

Proof. Indeed, a rational point is a fixpoint of $\mathbb{P}^1_K$ under $\bar{\sigma}$. But the equation for a fixpoint is precisely

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = 0, \quad \text{with} \quad (x_1, x_2, y_1, y_2) \neq (0,0,0,0).$$

So over $\mathbb{Q}$ there are no solutions. □

Corollary 3.2. $\mathbb{L}^2 \neq [V/G] \in K_0(\mathrm{Var}_{\mathbb{Q}})$.

Proof. The origin $x_1 = x_2 = y_1 = y_2 = 0$ in V is a fixpoint under G. Thus

$$[V/G] = [V^\times/G] + [\operatorname{Spec}\mathbb{Q}].$$

On the other hand, as we have seen in Claim 2.1, $V^\times/G \to \mathbb{P}^1_K/G$ is a locally trivial $\mathbb{G}_m$ bundle.

Here the trivialization of can be written down explicitly: $V^\times$ is the total space of the $\mathbb{G}_m$-bundle to the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_K}(-1)$, while $V^\times/H \to \mathbb{P}^1_K$ is the total space of the $\mathbb{G}_m$-bundle to the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_K}(-2) = \pi^*\mathcal{L}$, where $\mathcal{L} \in \operatorname{Pic}(\mathbb{P}^1_K/G)$. So $V^\times/G \to \mathbb{P}^1_K/G$ is the $\mathbb{G}_m$-bundle to the invertible sheaf $\mathcal{L}$. One concludes

$$[V/G] - [\operatorname{Spec}\mathbb{Q}] = [V^\times/G] = [\mathbb{G}_m] \cdot [\mathbb{P}^1_K/G] \in K_0(\mathrm{Var}_{\mathbb{Q}}).$$

On the other hand, one also has

$$\mathbb{L}^2 - [\operatorname{Spec}\mathbb{Q}] = [\mathbb{A}^2_{\mathbb{Q}} \setminus \{0\}] = [\mathbb{G}_m] \cdot [\mathbb{P}^1_{\mathbb{Q}}] \in K_0(\mathrm{Var}_{\mathbb{Q}}).$$

If $[V/G]$ was equal to $\mathbb{L}^2$ in $K_0(\mathrm{Var}_{\mathbb{Q}})$, then one would have the relation $[V^\times/G] = [\mathbb{A}^2_{\mathbb{Q}} \setminus \{0\}]$ in $K_0(\mathrm{Var}_{\mathbb{Q}})$, thus the relation

$$\Phi([V^\times/G]) = -\Phi([\mathbb{P}^1_K/G]) = \Phi([\mathbb{A}^2_{\mathbb{Q}} \setminus \{0\}]) = -\Phi([\mathbb{P}^1_{\mathbb{Q}}]) \ \text{ in } \ \mathbb{Z}[SB],$$

where $\Phi : K_0(\mathrm{Var}_{\mathbb{Q}}) \to \mathbb{Z}[SB]$ maps the class $[X]$ of a smooth projective $\mathbb{Q}$-variety X to its stably birational equivalence class.

This however contradicts Claim 3.1, as the existence of a rational point is compatible with the stably birational equivalence on smooth projective varieties over any infinite field k.

For sake of completeness let us recall the proof of this well known fact. If $\tau : V \dashrightarrow W$ is a birational map between two smooth projective varieties, and τ is well defined near $v \in V(k)$, then $\tau(v)$ is well defined and lies in $W(k)$. Else one blows up v. This yields an exceptional divisor $\mathbb{P}^{\dim_k V - 1}$. Since τ is well defined outside of codimension ≥ 2, and since k is infinite, there are rational points on the exceptional divisor on which τ is defined and one repeats the argument. □

4. Proof of Proposition 1.3

We first show 1). If V has k-dimension ≤ 2, we write the G-equivariant stratification $V = \{0\} \sqcup V^\times$. Furthermore, the projection $V^\times \to \mathbb{P}(V)$ is G-equivariant as well. Looijenga's argument shows here

$$[V^\times/G] = [\mathbb{G}_m] \cdot [\mathbb{P}(V)/G] \in K_0(\mathrm{Var}_k).$$

On the other hand, either

$$\mathbb{P}(V) = \operatorname{Spec} k = \mathbb{P}(V)/G \quad \text{or} \quad \mathbb{P}(V)/G \cong_k \mathbb{P}^1_k \cong_k \mathbb{P}(V).$$

Adding up, one finds $[V/G] = \mathbb{L}^2 \in K_0(\mathrm{Var}_k)$.

We now show 2). Instead of the decomposition $V = \bigoplus_{i=1}^r V_i \otimes T_i$ of V as a direct sum of irreducible G representations considered in the introduction, we will drop the condition that $\mathrm{Hom}_G(V_i, V_j) = \delta_{ij} \cdot k$ and choose a decomposition $V = \bigoplus_{i=1}^m V_i$ as a direct sum of irreducible representations. As usual we consider V as a variety and write

$$V = \prod_{i=1}^m V_i. \tag{4.1}$$

The monodromy group, that is the image of G in $GL_k(V)$, is still a p-order cyclic group. So we may assume

$$G \subset GL_k(V) \tag{4.2}$$

in the discussion.

Claim 4.1. There is a direct factor V_i of (4.1) such that $G \subset GL_k(V_i)$.

Proof. Since a p-power order cyclic group G contains a unique p-order cyclic subgroup $C(G)$, if $\{1\} \neq K_i := \mathrm{Ker}\big(G \to GL_k(V_i)\big)$ then $C(G) = C(K_i) \subset K_i$. We conclude by (4.2). □

We now change the notation: we set $U = V_i$ and $W = \bigoplus_{j \neq i} V_j$ with V_i constructed in Claim 4.1. So $V = U \oplus W$ equivariantly. We assume that the dimension of U is 2. If this is 1, the argument simplifies enormously and we don't detail. We define the G-equivariant stratifications

$$\begin{gathered} U = \{0\} \sqcup D^\times \sqcup U^{(2)} \\ V = (\{0\} \times_k W) \sqcup (D^\times \times_k W) \sqcup (U^{(2)} \times_k W). \end{gathered} \tag{4.3}$$

The strata are defined as follows. Write $\langle \sigma \rangle = G$. Let $F(T) \in k[T]$ be the minimal polynomial of σ as a linear map on U. Since U is irreducible, $F(T)$ is also the characteristic polynomial of σ on U. This defines the quadratic extension

$$K = k[T]/(F(T)). \tag{4.4}$$

The linear map $\sigma \otimes K \in GL(U \otimes K)$ has two conjugate eigenlines and

$$D = \{0\} \sqcup D^\times \subset U$$

is the k-irreducible curve defined by the union of the two lines. Further

$$U^{(2)} = U \setminus D.$$

By definition, G acts fixpoint free on $U^{(2)}$.

Claim 4.2. $[(U^{(2)} \times_k W)/G] = [(U^{(2)}/G) \times_k W] = [U^{(2)}/G] \cdot [W] \in K_0(\mathrm{Var}_k)$.

Proof. One has the G-equivariant projection $q : (U^{(2)} \times_k W)/G \to U^{(2)}/G$. Since $G \subset GL_k(U)$, for all points $x \in U^{(2)}$ with residue field $\kappa(x) \supset k$, one has $q^{-1}(x) \cong_{\kappa(x)} W \otimes_k \kappa(x)$. By construction, one has a fiber square

$$\begin{tikzcd} U^{(2)} \times_k W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (U^{(2)} \times_k W)/G \arrow[d, "q"] \\ U^{(2)} \arrow[r] & U^{(2)}/G. \end{tikzcd} \tag{4.5}$$

Since $U^{(2)} \to U^{(2)}/G$ is étale, q defines a local system in $H^1_{\text{ét}}(U^{(2)}/G, G_W)$ where G_W is the image of G in $GL_k(W)$. Then $(U^{(2)} \times_k W)/G$ is the total space of the torsor in $H^1_{\text{ét}}(U^{(2)}/G, GL_k(W))$ induced by $G_W \hookrightarrow GL_k(W)$. By flat descent [7, Lemma 4.10],

$$H^1_{\text{ét}}(U^{(2)}/G, GL_k(W)) = H^1_{\text{Zar}}(U^{(2)}/G, GL_k(W)).$$

Thus $(U^{(2)} \times_k W)/G \xrightarrow{q} U^{(2)}/G$, as the total space of a vector bundle, is Zariski locally trivial. We conclude

$$[(U^{(2)} \times_k W)/G] = [U^{(2)}/G] \cdot [W] \in K_0(\text{Var}_k). \tag{4.6}$$

□

So using (4.3) and Claim 4.2, we see

$$\begin{aligned}[V] - [V/G] = \big([W] - [W/G]\big) + \\ \big([D^\times \times_k W] - [(D^\times \times_k W)/G]\big) + ([U^{(2)}] - [U^{(2)}/G]) \cdot [W].\end{aligned} \tag{4.7}$$

The curve $D^\times$ is k-irreducible, but splits over K. Therefore $K \subset H^0(D^\times, \mathcal{O})$ is the algebraic closure of k and thus G acts on K.

Claim 4.3. The action of G on $\operatorname{Spec} K$ is trivial.

Proof. After the choice of a cyclic vector, σ is the matrix $\begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$ with $a, b \in k$. The curve $D^\times$ is k-affine. Its affine ring is

$$H^0(D^\times, \mathcal{O}) = k[X, Y, \frac{1}{X}]/\langle f(X,Y) \rangle$$

where the homogeneous polynomial $f(X,Y) = Y^2 - aXY - bX^2$ defines the irreducible polynomial $F(T) = T^2 - aT - b$ yielding the k-quadratic extension K. The inclusion of $K \subset H^0(D^\times, \mathcal{O})$ is k-linear and defined by $T \mapsto \frac{Y}{X}$. Furthermore, $\sigma(X) = Y,\ \sigma(Y) = bX + aY$, thus

$$\sigma(T) = \frac{\sigma(Y)}{\sigma(X)} = \frac{bX + aY}{Y} = \frac{b}{T} + a = T.$$

□

We can now analyze the second difference in (4.7). One has the G-equivariant fiber product

$$\begin{tikzcd} D^\times \times_k W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \times_k W \arrow[d] \\ D^\times \arrow[r] & \operatorname{Spec} K. \end{tikzcd}$$

Since $D^\times = \operatorname{Spec} K \times_k \mathbb{G}_m$, the morphism $D^\times \times_k W \to \operatorname{Spec} K \times_k W$ is a G-equivariant Zariski locally trivial $\mathbb{G}_m$-fibration. We first deduce

$$[D^\times \times_k W] = [\mathbb{G}_m] \cdot [\operatorname{Spec} K] \cdot [W].$$

From the induced fiber square

$$\begin{tikzcd} (D^\times \times_k W)/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (\operatorname{Spec} K \times_k W)/G \arrow[d] \\ (D^\times)/G \arrow[r] & (\operatorname{Spec} K)/G = \operatorname{Spec} K \end{tikzcd}$$

and $(D^\times)/G = (\operatorname{Spec} K \times_k \mathbb{G}_m)/G = \operatorname{Spec} K \times_k (\mathbb{G}_m/G) = \operatorname{Spec} K \times_k \mathbb{G}_m$, we deduce that $(D^\times \times_k W)/G \to (\operatorname{Spec} K \times_k W)/G$ is a Zariski locally trivial $\mathbb{G}_m$-fibration, and thus

$$[(D^\times \times_k W)/G] = [\mathbb{G}_m] \cdot [\operatorname{Spec} K] \cdot [W/G].$$

We conclude

$$[D^\times \times_k W] - [(D^\times \times_k W)/G] = [\mathbb{G}_m] \cdot [\operatorname{Spec} K] \cdot \big([W] - [W/G]\big). \tag{4.8}$$

We now analyze the third difference in (4.7). One has a G-equivariant projection $U^\times = U \setminus \{0\} \to \mathbb{P}(U)$. Here is $D^\times$ the inverse image of a K-valued point $\operatorname{Spec} K \to \mathbb{P}(U)$. On the complement, it yields the G-equivariant fibration $U^{(2)} \to \mathbb{P}(U) \setminus \operatorname{Spec} K$, which is a $\mathbb{G}_m$-bundle. So

$$[U^{(2)}] = [\mathbb{G}_m] \cdot ([\mathbb{P}(U)] - [\operatorname{Spec} K]).$$

Since $\mathbb{P}(U)/G$ is k-isomorphic to $\mathbb{P}^1_k$, the group G acts trivially on $\operatorname{Spec} K$, and $U^{(2)}/G \to (\mathbb{P}(U) \setminus \operatorname{Spec} K)/G$ is a $\mathbb{G}_m$-bundle, one has

$$\begin{aligned} [U^{(2)}/G] = [\mathbb{G}_m] \cdot ([\mathbb{P}(U)/G] - [\operatorname{Spec} K]) = \\ [\mathbb{G}_m] \cdot ([\mathbb{P}(U)] - [\operatorname{Spec} K]) = [U^{(2)}] \in K_0(\mathrm{Var}_k). \end{aligned} \tag{4.9}$$

Summing up, (4.7) reads

$$[V] - [V/G] = \big(1 + [\mathbb{G}_m] \cdot [\operatorname{Spec} K]\big) \cdot ([W] - [W/G]). \tag{4.10}$$

Now W has one less irreducible factor than V. We argue by induction on the number of irreducible factors, applying 1) to start the induction. This finishes the proof. $\square$

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