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REMARKS ON CYCLE CLASSES OF SECTIONS OF THE ARITHMETIC FUNDAMENTAL GROUP

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À Pierre Deligne, avec reconnaissance et admiration

Abstract. Given a smooth and separated $K(\pi, 1)$ variety X over a field k, we associate a "cycle class" in étale cohomology with compact supports to any continuous section of the natural map from the arithmetic fundamental group of X to the absolute Galois group of k. We discuss the algebraicity of this class in the case of curves over p-adic fields. Finally, an étale adaptation of Beilinson's geometrization of the pronilpotent completion of the topological fundamental group allows us to lift this cycle class in suitable cohomology groups.

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1. Introduction

Let X be a geometrically connected variety over a field k of characteristic 0. Denote by $\bar{k}$ an algebraic closure of k. Let $\bar{X} = X \otimes_k \bar{k}$ and $G_k = \mathrm{Gal}(\bar{k}/k)$.

Grothendieck considered in [4] the category $\mathsf{Et}(X)$ of étale covers of X (i.e., of finite étale morphisms of schemes $\pi\colon Y \to X$) and used it to define the fundamental group $\pi_1(X, x)$ of X based at a geometric point x of X as the automorphism group of the fiber functor $\mathsf{Et}(X) \to \mathsf{Sets}$, $\pi \mapsto \pi^{-1}(x)$. The structure morphism $\varepsilon\colon X \to \mathrm{Spec}(k)$ induces a map $\varepsilon_\star\colon \pi_1(X, x) \to G_k$, since $\pi_1(\mathrm{Spec}(k), x)$ is canonically isomorphic to G_k. Grothendieck proved that $\varepsilon_\star$ is onto and that its kernel identifies with $\pi_1(\bar{X}, x)$. Moreover, for any other geometric point x' of X, there exists an isomorphism $\pi_1(X, x) \simeq \pi_1(X, x')$ which is compatible with the projections to G_k. Any rational point $a \in X(k)$ therefore induces a section $a_\star\colon G_k \to \pi_1(X, x)$ of $\varepsilon_\star$, well-defined up to conjugacy by $\pi_1(\bar{X}, x)$ (indeed, a induces a canonical section of $\pi_1(X, a) \to G_k$).

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Let $G_{k(X)}$ denote the absolute Galois group of $k(X)$. The inclusion of the generic point of X induces a map $G_{k(X)} \to \pi_1(X, x)$ which is compatible with the projections to G_k and which is well-defined up to conjugacy by $\pi_1(\bar{X}, x)$. In [2, Section 15], Deligne proves that for $a \in X(k)$, the section $a_\star$ admits liftings to $G_{k(X)}$, as pictured below:

$$\begin{array}{ccc} G_{k(X)} & & \\ \downarrow & \nwarrow & \\ \pi_1(X, x) & \underset{a_\star}{\overset{\varepsilon_*}{\rightleftarrows}} & G_k \end{array}$$

In terms of Galois groups, this can be seen as follows. Assume for simplicity that X is a curve (Deligne writes "Par lassitude, nous ne traiterons que du cas où X est de dimension 1"). The choice of a local parameter t of X at a determines an isomorphism $k(X)_a \simeq k((t))$, where $k(X)_a$ denotes the completion of $k(X)$ at a; hence an embedding $k(X) \subset K$, where $K = \bigcup_{n \geqslant 1} k((t^{1/n}))$. The resulting map between absolute Galois groups $G_K \to G_{k(X)}$ then provides a lifting of $a_\star$. Indeed, the map $G_K \to G_k$ induced by the inclusion $k \subset K$ is an isomorphism.

The construction just described is easily seen to depend on the local parameter t only to order 1, i.e., it only depends on the choice of a tangent vector τ to X at a. Thus Deligne's *tangential base points* (a, τ) induce splittings of $G_{k(X)} \to G_k$. Equivalently, for any dense open $U \subseteq X$, one can say that the datum of a nonzero tangent vector τ on X at a enables one to define the fundamental group $\pi_1(U, (a, \tau))$ of U based at (a, τ) as the automorphism group of the fiber functor $\mathsf{Et}(U) \to \mathsf{Sets}$, $\pi \mapsto \pi^{-1}(\bar{\eta})$ where $\bar{\eta} = \operatorname{Spec}\big(\bigcup_{n \geqslant 1} \bar{k}((t^{1/n}))\big)$.

Deligne's tangential base points (a, τ) produce sections $G_k \to G_{k(X)}$ which factor through $G_{k(X)_a}$. For a given $a \in X(k)$, the set of all sections (up to conjugacy) which satisfy this property is naturally a torsor under the group $H^1(k, \hat{\mathbb{Z}}(1)) := \varprojlim_{n \geqslant 1} H^1(k, \boldsymbol{\mu}_n)$. Those sections which in addition come from a tangential base point form a subtorsor under the image of $H^1(k, \mathbb{Z}(1)) := k^\times$ in $H^1(k, \hat{\mathbb{Z}}(1))$; one could say that they are motivic, as opposed to profinite.

In this note, given a smooth and geometrically connected separated variety X of dimension d over a field k, under the assumption that X is a $K(\pi, 1)$ variety (see Definition 2.1 or [10, Appendix A]) we associate to any section $s\colon G_k \to \pi_1(X, x)$ a class in the étale cohomology group with compact supports $H^{2d}_c(X, \mathbb{Z}/N\mathbb{Z}(d))$ for any N invertible in k (see Theorem 2.6). We call it the cycle class of s. (Such a class had been considered by Mochizuki [9, Introduction, Structure of the Proof, (1)] in the case where X is proper and has dimension 1. Theorem 2.6 below, on the other hand, may be used to associate cycle classes, on open varieties, to sections s coming from tangential base points and even from "profinite" tangent vectors at infinity as alluded to in the previous paragraph.) The construction we give resembles for étale cohomology with torsion coefficients Bloch's decomposition of the diagonal for cohomology with nontorsion coefficients. According to the latter, if the Chow group of 0-cycles satisfies $CH_0(X \otimes_k k(X)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then, up to a cycle whose restriction to $X \otimes_k k(X)$ vanishes, some nonzero multiple of the diagonal Δ is rationally equivalent, on $X \times X$, to $\alpha \times X$ for some 0-cycle α on X. In Theorem 2.6

below, the $K(\pi, 1)$ assumption implies that up to a class whose pullback to $X \times X_s$ vanishes, where $X_s \to X$ denotes the k-form of the universal cover of $\bar{X}$ determined by s, the cohomology class of Δ is equal to $\alpha \times X$ for some $\alpha \in H_c^{2d}(X, \mathbb{Z}/N\mathbb{Z}(d))$.

If s is the section associated to a rational point $a \in X(k)$, then the cycle class of s in $H_c^{2d}(X, \mathbb{Z}/N\mathbb{Z}(d))$ coincides with the cycle class of a. As a consequence, in any situation in which one might hope that all sections come from rational points (for instance, Grothendieck's section conjecture predicts that this should be the case if X is a proper curve of genus $g \geqslant 2$ and k is a finitely generated extension of $\mathbb{Q}$), one first has to prove that the cycle class of any section s is actually the cycle class of an algebraic cycle. We consider this question in Section 3 under the assumption that X is a smooth proper curve over a p-adic field k. With these hypotheses, Koenigsmann [6] established that if $s\colon G_k \to \pi_1(X, x)$ is a section, then s comes from a rational point of X as soon as s lifts to a section of the natural map $G_{k(X)} \to G_k$ (this condition is necessary thanks to Deligne's construction). The proof builds upon model-theoretic results due to Pop. In Proposition 3.1 we show that in the situation of Koenigsmann's theorem, the liftings of the cycle class $\alpha \in H^2(X, \mathbb{Z}/N\mathbb{Z}(1))$ of s to $H_c^2(U, \mathbb{Z}/N\mathbb{Z}(1))$ given by Theorem 2.6 (for all dense open subsets $U \subseteq X$) may be used to give a direct proof of the weaker statement that α is the cycle class of a divisor of degree 1 on X (well-defined up to linear equivalence). We then investigate the algebraicity of the cycle class α of s in the absence of any birational hypothesis. Using Lichtenbaum's results about the period and the index of curves over p-adic fields, Stix [11] was able to show that if X has genus $g \geqslant 1$ and $\pi_1(X, x) \to G_k$ admits a section, then the index and the period of X are powers of p. In the remainder of Section 3, we prove that for any N which is prime to p, the cycle class of any section $s\colon G_k \to \pi_1(X, x)$ is an algebraic cycle class. From this we deduce in particular a new proof of Stix's theorem.

The last part of this note discusses a possible way to lift the cycle class of a section $s\colon G_k \to \pi_1(X, x)$ to cohomology groups which take into account the pronilpotent completion of $\pi_1(\bar{X}, x)$. Over the field of complex numbers, Beilinson [3, Section 3] constructed a cosimplicial scheme $\mathcal{P}_a(X)$ with the property that the Hopf algebra $\varinjlim_{n \geqslant 1} \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}, \mathbb{Q})$ arises from the cohomology of $\mathcal{P}_a(X)$, where $\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)$ denotes the topological fundamental group based at a point $a \in X(\mathbb{C})$ and I denotes the augmentation ideal. In Section 4 we adapt part of Beilinson's description of $\mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$ to the étale fundamental group. The purpose is to replace the rational point a by an abstract section $s\colon G_k \to \pi_1(X, x)$ of Grothendieck's arithmetic fundamental group. If X has dimension 1, this section s does not have to come from one of Deligne's tangential base points. The only assumption we need to make is that X is a $K(\pi, 1)$, for example X could be a smooth proper curve of genus $\geqslant 1$. We closely follow the topological description in [3, Section 3.3, Section 3.4]. There, the authors express in cohomological terms not the $\mathbb{Q}$-vector space $\mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$ but its dual. The latter turns out to coincide with the hypercohomology of the complex ${}_a\mathcal{K}_a$ described in *loc. cit.* It is unclear how to define an analogous complex if one replaces a by an abstract section s. However, under the $K(\pi, 1)$ assumption, one may replace a by the k-form of the universal cover of $\bar{X}$ defined by s to obtain a complex, albeit on a larger space, which gives the correct hypercohomology group (see Proposition 4.2).

If one dualizes, that is, if one comes back to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$ (see Section 4.1), one finds a complex which is more difficult to write down since it is defined only as an object of the derived category (see Definition 4.4). However, it turns out that the cohomology of this complex carries liftings of the cycle class associated to $s\colon G_k \to \pi_1(X, x)$ (see Proposition 4.8). We discuss in more detail the additional information contained in the first of these liftings.

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Notation. If M is an abelian group, m is an integer and ℓ is a prime number, we let $M \widehat{\otimes} \mathbb{Z}_\ell = \varprojlim_{n\geqslant 1} M/\ell^n M$ and $\mathrm{T}_\ell(M) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M)$, and we denote by $M[m]$ the m-torsion subgroup of M. Moreover we let $M \widehat{\otimes} \mathbb{Q}_\ell = (M \widehat{\otimes} \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\mathrm{V}_\ell(M) = \mathrm{T}_\ell(M) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

2. Cycle Classes of Sections

Let k be a field and $\bar{k}$ be a separable closure of k. We denote by X a geometrically connected separated variety over k endowed with a $\bar{k}$-point $x \in X(\bar{k})$, and we let $\bar{X} = X \otimes_k \bar{k}$.

Grothendieck [4] defines a *universal cover of* X *at* x to be a directed inverse system of connected pointed étale covers of (X, x) such that the inverse limit $\tilde{\pi}_x\colon \widetilde{X}_x \to X$ factors through every étale cover of X. (A universal cover of X at x exists and is unique up to a unique isomorphism.) The fundamental group $\pi_1(X, x)$ can be identified with the automorphism group of $\tilde{\pi}_x\colon \widetilde{X}_x \to X$. Closed subgroups of $\pi_1(X, x)$ then correspond to "sub-pro-étale covers" of $\widetilde{X}_x \to X$, that is, to factorisations $\widetilde{X}_x \to X' \to X$ of $\tilde{\pi}_x$ where X' is a connected pro-étale cover of X.

The morphism $\bar{X} \to X$ is a connected pro-étale cover. Moreover $\bar{X}$ is canonically endowed with a $\bar{k}$-point above x. Hence $\tilde{\pi}_x\colon \widetilde{X}_x \to X$ factors canonically through $\bar{X}$, so that $\widetilde{X}_x$ can be viewed as the universal cover of $\bar{X}$ at x. In general, there need not exist a pro-cover of X which, after extension of scalars from k to $\bar{k}$, becomes isomorphic to the universal cover of $\bar{X}$ at x. Such a pro-cover of X exists if and only if the natural map $\pi_1(X, x) \to G_k$ admits a (continuous, as will always be implied) section. More precisely, let $s\colon G_k \to \pi_1(X, x)$ be a section. Let us denote by $\pi_s\colon X_s \to X$ the sub-pro-étale cover of $\tilde{\pi}_x\colon \widetilde{X}_x \to X$ which corresponds to the subgroup $s(G_k) \subseteq \pi_1(X, x)$. The scheme $X_s \otimes_k \bar{k}$ is then canonically isomorphic to $\widetilde{X}_x$ over $\bar{X}$. In particular X_s is a directed inverse limit of étale covers of X each of which is geometrically connected over k. If s is the section associated (up to conjugacy by $\pi_1(\bar{X}, x)$) to a rational point $a \in X(k)$, then $X_s(k) \neq \varnothing$; more precisely, the set $X_s(k)$ then contains a canonical lifting $\tilde{a} \in X_s(k)$ of a.

Let Λ denote the ring $\mathbb{Z}/N\mathbb{Z}$ for some integer $N \geqslant 1$ which is invertible in k. For $m \in \mathbb{Z}$, we denote by $\Lambda(m)$ the m-th Tate twist of Λ (so $\Lambda(1) = \boldsymbol{\mu}_N$).

Definition 2.1. We say that the variety X *is a* $K(\pi, 1)$ if $\varinjlim H^i(Y, \Lambda) = 0$ for all $i \geqslant 1$ and all $N \geqslant 1$ such that N is invertible in k. The direct limit ranges over all factorisations $\widetilde{X}_x \to Y \to X$ of $\tilde{\pi}_x \colon \widetilde{X}_x \to X$ where Y is finite over X (for some fixed $x \in X(\bar{k})$). This property is purely geometric: it only depends on the scheme $\bar{X}$.

An extended discussion of the notion of $K(\pi, 1)$ varieties may be found in [10, Appendix A]. The equivalence between Definition 2.1 and [10, Definition A.1.2] is established in [10, Proposition A.3.1].

Examples 2.2. (i) Smooth proper curves of genus $g \geqslant 1$ as well as smooth affine curves are $K(\pi, 1)$'s.

(ii) If X is a $K(\pi, 1)$ variety, any étale cover of X is also a $K(\pi, 1)$.

(iii) According to Artin [1], on any smooth variety over a field of characteristic 0, the $K(\pi, 1)$ open subsets form a basis of the Zariski topology.

Under the assumption that X is a $K(\pi, 1)$, we shall now associate, to any section $s \colon G_k \to \pi_1(X, x)$, a "cycle class" in the étale cohomology group with compact supports $H^{2d}_c(X, \Lambda(d))$, where $d = \dim(X)$.

Definition 2.3. (i) If $\hat{V}$ is a variety over k and $V \subseteq \hat{V}$ is a dense open subset with complement D, we shall denote the étale cohomology group $H^m(\hat{V}, j_!\Lambda(n))$, where j stands for the inclusion $j \colon V \hookrightarrow \hat{V}$, by $H^m(\hat{V}, D, \Lambda(n))$. When $\hat{V}$ is a proper variety, this group depends only on V and is usually denoted $H^m_c(V, \Lambda(n))$.

(ii) If $s \colon G_k \to \pi_1(X, x)$ is a section, we shall say that an étale cover $\pi \colon Y \to X$ *appears in* $\pi_s \colon X_s \to X$ if the morphism π_s factors through π. Given an étale cover $Y \to X$ appearing in π_s, we shall say that an étale cover $Z \to Y$ *appears in* π_s if the composition $Z \to X$ appears in π_s. A property depending on an étale cover $Y \to X$ will be said to *hold if* $Y \to X$ *appears high enough in* π_s if there exists an étale cover $Y_0 \to X$ appearing in π_s such that all étale covers $Y \to X$ which appear in π_s and which factor through Y_0 satisfy the given property. Finally, given an étale cover $Y \to X$ appearing in π_s, we shall say that a property depending on an étale cover $Z \to Y$ *holds if* $Z \to Y$ *appears high enough in* π_s if there exists an étale cover $Z_0 \to Y$ appearing in π_s such that all étale covers $Z \to Y$ which appear in π_s and which factor through Z_0 satisfy the given property.

Proposition 2.4. *Let $\hat{V}$ be a variety over k and $V \subseteq \hat{V}$ be a dense open subset, with complement $D = \hat{V} \setminus V$. Let X be a geometrically connected variety over k. Let $x \in X(\bar{k})$. Let $s \colon G_k \to \pi_1(X, x)$ be a section of the natural map $\pi_1(X, x) \to G_k$. Assume X is a $K(\pi, 1)$. Then for any étale cover $\pi \colon Y \to X$ appearing high enough in π_s, the following conditions hold: for any $m \geqslant 0$ and any $n \in \mathbb{Z}$, the pullback map*

$$H^m(\hat{V}, D, \Lambda(n)) \longrightarrow H^m(\hat{V} \times Y, D \times Y, \Lambda(n))$$

induced by the first projection $\hat{V} \times Y \to \hat{V}$ *is injective and its image is equal to the image of the pullback map*

$$(1 \times \pi)^\star \colon H^m(\hat{V} \times X,\, D \times X,\, \Lambda(n)) \longrightarrow H^m(\hat{V} \times Y,\, D \times Y,\, \Lambda(n)).$$

Proof. Let j denote the inclusion $j \colon V \times Y \hookrightarrow \hat{V} \times Y$. Consider the Leray spectral sequence for the first projection $p \colon \hat{V} \times Y \to \hat{V}$ and the étale sheaf $j_!\Lambda(n)$:

$$E_2^{a,b}(Y) := H^a(\hat{V},\, R^b p_\star(j_!\Lambda(n))) \Longrightarrow E^{a+b}(Y) := H^{a+b}(\hat{V} \times Y,\, j_!\Lambda(n)) \qquad (2.1)$$

We are interested in determining the kernel and the image of the natural maps $E_2^{m,0}(Y) \to E^m(Y)$ under the assumption that Y appears high enough in π_s, since $E_2^{m,0}(Y) = H^m(\hat{V},\, D,\, \Lambda(n))$ and $E^m(Y) = H^m(\hat{V} \times Y,\, D \times Y,\, \Lambda(n))$. Let us denote by $(F^i E^m(Y))_{i \geqslant 0}$ the descending filtration on $E^m(Y)$ determined by (2.1), so that $F^i E^m(Y)/F^{i+1}E^m(Y) = E_\infty^{i,m-i}(Y)$.

Lemma 2.5. *Let* $Y \to X$ *be an étale cover appearing in* π_s*. If* $Z \to Y$ *is an étale cover which appears high enough in* π_s*, then for every* $a \geqslant 0$ *and every* $b \geqslant 1$*, the natural map* $E_2^{a,b}(Y) \to E_2^{a,b}(Z)$ *is zero.*

Proof. The étale sheaf $R^b p_\star(j_!\Lambda(n))$ on $\hat{V}$ coincides with the extension by zero from V to $\hat{V}$ of the pullback, by the structure morphism $V \to \mathrm{Spec}(k)$, of the étale sheaf defined by the G_k-module $H^b(\bar{Y},\, \Lambda(n)) = H^b(\bar{Y},\, \Lambda) \otimes_\Lambda \Lambda(n)$, where $\bar{Y} = Y \otimes_k \bar{k}$. As a consequence we need only check that there exists an étale cover $Z \to Y$ appearing in π_s such that for every $b \geqslant 1$, the pullback map $H^b(\bar{Y},\, \Lambda) \to H^b(\bar{Z},\, \Lambda)$ is zero. Since X (and therefore Y) is a $K(\pi,\, 1)$, there exists an étale cover $\bar{Z} \to \bar{Y}$ such that for every $b \geqslant 1$, the pullback map $H^b(\bar{Y},\, \Lambda) \to H^b(\bar{Z},\, \Lambda)$ is zero. After replacing $\bar{Z}$ with an étale cover of $\bar{Z}$, we may assume that the composition $\bar{Z} \to \bar{Y} \to Y$ is Galois. This implies that there exists an étale cover $Z \to Y$ which appears in π_s and whose base change to $\bar{Y}$ is $\bar{Z} \to \bar{Y}$. □

It follows from Lemma 2.5 that for any $Y \to X$ appearing in π_s and any $i < m$, if $Z \to Y$ appears high enough in π_s then the image of the natural map $F^i E^m(Y) \to F^i E^m(Z)$ is contained in $F^{i+1}E^m(Z)$. By iterating, we see that if $Y \to X$ appears high enough in π_s, then the image of the natural map $E^m(X) \to E^m(Y)$ is contained in $F^m E^m(Y)$. Now $F^m E^m(Y)$ is the image of the map $E_2^{m,0}(Y) \to E^m(Y)$. Hence the second assertion of the proposition is established.

To prove that the map $E_2^{m,0}(Y) \to E^m(Y)$ is injective if Y appears high enough in π_s, it suffices to check (by iteration again) that for every $p \geqslant 2$, the natural map $E_p^{m,0}(Y) \to E_{p+1}^{m,0}(Y)$ is injective if Y appears high enough in π_s. The kernel of the latter map is a quotient of $E_p^{m-p,p-1}(Y)$. Hence the result again follows from Lemma 2.5 (since $p - 1 \geqslant 1$). □

Construction–Theorem 2.6. *Let* X *be a smooth, geometrically connected and separated variety of dimension* d *over* k*. We assume that* X *is a* $K(\pi,\, 1)$*. Let* $x \in X(\bar{k})$ *and let* s *be a section of the natural map* $\pi_1(X,\, x) \to G_k$*. Let* $N \geqslant 1$ *be an integer invertible in* k*. Put* $\Lambda = \mathbb{Z}/N\mathbb{Z}$*. To* s *we associate a class* $\alpha(\Lambda) \in H_c^{2d}(X,\, \Lambda(d))$*. More generally, to* s *and to any étale cover* $Y \to X$ *appearing in* π_s*, we associate a class* $\alpha(\Lambda,\, Y) \in H_c^{2d}(Y,\, \Lambda(d))$*.*

All these classes are compatible as Y and Λ vary, in the sense that for any prime number ℓ which is invertible in k, they define an element $\tilde{\alpha}$ of the inverse limit

$$\varprojlim_{m\geqslant 1} \varprojlim_{Y\to X} H_c^{2d}(Y, \mathbb{Z}/\ell^m\mathbb{Z}(d))$$

where the second limit ranges over all factorisations $X_s \to Y \to X$ of π_s with Y finite over X, and where the transition morphism associated to $Z \to Y$ is the trace map $H_c^{2d}(Z, \mathbb{Z}/\ell^m\mathbb{Z}(d)) \to H_c^{2d}(Y, \mathbb{Z}/\ell^m\mathbb{Z}(d))$.

These classes have degree 1, in the sense that for any étale cover $Y \to X$ appearing in π_s, the image of $\alpha(\Lambda, Y)$ in $H_c^{2d}(\bar{Y}, \Lambda(d)) = \Lambda$ is equal to $1 \in \Lambda$.

Moreover, if s is the section associated (up to conjugacy) to a rational point $a \in X(k)$, then $\tilde{\alpha}$ is the cycle class of the rational point $\tilde{a} \in X_s(k)$.

Proof. Let $X \subseteq \hat{X}$ be a compactification of X and let $D = \hat{X} \setminus X$. Since X is smooth and separated over k, the diagonal embedding $X \hookrightarrow \hat{X} \times X$ defines a class $[\Delta] \in H^{2d}(\hat{X} \times X, D \times X, \Lambda(d))$. According to Proposition 2.4, if $\pi\colon Z \to X$ is an étale cover which appears high enough in π_s, the inverse image $(1 \times \pi)^\star[\Delta]$ of $[\Delta]$ in $H^{2d}(\hat{X} \times Z, D \times Z, \Lambda(d))$ comes, by pullback, from a unique element of $H^{2d}(\hat{X}, D, \Lambda(d)) = H_c^{2d}(X, \Lambda(d))$. It is this element which we denote $\alpha(\Lambda)$. It does not depend on the choice of Z. It maps to 1 in $H_c^{2d}(\bar{X}, \Lambda(d)) = \Lambda$ because for any $z \in Z(\bar{k})$, the class $(1\times\pi)^\star[\Delta]$ obviously maps to 1 by $H^{2d}(\hat{X}\times Z, D\times Z, \Lambda(d)) \to H^{2d}(\hat{X} \times \{z\}, D \times \{z\}, \Lambda(d)) = H_c^{2d}(\bar{X}, \Lambda(d)) = \Lambda$.

For any étale cover $Y \to X$ appearing in π_s, the section s induces a section of the natural map $\pi_1(Y, y) \to G_k$ for some $y \in Y(\bar{k})$ above x. By applying the previous construction to Y and to this new section, we therefore obtain a degree 1 class $\alpha(\Lambda, Y) \in H_c^{2d}(Y, \Lambda(d))$.

That these classes are compatible if Λ and Y are allowed to vary follows from the fact that the construction of $\alpha(\Lambda)$ can be carried out with a given étale cover $\pi\colon Z \to X$ as soon as it appears high enough in π_s.

Suppose now that s is associated to a rational point $a \in X(k)$. To prove that $\tilde{\alpha}$ is the cycle class of $\tilde{a}$, it suffices to prove that $\alpha(\Lambda)$ is the cycle class of a (one can then apply the argument to every étale cover $Y \to X$ which appears in π_s). Let $\pi\colon Z \to X$ be as in the construction of $\alpha(\Lambda)$. Let $a_Z \in Z(k)$ denote the image of $\tilde{a} \in X_s(k)$. Restriction to $\hat{X} \times \{a_Z\} \subseteq \hat{X} \times Z$ defines a map $H^{2d}(\hat{X} \times Z, D \times Z, \Lambda(d)) \to H^{2d}(\hat{X}, D, \Lambda(d))$ which is a retraction of the natural map in the other direction. Therefore $\alpha(\Lambda)$ is the image of $(1\times\pi)^\star[\Delta]$ by this map. Now $(1\times\pi)^\star[\Delta]$ is the cycle class of the graph of π; hence its restriction to $\hat{X} \times \{a_Z\}$ is the cycle class of $\pi(a_Z) = a$. □

Remark 2.7. Let us not assume that X is a $K(\pi, 1)$. Let s be a section of the natural map $G_{k(X)} \to G_k$, where $G_{k(X)}$ denotes the absolute Galois group of $k(X)$. Then s determines (up to conjugacy) a section of $\pi_1(U, u) \to G_k$ for any dense open $U \subseteq X$ and any $u \in U(\bar{k})$. In particular, by choosing U to be small enough, one may assume that U is a $K(\pi, 1)$ and apply Theorem 2.6 to produce a well-defined class $\alpha(\Lambda, U) \in H_c^{2d}(U, \Lambda(d))$. The image of $\alpha(\Lambda, U)$ in $H_c^{2d}(X, \Lambda(d))$ does not depend on the choice of U. We shall again denote it by $\alpha(\Lambda)$.

3. Algebraicity of Cycle Classes over p-Adic Fields

Building upon earlier results of Pop, Koenigsmann [6] showed that if X is a smooth and geometrically connected proper curve over a p-adic field k, then every section s of the natural map $G_{k(X)} \to G_k$ determines a unique rational point $a \in X(k)$ (in the sense that the section of $\pi_1(X, x) \to G_k$ induced by s is associated to a). Koenigsmann's proof is model-theoretic. We observe here that Theorem 2.6 gives a geometric understanding of the "abelian" part of the rational point a (i.e., of a considered as a divisor of degree 1 on X):

Proposition 3.1. *Let X be a smooth proper geometrically connected curve over a p-adic field k. Let s be a section of the natural map $G_{k(X)} \to G_k$. Let $\alpha \in H^2(X, \hat{\mathbb{Z}}(1))$ denote the inverse limit of the classes $\alpha(\mathbb{Z}/n\mathbb{Z}) \in H^2(X, \boldsymbol{\mu}_n)$ constructed in Theorem 2.6 and Remark 2.7. Then α is the cycle class of a degree 1 divisor on X (uniquely determined up to linear equivalence).*

Proof. Multiplication by n on $\mathbf{G}_\mathrm{m}$ induces an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/n\mathrm{Pic}(X) \longrightarrow H^2(X, \boldsymbol{\mu}_n) \longrightarrow \mathrm{Br}(X),$$

where $\mathrm{Br}(X) = H^2(X, \mathbf{G}_\mathrm{m})$. According to Remark 2.7, the class $\alpha(\mathbb{Z}/n\mathbb{Z})$ belongs to the image of the map $H^2_c(U, \boldsymbol{\mu}_n) \to H^2(X, \boldsymbol{\mu}_n)$ for every dense open $U \subseteq X$. Hence its image in $\mathrm{Br}(X)$ belongs to the image of $H^2_c(U, \mathbf{G}_\mathrm{m}) \to H^2(X, \mathbf{G}_\mathrm{m})$ for every dense open $U \subseteq X$. In other words the image of $\alpha(\mathbb{Z}/n\mathbb{Z})$ in $\mathrm{Br}(X)$ belongs to the right kernel of the natural pairing $\mathrm{Pic}(X) \times \mathrm{Br}(X) \to \mathrm{Br}(k)$. By Lichtenbaum–Tate duality [7] this right kernel is zero; hence finally $\alpha(\mathbb{Z}/n\mathbb{Z}) \in \mathrm{Pic}(X)/n\mathrm{Pic}(X)$. On the other hand, the image of α in $H^2(\bar{X}, \hat{\mathbb{Z}}(1)) = \hat{\mathbb{Z}}$ is equal to 1. Therefore α belongs to the inverse image of $1 \in \hat{\mathbb{Z}}$ by the degree map $\varprojlim_{n \geqslant 1}(\mathrm{Pic}(X)/n\mathrm{Pic}(X)) \to \hat{\mathbb{Z}}$, which means, since k is p-adic, that α is the image of a unique element of $\mathrm{Pic}(X)$ (see [8, I.3.3]). □

Let now X be a smooth proper geometrically connected curve of genus $g \geqslant 2$ over a p-adic field k. It is an open question whether any section $s\colon G_k \to \pi_1(X, x)$ coincides, up to conjugacy, with the section associated to a rational point of X (p-adic analogue of Grothendieck's section conjecture). For such a statement to hold, it is necessary that for any section s, the cycle class constructed in Theorem 2.6 be algebraic (i.e., be the cycle class of an algebraic cycle). Proposition 3.1 (and more generally Koenigsmann's theorem) asserts that such is the case if s admits a lifting to $G_{k(X)}$. In Corollary 3.4 below we prove that such is also the case in general provided one restricts attention to cycle classes in ℓ-adic étale cohomology with $\ell \neq p$. From this it follows (Corollary 3.6) that the existence of a section $s\colon G_k \to \pi_1(X, x)$ implies the existence, on X, of a divisor whose degree is a power of p (in other words, the index I of X is a power of p).

Corollary 3.6 was already known, see Stix [11] (moreover, Corollary 3.4 may in fact be deduced from Corollary 3.6). Quite generally, Lichtenbaum [7] proved that the period P and the index I of a curve X as above over a p-adic field k satisfy the divisibility relations $P\,|\,I\,|\,2P$ and that moreover, if $I = 2P$, then $(g-1)/P$ is an odd integer. If one believes that every section $s\colon G_k \to \pi_1(X, x)$ comes from a rational point, then one should have $I = P = 1$ as soon as a section exists. Stix [11, Theorem 2] uses Lichtenbaum's result to show, assuming the existence of a section, that $I = P$ if $p > 2$ and that in any case I and P are powers of p. The proof we give below for Corollary 3.4 and Corollary 3.6, however, does not appeal to Lichtenbaum's result; it is based on a study of the cycle classes associated to sections of $\pi_1(X, x) \to G_k$.

We henceforth denote by X a smooth proper geometrically connected $K(\pi, 1)$ variety of dimension d over a field k. For any integers m, n and N, we let $(F^i H^m(X, \Lambda(n)))_{i\geqslant 0}$, where $\Lambda = \mathbb{Z}/N\mathbb{Z}$, denote the descending filtration on $H^m(X, \Lambda(n))$ determined by the Leray spectral sequence for the structure morphism $\varepsilon\colon X \to \mathrm{Spec}(k)$ and the étale sheaf $\Lambda(n)$. Let ℓ be a prime number invertible in k. We set

$$H^m(X, \mathbb{Z}_\ell(n)) = \varprojlim_{q\geqslant 1} H^m(X, \mathbb{Z}/\ell^q\mathbb{Z}(n)),$$
$$F^i H^m(X, \mathbb{Z}_\ell(n)) = \varprojlim_{q\geqslant 1} F^i H^m(X, \mathbb{Z}/\ell^q\mathbb{Z}(n))$$

and we shall write $\mathbb{Q}_\ell$ coefficients to denote these $\mathbb{Z}_\ell$-modules tensored with $\mathbb{Q}_\ell$. Any section $s\colon G_k \to \pi_1(X, x)$ induces a retraction of $\varepsilon^\star\colon H^{2d}(k, \mathbb{Z}_\ell(d)) \to H^{2d}(X, \mathbb{Z}_\ell(d))$ according to Proposition 2.4 applied to $V = \hat{V} = \mathrm{Spec}(k)$. When a section s is given, we denote this retraction by $r\colon H^{2d}(X, \mathbb{Z}_\ell(d)) \to H^{2d}(k, \mathbb{Z}_\ell(d))$ and we let $\alpha \in H^{2d}(X, \mathbb{Z}_\ell(d))$ denote the cycle class of s (given by Theorem 2.6). We keep the notation π_s from Section 2.

Proposition 3.2. *Let $s\colon G_k \to \pi_1(X)$ be a section. Assume*

(i) *$F^1 H^{2d}(Y, \mathbb{Q}_\ell(d)) = F^{2d} H^{2d}(Y, \mathbb{Q}_\ell(d))$ for all $Y \to X$ appearing in π_s;*
(ii) *$H^{2d}(k, \mathbb{Z}_\ell(d))$ is a finitely generated free $\mathbb{Z}_\ell$-module.*

Then $r(\alpha) = 0$.

Proof. Let $\pi\colon Y \to X$ be an étale cover appearing in π_s, and let δ denote its degree. Let $\alpha_Y \in H^{2d}(Y, \mathbb{Z}_\ell(d))$ be the cycle class of s on Y (see Theorem 2.6). The retraction r factors as $r = r_Y \circ \pi^\star$ where $r_Y\colon H^{2d}(Y, \mathbb{Z}_\ell(d)) \to H^{2d}(k, \mathbb{Z}_\ell(d))$ is the retraction of $(\varepsilon \circ \pi)^\star\colon H^{2d}(k, \mathbb{Z}_\ell(d)) \to H^{2d}(Y, \mathbb{Z}_\ell(d))$ obtained by applying Proposition 2.4 to Y instead of X.

Let $\beta = \delta\alpha_Y - \pi^\star\alpha \in H^{2d}(Y, \mathbb{Z}_\ell(d))$. Since the image of α (resp. α_Y) in $H^{2d}(\bar{X}, \mathbb{Z}_\ell(d)) = \mathbb{Z}_\ell$ (resp. in $H^{2d}(\bar{Y}, \mathbb{Z}_\ell(d)) = \mathbb{Z}_\ell$) is equal to 1, one has $\beta \in F^1 H^{2d}(Y, \mathbb{Z}_\ell(d))$. Hence, by (i), the existence of an integer $N > 0$ such that $N\beta \in F^{2d} H^{2d}(Y, \mathbb{Z}_\ell(d))$. Now $F^{2d} H^{2d}(Y, \mathbb{Z}_\ell(d)) = \pi^\star\varepsilon^\star H^{2d}(k, \mathbb{Z}_\ell(d))$, so that there exists $c \in H^{2d}(k, \mathbb{Z}_\ell(d))$ such that $N\beta = \pi^\star\varepsilon^\star c$. Let us apply $\pi_\star$ and then r to this equality. One has $\pi_\star\beta = 0$ since $\pi_\star\alpha_Y = \alpha$; hence we find $\delta c = 0$. By (ii) this implies $c = 0$, so that $N\beta = 0$. It follows that $Nr_Y(\beta) = 0$ and then, by (i), that $r_Y(\beta) = 0$. In view of the equality $r(\alpha) = r_Y(\pi^\star(\alpha))$, the vanishing of $r_Y(\beta)$ means that $r(\alpha) = \delta r_Y(\alpha_Y)$. In particular $r(\alpha)$ is divisible by δ.

The integer δ can be chosen at the beginning of the argument to be divisible by arbitrarily large powers of ℓ. Therefore $r(\alpha)$ is infinitely divisible in $H^{2d}(k, \mathbb{Z}_\ell(d))$, which, by (ii), finally implies $r(\alpha) = 0$. □

Corollary 3.3. *Let X be a smooth proper geometrically connected curve of genus $g \geqslant 1$ over a p-adic field k. Let $s\colon G_k \to \pi_1(X, x)$ be a section, and ℓ be a prime number different from p. Then $r(\alpha) = 0$.*

Proof. Condition (ii) in Proposition 3.2 is satisfied since $H^2(k, \mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell$. Condition (i) amounts to the cohomology group $H^1(k, V)$ being zero, where $V = H^1(\bar{X}, \mathbb{Q}_\ell(1))$. There are several ways to see this.

One may apply Deligne's theory of weights. Let k^u be the maximal unramified extension of k, let R (resp. R^u) be the ring of integers of k (resp. k^u), let $\mathcal{X}$ be a proper regular model of X over R, and let $\bar{S}$ denote the geometric special fiber of $\mathcal{X}$. Put $\mathcal{X}^u = \mathcal{X} \otimes_R R^u$ and $X^u = X \otimes_k k^u$. The eigenvalues of the geometric Frobenius acting on $H^0(k^u, V)$ have weights in $\{-2, -1\}$ according to [5, Exp. IX, Cor. 4.4]. The kernel of the natural map $H^2(X^u, \mathbb{Q}_\ell(1)) \to H^3_{\bar{S}}(\mathcal{X}^u, \mathbb{Q}_\ell(1))$ has dimension 1 and weight 0 (since it is the cokernel of the intersection matrix of $\bar{S}$ tensored with $\mathbb{Q}_\ell$). On the other hand, $H^3_{\bar{S}}(\mathcal{X}^u, \mathbb{Q}_\ell(1))$ has weights in $\{1, 2\}$; hence $H^2(X^u, \mathbb{Q}_\ell(1))$ has weights in $\{0, 1, 2\}$ and 0 appears only with multiplicity 1. Now $H^2(X^u, \mathbb{Q}_\ell(1))$ is an extension of $\mathbb{Q}_\ell$ by $H^1(k^u, V)$. Therefore $H^1(k^u, V)$ has weights in $\{1, 2\}$. We conclude that 0 does not appear as a weight of $H^i(k^u, V)$ for any $i \in \{0, 1\}$, which implies, by inflation-restriction, that $H^1(k, V) = 0$.

One may also argue as follows. Let J denote the Jacobian variety of X. Multiplication by n on J yields a short exact sequence

$$0 \longrightarrow J(k) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \longrightarrow H^1(k, T) \longrightarrow \mathrm{T}_\ell(H^1(k, J)) \longrightarrow 0,$$

where $T = H^1(\bar{X}, \mathbb{Z}_\ell(1))$ (the notations T_ℓ and $\widehat{\otimes}$ were defined in Section 1). According to Mattuck, there exists a subgroup of finite index in $J(k)$ isomorphic to R^g (see [8, I.3.3]). The group on the left-hand side of the above exact sequence is therefore finite. On the other hand, Tate duality [8, I.3.4] implies that the group on the right-hand side is $\mathbb{Z}_\ell$-dual to $J(k)$, so that it even vanishes. Thus $H^1(k, T)$ is finite and hence $H^1(k, V) = 0$. □

From Corollary 3.3 we now deduce the main corollary of Proposition 3.2:

Corollary 3.4. *Let X be a smooth proper geometrically connected curve of genus $g \geqslant 2$ over a p-adic field k. Let $s\colon G_k \to \pi_1(X, x)$ be a section, and ℓ be a prime number different from p. Let $\alpha \in H^2(X, \mathbb{Z}_\ell(1))$ denote the cycle class of s. Then α is an algebraic cycle class, i.e., it belongs to the image of the cycle class map $\mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$.*

Proof. Kummer theory yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \longrightarrow H^2(X, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X)) \longrightarrow 0. \tag{3.1}$$

Since $T_\ell(\mathrm{Br}(X))$ is torsion-free, one may as well work with the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Q}_\ell \longrightarrow H^2(X, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell(\mathrm{Br}(X)) \longrightarrow 0; \tag{3.2}$$

it is then enough to show that $\alpha \in H^2(X, \mathbb{Q}_\ell(1))$ lies in the image of $\mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Q}_\ell$. Let $\omega \in H^2(X, \mathbb{Q}_\ell(1))$ denote the cycle class of the canonical sheaf. Then the class $\alpha - \omega/(2g-2)$ has degree 0, i.e., it vanishes in $H^2(\bar{X}, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$. Therefore it belongs to the image of $\varepsilon^\star\colon H^2(k, \mathbb{Q}_\ell(1)) \to H^2(X, \mathbb{Q}_\ell(1))$, since $H^1(k, V) = 0$ (in the notation of the proof of Corollary 3.3). Now Corollary 3.3 and Lemma 3.5 below imply that it also belongs to the kernel of $r\colon H^2(X, \mathbb{Q}_\ell(1)) \to H^2(k, \mathbb{Q}_\ell(1))$. As a consequence, it vanishes; hence $\alpha = \omega/(2g-2)$ in $H^2(X, \mathbb{Q}_\ell(1))$, which establishes the corollary. □

Lemma 3.5. *Let X be a smooth proper geometrically connected curve of genus $g \geqslant 1$ over a field k. Let $s\colon G_k \to \pi_1(X, x)$ be a section. Let $\Lambda = \mathbb{Z}/N\mathbb{Z}$, where N is invertible in k. We still denote by $\alpha \in H^2(X, \Lambda(1))$ the cycle class of s, by $\omega \in H^2(X, \Lambda(1))$ the cycle class of the canonical sheaf, and by $r\colon H^2(X, \Lambda(1)) \to H^2(k, \Lambda(1))$ the retraction induced by s. Then $r(\alpha + \omega) = 0$.*

Proof. Let $\pi\colon Y \to X$ be an étale cover appearing high enough in π_s. Denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the two projections, and by $i\colon Y \to X \times Y$ the embedding $\pi \times 1$. Let $\gamma \in H^2(X \times Y, \Lambda(1))$ be the cycle class of the graph of π. The adjunction formula reads $i^\star(\gamma + p^\star\omega + q^\star\omega_Y) = \omega_Y$, where $\omega_Y \in H^2(Y, \Lambda(1))$ denotes the cycle class of the canonical sheaf of Y. It follows that $i^\star\gamma = -\pi^\star\omega$. Now by the definition of α, one has $p^\star\alpha = \gamma$, hence $\pi^\star\alpha = i^\star\gamma$. Thus $\pi^\star(\alpha+\omega) = 0$, which proves the lemma. □

Corollary 3.6 (see Stix [11, Theorem 2])**.** *Let X be a smooth proper geometrically connected curve of genus $g \geqslant 2$ over a p-adic field k. Assume the natural map $\pi_1(X, x) \to G_k$ admits a section. Then the index of X is a power of p (i.e., there exists a divisor on X whose degree is a power of p).*

Proof. It suffices to show that for any prime number ℓ different from p, there exists a divisor on X whose degree is prime to ℓ. Fix a prime $\ell \neq p$. Choose a section $s\colon G_k \to \pi_1(X, x)$. Denote by $\alpha \in H^2(X, \mathbb{Z}_\ell(1))$ the ℓ-adic cycle class of s and by $\alpha_1 \in H^2(X, \mathbb{Z}/\ell\mathbb{Z}(1))$ the mod ℓ cycle class of s. According to Corollary 3.4, α is algebraic; therefore so is α_1. In other words $\alpha_1 \in \operatorname{Pic}(X)/\ell\operatorname{Pic}(X)$. Now any divisor on X whose class in $\operatorname{Pic}(X)/\ell\operatorname{Pic}(X)$ equals α_1 has degree congruent to 1 modulo ℓ, since the image of α_1 in $H^2(\bar{X}, \mathbb{Z}/\ell\mathbb{Z}(1)) = \mathbb{Z}/\ell\mathbb{Z}$ is equal to 1. □

Remarks 3.7. (i) Let X and s be as in Corollary 3.4. Let ℓ be a prime number (possibly equal to p). Lichtenbaum–Tate duality asserts that $\operatorname{Pic}(X) \widehat{\otimes} \mathbb{Q}_\ell$ is a maximal totally isotropic subspace of $H^2(X, \mathbb{Q}_\ell(1))$ with respect to the cup-product pairing (which takes values in $H^4(X, \mathbb{Q}_\ell(2)) = \mathbb{Q}_\ell$). If $\ell \neq p$ then this subspace is simply the line generated by ω; hence in this case, in order to prove that a class of $H^2(X, \mathbb{Q}_\ell(1))$ is algebraic, it suffices to check that it is orthogonal to ω. This is precisely what we do in the proof of Corollary 3.4. Indeed the condition $\omega \smile \alpha = 0$ is equivalent to $r(\omega) = 0$. More generally it follows from the definition of r that $r(x) = x \smile \alpha$ for any $x \in H^2(X, \mathbb{Q}_\ell(1))$.

(ii) One cannot expect the statements of Corollaries 3.4 and 3.6 to hold in the case of genus 1 curves without the assumption that $\ell \neq p$ (and in fact they fail). However, for $\ell \neq p$, these statements do hold for genus 1 curves. Indeed, in [11, Theorem 2], the genus g is only assumed to be $\geqslant 1$. Jakob Stix points out to us that the proof of Corollary 3.4 given above can be adapted to cover uniformly the $g = 1$ and $g > 1$ cases in the following way. Let $\lambda \in \operatorname{Pic}(X) \widehat{\otimes} \mathbb{Q}_\ell$ be the class which has degree 1 (when $g > 1$, it is given by $\lambda = \omega/(2g-2)$). Then $\lambda - \alpha \in H^2(X, \mathbb{Q}_\ell(1))$ has degree 0, hence it belongs to the image of $\varepsilon^\star$, hence $(\lambda - \alpha) \smile (\lambda - \alpha) = 0$. In view of Corollary 3.3 and of Remark 3.7 (i), this implies that $\alpha \smile \lambda = 0$ and hence that $\alpha = \lambda$ in $H^2(X, \mathbb{Q}_\ell(1))$.

(iii) Let X, s and ℓ be as in Corollary 3.4, and let $\mathcal{X}$ denote a proper regular model of X over the ring of integers of k. It follows from Corollary 3.4 that the ℓ-adic cycle class $\alpha \in H^2(X, \mathbb{Z}_\ell(1))$ admits liftings to $H^2(\mathcal{X}, \mathbb{Z}_\ell(1))$. Indeed the restriction map $\mathrm{Pic}(\mathcal{X}) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to \mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ is surjective, and its composition with the cycle class map $\mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$ factors through $H^2(\mathcal{X}, \mathbb{Z}_\ell(1))$. Now if the p-adic analogue of Grothendieck's section conjecture holds true, then α should even admit a canonical lifting to $H^2(\mathcal{X}, \mathbb{Z}_\ell(1))$; to wit, if s comes from a rational point $a \in X(k)$, then the cycle class in $H^2(\mathcal{X}, \mathbb{Z}_\ell(1))$ of the closure of a in $\mathcal{X}$ is a lifting of α. Starting from an arbitrary section s, we are unable to construct such a canonical lifting in general. We can do it only under the assumption that none of the irreducible components of the geometric special fiber of $\mathcal{X}$ is rational (using a variant of the arguments employed in the proof of Theorem 2.6).

(iv) Corollary 3.6 has an analogue over the real numbers. Namely, many authors have noticed that the following statement holds: let X be a smooth proper geometrically connected curve of genus $g \geqslant 1$ over the field $k = \mathbb{R}$ of real numbers; assume $\pi_1(X, x) \to G_k$ admits a section; then $X(k) \neq \varnothing$. (This is the "real analogue of the weak section conjecture". See [11, Appendix A] for a summary.) To the best of our knowledge, all previous proofs rely on nontrivial theorems in real algebraic geometry (by Witt, Artin, Verdier, or Cox). Here we remark that Tsen's theorem (according to which the Brauer group of a complex curve vanishes) suffices. Indeed it implies that $\mathrm{Br}(X)$ has exponent 2, and hence that $\mathrm{T}_2(\mathrm{Br}(X)) = 0$. Now let $s\colon G_k \to \pi_1(X, x)$ be a section. According to (3.1), the cycle class of s in $H^2(X, \mathbb{Z}_2(1))$ belongs to $\mathrm{Pic}(X) \mathbin{\widehat{\otimes}} \mathbb{Z}_2$. As in the proof of Corollary 3.6 we conclude that X has odd index and therefore that $X(k) \neq \varnothing$.

4. Nilpotent Completion and Liftings of Cycle Classes of Sections

4.1. Beilinson's construction. Let X be a complex manifold and let $a \in X$. Denote by I the augmentation ideal of the group algebra $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]$. For any $n \geqslant 1$, Beilinson constructed a complex of sheaves of $\mathbb{Q}$-vector spaces on X^n whose n-th hypercohomology group is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$. His construction is described in [3, Section 3]. We recall it briefly. Consider the manifold $X \times X^n \times X$ with coordinates $(t_0, \ldots, t_{n+1})$. For $i \in \{0, \ldots, n\}$, let A_i denote the submanifold defined by $t_i = t_{i+1}$. For $J \subseteq \{0, \ldots, n\}$, let $A_J = \bigcap_{j \in J} A_j$ and let $\mathbb{Q}_{A_J}$ denote the direct image of the constant sheaf $\mathbb{Q}$ by the inclusion $A_J \hookrightarrow X \times X^n \times X$. Let $\mathcal{B}(n)$ denote the complex of sheaves on $X \times X^n \times X$ defined by $\mathcal{B}(n)^p = \bigoplus_{J \subseteq \{0,\ldots,n\}, \#J = p} \mathbb{Q}_{A_J}$ for $p \in \{0, \ldots, n\}$ and $\mathcal{B}(n)^p = 0$ for all other p; the differential $\mathcal{B}(n)^p \to \mathcal{B}(n)^{p+1}$ is the sum, over all J and all c such that $c \notin J$, of $(-1)^{\#\{m \in J; m < c\}}$ times the restriction map $\mathbb{Q}_{A_J} \to \mathbb{Q}_{A_{J \cup \{c\}}}$. For $(b, a) \in X \times X$, let i_{ba} denote the closed immersion $X^n = \{b\} \times X^n \times \{a\} \hookrightarrow X \times X^n \times X$ and let j_{ba} denote the open immersion $X^n \setminus i_{ba}^{-1}\big(\bigcup_{i=0}^n A_i\big) \hookrightarrow X^n$. Set $\mathcal{B}(n)_{ba} = i_{ba}^\star \mathcal{B}(n)$. If $b \neq a$, then $\mathcal{B}(n)_{ba}$ is quasi-isomorphic to $j_{ba!}\mathbb{Q}$. On the other hand, if $b = a$, the defect of exactness of $\mathcal{B}(n)$ at $\mathcal{B}(n)^n$ provides a map $\mathcal{B}(n) \to \mathbb{Q}_{A_{\{0,\ldots,n\}}}[-n]$ which in hypercohomology induces a map $\theta_{aa}\colon H^n(X^n, \mathcal{B}(n)_{aa}) \to \mathbb{Q}$. Proposition 3.4 of *loc. cit.* then asserts that the $\mathbb{Q}$-vector space $H^n(X^n, \mathcal{B}(n)_{aa})$, endowed with θ_{aa}, is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$, endowed with the map $\mathbb{Q} \to \mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ which sends 1 to 1.

4.2. Replacing the base point by the universal cover. We first remark that when X is a $K(\pi, 1)$, Beilinson's construction can be reformulated in terms of the (topological) universal cover of X at a instead of the point a itself. Let X be as above. For $a \in X$, let $\tilde{\pi}_a^{\mathrm{top}}\colon (\widetilde{X}_a^{\mathrm{top}}, \tilde{a}) \to (X, a)$ denote the (topological) universal pointed cover of the pointed space (X, a). Its fiber above a is $\pi_1^{\mathrm{top}}(X, a)$.

Definition 4.1. (i) We say that X is *topologically a* $K(\pi, 1)$ if $H^i(\widetilde{X}_a^{\mathrm{top}}, \mathbb{Z}) = 0$ for all $i \geqslant 1$.

(ii) Let $\pi\colon Y \to X$ be a topological cover. Let $p\colon Y \times X^n \times Y \to X \times X^n \times X$ denote the map $\pi \times 1 \times \pi$. We put $\mathcal{B}(n)(Y) = p^\star \mathcal{B}(n)$. The defect of exactness of the complex $\mathcal{B}(n)(Y)$ at $\mathcal{B}(n)(Y)^n$ provides a map $\mathcal{B}(n)(Y) \to \iota_\star \mathbb{Q}[-n]$ where $\iota\colon Y \to Y \times X^n \times Y$ is the closed immersion $1 \times \pi^n \times 1$. We denote by $\theta_\pi\colon H^n(Y \times X^n \times Y, \mathcal{B}(n)(Y)) \to \mathbb{Q}$ the linear form it induces in hypercohomology.

Proposition 4.2. *Assume X is topologically a $K(\pi, 1)$. For any $a \in X$, the $\mathbb{Q}$-vector space $H^n(X^n, \mathcal{B}(n)_{aa})$, endowed with the linear form θ_{aa}, is canonically isomorphic to $H^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$ endowed with $\theta_{\tilde{\pi}_a^{\mathrm{top}}}$.*

Proof. Let $i\colon X^n \hookrightarrow \widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}$ denote the closed immersion $\{\tilde{a}\} \times 1 \times \{\tilde{a}\}$. The inverse image of $\mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}})$ by i is equal to $\mathcal{B}(n)_{aa}$. As a consequence, to establish the proposition it suffices to check that the restriction map

$$H^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}})) \longrightarrow H^n(X^n, i^\star \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$$

is an isomorphism. For this, in view of the definition of $\mathcal{B}(n)$, it suffices to check that the restriction map

$$H^m(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, p^\star \mathbb{Q}_{A_J}) \longrightarrow H^m(X^n, i^\star p^\star \mathbb{Q}_{A_J})$$

is an isomorphism for any m and for any $J \subsetneq \{0, \ldots, n\}$, where p is as in Definition 4.1 (ii). Now this is a direct consequence of the hypothesis that X is topologically a $K(\pi, 1)$ together with the Künneth formula. □

As a consequence, the $\mathbb{Q}$-vector space $H^n(\widetilde{X}_a^{\mathrm{top}} \times X^n \times \widetilde{X}_a^{\mathrm{top}}, \mathcal{B}(n)(\widetilde{X}_a^{\mathrm{top}}))$, endowed with $\theta_{\tilde{\pi}_a^{\mathrm{top}}}$, is canonically dual to $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$, endowed with the map $\mathbb{Q} \to \mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ which sends 1 to 1.

4.3. In the algebraic setting. Let X be a smooth geometrically irreducible separated variety of dimension d over a field k with separable closure $\bar{k}$. Let N be an integer invertible in k and let $\Lambda = \mathbb{Z}/N\mathbb{Z}$. Fix $n \geqslant 1$. For $J \subseteq \{0, \ldots, n\}$, let $A_J \subseteq X \times X^n \times X$ be, as in Section 4.1, the subvariety defined by the equations $t_j = t_{j+1}$ for $j \in J$, and let Λ_{A_J} denote the direct image of the constant étale sheaf Λ by the inclusion $A_J \hookrightarrow X \times X^n \times X$. Let $\mathcal{B}^e(n)$ (where e stands for "étale") denote the complex of étale sheaves on $X \times X^n \times X$ defined in the same way as $\mathcal{B}(n)$ was defined in Section 4.1, except that $\mathbb{Q}_{A_J}$ is now replaced by Λ_{A_J}. For $a \in X(\bar{k})$, we denote by $\mathcal{B}^e(n)_{aa}$ the inverse image of $\mathcal{B}^e(n)$ by the natural map $\bar{X}^n = \{a\} \times \bar{X}^n \times \{a\} \to X \times X^n \times X$, where $\bar{X} = X \otimes_k \bar{k}$. The natural map $\mathcal{B}^e(n) \to \Lambda_{A_{\{0,\dots,n\}}}[-n]$ induces a linear form $\theta_{aa}\colon H^n(\bar{X}^n, \mathcal{B}^e(n)_{aa}) \to \Lambda$. Finally, if $\pi\colon Y \to X$ is an étale cover, we denote by $p\colon Y \times X^n \times Y \to X \times X^n \times X$ the map $\pi \times 1 \times \pi$. Definition 4.1 (ii) still makes sense. It yields a linear form $\theta_\pi\colon H^n(Y \times X^n \times Y, \mathcal{B}^e(n)(Y)) \to \Lambda$, where $\mathcal{B}^e(n)(Y) = p^\star\mathcal{B}^e(n)$.

With these definitions in hand, we may formulate a statement analogous to Proposition 4.2. Its proof, which we omit, is also entirely analogous.

Proposition 4.3. *Assume X is a $K(\pi, 1)$. For any $a \in X(\bar{k})$, the Λ-module $H^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$, endowed with θ_{aa}, is canonically isomorphic to*

$$\varinjlim H^n(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{B}^e(n)(\bar{Y})) \tag{4.1}$$

endowed with the linear form $\varinjlim \theta_\pi$. The direct limit ranges over all factorisations $\widetilde{X}_a \to \bar{Y} \to \bar{X}$ of $\tilde{\pi}_a\colon \widetilde{X}_a \to \bar{X}$ with $\bar{Y}$ finite over $\bar{X}$.

We now turn to the Λ-module dual to $H^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$.

Definition 4.4. Let $n \geqslant 1$. We define an object $\mathcal{C}(n)$ in the bounded derived category of étale sheaves of Λ-modules on $X \times X^n \times X$ by the formula

$$\mathcal{C}(n) = \tau_{\leqslant n(2d-1)} Rj_\star\Lambda(nd)$$

where j denotes the open immersion $j\colon (X \times X^n \times X) \setminus \left(\bigcup_{i=0}^n A_i\right) \hookrightarrow X \times X^n \times X$. For any étale cover $\pi\colon Y \to X$, we set $\mathcal{C}(n)(Y) = p^\star\mathcal{C}(n)$.

For later use, we note that $R^q j_\star\Lambda(nd) = 0$ if q is not divisible by $2d-1$ and that

$$R^q j_\star\Lambda(nd) = \bigoplus_{\substack{J \subseteq \{0,\dots,n\} \\ \#J = m}} \Lambda_{A_J}((n-m)d)$$

if $q = m(2d-1)$ for some integer m. In particular there are natural distinguished triangles

$$\tau_{\leqslant (n-1)(2d-1)} Rj_\star\Lambda(nd) \longrightarrow \mathcal{C}(n) \longrightarrow \bigoplus_{\substack{J \subseteq \{0,\dots,n\} \\ \#J = n}} \Lambda_{A_J}[-n(2d-1)] \xrightarrow{+1} \tag{4.2}$$

and

$$\mathcal{C}(n) \longrightarrow Rj_\star\Lambda(nd) \longrightarrow \Lambda_{A_{\{0,\dots,n\}}}(-d)[-(n+1)(2d-1)] \xrightarrow{+1}. \tag{4.3}$$

The following proposition shows that in the algebraic setting, the n-th hypercohomology group of $\mathcal{C}(n)$ twisted by $2d$ plays a rôle analogous to that of the vector space $\mathbb{Q}[\pi_1^{\mathrm{top}}(X, a)]/I^{n+1}$ in Beilinson's original construction. We shall not pursue this analogy further. Instead, we shall use the complexes $\mathcal{C}(n)$ in Section 4.4 to define liftings of the class $\alpha(\Lambda)$ associated in Theorem 2.6 to a section of $\pi_1(X, a) \to G_k$.

Proposition 4.5. *Assume X is a $K(\pi, 1)$. For any $a \in X(\bar{k})$, the Λ-module*

$$\varprojlim H_c^{2(n+2)d-n}(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{C}(n)(\bar{Y}) \otimes_\Lambda \Lambda(2d)),$$

where the inverse limit ranges over all factorisations $\widetilde{X}_a \to \bar{Y} \to \bar{X}$ *of* $\tilde{\pi}_a$ *with* $\bar{Y}$ *finite over* $\bar{X}$*, and where the transition maps are the trace maps, is canonically dual to* $H^n(\bar{X}^n, \mathcal{B}^e(n)_{aa})$*.*

Proof. There is a canonical quasi-isomorphism $\mathcal{C}(n) = R\mathcal{H}\mathrm{om}(\mathcal{B}^e(n), \Lambda(nd))$. (More generally, applying $R\mathcal{H}\mathrm{om}(-, \Lambda(nd))$ to the distinguished triangle

$$\Lambda_{A_{\{0,\dots,n\}}}[-n-1] \longrightarrow j_!\Lambda \longrightarrow \mathcal{B}^e(n) \xrightarrow{+1}$$

yields (4.3).) Hence, by Poincaré duality, a perfect pairing of Λ-modules

$$H_c^{2(n+2)d-n}(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{C}(n)(\bar{Y})) \times H^n(\bar{Y} \times \bar{X}^n \times \bar{Y}, \mathcal{B}^e(n)(\bar{Y})) \longrightarrow \Lambda(-2d)$$

for any étale cover $\bar{Y} \to \bar{X}$. We conclude by applying Proposition 4.3. □

4.4. Liftings of the classes $\alpha(\Lambda)$. The aim of Section 4.4 is to use the definitions of Section 4.3 to construct liftings of the classes $\alpha(\Lambda)$ associated in Theorem 2.6 to sections of $\pi_1(X, x) \to G_k$. We retain the notations and hypotheses of Section 4.3. In addition, we assume throughout that X is a $K(\pi, 1)$ and we fix a section s of the natural map $\pi_1(X, x) \to G_k$ for some $x \in X(\bar{k})$. For simplicity we also assume that X is proper.

First we remark that (4.2) yields, for $n = 1$, a map

$$H^0(A_0, \Lambda) \oplus H^0(A_1, \Lambda) \longrightarrow H^{2d}(X \times X \times X, \Lambda(d))$$

and hence two classes in $H^{2d}(X \times X \times X, \Lambda(d))$. Concretely, these two classes are the classes of the algebraic cycles $A_i \subset X \times X \times X$ for $i \in \{0, 1\}$. Let c_0, c_1 denote their images in $\varinjlim H^{2d}(Y \times X \times Y, \Lambda(d))$, where the direct limit ranges over all factorisations $X_s \to Y \to X$ of $\pi_s \colon X_s \to X$ with Y finite over X. As in Proposition 2.4, we have $\varinjlim H^{2d}(Y \times X \times Y, \Lambda(d)) = H^{2d}(X, \Lambda(d))$. Now it follows from the construction of $\alpha(\Lambda)$ that c_0 and c_1 coincide, via this equality, with $\alpha(\Lambda) \in H^{2d}(X, \Lambda(d))$. Hence $\alpha(\Lambda)$ can be read off of (4.2) for $n = 1$; which begs us to also consider this triangle for $n > 1$.

Definition 4.6. Let $n \geqslant 1$. We define $\mathcal{C}'(n)$ by the formula

$$\mathcal{C}'(n) = \tau_{\leqslant n(2d-1)} Rj'_\star \Lambda(nd)$$

where j' denotes the open immersion $j' \colon (X \times X^n \times X) \setminus \left(\bigcup_{i=0}^{n-1} A_i\right) \hookrightarrow X \times X^n \times X$. For any étale cover $\pi \colon Y \to X$, we set $\mathcal{C}'(n)(Y) = p^\star \mathcal{C}'(n)$.

The complexes $\mathcal{C}(n)$ and $\mathcal{C}'(n)$ are related by a distinguished triangle

$$\mathcal{C}'(n) \longrightarrow \mathcal{C}(n) \xrightarrow{\mathrm{res}} i_\star \mathcal{C}(n-1)[-(2d-1)] \xrightarrow{+1}, \tag{4.4}$$

where i denotes the closed immersion $i \colon X \times X^{n-1} \times X \simeq A_n \hookrightarrow X \times X^n \times X$ and res is the residue map. (We take the convention that $\mathcal{C}(0) = \Lambda$.)

Definition 4.7. For any $n \geqslant 1$, we define

$$c_{1,\dots,n} \in \varinjlim H^{n(2d-1)+1}(Y \times X^n \times Y, \tau_{\leqslant (n-1)(2d-1)} p^\star Rj_\star \Lambda(nd)),$$

where the direct limit ranges over all factorisations $X_s \to Y \to X$ of $\pi_s \colon X_s \to X$ with Y finite over X, to be the class of the image of 1 by the map

$$H^0(A_{\{1,\dots,n\}}, \Lambda) \longrightarrow H^{n(2d-1)+1}(X \times X^n \times X, \tau_{\leqslant(n-1)(2d-1)} Rj_\star \Lambda(nd))$$

stemming from (4.2).

Proposition 4.8. *The class $c_{1,\dots,n}$ is a lifting of $\alpha(\Lambda) \in H^{2d}(X, \Lambda(d))$ (by an "iterated residue" map).*

Proof. Let $n \geqslant 2$. For any $Y \to X$ appearing in π_s, the residue map in (4.4), together with (4.2), gives rise to a commutative square

$$\begin{array}{ccc}
\displaystyle\bigoplus_{\substack{J \subseteq \{0,\dots,n\} \\ \#J = n}} H^0(A_J, \Lambda) & \longrightarrow & H^{n(2d-1)+1}(Y \times X^n \times Y, \tau_{\leqslant(n-1)(2d-1)} p^\star Rj_\star \Lambda(nd)) \\
\downarrow & & \downarrow \\
\displaystyle\bigoplus_{\substack{J \subseteq \{0,\dots,n-1\} \\ \#J = n-1}} H^0(A_J, \Lambda) & \longrightarrow & H^{(n-1)(2d-1)+1}(Y \times X^{n-1} \times Y, \tau_{\leqslant(n-2)(2d-1)} p^\star Rj_\star \Lambda((n-1)d)),
\end{array}$$

where the vertical map on the left sends $1 \in H^0(A_J, \Lambda)$ to $1 \in H^0(A_{J \setminus \{n\}}, \Lambda)$ if $n \in J$, to 0 otherwise, and the vertical map on the right is the residue along $Y \times X^{n-1} \times Y = p^{-1}(A_n) \subset Y \times X^n \times Y$. As a consequence, for every $n \geqslant 2$, the class $c_{1,\dots,n}$ maps to $c_{1,\dots,n-1}$ by the residue map. Since $c_1 = \alpha(\Lambda)$, this proves the proposition. □

As an example, let us consider the case $n = 2$ in more detail. The class $c_{1,2}$ lives in $L := \varinjlim H^{4d-1}(Y \times X^2 \times Y, \tau_{\leqslant 2d-1} p^\star Rj_\star \Lambda(2))$. One may consider its residues along three subvarieties of $\varprojlim(Y \times X^2 \times Y)$, namely $\varprojlim p^{-1}(A_i)$ for $i \in \{0, 1, 2\}$. These three residue maps fit into the exact sequence

$$L \longrightarrow \bigoplus_{i \in \{0,1,2\}} H^{2d}(X, \Lambda(d)) \longrightarrow H^{4d}(X^2, \Lambda(2d)) \tag{4.5}$$

obtained by applying $\varinjlim H^{4d-1}(Y \times X^2 \times Y, p^\star -)$ to the distinguished triangle

$$\tau_{\leqslant 2d-1} Rj_\star \Lambda(2d) \longrightarrow \bigoplus_{i \in \{0,1,2\}} \Lambda_{A_i}(d)[-(2d-1)] \longrightarrow \Lambda(2d)[1] \xrightarrow{+1} .$$

A simple calculation reveals that up to a sign, the image of $c_{1,2}$ in the middle group of (4.5) is $(0, -\alpha, \alpha)$ and the second map of (4.5) is given by $(x, y, z) \mapsto \alpha \boxtimes x + \Delta_\star y + z \boxtimes \alpha \in H^{4d}(X^2, \Lambda(2d))$. Here α stands for $\alpha(\Lambda)$, and $\boxtimes$ and $\Delta_\star$ respectively denote the exterior product and the Gysin map associated to the diagonal embedding $X \subset X^2$. Since the image of $c_{1,2}$ in the right-hand side group of (4.5) vanishes, we deduce that the cycle class α satisfies the nontrivial relation

$$\alpha \boxtimes \alpha = \Delta_\star \alpha \tag{4.6}$$

in $H^{4d}(X^2, \Lambda(2d))$. Hence we see that the class $c_{1,2}$ contains more information about α than c_1 alone. More generally, one might hope to use all of the liftings of α defined in Proposition 4.8 in order to "rigidify" the 0-cycle of degree 1 constructed up to linear equivalence in Proposition 3.1 (so as to show that it lies in $X(k) \subset \mathrm{Pic}^1(X)$).

Remarks 4.9. (i) Cycle classes of rational points satisfy (4.6); the situation for cycle classes in $H^{2d}(X, \Lambda(d))$ of arbitrary 0-cycles of degree 1 is less clear.

(ii) Assume X is a curve. Denote by $[\Delta] \in H^2(X^2, \Lambda(1))$ the class of the diagonal and by $\omega \in H^2(X, \Lambda(1))$ the class of the canonical sheaf. Then $\Delta_\star\alpha = p^\star\alpha \smile [\Delta]$, where $p\colon X \times X \to X$ denotes the first projection, and hence $\Delta^\star\Delta_\star\alpha = \alpha \smile (-\omega)$ by adjunction. Applying $\Delta^\star$ to (4.6) therefore yields the equality $\alpha \smile (\alpha + \omega) = 0$, which we had already encountered in Lemma 3.5 (see Remark 3.7 (i)).

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