PARTIAL CONNECTION FOR p-TORSION LINE BUNDLES IN CHARACTERISTIC $p > 0$

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To S. S. Chern, in memoriam

Abstract. The aim of this brief note is to give a construction for p-torsion line bundles in characteristic $p > 0$ which plays a similar rôle as the standard connection on an n-torsion line bundle in characteristic 0.

1. Introduction

In [3] (see also [4]) we gave an algebraic construction of characteristic classes of vector bundles with a flat connection (E, ∇) on a smooth algebraic variety X defined over a field k of characteristic 0. Their value at the generic point $\mathrm{Spec}(k(X))$ was studied and redefined in [1], and then applied in [2] to establish a Riemann-Roch formula. One way to understand Chern classes of vector bundles (without connection) is via the Grothendieck splitting principle: if the receiving groups $\oplus_n H^{2n}(X, n)$ of the classes form a cohomology theory which is a ring and is functorial in X, then via the Whitney product formula it is enough to define the first Chern class. Indeed, on the flag bundle $\pi : \mathrm{Flag}(E) \to X$, $\pi^*(E)$ acquires a complete flag $E_i \subset E_{i+1} \subset \pi^*(E)$ with E_{i+1}/E_i a line bundle, and $\pi^* : H^{2n}(X, n) \to H^{2n}(\mathrm{Flag}(E), n)$ is injective, so it is enough to construct the classes on $\mathrm{Flag}(E)$. However, if ∇ is a connection on E, $\pi^*(\nabla)$ does not stabilize the flag E_i. So the point of [3] is to show that there is a differential graded algebra $A^\bullet$ on $\mathrm{Flag}(E)$, together with a morphism of differential graded algebras $\Omega^\bullet_{\mathrm{Flag}(E)} \xrightarrow{\tau} A^\bullet$, so that $R\pi_* A^\bullet \cong \Omega^\bullet_X$ and so that the operator defined by the composition $\pi^*(E) \xrightarrow{\pi^*(\nabla)} \Omega^1_{\mathrm{Flag}(E)} \otimes_{\mathcal{O}_{\mathrm{Flag}(E)}} \pi^*(E) \xrightarrow{\tau\otimes 1} A^1 \otimes_{\mathcal{O}_{\mathrm{Flag}(E)}} \pi^*(E)$ stabilizes E_i. We call the induced operator $\nabla_i : E_i \to A^1 \otimes_{\mathcal{O}_{\mathrm{Flag}(E)}} E_i$ a (*flat*) *τ-connection*. So it is a k-linear map which fulfills the *τ-Leibniz*

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formula

$$\nabla_i(\lambda \otimes e) = \tau d(\lambda) \otimes e + \lambda \nabla_i(e) \tag{1.1}$$

for λ a local section of $\mathcal{O}_{\mathrm{Flag}(E)}$ and e a local section of E_i. It is flat when $0 = \nabla_i \circ \nabla_i \in H^0(X, A^2 \otimes_{\mathcal{O}_X} \mathcal{E}nd(E))$, with the appropriate standard sign for the derivation of forms with values in E_i. The last point is then to find the correct cohomology which does not get lost under π^*. It is a generalization of the classically defined group

$$\mathbb{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots) \tag{1.2}$$

of isomorphism classes of rank one line bundles on X with a flat connection.

A typical example of such a connection is provided by a torsion line bundle: if L is a line bundle on X which is n-torsion, that is which is endowed with an isomorphism $L^n \cong \mathcal{O}_X$, then the isomorphism yields an $\mathcal{O}_X$-étale algebra structure on $\mathcal{A} = \oplus_0^{n-1} L^i$, hence a finite étale covering $\sigma : Y = \mathrm{Spec}_{\mathcal{O}_X} \mathcal{A} \to X$, which is a principal bundle under the group scheme μ_n of n-th roots of unity, thus is Galois cyclic as soon as $\mu_n \subset k^\times$. Since the μ_n-action commutes with the differential $d_Y : \mathcal{O}_Y \to \Omega^1_Y = \sigma^* \Omega^1_X$, it defines a flat connection $\nabla_L : L \to \Omega^1_X \otimes_{\mathcal{O}_X} L$. Concretely, if $g_{\alpha,\beta} \in \mathcal{O}_X^\times$ are local algebraic transition functions for L, with trivialization

$$g^n_{\alpha,\beta} = u_\beta u_\alpha^{-1}, u_\alpha \in \mathcal{O}_X^\times, \tag{1.3}$$

then

$$(g_{\alpha,\beta}, \frac{1}{n}\frac{du_\alpha}{u_\alpha}) \in \big(\mathcal{C}^1(\mathcal{O}_X^\times) \times \mathcal{C}^0((\Omega^1_X)_{\mathrm{clsd}})\big)_{\mathrm{dlog}-\delta} \tag{1.4}$$

$$\frac{dg}{g} = \delta(\frac{du}{u})$$

is a Cech cocyle for the class

$$(L, \nabla_L) \in \mathbb{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots). \tag{1.5}$$

Clearly (1.4) is meaningless if the characteristic p of k is positive and divides n. The purpose of this short note is to give an Ersatz of this canonical construction in the spirit of the τ-connections explained above when p divides n.

2. A partial connection for p-torsion line bundles

Let X be a scheme of finite type over a perfect field k of characteristic $p > 0$. Let L be a n-torsion line bundle on X, thus endowed with an isomorphism

$$\theta : L^n \cong \mathcal{O}_X. \tag{2.1}$$

Then θ defines an $\mathcal{O}_X$-algebra structure on $\mathcal{A} = \oplus_0^{n-1} L^i$ which is étale if and only if $(p, n) = 1$. It defines the principal μ_n-covering

$$\sigma : Y = \mathrm{Spec}_{\mathcal{O}_X}\mathcal{A} \to X \tag{2.2}$$

which is étale if and only if $(p, n) = 1$, else decomposes into

$$\sigma : Y \xrightarrow{\iota} Z \xrightarrow{\sigma'} X \tag{2.3}$$

with σ' étale and ι purely inseparable. More precisely, if $n = m \cdot p^r, (m, p) = 1$, and $M = L^{p^r}$, θ defines an $\mathcal{O}_X$-étale algebra structure on $\mathcal{B} = \oplus_0^{m-1} M^i$, which defines $\sigma' : Z = \mathrm{Spec}_{\mathcal{O}_X}\mathcal{B} \to X$ as an (étale) μ_m-principal bundle. The isomorphism θ also defines an isomorphism $(L')^{p^r} \cong \mathcal{O}_Z$ as it defines the isomorphism $(\sigma')^*(M) \cong \mathcal{O}_Z$, where $L' = (\sigma')^*(L)$. So $\mathcal{C} = \oplus_0^{p^r-1}(L')^i$ becomes a finite purely inseparable $\mathcal{O}_Z$-algebra defining the principal μ_{p^r}-bundle $\iota : Y = \mathrm{Spec}_{\mathcal{O}_Z}\mathcal{C} \to Z$.

If $(n, p) = 1$, that is if $r = 0$, the formulae (1.3), (1.4) define (L, ∇) as in (1.5). We assume from now on that $(n, p) = p$. Then, as is well known, as a consequence of (1.3) one sees that the form

$$\omega_L := \frac{du_\alpha}{u_\alpha} \in \Gamma(X, \Omega^1_X)^{\mathrm{Cartier}=1}_{\mathrm{clsd}} \tag{2.4}$$

is globally defined and Cartier invariant. Let e_α be local generators of L, with transition functions $g_{\alpha,\beta}$ with $e_\alpha = g_{\alpha,\beta} e_\beta$. The isomorphism θ yields a trivialization

$$\sigma^* L \cong \mathcal{O}_Y \tag{2.5}$$

thus local units v_α on Y with

$$v_\alpha \in \mathcal{O}_Y^\times,\ g_{\alpha,\beta} = v_\beta v_\alpha^{-1} \tag{2.6}$$
$$\text{so that } 1 = v_\alpha \sigma^*(e_\alpha) = v_\beta \sigma^*(e_\beta).$$

Definition 2.1. One defines the $\mathcal{O}_X$-coherent sheaf Ω^1_L as the subsheaf of $\sigma_*\Omega^1_Y$ spanned by $\mathrm{Im}(\Omega^1_X)$ and $\frac{dv_\alpha}{v_\alpha}$.

Lemma 2.2. *Ω^1_L is well defined and one has the exact sequence*

$$0 \to \mathcal{O}_X \xrightarrow{\cdot\omega_L} \Omega^1_X \xrightarrow{\sigma^*} \Omega^1_L \xrightarrow{s} \mathcal{O}_X \to 0 \tag{2.7}$$
$$s(\frac{dv_\alpha}{v_\alpha}) = 1.$$

Proof. The relation (2.6) implies

$$\frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}} = \frac{dv_\beta}{v_\beta} - \frac{dv_\alpha}{v_\alpha} \tag{2.8}$$

$$\text{so } \frac{dv_\beta}{v_\beta} \equiv \frac{dv_\alpha}{v_\alpha} \ \in \sigma_*\Omega^1_Y/\mathrm{Im}(\Omega^1_X).$$

Hence the sheaf Ω^1_L is well defined. If e'_α is another basis, then one has $e_\alpha = w_\alpha e'_\alpha$ for local units $w_\alpha \in \mathcal{O}_X^\times$. The new v_α are then multiplied by local units in $\mathcal{O}_X^\times$, so the surjection s is well defined. It remains to see that $\mathrm{Ker}(\sigma^*) = \mathrm{Im}(\cdot\omega_L)$. By definition, on the open of X on which L has basis e_α, one has

$$Y = \mathrm{Spec}\ \mathcal{O}_X[v_\alpha]/(v_\alpha^n - u_\alpha). \tag{2.9}$$

This implies $\Omega^1_Y = \langle \mathrm{Im}(\Omega^1_X), dv_\alpha\rangle_{\mathcal{O}_Y}/\langle du_\alpha\rangle_{\mathcal{O}_Y}$ on this open and finishes the proof.

□

Remarks 2.3. 1) Assume for example that X is a smooth projective curve of genus g, and $n = p$. Recall that $0 \neq \omega_L \in \Gamma(X, \Omega^1_X)$. In particular, if $g \geq 2$, necessarily $0 \neq \Omega^1_X/\mathcal{O}_X \cdot \omega_L$ is supported in codimension 1. So Ω^1_L contains a non-trivial torsion subsheaf.

2) The sheaf Ω^1_L lies in $\sigma_*\Omega^1_Y$ but is not equal to it. Indeed, on the smooth locus of X (assuming X is reduced) the torsion free quotient of Ω^1_L has rank equal to the dimension of X, while $\sigma_*\Omega^1_Y$ has rank $n\cdot$ dimension (X) on the étale locus of σ (which is non-empty if L itself is not a p-power line bundle).

3) The class in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X)$ defined by (2.7) vanishs. Indeed, let us decompose (2.7) as an extension of $\mathcal{O}_X$ by $\Omega^1_X/\mathcal{O}_X \cdot \omega_L$, followed by an extension of $\Omega^1_X/\mathcal{O}_X \cdot \omega_L$ by $\mathcal{O}_X \cdot \omega_L$. The first extension class in $H^1(X, \Omega^1_X/\mathcal{O}_X \cdot \omega_L)$ has cocycle $\frac{dv_\beta}{v_\beta} - \frac{dv_\alpha}{v_\alpha} = \frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}}$ (see (2.8)), thus is the image of the Atiyah class of L in $H^1(X, \Omega^1_X)$. Thus the second boundary to $H^2(X, \mathcal{O}_X)$ dies.

Definition 2.4. We set $\Omega^0_L := \mathcal{O}_X$ and for $i \geq 1$ we define the $\mathcal{O}_X$-coherent sheaf Ω^i_L as the subsheaf of $\sigma_*\Omega^i_Y$ spanned by $\mathrm{Im}(\Omega^i_X)$ and $\frac{dv_\alpha}{v_\alpha} \wedge \mathrm{Im}(\Omega^{i-1}_X)$.

Proposition 2.5. *The sheaf Ω^i_L is well defined. One has an exact sequence*

$$0 \to \omega_L \wedge \Omega^{i-1}_X \to \Omega^i_X \xrightarrow{\sigma^*} \Omega^i_L \xrightarrow{s} \Omega^{i-1}_X \to 0 \tag{2.10}$$

$$s(\frac{dv_\alpha}{v_\alpha} \wedge \beta) = \beta.$$

Furthermore, the differential $\sigma_*(d_Y)$ on $\sigma_*\Omega^\bullet_Y$ induces on $\oplus_{i\geq 0}\Omega^i_L$ the structure of a differential graded algebra $(\Omega^\bullet_L, d_L)$ so that $\sigma^* : (\Omega^\bullet_X, d_X) \to (\Omega^\bullet_L, d_L)$ is a morphism of differential graded algebras.

Proof. One proves (2.10) as one does (2.7). One has to see that $\sigma_*(d_Y)$ stabilizes $\Omega^\bullet_L$. As $0 = d_X(\omega_L) \in \Omega^2_X$, $0 = d_Y(\frac{dv_\alpha}{v_\alpha}) \in \sigma_*\Omega^2_Y$, (2.10) extends to an exact sequence of complexes

$$0 \to (\omega_L \wedge \Omega^{\bullet-1}_X, -1 \wedge d_X) \to (\Omega^\bullet_X, d_X) \xrightarrow{\sigma^*} (\Omega^\bullet_L, d_L) \xrightarrow{s} (\Omega^{\bullet-1}_X, -d_X) \to 0. \tag{2.11}$$

This finishes the proof. □

Remark 2.6. As $\frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}} \in (\Omega^1_X)_{\rm clsd}$ the same proof as in Remark 2.3, 3) shows that the extension class $\mathrm{Ext}^2(\Omega^{\bullet-1}_X, \omega_L \wedge \Omega^{\bullet-1}_X)$ defined by (2.11) dies.

In order to tie up with the notations of the Introduction, we set

$$\tau = \sigma^* : \Omega^\bullet_X \to \Omega^\bullet_L. \tag{2.12}$$

Proposition 2.7. *The formula $\nabla(e_\alpha) = -\frac{dv_\alpha}{v_\alpha} \otimes e_\alpha \in \Omega^1_L \otimes_{\mathcal{O}_X} L$ defines a flat τ-connection ∇_L on L. So (L, ∇_L) is a class in $\mathbb{H}^1(X, \mathcal{O}^\times_X \xrightarrow{\tau d\log} \Omega^1_L \xrightarrow{d_L} \Omega^2_L \xrightarrow{d_L} \cdots)$, the group of isomorphism classes of line bundles with a flat τ-connection.*

Proof. Formula (2.6) implies that this defines a τ-connection. Flatness is obvious. A Cech cocycle for (L, ∇_L) is $(g_{\alpha,\beta}, \frac{dv_\alpha}{v_\alpha})$. □

Remarks 2.8. 1) The same formal definitions 2.1 and 2.4 of $\Omega^\bullet_L$ when $(n,p) = 1$ yield $(\Omega^\bullet_L, d_L) = (\Omega^\bullet_X, d_X)$, and the flat τ-connection becomes the flat connection defined in (1.4) and (1.5). So Proposition 2.7 is a direct genealization of it.

2) Let X be proper reduced over a perfect field k, irreducible in the sense that $H^0(X, \mathcal{O}_X) = k$, and admitting a rational point $x \in X(k)$. A generalization of torsion line bundles to higher rank bundles is the notion of Nori finite bundles, that is bundles E which are trivialized over principal bundle $\sigma : Y \to X$

under a finite flat group scheme G (see [6] for the original definition and also [5] for a study of those bundles). So for the n-torsion line bundles considered in this section, $G \cong \mu_n$. If the characteristic of k is 0, then again σ is étale, the differential $d_Y : \mathcal{O}_Y \to \sigma^*\Omega^1_X = \Omega^1_Y$ commutes with the action of G, inducing a connection $\nabla_E : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ and characteristic classes in our groups $\mathbb{H}^i(X, \mathcal{K}^m_i \xrightarrow{d\log} \Omega^i_X \xrightarrow{d} \Omega^{i+1}_X \cdots)$ (see [3]). If the characteristic of k is $p > 0$, then σ is étale if and only if G is smooth (which here means étale), in which case one can also construct those classes. If G is not étale, thus contains a non-trivial local subgroupscheme, then one should construct as in Proposition 2.5 a differential graded algebra $(\Omega^\bullet_E, d_E)$ with a map $(\Omega^\bullet_X, d_X) \xrightarrow{\tau} (\Omega^\bullet_E, d_E)$, so that E is endowed naturally with a flat τ-connection $\nabla_E : E \to \Omega^1_E \otimes_{\mathcal{O}_X} E$. The techniques developed in [3] should then yield classes in the groups $\mathbb{H}^i(X, \mathcal{K}^m_i \xrightarrow{\tau d\log} \Omega^i_E \xrightarrow{d_E} \Omega^{i+1}_E \cdots)$.

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