Characteristic 0 and p analogies, and some motivic cohomology

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Introduction

The purpose of this survey is to explain some recent results about analogies between characteristic 0 and characteristic $p > 0$ geometry, and to discuss an infinitesimal variant of motivic cohomology.

Homotopy invariance for motivic cohomology implies, in particular, that the Picard group of the affine line over a filed k is trivial, i.e. $\mathrm{Pic}(\mathbb{A}^1_k) = 0$. However, if instead of considering the Picard group, we consider the group of isomorphism classes of pairs $(\mathcal{L}, t)$ consisting of a line bundle $\mathcal{L}$ on $\mathbb{A}^1_k$, and an isomorphism $t : \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1}/\mathfrak{m}^n)} \xrightarrow{\cong} \mathcal{L}_{\mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1}/\mathfrak{m}^n)}$ where $\mathfrak{m}$ is the maximal ideal at the origin, then one obtains the group of 1 units in $k[[t]]/(t^n)$, that is $\{\text{isom. classes } (\mathcal{L}, t)\} \cong \big(1 + tk[[t]]/(1 + t^n k[[t]])\big)^\times$. This group is indeed a ring, namely the big ring of Witt vectors $\mathbb{W}_{n-1}(k)$ of length $n-1$ over k.

On the other hand, the theory of additive higher Chow groups, or Chow groups with modulus condition 2, developed in [10] and [11], allows to realize absolute differential forms $\Omega^i_{k/\mathbb{Z}}$ of k as a group of 0-cycles (Theorems 2.1 and 2.3). In section 2 these groups of 0-cycles and their higher modulus n generalization are presented. Theorem 2.7 – the main result of section 2 – asserts that these groups of 0-cycles with higher modulus n compute big Witt differential forms $\mathbb{W}_{n-1}\Omega^i_k$. Cutting out the p-isotypical component in characteristic $p > 0$ by suitable correspondences, and extending Theorem 2.7 to smooth local rings over a perfect field would then describe crystalline cohomology of a smooth proper variety over a perfect field of characteristic $p > 0$ as the hypercohomology of a complex of sheaves of 0-cycles.

If X is a smooth complex variety, the Riemann-Hilbert correspondence establishes an equivalence of categories between holonomic D_X-modules (coherent) and constructible sheaves (topological). Emerton and Kisin developed

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in [22] for characteristic $p > 0$ a correspondence with properties analogous to the complex theory. The "topological like" side consists of sheaves of finite dimensional $\mathbb{F}_p$-vector spaces which are locally constant in the étale topology, the "D-module like" side consists of locally finitely generated unit $\mathcal{O}_{X,F}$ modules (see Theorem 3.1), which are modules on which the Frobenius acts in a certain way (unit). Even if the unit condition is quite restrictive – as will be explaind shortly, it essentially corresponds to "slope zero" – the theory has a vaste range of applications and many objects naturally carry such a structure. For example $\mathcal{O}_X$ itself is such a unit $\mathcal{O}_{X,F}$ module. In section 3 we investigate singularities of Y (at a point $x \in Y \subseteq X$, where X is smooth) from this viewpoint and obtain striking analogies for local invariants over the complex numbers and in characteristic $p > 0$. Postponing the definition of these invariants, which were introduced by Lyubeznik in [43], the main result (Theorem 3.1) of section 3 asserts that these invariants can be expressed in terms of étale cohomology $H^i_x(Y_{\text{ét}}, \mathbb{F}_p)$ in characteristic $p > 0$ and respectively in terms of singular cohomology $H^i_x(Y_{\text{an}}, \mathbb{C})$ over the complex numbers, by virtually the same expression. This extends earlier results in the isolated singular analytic case of [43] and [33]. This extension is made possible by the similarity between the two correspondences which allows to essentially treat both settings (char. $p > 0$ and char. 0) formally as one.

In section 1, we review recent results on congruences modulo q-powers for the number of $\mathbb{F}_q$-rational points of algebraic varieties. They are all based on Deligne's philosophy which predicts a deep analogy between the level of congruences for varieties defined over $\mathbb{F}_q$ and the Hodge level for varieties over the complex numbers. If a variety has Hodge level $\geq \kappa$ over the complex numbers, one expects that "over" $\mathbb{F}_q$ it will have the same number of rational points as $\mathbb{P}^n$ modulo q^κ. Of course, the challenge is to make precise "over" as it can't be the same variety. Divisibility of eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology is one method to show the existence of congruences, slope computation in crystalline cohomology, or, in the singular case, in Berthelot's rigid cohomology is another one. Of course, ideally one would wish to prove a motivic statement which would imply all those results at once. But it is often beyond reach. The main results are Theorem 1.5, which in particular gives a positive answer to the Lang-Manin conjecture asserting that Fano varieties over a finite field have a rational point, Theorem 1.7, asserting that the mod p reduction of a regular model of a smooth projective variety defined over a local field, the ℓ-adic cohomology of which is supported in codimension 1, carries one rational point modulo q, Theorem 1.11, asserting that two theta divisors on an abelian variety defined over a finite field carry the same number of points modulo q. Theorem 1.11 answers positively the finite consequence of a conjecture of Serre, and appears as a consequence of the slope Theorem 1.10 asserting that the slope < 1 piece of rigid cohomology is computed by Witt vector cohomology. Theorem 1.5 can be proven using either ℓ-adic cohomology or crystalline cohomology. Indeed, the geometric result behind is that Fano varieties are rationally connected and therefore their Chow group of 0-cycles satisfies base change. Theorem 1.7 relies on a generalization to local fields of Deligne's integrality theorem over finite fields (Theorem 1.8).

1 Hodge type over the complex numbers and congruences for the number of rational points over a finite field

1.1 Deligne's integrality theorem over a finite field

Deligne developed the theory of weights for complex varieties via the weight filtration in his mixed Hodge theory [17], and, for varieties defined over finite fields, via the absolute values with respect to any complex embedding of the eigenvalues of the geometric Frobenius operator acting on ℓ-adic cohomology [18]. The philosophy of motives, as conceived by him and Grothendieck, predicts a closed analogy between those two concepts of weights. It has led to many very fundamental results, the first of which being Deligne's proof of the Weil conjecture for Hodge level one complete intersections and for $K3$ surfaces (Invent. math. **15** (1972)). On the other hand, Deligne shows the fundamental integrality theorem

Theorem 1.1 (Deligne [19], Corollaire 5.5.3). *Let X be a scheme of finite type defined over $\mathbb{F}_q$. Then the eigenvalues of the geometric Frobenius acting on compactly supported ℓ-adic cohomology $H^i_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ are algebraic integers.*

If X is defined over the complex numbers, its Hodge filtration F^j satisfies $\mathrm{gr}^j_F H^i_c(X) = 0$ for $j < 0$ (see Hodge III [17]). In Deligne's motivic philosophy, the integrality theorem 1.1 is analogous to the Hodge filtration starting in degrees ≥ 0 in Hodge theory. This analogy has been less studied in the past than the weight analogy. The purpose of this section is to demonstrate on some examples how the analogy works between the F-filtration in Hodge theory and the integrality over a finite field for ℓ-adic cohomology. It has led in very recent years to a series of results on congruences for the number of points on varieties defined over finite fields.

Theorem 1.1 implies

Theorem 1.2. *Let X be a scheme of finite type defined over $\mathbb{F}_q$. Then the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ are algebraic integers.*

Strictly speaking, Deligne shows this for X smooth via duality, but applying de Jong's alteration theorem, one easily reduces the theorem to the smooth case as in [20], Corollary 0.3.

1.2 Divisibility and rational points

The Grothendieck-Lefschetz trace formula [34]

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{\infty} (-1)^i \text{Trace Frobenius}|H^i_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell) \tag{1}$$

together with Theorem 1.1 implies that if the eigenvalues of the geometric Frobenius are not only algebraic integers, i.e. $\in \bar{\mathbb{Z}}$, but also, for $i \geq 1$, they are q-divisible as algebraic integers, i.e. $\in q \cdot \bar{\mathbb{Z}}$, then one has

$$|X(\mathbb{F}_q)| \equiv \dim\, H^0_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}) \bmod q. \tag{2}$$

So we conclude

$$\begin{gathered} X \text{ proper and geometrically connected} \\ \text{with eigenvalues of geom. Frob. } \in q \cdot \bar{\mathbb{Z}} \\ \Longrightarrow |X(\mathbb{F}_q)| \equiv 1 \bmod q. \end{gathered} \tag{3}$$

Purity, for which the smoothness condition is definitely necessary, together with Theorem 1.2 implies

Theorem 1.3. *Let X be a smooth scheme of finite type defined over $\mathbb{F}_q$. Then the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^i_{A \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ with supports along A are algebraic integers divisible by q^κ, if $A \subset X$ is a closed subscheme of codimension $\geq \kappa$.*

(See [26], Lemma 2.1 for the analogous proof in crystalline cohomology). If X is no longer smooth, the conclusion of Theorem 1.3 is no longer true (see [20], Remark 0.5). But it remains true for generic hyperplanes for example, as purity is then true (see [27], Theorem 2.1). In Deligne's philosophy, Theorem 1.3 is analogous to the Hodge level statement

Theorem 1.4. *Let X be a smooth scheme of finite type defined over $\mathbb{C}$. Then the graded pieces for the Hodge filtration F on de Rham cohomology with support fulfill $\mathrm{gr}^a_F H^i_A(X) = 0$ for $a < \kappa$, if $A \subset X$ is a closed subscheme of codimension $\geq \kappa$.*

The proof, based on purity, is the same as for Theorem 1.3.

1.3 Chow group of 0-cycles, coniveau and divisibility for smooth proper varieties

So if X is smooth proper geometrically connected over $\mathbb{F}_q$, Theorem 1.3 implies that (3) holds if $H^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ is supported in codimension ≥ 1 for $i \geq 1$. An equivalent terminology is to say that $H^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ has coniveau ≥ 1. According to Bloch's decomposition of the diagonal ([8], Appendix to Lecture 1), this is the case if the Chow group of 0-cycles over a field containing the field of rational functions $\mathbb{F}_q(X)$ is trivial. We conclude

Theorem 1.5 ([26], Corollary 1.2, Corollary 1.3). *Let X be a smooth, proper, geometrically connected variety over $\mathbb{F}_q$. Assume that $CH_0(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q(X)}) = \mathbb{Q}$. Then $|X(\mathbb{F}_q)| \equiv 1 \bmod q$. In particular, Fano varieties have a rational point, as conjectured by Lang [42] and Manin [45].*

Originally, Bloch decomposed the diagonal in order to recover Mumford's theorem (and its variants) asserting in its simplest form that if X is a smooth projective surface over $\mathbb{C}$, and if $\mathrm{gr}^0_F H^2(X) \neq 0$, then it can't be true that $CH_0(X)_{\deg=0} \cong \mathrm{Alb}(X)$. Furthermore, that Fano varieties, that is smooth projective geometrically irreducible varieties X so that the inverse of the dualizing sheaf ω_X^{-1} is ample, are rationally connected over any algebraically closed field, is a consequence of Mori's break and bend theory, and has been proven independently by Kollár-Miyaoka-Mori and Campana (see [41] and references there).

1.4 Singular varieties defined by equations: Ax-Katz' theorem and divisibility

If X is no longer smooth, not only one can't apply Theorem 1.3 to get divisibility of eigenvalues, but also Bloch's decomposition of the diagonal does not work. Indeed, the diagonal has only a homology class, so it does not act on cohomology, while Grothendieck-Lefschetz trace formula (1) allowing to count points needs cohomology. Yet, Deligne's philosophy on the analogy between Hodge type over $\mathbb{C}$ and eigenvalue divisibility over $\mathbb{F}_q$, is still at disposal. There is an instance where one can directly generalize the Leitfaden sketched for the proof of Theorem 1.5, by refining the motivic cohomology used. Rather than considering $CH_0(X)$ of the singular variety, one embedds $X \subset \mathbb{P}^n$ in a projective space, and considers relative motivic cohomology $H^{2n}(\mathbb{P} \times U, Y \times U, n)$ ([12]), section 1), where $\mathbb{P} \xrightarrow{\pi} \mathbb{P}^n$ is an alteration so that $Y = \pi^{-1}(X)$ is a normal crossings divisor and $U = \mathbb{P}^n \setminus X$. Then the graph of π has a cycle class in this relative motivic group and one shows ([12], Theorem 1.2) that if X is a hypersurface in $\mathbb{P}^n$ of degree $\leq n$, then this class decomposes in a suitable sense, generalizing Bloch's decomposability notion in $H^{2\dim(X)}(X, \dim(X)) \cong CH_0(X)$ when X is smooth. This implies immediately Hodge type ≥ 1 over $\mathbb{C}$ as well as eigenvalue q-divisibility over $\mathbb{F}_q$, and yields a motivic proof of Ax' theorem asserting the congruence (3) for hypersurfaces of degree $\leq n$ in $\mathbb{P}^n$ over $\mathbb{F}_q$. Ax' theorem generalizes the mod p congruence due to Chevalley-Warning.

Another instance for which one can make Deligne's philosophy work concerns closed subsets X of $\mathbb{P}^n$ defined by equations of degrees $d_1 \geq \ldots \geq d_r$, without any other assumption. Those equations could be chosen in a highly non-optimal way. For example, one could take r times the same equation. Ax and Katz in [1], [40], assign to X a level $\kappa := \max\{0, [\frac{n-d_2-\ldots-d_r}{d_1}]\}$, and show that $|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \bmod q^\kappa$. On the other hand, one can compute ([24] and [25]) that over $\mathbb{C}$, $\mathrm{gr}_F^a H_c^i(\mathbb{P}^n \setminus X) = 0$ for $a < \kappa$. Thus one expects eigenvalue divisibility. Indeed one has

Theorem 1.6 ([27], Theorem 1.1, [28], Theorem 2.1). *Let $X \subset \mathbb{P}^n$ be a closed subset defined by equations of degrees $d_1 \geq \ldots \geq d_r$. Then the eigenvalues of the geometric Frobenius acting on $H_c^i((\mathbb{P}^n \setminus X) \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ are in $q^\kappa \cdot \bar{\mathbb{Z}}$.*

According to (1), Theorem 1.6 implies Ax-Katz' theorem. However, the proof of Theorem 1.6 is not good as it uses Ax-Katz' theorem. One would like to understand a motivic proof in the spirit of [12]. We are very far from it, as, even if X is a smooth hypersurface of low degree, we do not know how to compute its Chow groups of higher dimensional cycles.

1.5 Singular varieties in families

Singular varieties which are degenerations of smooth ones have more structure. Fakhruddin and Rajan ([31], Corollary 1.2) generalize the motivic method of Theorem 1.5 in a relative situation: if $f : X \to S$ is a proper dominant morphism of smooth irreducible varieties over a finite field k with $CH_0(X \times_S \overline{k(X)}) = \mathbb{Q}$, then for any $s \in S(k)$, one has $|f^{-1}(s)| \equiv 1 \bmod |k|$. Similarly on the Hodge side one proves ([29], Theorem 1.1) that if $f : X \to S$ is a proper morphism with S a smooth connected curve and X smooth, then if $\mathrm{gr}_F^0 H^i(f^{-1}(s_0)) = 0$ for some s_0 in the smooth locus of f and all $i \geq 1$, then $\mathrm{gr}_F^0 H^i(f^{-1}(s)) = 0$ for all s and all $i \geq 1$. Those two statements, the first one in equal characteristic $p > 0$ with its strong motivic assumption, the second one in equal characteristic 0 with its minimal Hodge type assumption, suggest, using Deligne's philosophy, that the mod p reduction of a smooth projective variety in characteristic zero with $\mathrm{gr}_F^0 H^i(X) = 0$ for all $i \geq 1$ has eigenvalue q-divisibility, and therefore by (1), its number of rational points is congruent to one modulo q. One shows

Theorem 1.7 ([30], Theorem 1.1, Section 4, Proof of Theorem 1.1). *Let X be a smooth projective variety over a local field K with finite residue field $\mathbb{F}_q$. Assume $H^i(X \times_K \bar{K}, \mathbb{Q}_\ell)$ lives in codimension ≥ 1 for all $i \geq 1$. Then the eigenvalues of the geometric Frobenius acting on $H^i(Y \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ for $i \geq 1$, where Y is the mod p reduction of a projective regular model, are lying in $q \cdot \bar{\mathbb{Z}}$. In particular, $|Y(\mathbb{F}_q)| \equiv 1$ mod q.*

If the local field K has equal characteristic $p > 0$, one expects Theorem 1.7 to be optimal. However, if the local field K has unequal characteristic, one would wish, following Deligne's philosophy, to replace the assumption on the coniveau of ℓ-adic cohomology by $\mathrm{gr}^0_F H^i(X) = 0$ for all $i \geq 1$. Due to the comparison of étale and de Rham cohomology, and due to the Hodge conjecture in codimension 1, those two assumptions are equivalent for surfaces. In general, Grothendieck's generalized Hodge conjecture in codimension 1 predicts that if $\mathrm{gr}^0_F H^i(X) = 0$ then $H^i(X)$ lives in codimension ≥ 1. Thus those two conditions are expected to be equivalent in general. So in unequal characteristic, Theorem 1.7 is optimal for surfaces, but in higher dimension, in absence of a proof of the generalized Hodge conjecture in codimension 1, one would wish to have another proof.

On the other hand, a generalization of Bloch's decomposition of the diagonal implies that the motivic assumption $CH_0(X_0 \otimes_{K_0} \Omega) = \mathbb{Q}$, where $K_0 \subset K$ is a subfield of finite type over the prime field over which X is defined, i.e. $X = X_0 \times_{K_0} K$, and Ω contains $K_0(X_0)$, so for example, $\Omega = \bar{K}$ in unequal characteristic, implies the coniveau assumption of the theorem. So the motivic assumption implies the following direct corollary of Theorem 1.7: Let X be a smooth projective variety over a local field K with finite residue field $\mathbb{F}_q$. Assume $CH_0(X_0 \times_{K_0} \Omega) = \mathbb{Q}$. Then the eigenvalues of the geometric Frobenius acting on $H^i(Y \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ for $i \geq 1$, where Y is the mod p reduction of a projective regular model, are lying in $q \cdot \bar{\mathbb{Z}}$. In particular, $|Y(\mathbb{F}_q)| \equiv 1 \bmod q$ ([30], Corollary 1.2).

It is to be noted that for surfaces in characteristic zero, we are very far from knowing a positive answer to Bloch's conjecture, asserting that $\mathrm{gr}^0_F H^i(X) = 0$ for $i \geq 1$ implies the motivic condition. Thus the range of applicability of Theorem 1.7 is much larger than the one of its corollary.

The proof of Theorem 1.7 relies on the specialization map, on Gabber's purity theorem [32], Theorem 2.1.1, de Jong's alteration [16], Theorem 6.5, and on the direct generalization of Deligne's integrality theorem 1.1 to the local field case

Theorem 1.8 ([20], Theorem 0.2, Corollary 0.3). *Let X be a scheme of finite type defined over a local field K with finite residue field. Then the eigenvalue of a lifting of the geometric Frobenius in the Galois group* $\mathrm{Gal}(\bar{K}/K)$ *of the local field acting on $H^i_c(X \times_K \bar{K})$ and $H^i(X \times_K \bar{K})$ are algebraic integers for all i.*

For the proof of Theorem 1.7, one needs a form of the integrality theorem over local fields which is weaker than the one stated in Theorem 1.8, and which can be proven directly using alterations. Theorem 1.8 itself is a corollary of a more general integrality theorem for ℓ-adic sheaves.

As a corollary, one has, as in Theorem 1.3 over finite field, the divisibility statement for smooth varieties

Theorem 1.9 ([20], Corollary 0.4). *Let X be a smooth scheme of finite type defined over a local field K with finite residue field $\mathbb{F}_q$. Then the eigenvalues* *of a lifting of the geometric Frobenius in the Galois group* $\mathrm{Gal}(\bar{K}/K)$ *of the local field acting on* $H^i_{A\times_K\bar{K}}(X\times_K\bar{K},\mathbb{Q}_\ell)$ *are algebraic integers divisible by* q^κ*, if* $A\subset X$ *is a closed subscheme of codimension* $\geq\kappa$*.*

1.6 Witt vector cohomology

Intuitively, if instead of ℓ-adic cohomology, we consider crystalline or rigid cohomology, we expect a more direct link between Hodge type and slopes, and consequently congruences for the number of points if the ground field is finite. The theorem of Bloch and Illusie asserts that if X is smooth proper over a perfect field k of characteristic $p>0$, $W=W(k)$, $K=\mathrm{Frac}(W)$, then there is a functorial isomorphism ([7], III, 3.5 and [38], II, 3.5)

$$H^i(X/K)^{<1} \xrightarrow{\cong} H^i(X,W\mathcal{O}_X)_K. \tag{4}$$

Here $H^i(X/K)^{<1}$ denotes the maximal subspace of crystalline cohomology on which Frobenius acts with slopes <1, the subscript $_K$ denotes tensorisation with K, and the right hand side is Witt vector cohomology, as considered by Serre in [49]. On the other hand, over $\mathbb{C}$, one has a functorial surjective map ([24], Proposition 1.2, [29], Proof of Theorem 1.1)

$$H^i(X,\mathcal{O}_X) \xrightarrow{\text{surj}} \mathrm{gr}^F_0 H^i(X). \tag{5}$$

This gives an upper bound on the Hodge type in the sense that if the left hand side, which is usually easy to compute as this is coherent cohomology, dies, then so does the right hand side, which is a topological invariant. A weak version of (5) allows one for example to remark that if $\Theta\subset A$ is a theta divisor on an abelian variety A of dimension g over $\mathbb{C}$ (i.e. effective, ample, with $h^0(A,\mathcal{O}_A(\Theta))=1$), then one has an isomorphism

$$\mathrm{gr}^F_0 H^g_c(A\setminus\Theta) \xrightarrow{\text{iso}} \mathrm{gr}^F_0 H^g(A). \tag{6}$$

There is a generalization of (4) and (5)

Theorem 1.10 ([2], Theorem 1.1). *Let* X *be a proper scheme over a perfect field* k *of characteristic* $p>0$*, then* (4) *holds true, where the left hand side is replaced by Berthelot's rigid cohomology. So Witt vector cohomology computes the slope* <1 *piece of rigid cohomology.*

Then the finite field version of (6) asserts

Theorem 1.11 ([2], Theorem 1.4). *Let* A *be an abelian variety defined over* $\mathbb{F}_q$*, and let* Θ,Θ' *be two theta divisors. Then* $|\Theta(\mathbb{F}_q)|\equiv|\Theta'(\mathbb{F}_q)|$ *mod* q*.*

This answers positively the finite field consequence of a conjecture of Serre asserting that the difference of the motives of Θ and Θ' should be divisible by the Lefschetz motive over any field.

Remarks 1.12. As concluding remarks to this section, let us first observe that the motivic philosophy discussed here leads to various questions anchored directly in geometry. As an example, as already mentioned in [26], section 3, Gorenstein Fano varieties X in characteristic 0 fulfill $\mathrm{gr}^0_F H^i(X) = 0$ for $i \geq 1$. This suggests the existence of a good definition of rational singularities in characteristic $p > 0$ so that a Gorenstein Fano variety over $\mathbb{F}_q$ would have one rational point modulo q. This would also require a generalization of Mori's break and bend method to varieties with this type of mild singularities.

Next, let us remark that we did not discuss in this survey higher congruences, that is congruences modulo $q^\kappa, \kappa \geq 2$. Indeed, Theorem 1.3 implies that if X is projective smooth over $\mathbb{F}_q$, then if $H^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ is supported in codimension $\geq \kappa$ up to the class of the j-th self-product of the polarization if $i = 2j$, then (1) yields $|X(\mathbb{F}_q)| \equiv |\mathbb{P}^n(\mathbb{F}_q)| \bmod q^\kappa$. However, following the Leitfaden explained in subsection 1.3, the coniveau κ condition is implied by triviality of $CH_i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q(X)})$, for all $i < \kappa$, and this is a condition we can basically never check.

Finally, by Theorem 1.10, Witt vector cohomology computes the slope < 1 part of rigid cohomology. In particular, it is a topological cohomology theory. We do not know the relation between rigid and ℓ-adic cohomology if X is proper but not smooth. On the other hand, the slope $= 0$ part of crystalline cohomology, when X is smooth, is easier to understand as it is described by coherent cohomology. This viewpoint is developed in section 3.

2 Additive higher Chow groups with higher modulus of type (n,n) over a field

2.1 Additive higher Chow groups

Let k be a field and X an equidimensional k-scheme. In [9] Bloch develops a theory of higher Chow groups, which are isomorphic to the motivic cohomology groups $\mathrm{CH}^p(X, n) \cong H^{2p-n}_{\mathcal{M}}(X, \mathbb{Z}(p))$, $p, n \geq 0$ (see [51]), by using the scheme $\Delta^n = \operatorname{Spec} k[t_0, \dots, t_n]/(\sum_{i=0}^n t_i - 1)$ or, in the cubical definition, the scheme $(\mathbb{P}^1 \setminus \{1\})^n$. Replacing in the simplicial definition $\sum t_i = 1$ by $\sum t_i = \lambda$ yields the same groups as long as $\lambda \in k^\times$. The degenerate case $\lambda = 0$ is investigated in [10]. One obtains a theory of additive higher Chow groups, $\mathrm{SH}^p(X, n)$, $p \geq 0$, $n \geq 1$. In analogy to the theorem of Nesterenko-Suslin and Totaro (see [47], [50]) $\mathrm{CH}^n(k, n) \cong K^M_n(k)$, it is shown

Theorem 2.1 ([10], Theorem 5.3). *Let k be field with $\mathrm{char}\, k \neq 2$, then there is an isomorphism of groups*

$$SH^n(k, n) \cong \Omega^{n-1}_{k/\mathbb{Z}}.$$

The proof is in the spirit of the proofs in [47], [50]. We will sketch the proof for the corresponding statement with $\mathrm{SH}^n(k, n)$ replaced by a cubical version of the higher additive Chow groups. This cubical version is defined in [11], so far only for a field and on the level of 0-cycles. The definition is as follows. Consider the k-scheme $X_n = \mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{1\})^n$ with coordinates $(x, y_1, \ldots, y_n)$ and denote the union of all faces by $Y_n = \bigcup_{i=1}^n (y_i = 0, \infty)$. Now denote by $\mathrm{Z}_0(k, n-1)$ the free abelian group generated on all closed points of $\mathbb{A}^1 \setminus \{0\} \times (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-1}$. Let $\mathrm{Z}_1(k, n; 2)$ be the free abelian group generated on all irreducible curves $C \subset X_n \setminus Y_n$ satisfying the following properties

1. (*Good position*) $\partial_i^j [C] = (y_i = j).[C] \in \mathrm{Z}_0(k, n-1)$, for $i = 1, \ldots, n, j = 0, \infty$.
2. (*Modulus 2 condition*) Let $\nu : \widetilde{C} \to \mathbb{P}^1 \times (\mathbb{P}^1)^n$ be the normalization of the compactification of C, then in $\mathrm{Z}_0(\widetilde{C})$

$$2 \operatorname{div}(\nu^* x) \le \sum_{i=1}^n \operatorname{div}(\nu^* y_i - 1). \tag{7}$$

Because of (i) one has a complex

$$\partial = \sum_{i=1}^n (-1)^i (\partial_i^0 - \partial_i^\infty) : \mathrm{Z}_1(k, n; 2) \to \mathrm{Z}_0(k, n-1) \to 0.$$

Definition 2.2 ([11], Definition 6.2). The additive higher Chow groups of type (n, n) and modulus 2 of a field k are given by the homology of the above complex, i.e.

$$\mathrm{TH}^n(k, n; 2) = \frac{\mathrm{Z}_0(k, n-1)}{\partial \mathrm{Z}_1(k, n; 2)}.$$

It is shown in [11, Theorem 6.4] (cf. Theorem 2.1)

Theorem 2.3. *Let k be a field with $\operatorname{char} k \neq 2, 3$, then the map*

$$TH^n(k, n; 2) \xrightarrow{\simeq} \Omega^{n-1}_{k/\mathbb{Z}}, \tag{8}$$

$$[P] \mapsto \mathrm{Tr}_{k(P)/k} \left(\frac{1}{x(P)} \frac{dy_1(P)}{y_1(P)} \cdots \frac{dy_{n-1}(P)}{y_{n-1}(P)} \right)$$

is an isomorphism of groups. Furthermore, the inclusion $\iota : (\mathbb{P}^1 \setminus \{1\})^{n-1} \hookrightarrow X_{n-1}$, $(y_1, \ldots, y_{n-1}) \mapsto (1, y_1, \ldots, y_{n-1})$ induces a commutative diagram

$$\begin{tikzcd} K^M_{n-1}(k) \arrow[r, "\simeq"] \arrow[d, "\mathrm{dlog}"'] & CH^{n-1}(k, n-1) \arrow[d, "\iota_*"] \\ \Omega^{n-1}_{k/\mathbb{Z}} \arrow[r, "\simeq"] & TH^n(k, n; 2) \end{tikzcd}$$

defined on elements by

$$\begin{tikzcd} \{a_1,\dots,a_{n-1}\} \arrow[r, mapsto] \arrow[d, mapsto] & (a_1,\dots,a_{n-1}) \arrow[d, mapsto] \\ \frac{da_1}{a_1}\wedge\dots\wedge\frac{da_{n-1}}{a_{n-1}} \arrow[r, mapsto] & (1,a_1,\dots,a_{n-1}). \end{tikzcd}$$

The idea of the proof is the following. One first shows, that the map (8) is well defined, using the reciprocity law for rational differential forms on a non-singular projective curve. Here one explicitly uses the modulus condition (7). Then one constructs an inverse map with the help of a representation of the absolute differential forms of k by generators and relations. Showing this map to be well defined is equivalent to finding 1-cycles in $\mathrm{Z}_1(k,n;2)$, whose boundary yield the relations. This map is easily seen to be injective and the surjectivity follows from the fact, that the trace on the absolute differentials corresponds to the pushforward on the additive Chow groups.

2.2 Higher modulus

The cubical definition allows one to generalize the definition of the additive higher Chow groups (of type (n,n)) to higher modulus, i.e. replace the 2 in equation (7) by an integer $m \geq 2$. The resulting groups are denoted by $\mathrm{TH}^n(k,n;m)$. (Notice that one has $\mathrm{TH}^n(k,n;0) = \mathrm{CH}^n(\mathbb{A}^1\setminus\{0\},n-1) = 0 = \mathrm{TH}^n(k,n;1)$.) Up to now it is not clear how to formulate this higher modulus condition in the simplicial setup.

The attempt to generalize Theorem 2.3 leads to the following considerations. Let $\mathrm{Pic}(\mathbb{A}^1, m\{0\})$ be the relative Picard group of $\mathbb{A}^1$ with modulus $m\{0\}$. Then there is a natural surjective map

$$\mathrm{Pic}(\mathbb{A}^1, m\{0\}) \longrightarrow \mathrm{TH}^1(k,1;m). \tag{9}$$

For this one observes that, if two 0-cycles $\mathrm{div} f$ and $\mathrm{div} g$ in the left hand side are equal, then the curve $C \subset \mathbb{A}^1 \times \mathbb{P}^1 \setminus \{1\}$ defined by $fy = g$ satisfies the modulus condition

$$m(x=0).\bar{C} \leq (y-1).\bar{C},$$

with $\bar{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ the closure of C. Hence $\mathrm{div} f = \mathrm{div} g$ also in the right hand side. By Theorem 2.3 this is an isomorphism for $m = 2$ and one can show that it is an isomorphism for all $m \geq 2$. On the other hand we may identify $\mathrm{Pic}(\mathbb{A}^1, m\{0\})$ as a group with the additive group of the ring of big Witt vectors of length $m-1$ of k via

$$\mathrm{Pic}(\mathbb{A}^1, m\{0\}) \cong \left(\frac{1+tk[t]}{1+t^{m+1}k[t]}\right)^{\times} \cong \mathbb{W}_{m-1}(k). \tag{10}$$

This together with Theorem 2.3 leads to the prediction, that $\mathrm{TH}^n(k,n;m)$ is isomorphic to the group of generalized degree $n-1$ Witt differential forms of length $m-1$. These groups form the generalized de Rham-Witt complex of Hesselholt-Madsen generalizing the p-typical de Rham-Witt complex of Bloch-Deligne-Illusie (see [7], [38]). Before stating the generalization of Theorem 2.3 to the case of higher modulus, we describe the de Rham-Witt complex and some of his properties.

Definition 2.4 (see [36], [48], cf. [38]). Let A be a ring. A *Witt complex over* A is a projective system of differential graded $\mathbb{Z}$-algebras

$$((E_m)_{m\in\mathbb{N}}, \mathrm{R} : E_{m+1} \to E_m)$$

together with families of homomorphisms of graded rings

$$(\mathrm{F}_n : E_{nm+n-1} \to E_m)_{m,n\in\mathbb{N}}$$

and homomorphisms of graded groups

$$(V_n : E_m \to E_{nm+n-1})_{m,n\in\mathbb{N}}$$

satisfying the following relations, for all $n, r \in \mathbb{N}$

(i) $\mathrm{RF}_n = \mathrm{F}_n\mathrm{R}^n$, $\mathrm{R}^n\mathrm{V}_n = \mathrm{V}_n\mathrm{R}$, $\mathrm{F}_1 = \mathrm{V}_1 = \mathrm{id}$, $\mathrm{F}_n\mathrm{F}_r = \mathrm{F}_{nr}$, $\mathrm{V}_n\mathrm{V}_r = \mathrm{V}_{nr}$.
(ii) $\mathrm{F}_n\mathrm{V}_n = n$, and if $(n,r) = 1$, then $\mathrm{F}_r\mathrm{V}_n = \mathrm{V}_n\mathrm{F}_r$ on E_{rm+r-1}.
(iii) $\mathrm{V}_n(\mathrm{F}_n(x)y) = x\mathrm{V}_n(y)$, for $x \in E_{nm+n-1}$, $y \in E_m$.
(iv) $\mathrm{F}_n d\mathrm{V}_n = d$, with d the differential on E_m and E_{nm+n-1} respectively.

Furthermore, there is a homomorphism of projective systems of rings

$$(\lambda : \mathbb{W}_m(A) \to E^0_m)_{m\in\mathbb{N}},$$

under which the Frobenius and the Verschiebung maps on the big Witt vectors correspond to F_n and V_n, $n \in \mathbb{N}$ on E^0 and satisfies

(v) $\mathrm{F}_n d\lambda([a]) = \lambda([a]^{n-1})d\lambda([a])$, for $a \in A$.

A morphism of Witt complexes over A is a morphism of projective systems of dga's compatible with all the structures. This yields a category of Witt complexes over A.

Theorem 2.5 (see [36], [48], cf. [38]). *The category of Witt complexes has an initial object, called the de Rham-Witt complex of A and denoted by* $(\mathbb{W}_m\Omega^{\cdot}_A)_{m\in N}$.

Hesselholt-Madsen prove this using the Freyd adjoint functor theorem. In case A is a $\mathbb{Z}_{(p)}$-algebra, $p \neq 2$ a prime, $\mathbb{W}_m\Omega^{\cdot}_A$ may be constructed following Illusie as a quotient of $\Omega^{\cdot}_{\mathbb{W}_m(A)}$. It follows

$$\mathbb{W}_1\Omega^{\cdot}_A = \Omega^{\cdot}_{A/\mathbb{Z}} \quad \mathbb{W}_m\Omega^0_A = \mathbb{W}_m(A).$$

Remark 2.6. Let A be a $\mathbb{Z}_{(p)}$-algebra, $p \neq 2$ a prime, and denote by $(\mathrm{W}_n \Omega^{\cdot}_A)_{n\in\mathbb{N}_0}$ the p-typical de Rham-Witt complex of Bloch-Deligne-Illusie-Hesselholt-Madsen, then one has

$$\mathbb{W}_m \Omega^{\cdot}_A \cong \prod_{(j,p)=1} \mathrm{W}_{n(j)} \Omega^{\cdot}_A, \quad n(j) \text{ given by } jp^{n(j)} \leq m < jp^{n(j)+1}. \tag{11}$$

If X is a smooth variety over a perfect field, Bloch and Illusie also define $\mathrm{W}_n \Omega^{\cdot}_X$, which is a complex of coherent sheaves on the scheme $\mathrm{W}_n(X)$. Its hypercohomology equals the crystalline cohomology. This is used for example to derive (4).

The generalization of Theorem 2.3 to the case of higher modulus is given by the

Theorem 2.7 ([48]). *Let k be a field with* char $\neq 2$*. Then the projective system $(\bigoplus_{n\geq 1} TH^n(k,n;m+1) \to \bigoplus_{n\geq 1} TH^n(k,n;m))_{m\geq 2}$ can be equipped with the structure of a Witt complex over k and the natural map $\mathbb{W}_* \Omega^{\cdot}_k \to \bigoplus_{n\geq 0} TH^{n+1}(k,n+1;*+1)$ induced by the universality of the de Rham-Witt complex is an isomorphism. In particular*

$$\mathbb{W}_{m-1} \Omega^{n-1}_k \cong TH^n(k,n;m).$$

In the following the Witt complex structure of

$$T_m := \bigoplus_{n\geq 0} \mathrm{TH}^{n+1}(k,n+1;m+1)$$

is described. The multiplication of a graded commutative ring on T_m is induced by the exterior product of cycles followed by a pushforward, which is induced by the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. The differential is induced by pushing forward via the diagonal $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ and then restricting to $\mathbb{A}^1 \times \mathbb{P}^1 \setminus \{1\}$. F_r (resp. V_r) is induced by pushing forward (resp. pulling back) via $\mathbb{A}^1 \to \mathbb{A}^1$, $a \to a^r$. And finally the map $\mathbb{W}_m(k) \to T^0_m = \mathrm{TH}^1(k,1;m+1)$ is given by the composition of (10) and (9). The relations (i)-(iii) and (v) in Definition 2.4 are already satisfied on the level of cycles. That T_m is a dga with (iv), follows from the result of Nesterenko-Suslin and Totaro, since one has a surjective map

$$\bigoplus_{P_0\in\mathbb{A}^1\setminus\{0\}} CH^{n-1}(k(P_0),n-1) \to \mathrm{TH}^n(k,n;m)$$

induced by the inclusions $\{P_0\} \times (\mathbb{P}^1 \setminus \{1\})^{n-1} \hookrightarrow \mathbb{A}^1 \setminus \{0\} \times (\mathbb{P}^1 \setminus \{1\})^{n-1}$. Thus T is a Witt complex and this gives the natural map from Theorem 2.7. In [48] a trace map for arbitrary field extensions on the de Rham-Witt groups is constructed as well as a residue symbol for closed points of a smooth projective curve on the rational Witt differentials of this curve and a reciprocity law is proven, generalizing the corresponding notions and statements on Kähler differentials (and the one obtained by Witt in [53]). Using this results one can generalize the proof of Theorem 2.3 to obtain, that the map

$$\begin{gathered}\mathrm{TH}^n(k,n;m) \to \mathbb{W}_{m-1}\Omega_k^{n-1}\\ [P] \mapsto \mathrm{Tr}_{k(P)/k}\left(\frac{1}{[x(P)]}\frac{d[y_1(P)]}{[y_1(P)]}\cdots\frac{d[y_{n-1}(P)]}{[y_{n-1}(P)]}\right)\end{gathered}$$

gives the inverse map to the one obtained by the universality of the de Rham-Witt complex. Here $[-] : k(P) \to \mathbb{W}_{m-1}(k(P))$ is the Teichmüller lift.

Finally we explain how to describe the p-typical de Rham-Witt complex over a field k via the additive higher Chow groups. Denote by

$$f = \exp(\sum_{i=0}^{\infty} -\frac{t^{p^i}}{p^i}) \in 1 + t\mathbb{Z}_{(p)}[[t]]$$

the inverse of the Artin-Hasse eponential and by

$$f_m \in (1 + \oplus_{i=1}^m t^i\mathbb{Z}_{(p)})$$

the truncation of f. Write ϵ_m for the 0-cycle $[\mathrm{div} f_m] \in \mathrm{TH}^1(k,1;m+1)$. Then it follows from the description of the additive higher Chow groups as the de Rham-Witt complex and from (11), that one has

$$\mathrm{W}_r\Omega_k^{n-1} \cong \mathrm{TH}^n(k,n;p^r+1) * \epsilon_{p^r},$$

where we denote by $*$ the multiplication in the additive higher Chow groups explained above.

3 Riemann–Hilbert type correspondences and applications to local cohomology invariants

This section starts with a very rough introduction to some aspects of the Riemann–Hilbert correspondence (over $\mathbb{C}$) and, shortly thereafter, to a positive characteristic analog, developed recently by Emerton and Kisin [22]. We start at a basic level – merely motivating the correspondence with the help of a fundamental example – but progress quickly to a nontrivial construction which is central to our applications: the intermediate extension.

The aim is to show how these correspondences can be put to good use in order to study singularities. Concretely we obtain a new characterization of invariants arising from local cohomology in terms of étale cohomology. One central interesting aspect of our approach is that the treatment is, up to the use of the respective correspondence, independent of the characteristic.

The result we will discuss is a description of Lyubeznik's local cohomology invariants in any characteristic [43] which are originally defined for a quotient $A = R/I$ of a n-dimensional regular local ring $(R, \mathfrak{m})$ to be

$$\lambda_{a,i} \stackrel{\text{def}}{=} e(H^a_{\mathfrak{m}}(H^{n-i}_I(R)))$$

where $e(\_\_)$ denotes the D–module multiplicity (see Section 3.2). In [43] these are shown to be independent of the representation of A as the quotient of a regular local ring. In [4] and [5] these invariants were described in many cases in terms of étale cohomology; we discuss these results here as an application of the aforementioned correspondences:

Theorem 3.1. *Let $A = \mathcal{O}_{Y,x}$ for Y a closed k–subvariety of a smooth variety X. If for $i \neq d$ the modules $H^{n-i}_{[Y]}(\mathcal{O}_X)$ are supported at the point x then*

1. *For $2 \leq a \leq d$ one has*

$$\lambda_{a,d}(A) - \delta_{a,d} = \lambda_{0,d-a+1}(A)$$

and all other $\lambda_{a,i}(A)$ vanish.

2. $$\lambda_{a,d}(A) - \delta_{a,d} = \begin{cases} \dim_{\mathbb{F}_p} H^{d-a+1}_{\{x\}}(Y_{\text{ét}}, \mathbb{F}_p) & \text{if } \operatorname{char} k = p \\ \dim_{\mathbb{C}} H^{d-a+1}_{\{x\}}(Y_{\text{an}}, \mathbb{C}) & \text{if } k = \mathbb{C} \end{cases}$$

where $\delta_{a,d}$ is the Kroneker delta function.

The apparent analogy between the situation over $\mathbb{C}$ and over $\mathbb{F}_p$ suggested by this result is somewhat misleading. As we briefly discuss at the end of this section, étale cohomology with $\mathbb{F}_p$–coefficients is only a very small part (the slope zero part) of, say, crystalline cohomology. For crystalline cohomology on the other hand there are comparison results to singular cohomology (via de Rham theory) hence Lyubeznik's invariants really capture a different type of information in characteristic 0 than they do in characteristic $p > 0$.

3.1 Riemann–Hilbert and Emerton–Kisin correspondence

Let us first fix the following notation: Throughout this section X will be a smooth scheme over a field k of dimension n. Mostly k will be either $\mathbb{C}$, the field of complex numbers, or $\mathbb{F}_q$, the finite field with $q = p^e$ elements, as in Section 1.

The Riemann–Hilbert correspondence

Now let $k = \mathbb{C}$. In its simplest incarnation the Riemann–Hilbert correspondence asserts a one to one map between

$$\left\{\begin{matrix} \text{local systems of} \\ \mathbb{C}\text{-vectorspaces} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{locally free coherent } \mathcal{O}_X\text{–modules} \\ \text{with regular singular integrable connection} \end{matrix}\right\}$$

This correspondence grew out of Hilbert's 21st problem, motivated by work of Riemann, to find, given a monodromy action at some points, a Fuchsian (regular singular) differential equation with the prescribed monodromy action at the singular points.

The correspondence is given via de Rham theory: A connection is a k–linear map $\nabla : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ that satisfies the Leibniz rule $\nabla(rm) = r\nabla(m) + dr \otimes m$. One can extend ∇ in an (up to signs) obvious way to get a sequence of maps (each denoted also by ∇)

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{M} \xrightarrow{\nabla} \Omega^2_X \otimes \mathcal{M} \xrightarrow{\nabla} \ldots \longrightarrow \Omega^n_X \otimes \mathcal{M}$$

and the connection is called integral if this sequence is a complex (i.e. $\nabla^2 = 0$), called the de Rham complex $\mathrm{dR}(\mathcal{M})$ associated to the connection ∇. Now, given such integrable connection its *horizontal sections*

$$\mathcal{M}^{\nabla} = \ker \nabla = H^0(\mathrm{dR}(\mathcal{M}))$$

is a local system.

The most trivial, but at the same time most important example, is $\mathcal{M} = \mathcal{O}_X$ with $\nabla = d$ the universal differential $\mathcal{O}_X \xrightarrow{d} \Omega^1_X$. This is the $\mathcal{O}_X$–module associated to the system of differential equations $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \cdot f = 0$ where $x_1, \ldots, x_n$ are local coordinates at some point of X. The corresponding local system (the solutions to the differential equation) is of course the constant local system $\mathbb{C}$. In fact more is true: By the Poincaré Lemma, the (analytic) de Rham complex is a resolution of the constant sheaf $\mathbb{C}$ and hence we can rephrase this by saying that $\mathbb{C}$ is quasi-isomorphic to the de Rham complex.

In view of Grothendieck's philosophy the above correspondence is flawed since the categories on either side are not closed under any reasonable functors. For example, the pushforward of a local system is generally not a local system as the inclusion of a point $\{x\} \xrightarrow{i} \mathbb{A}^1$ readily illustrates ($i_*\mathbb{C}$ is a skyscraper sheaf). However it is a constructible sheaf (that is one that is locally constant on each piece of a suitable stratification of X), and in fact on constructible sheaves all the functors one would like to have are defined.

On the other side of the correspondence, modules with integrable connection are replaced by modules over the ring of differential operators D_X. The conditions one has to impose are holonomicity (which is the crucial finiteness condition) and a further condition (namely that $\mathcal{M}$ is regular singualar) which will not be considered here. Without giving the precise definition, a holonomic D_X–module is one of minimal possible dimension, and the category they form enjoys a strong finiteness condition:

Proposition 3.2 ([15]). *The category of holonomic D_X–modules is abelian and closed under extensions. Every holonomic D_X module has finite length.*

The correspondence should again be given via de Rham theory. However, it quickly becomes clear that one has to pass to the derived category. We give the following simple example as a small indication that the passage to the derived category cannot be avoided.

Example 3.3. Let $X = \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[x]$ such that $D_X = k[x, \frac{\partial}{\partial x}]$. Let $\mathcal{M} = H^1_{\{0\}}(\mathcal{O}_{\mathbb{A}^1})$ denote the cokernel of the following injection of D_X–modules

$$\mathbb{C}[x] \longrightarrow \mathbb{C}[x, x^{-1}].$$

Then, as a $\mathbb{C}$-vectorspace $\mathcal{M}$ has a basis consisting of $\frac{1}{x^i}$ for $i > 0$. The de Rham complex associated to $\mathcal{M}$ is given

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_{\mathbb{A}^1} \otimes \mathcal{M}$$

where the map sends a basis element $\frac{1}{x^i} \longrightarrow dx \otimes \frac{-i}{x^{i+1}}$. Hence one immediately sees that ∇ is injective and that its cokernel is generated by $dx \otimes \frac{1}{x}$, which is the skyscraper sheaf supported at 0. Hence we have an quasi-isomorphism

$$i_*\mathbb{C}_{\{0\}} \simeq \mathrm{dR}(H^1_{\{0\}}(\mathcal{O}_{\mathbb{A}^1}))[1]. \tag{12}$$

Note the appearance of the shift [1] which is an indication that one cannot avoid to pass to the derived category.

The ultimate generalization of the basic version above is the Riemann–Hilbert correspondence as proved by Mebkhout [46], Kashiwara [39], Beilinson and Bernstein [15]:

Theorem 3.4. *Let X be a smooth $\mathbb{C}$–variety. On the level of bounded derived categories, there is an equivalence between*

$$\left\{\begin{matrix}\textit{constructible sheaves of}\\ \mathbb{C}\textit{-vectorspaces}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{holonomic } D_X\textit{–modules}\\ \textit{which are regular singular}\end{matrix}\right\}$$

The correspondence is given by sending a complex $\mathcal{M}^\bullet$ of D_X–modules to $\mathrm{dR}(\mathcal{M}^\bullet) = \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, \mathcal{M})$.

This equivalence preserves the six standard functors *f^* and f_*, their duals $f_!$ and $f^!$ and also $\otimes$ and* Hom.

Via duality on X the de Rham functor is related to the functor $\mathrm{Sol}(\_\_) = \mathbf{R}\mathrm{Hom}(\_\_, \mathcal{O}_X)$. This functor yields therefore an anti-equivalence, and it is this anti-equivalence which can be obtained in positive characteristic.

Emerton–Kisin correspondence

In positive characteristic a naive approach via de Rham theory does not work due to the failure of the Poincaré lemma. At least one should expect that the correspondence respects the relationship “$\mathbb{F}_p$ corresponds to $\mathcal{O}_X$” in some way or another. But due to the fact that in characteristic p there are many functions with derivative zero (namely all pth powers), the de Rham theory is ill behaved. For the same reason, D–module theory in positive characteristic is also quite different from the one in characteristic zero.

So what is the correct counterpart for constructible sheaves of $\mathbb{F}_p$–vector-spaces? The solution arises from the Artin–Schreyer sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathcal{O}_X \xrightarrow{x \mapsto x^p} \mathcal{O}_X \longrightarrow 0.$$

This is an exact sequence in the étale topology. Hence if one views the Riemann-Hilbert correspondence as a vast generalization of de Rham theory (in terms of the Poincaré lemma), then the correspondence of Emerton–Kisin – to be outlined shortly – is an analogous generalization of Artin–Schreyer theory.

Hence, the basic objects one studies in this correspondence is not D_X–modules but rather quasi-coherent $\mathcal{O}_X$–modules $\mathcal{M}$ which are equipped with an action of the Frobenius F. That is, we have an $\mathcal{O}_X$–linear map $F_{\mathcal{M}} : \mathcal{M} \longrightarrow F_*\mathcal{M}$. By adjunction, such a map is the same as a map $\theta_{\mathcal{M}} : F^*\mathcal{M} \longrightarrow \mathcal{M}$. Such objects have been studied in various forms for a long time, see for example [35] or [21, Exp. XXII].

Definition 3.5. An $\mathcal{O}_{X,F}$*–module* $(\mathcal{M}, \theta)$ is a quasi-coherent $\mathcal{O}_X$–module $\mathcal{M}$ together with a $\mathcal{O}_X$–linear map

$$\theta : F^*\mathcal{M} \longrightarrow \mathcal{M}$$

If θ is an isomorphism $(\mathcal{M}, \theta)$ is called *unit*. If is called *finitely generated* if $\mathcal{M}$ is finitely generated when viewed as a module over the non-commutative ring $\mathcal{O}_{X,F}$.

The following two are the essential examples of finitely generated unit $\mathcal{O}_{X,F}$–modules.

Example 3.6. For simplicity assume that $X = \operatorname{Spec} R$ is affine. The natural identification $F^*\mathcal{O}_X \cong \mathcal{O}_X$ gives $\mathcal{O}_X$ the structure of a finitely generated unit $\mathcal{O}_{X,F}$–module.

Let R_f be the localization of R at a single element $f \in R$. The natural map

$$F^*R_f = R \otimes_F R_f \longrightarrow R_f$$

sending $r \otimes \frac{a}{b}$ to $r\frac{a^p}{b^p}$ has a natural inverse given by sending $\frac{a}{b} \longrightarrow ab^{p-1} \otimes \frac{1}{b}$. Hence R_f is naturally a unit module. In fact, R_f is even finitely generated as a unit $R[F]$–module, generated by the element $\frac{1}{f}$, since $F(\frac{1}{f}) = \frac{1}{f^p}$.

The main result of [44] about finitely genrated unit $\mathcal{O}_{X,F}$–modules, which makes them a suitable analog of holonomic D–modules, is

Theorem 3.7. *Let X be smooth. In the abelian category of (locally) finitely generated $\mathcal{O}_{X,F}$–modules, every object has finite length.*

Example 3.8. Considering a Cech resolution to compute coherent cohomology with support in some subscheme Z, the easy *abelian category* part of the preceding theorem and the preceding examples imply that $H^i_Z(\mathcal{O}_X)$ is naturally a f.g. (finitely generated) unit $\mathcal{O}_{X,F}$–module.

Now the correspondence that is proven in [22] can be summarized as follows:

Theorem 3.9. *For X smooth, there is an anti-equivalence on the level of derived categories*

$$\left\{\begin{matrix}\textit{constructible sheaves of} \\ \mathbb{F}_p\textit{–vector-spaces on } X_{\text{ét}}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{locally finitely generated} \\ \mathcal{O}_{X,F}\textit{–modules}\end{matrix}\right\}$$

The correspondence is given by sending a (complex of) $\mathcal{O}_{X,F}$–modules $\mathcal{M}^\bullet$ to the constructible sheaf $\mathrm{Sol}(\mathcal{M}^\bullet) = \mathbf{R}\mathrm{Hom}(\mathcal{M}, \mathcal{O}_X)$ *and is roughly dual to the naive approach of taking the fixed points of the Frobenius (Artin-Schreyer sequence) alluded to above.*

The correspondence preserves certain functors, namely $f^!$, f_* (and $\otimes$) on the right hand side correspond to f^*, $f_!$ on the left hand side.

Intermediate extensions

From now on we treat the two situations – characteristic zero regular singular holonomic D_X–modules with the Riemann–Hilbert correspondence on one hand and positive characteristic finitely generated unit $\mathcal{O}_{F,X}$–modules with the Emerton–Kisin correspondence on the other – formally as one. The crucial property they both share is the fact that all modules have finite length. This is key to the following construction.

Due to the lack of duality in positive characteristic there is no analog of the functor $j_!$ available. However there is an adequate substitute, which still exists in our context [23, 6]. The substitute we have in mind is the intermediate extension $j_{!*}$, usually constructed as the image of the “forget supports map” from $j_! \longrightarrow j_*$. This usual definition does not work since $j_!$ is not available. Nevertheless it turns out that all one needs to define $j_{!*}$ is the fact that the modules have finite length:

Proposition 3.10 ([6],[23]). *Let $j : U \subseteq X$ be a locally closed immersion of smooth $\mathbb{F}_p$–schemes (resp. $\mathbb{C}$–schemes). Let $\mathcal{M}$ be a finitely generated unit $\mathcal{O}_{U,F}$–module. Then there is a* unique *submodule $\mathcal{N}$ of $R^0 j_* \mathcal{M}$ minimal with respect to the property that $f^! \mathcal{N} = \mathcal{M}$. This submodule $\mathcal{N}$ is called the* intermediate extension *and is denoted by $j_{!*}\mathcal{M}$.*

Proof. The key point is the fact that $\mathcal{M}$ has finite length which ensures the existence of *minimal* modules with the desired property (any decreasing chain is eventually constant). Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two modules with the desired property. Since $f^! \mathcal{N}_i = \mathcal{M}$ their intersection cannot be zero. On the other hand the exact sequence

$$0 \longrightarrow \mathcal{N}_1 \cap \mathcal{N}_2 \longrightarrow R^0 j_* \mathcal{M} \longrightarrow R^0 j_* \mathcal{M}/\mathcal{N}_1 \oplus R^0 j_* \mathcal{M}/\mathcal{N}_2$$

shows that their intersection is also a finitely generated unit $\mathcal{O}_{X,F}$–module (resp. r.s. holonomic D_X–module) as it is the kernel of a map of such modules. By minimality one has $\mathcal{N}_1 = \mathcal{N}_2$ showing uniqueness.

Under the correspondence the intermediate extensions behave well: In the situation as above we have

$$\mathrm{Sol}(j_{!*}\mathcal{M}) = \mathrm{Image}(j_! \,\mathrm{Sol}(\mathcal{M}) \longrightarrow R^0 j_* \,\mathrm{Sol}(\mathcal{M}))$$

such that they do in fact correspond to the intermediate extensions on the constructible side, where they can be identified as the image of the "forget supports" map.

For the basic computations that follow we list some key properties both correspondences enjoy:

1.
$$\mathrm{Sol}(\mathcal{O}_X) = \begin{cases} \mathbb{F}_p[n] & \text{if } k = \mathbb{F}_p \\ \mathbb{C}[n] & \text{if } k = \mathbb{C} \end{cases}$$
2. There are functors $f^!$ and f_* which behave under the correspondence in the expected way.
3. For $Y \subseteq X$ a subvariety, local cohomology is defined in the categories and satisfies the triangle (a highbrow way of writing the long exact sequence for cohomology with supports)
$$\mathbf{R}\Gamma_{[Y]}\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet \longrightarrow \mathbf{R}j_*j^*\mathcal{M}^\bullet \xrightarrow{+1}$$
where j is the open inclusion of the complement of Y into X.
4. Again, let $i : Y \hookrightarrow X$ be the inclusion of a closed subset, then
$$\mathrm{Sol} \circ \mathbf{R}\Gamma_{[Y]} \cong \mathbf{R}i_! i^* \circ \mathrm{Sol}\,.$$
This follows via the preceding two items and the triangle $\mathbf{R}j_! j^! \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathbf{R}i_! i^{-1}\mathcal{L} \xrightarrow{+1}$ for a (complex of) constructible sheaves $\mathcal{L}$.
5. The preceding items allow us to compute (see also equation (12) on page 17)
$$\mathrm{Sol}(\mathbf{R}\Gamma_Y(\mathcal{O}_X)) = i_! i^* \mathbb{F}_p[n] = i_! \mathbb{F}_p|_Y[n]$$
in positive characteristic and respectively $i_!\mathbb{C}|_Y[n]$ in characteristic zero.

3.2 Lyubeznik's local cohomology invariants

Let $A = R/I$ for I an ideal in a regular (local) ring (R, m) of dimension n and containing a field k. The main results of [43, 37] state that the local cohomology module $H^a_m(H^{n-i}_I(R))$ is injective and supported at m. Therefore it is a finite direct sum of $e = e(H^a_m(H^{n-i}_I(R)))$ many copies of the injective hull $E_{R/m}$ of the residue field of R. Lyubeznik shows in [43] that this number

$$\lambda_{a,i}(A) \stackrel{\mathrm{def}}{=} e(H^a_m(H^{n-i}_I(R)))$$

does not depend on the auxiliary choice of R and I. At the same time, this number $e(\mathcal{M})$ is the multiplicity of the holonomic D_X–module $\mathcal{M}$, respectively the finitely generated unit $\mathcal{O}_{X,F}$–module $\mathcal{M}$.

If A is a complete intersection, these invariants are essentially trivial (all are zero except $\lambda_{d,d} = 1$ where $d = \dim A$). In general $\lambda_{a,i}$ can only be nonzero in the range $0 \leq a, i \leq d$. These invariants were first introduced by Lyubeznik in [43] and further studied by Walther in [52]. In [33] Garcia-López and Sabbah show Theorem 3.1 in the case of an isolated complex singularity.

We now indicate briefly the proof of 3.1. As shown in [4] the condition imposed on the singularities in the Theorem 3.1 easily implies (via the spectral sequence $E_2^{a,j} = H^a_{[x]} H^j_{[Y]}(\mathcal{O}_X) \Rightarrow H^{a+j}_{[x]}(\mathcal{O}_X)$) part one. Part two is the point where the correspondences enter into the picture: The idea is of course to use that

$$e(H^a_m(H^{n-d}_Y(\mathcal{O}_X))) = \dim \mathrm{Sol}(H^a_{\{x\}}(H^{n-d}_Y(\mathcal{O}_X)))$$

by the correspondence, and then to compute the right hand side. As it is written here the right hand side is however not computable. The trick now is to replace $H^{n-d}_I(R)$ by $j_{!*}H^{n-d}_Y(\mathcal{O}_X)|_{X-\{x\}}$, which is easily checked to not affect our computation (long exact sequence for $\Gamma_{\{x\}}$). Now, the assumption on the singularity that for $i \neq d$ the module $H^{n-i}_Y(\mathcal{O}_X)$ is supported at x can simply be rephrased as $H^{n-d}_Y(\mathcal{O}_X)|_{X-\{x\}} = \mathbf{R}\Gamma_{Y-\{x\}}(\mathcal{O}_{X-\{x\}})[n-d]$.

Using that Sol commutes with $j_{!*}$ and the fact that

$$\mathrm{Sol}(\mathbf{R}\Gamma_{Y-\{x\}}(\mathcal{O}_{X-\{x\}})) = i_!(\mathbb{F}_p)_{Y-\{x\}}[n]$$

one obtains that

$$e(H^a_m(H^{n-d}_Y(\mathcal{O}_X))) = \dim(H^{-a}j_{!*}i_!(\mathbb{F}_p)_{Y-\{x\}}[d]).$$

To compute the right hand side is now a feasible task that yields the desired result (feasible due to the fact that j is just the inclusion of the complement of a point, which makes it possible to effectively understand and calculate $j_{!*}$; see [4, 5] for details).

3.3 Comparison via crystalline cohomology

We close this section with some remarks regarding the behaviour of these invariants under reduction to positive characteristic. There are by now classical examples that show that local cohomology does not behave well under reduction so one would expect that the invariants $\lambda_{a,i}$ do not behave well either. On a superficial level, glancing at Theorem 3.1, one might however suspect a complete analogy between positive and zero characteristic.

However, this is not true. The difference stems from the difference between the cohomology theories which describe $\lambda_{a,i}$. In positive characteristic, this is étale cohomology with coefficients in $\mathbb{F}_p$, in characteristic zero however it is (topological) cohomology. Under reduction mod p the former only constitutes a very small part of the latter, namely the part of slope zero.

For simplicity consider the situation of $Y \subseteq \mathbb{P}^n$ a smooth projective variety. Now. the local ring A of the cone of this projective embedding has an isolated singularity at its vertex and one can study the invariants $\lambda_{a,i}(A)$. Theorem 3.1 shows that $\lambda_{a,d}(A)$ are described (excision) by $H^{d-a}(Y_{\text{ét}}, \mathbb{F}_p)$ and $H^{d-a}(Y_{\text{an}}, \mathbb{C})$ in positive and zero characteristic respectively. So far everything appears in complete analogy.

But let us now consider reduction mod p and let Y be the special fiber of the smooth family $\mathcal{Y} \longrightarrow \operatorname{Spec} W(k)$, where $W(k)$ is the ring of Witt vectors over k and K denotes its field of fractions. Via the comparison results between topological cohomology, de Rham cohomology and Berthelot's crystalline cohomology (see [3])

$$H^{d-a}(\mathcal{Y}_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\cong} H^{d-a}_{\text{dR}}(\mathcal{Y}_K) \cong H^{d-a}(Y/K)$$

it turns out that $H^{d-a}(Y_{\text{ét}}, \mathbb{F}_p)$ is only a very small part of the crystalline cohomology $H^{d-a}(Y/K)$, namely the part on which the Frobenius acts with eigenvalue zero. This is an even smaller part then the part $H^{d-a}(Y/K)^{<1}$ which was of great importance in section 1.6. From this point of view it is not surprising to find the characteristic zero side to capture much more information than the positive characteristic side. Even though the positive characteristic side is therefore lacking the topological insight, the information one obtains is still very valuable. For example it is used very effectively to study L–functions in [22] and also [13], which uses a similar correspondence for that purpose.

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