Reflexive modules on quotient surface singularities

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Introduction

D. Mumford in characteristic 0 [10] and J. Lipman [8] in characteristic $p>0$ proved that a surface singularity X is rational if and only if X has finitely many rank one reflexive modules up to isomorphism. This implies a characterization of quotient surface singularities as those ones which have finitely many indecomposable (with respect to direct sum) reflexive modules (1. 2).

For rational double points on $\mathbb{C}$, J. Mc Kay [9] gave a one-to-one correspondence between vertices of the extended diagram associated to the finite subgroup $G \subset SL(2, \mathbb{C})$ defining X as a quotient $\mathbb{C}^2/G$, irreducible representations of G and indecomposable reflexive modules on X. Trying to understand geometrically this correspondence, G. Gonzalez-Sprinberg and J. L. Verdier [5] associated geometrically to each indecomposable reflexive module the first Chern class of the pull-back on the minimal resolution $f: \tilde{X} \to X$. They show that the Chern class intersects exactly one rational curve of the exceptional locus, and that it determines the module. After this, H. Knörrer [7] reinterpreted this geometrical construction group-theoretically. This, in a sense, completes a cycle, as the work of J. Mc Kay was from the point of view of representation theory. Finally M. Artin and J. L. Verdier [1] gave an elegant and theoretical proof of the fact that the first Chern class determines the module.

This led H. Knörrer and J. L. Verdier to ask whether it remains true that for a general quotient surface singularity X an indecomposable reflexive module is determined by its first Chern class.

We give here a negative answer to that question (3. 3). To this aim we first "rewrite" the construction of Artin-Verdier for a general rational singularity (2. 8). This exhibits a sort of "obstruction" to a positive answer (2. 9. 3). In § 3 we give a numerical criterion for invertible sheaves which allows to compute this obstruction. The method in the invertible case relies on the techniques of cyclic covers we developed in [4]. In the example (3. 3) one has two indecomposable sheaves of different ranks with the same first Chern class. A bound for the rank is given in (2. 10).

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§ 1. Quotient surface singularities

(1. 1) Definitions and notations. In this article a surface singularity X will always be the spectrum of a two dimensional local ring defined over $\mathbb{C}$ or a two dimensional complex analytic germ.

As usual one says that X is a *rational* singularity if $R^1 f_* \mathcal{O}_{\tilde{X}} = 0$ where $f: \tilde{X} \to X$ is any desingularization and $\mathcal{O}_{\tilde{X}}$ is the structure sheaf. One says that X is a *quotient* singularity if X is isomorphic to W/G where G is a finite group acting linearly on the spectrum W of a regular ring $\mathcal{O}_W$ (or on the smooth germ W). The well known theorem of Chevalley allows to assume that G acts fixpoint free outside of the closed point. It is also well known that a quotient singularity is rational.

We denote by U the regular locus of X.

Lemma (1. 2). *X is a quotient singularity if and only if each normal cover of X, which is étale over U, has rational singularities.*

Proof. If X is a quotient singularity denote by $\mu: W \to X$ the quotient map, which is étale over U. Let $\pi: Y \to X$ be a cover of X which is étale over U. Then $Y \times_X W \to W$ is étale outside of codimension 2. By "purity of branched locus" the normalization W' of $Y \times_X W$ is étale over W and therefore is smooth. This implies that Y has rational singularities [3]. This proves the "only if" part.

If each cover of X, which is étale over U, has rational singularities, then X itself is a rational singularity. Therefore the dualizing module ω_X has finite order [10] and defines the canonical cover $\mu: X' \to X$ ([4], (1. 4)) and ([13], (4. 6)). Now X' has Gorenstein and rational singularities. This implies that X' has only rational double points as singularities. Those are quotients. Therefore X has quotient singularities.

(1. 3). A module M on $\mathcal{O}_X$ is *indecomposable* if it is not the direct sum of two non trivial modules. It is *reflexive* if the natural map from M to its double dual M^{vv} is an isomorphism. In other words it says that $M = i_* M|_U$, where $i: U \to X$ is the embedding of the regular locus. If X is smooth (and two dimensional) all the reflexive modules are trivial, that means are isomorphic to a direct sum of copies of $\mathcal{O}_X$.

From (1. 2) one obtains an elementary and geometrical proof of the following known result.

Corollary (1. 3). *X is a quotient singularity if and only if it has finitely many indecomposable reflexive modules up to isomorphism.*

Proof. If X has finitely many indecomposable reflexive modules then the dualizing module ω_X has finite order and defines the canonical cover $\pi: X' \to X$ ([4], (1. 4)) and ([13], (4. 6)). For each reflexive module M on X', $\pi_* M$ is reflexive on X. In particular, X' can only have finitely many rank one modules. This implies that X' has rational singularities [10] and, as before, has rational double points. Therefore X has quotient singularities. This proves the "if part".

For the "only if" part one just has to verify that each indecomposable reflexive module occurs as a direct summand of the pull down of the structure sheaf of the smooth cover. See Herzog's proof ([6], (1. 7)).

Remark (1. 4). Along the same line one obtains a one-to-one correspondence between isomorphism classes of indecomposable reflexive modules (which are defined exactly by the direct factors of the decomposition of $\mu_* \mathcal{O}_W$) and isomorphism classes of G-indecomposable free G-modules in W (where G is the Galois group of μ). Those are in one-to-one correspondence with the isomorphism classes of irreducible representations of G (as described in [5]): if F is a free G-module on W then $F/m \cdot F$ is a representation of G, where m is the maximal ideal of the point fixed by G. If H is a representation of G then $\mathcal{O}_W \otimes_{\mathbb{C}} H$ is a free G-module on W.

§ 2. Reflexive modules on rational surface singularities

In this chapter we assume X to be a rational surface singularity. We denote by $i: U \to X$ the embedding of the regular locus and by $f: \tilde{X} \to X$ any desingularization.

Lemma (2. 1). *If $\mathcal{M}$ is a locally free sheaf on $\tilde{X}$ such that $R^1 f_* \mathcal{M}^v \otimes \omega_{\tilde{X}} = 0$, then $f_* \mathcal{M}$ is a reflexive module.*

Proof. For each torsion free sheaf $\mathcal{M}$ on $\tilde{X}$ one has the exact sequence

$$0 \longrightarrow f_* \mathcal{M} \longrightarrow i_* \mathcal{M}|_U \longrightarrow H^1_E(\mathcal{M}) \longrightarrow R^1 f_* \mathcal{M}.$$

If $\mathcal{M}$ is locally free then $H^1_E(\mathcal{M})$ is Serre-dual to $R^1 f_* \mathcal{M}^v \otimes \omega_{\tilde{X}}$.

Lemma and definition (2. 2). *Let $\mathcal{M}$ be a sheaf on $\tilde{X}$. There exists a reflexive module M on X such that $\mathcal{M} = f^* M/\text{torsion}$ if and only if the following conditions are fulfilled:*

i) *$\mathcal{M}$ is locally free.*

ii) *$\mathcal{M}$ is generated by its global sections.*

iii) $R^1 f_* \mathcal{M}^v \otimes \omega_{\tilde{X}} = 0$.

In this case $\mathcal{M}$ is said to be full. Moreover one has $f_* \mathcal{M} = M$, and $f_*(\mathcal{M}^v) = M^v$.

Proof. Assume that i), ii) and iii) are fulfilled. By (2. 1) one knows that the module of global sections $M = f_* \mathcal{M}$ is reflexive. By ii), the natural map $f^* M \to \mathcal{M}$ is surjective. Its kernel is the torsion part. This proves the "if part".

Assume that $\mathcal{M} = f^* M/\text{torsion}$ where M is a reflexive module. Then i) and ii) are proved in ([1], Lemma (1. 1)), as well as the vanishing of $R^1 f_* \mathcal{M}$. Therefore one has $H^1_E(\mathcal{M}) = 0$ in the sequence (2. 1). Since $\mathcal{M}$ is locally free, this is Serre-dual to iii). This proves the "only if" part.

Since $\mathcal{M}$ is generated by global sections and $\omega_{\tilde{X}}$ has no relative cohomology by the well known theorem of Grauert-Riemenschneider, one has $R^1 f_* \mathcal{M} \otimes \omega_{\tilde{X}} = 0$. Therefore $f_*(\mathcal{M}^v) = M^v$.

(2. 3). In the sequel we will study full sheaves on $\tilde{X}$ instead of studying reflexive modules on X. In order to apply it to rational surface singularities in general, we have to rewrite Artin-Verdier's construction. We choose the dual language (see [1], Lemma (1. 2)).

For a full sheaf $\mathcal{M}$ of rank r take r generic sections to define its first Chern class D as the discriminant of the r sections. The curve D is smooth and meets the exceptional locus of f transversally. This defines the exact sequence

$$0 \longrightarrow \mathcal{M}^v \longrightarrow \bigoplus_1^r \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0. \tag{2.3.1}$$

Taking global sections is not right exact. Let $\mathscr{C}$ be the image of the r sections given by (2. 3. 1) in $f_* \mathcal{O}_D$. The sequence

$$0 \longrightarrow M^v \longrightarrow \bigoplus_1^r \mathcal{O}_X \longrightarrow \mathscr{C} \longrightarrow 0 \tag{2.3.2}$$

is exact.

Lemma (2. 4). *Assume that $\mathcal{M}$ does not have $\mathcal{O}_{\tilde{X}}$ as a direct summand. Then the $\mathcal{O}_X$-submodule $\mathscr{C}$ of $f_* \mathcal{O}_D$ determines $\mathcal{M}$ up to isomorphism. The module M^v is the module of relations of a minimal set of generators of $\mathscr{C}$.*

Proof. The sequence (2. 3. 2) expresses M^v as the module of relations of r sections generating $\mathscr{C}$. Of course one has $r \geqq$ minimal number of generators of $\mathscr{C}$. Assume that this inequality is strict. Take $\mathbf{x} = (x_1, \ldots, x_s)$ a minimal set of generators among the generators $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_s, y_{s+1}, \ldots, y_r)$ given by (2. 3. 2). There is a matrix $\mathbf{a}$ with coefficients in $\mathcal{O}_X$ verifying $\mathbf{y} = \mathbf{a} \cdot \mathbf{x}$. The map $\bigoplus_1^r \mathcal{O}_X \to \mathscr{C}$, which is given by

$$(\boldsymbol{\lambda}, \boldsymbol{\eta}) \longrightarrow (\boldsymbol{\lambda} \cdot \mathbf{x} + \boldsymbol{\eta} \cdot \mathbf{y}),$$

factors over the map $\bigoplus_1^s \mathcal{O}_X \to \mathscr{C}$, which is given by $\boldsymbol{\mu} \to \boldsymbol{\mu} \cdot \mathbf{x}$, where $\boldsymbol{\mu} = \boldsymbol{\lambda} + {}^t\mathbf{a} \cdot \boldsymbol{\eta}$. This gives rise to a splitting $M^v = N^v \oplus \bigoplus_1^{r-s} \mathcal{O}_X$ defined by $(\boldsymbol{\lambda}, \boldsymbol{\eta}) \to (\boldsymbol{\mu}, \boldsymbol{\eta})$, where N^v is the module of relations of the minimal set of generators $\mathbf{x}$. Two different minimal sets of generators differ by a matrix with coefficients in $\mathcal{O}_X$ which is invertible on X. Along the same line one sees that M^v is unique up to isomorphism.

Remark (2. 5). If $R^1 f_* \mathcal{M}^v = 0$, then $f_* \mathcal{O}_D / \mathscr{C} = R^1 f_* \mathcal{M}^v = 0$. In this case D determines the full sheaf $\mathcal{M}$ up to $\mathcal{O}_{\tilde{X}}$ factors. For example if X is *Gorenstein*, then (2. 2. iii)) just says that $R^1 f_* \mathcal{M}^v = 0$. Therefore all full sheaves are determined by their first Chern classes. On the other hand if D meets the exceptional locus only in one point (call D *irreducible* in that case), the full sheaf determined by D has to be indecomposable (because each factor gives a contribution).

By the unicity of the full sheaf attached to a Chern class, those irreducible Chern classes D describe exactly the irreducible non trivial full sheaves. This is exactly the result of Artin-Verdier [1].

(2. 6). Let D be any curve on $\tilde{X}$ which is transversal to the exceptional locus of f and let $\mathcal{O}_{f_* D}$ be the image of $\mathcal{O}_X$ in $f_* \mathcal{O}_D$. Let $\mathscr{C}$ be an $\mathcal{O}_X$-submodule of $f_* \mathcal{O}_D$ verifying $\mathcal{O}_{f_* D} \subset \mathscr{C}$. Then there is a surjection $f^* \mathscr{C} \to \mathcal{O}_D$. Especially, if we take r generating sections $\bigoplus_1^r \mathcal{O}_X \to \mathscr{C}$, we obtain a surjection

$$\bigoplus_1^r \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_D. \tag{2.6.1}$$

Lemma (2. 7). *Let D be as in* (2. 6). *Let $\mathscr{C}$ be an $\mathcal{O}_X$-submodule of $f_*\mathcal{O}_D$ verifying*

i) $\mathcal{O}_{f_*D} \subset \mathscr{C}$;

ii) *the sequence* (2. 6. 1) *tensorized with $\omega_{\tilde{X}}$ is right exact after applying f_*.*

Then the kernel of (2. 6. 1) *is the dual of a full sheaf.*

Proof. Call $\mathscr{N}$ the kernel of (2. 6. 1). Since $\mathscr{E}xt^1_{\tilde{X}}(\mathcal{O}_D, \mathcal{O}_{\tilde{X}})$ is isomorphic to its double dual one sees that $\mathscr{N}$ is isomorphic to its double dual. Therefore $\mathscr{N}$ is locally free. The condition ii) implies that $R^1 f_* \mathscr{N} \otimes \omega_{\tilde{X}} = 0$. On the other hand, one has the exact sequences

$$0 \longrightarrow \bigoplus_1^r \mathcal{O}_{\tilde{X}} \longrightarrow \mathscr{N}^v \longrightarrow \mathcal{O}_D \longrightarrow 0 \quad \text{and} \quad \bigoplus_1^r \mathcal{O}_{\tilde{X}} \longrightarrow f^* f_* \mathscr{N}^v/\text{torsion} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Therefore $\mathscr{N}^v = f^* f_* \mathscr{N}^v/\text{torsion}$. This implies that $\mathscr{N}^v$ is generated by global sections. The conditions of (2. 2) being fulfilled, $\mathscr{N}^v$ is full.

Putting together (2. 4) and (2. 7) one obtains

Proposition (2. 8). *The full sheaves on $\tilde{X}$ without $\mathcal{O}_{\tilde{X}}$ as summand are in one-to-one correspondence with the $\mathcal{O}_X$-submodules $\mathscr{C}$ of $f_*\mathcal{O}_D$, for some D as in* (2. 6) *and some $\mathscr{C}$ as in* (2. 7) *with the properties* i) *and* ii).

Proof. One only has to verify that for a full sheaf $\mathscr{M}$, its associated $\mathscr{C}$ verifies (2. 7) i). This comes from the following diagram of exact sequences:

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & \bigoplus_1^{r-1} \mathcal{O}_{\tilde{X}} \arrow[u] \arrow[r, "\sim"] & \bigoplus_1^{r-1} \mathcal{O}_{\tilde{X}} \arrow[u] & & \\
0 \arrow[r] & \mathscr{M}^v \arrow[u] \arrow[r] & \bigoplus_1^{r} \mathcal{O}_{\tilde{X}} \arrow[u] \arrow[r] & \mathcal{O}_D \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{L}^{-1} \arrow[u] \arrow[r] & \mathcal{O}_{\tilde{X}} \arrow[u] \arrow[r] & \mathcal{O}_D \arrow[u, equal] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $\mathscr{L} = \bigwedge^r \mathscr{M}$. One applies f_* and finds that $\mathcal{O}_{f_*D}$ is included in $\mathscr{C}$.

Remarks (2. 9). (2. 9. 1) The construction (2. 8) does not say whether the full sheaf $\mathscr{M}$ associated to some $\mathscr{C}$ is indecomposable or not, whereas in the Gorenstein case $\mathscr{M}$ is indecomposable exactly when D is irreducible (2. 5).

(2. 9. 2) Nevertheless one can say the following. If D is not irreducible write $D=D_1+\cdots+D_k$, $k\geqq 2$, where the D_i are irreducible (this is possible in the henselization of X). Take two complement subsets I and J of $\{1,\ldots,k\}$ and write $D=D_I+D_J$ in the obvious way. Then $\mathscr{C}$ contains $\mathscr{C}_I\oplus\mathscr{C}_J$, where $\mathscr{C}_I=f_*\mathcal{O}_{D_I}\cap\mathscr{C}$ and similarly for J. Then the full sheaf $\mathscr{M}$ associated to $\mathscr{C}$ is irreducible if and only if for each non trivial subdivision $\{I,J\}$ of $\{1,\ldots,k\}$ $\mathscr{C}$ is not isomorphic to $\mathscr{C}_I\oplus\mathscr{C}_J$.

(2. 9. 3) The construction (2. 8) is obviously compatible with inclusions in the following sense. Let $\mathscr{C}$ be as in (2. 7) (verifying i) and ii)) and let $\mathscr{C}'$ be any $\mathcal{O}_X$-submodule of $f_*\mathcal{O}_D$ verifying $\mathscr{C}\subset\mathscr{C}'$. Then $\mathscr{C}'$ fulfills the conditions i) and ii) too. Complete the minimal set of r generating sections of $\mathscr{C}$ to r' generating sections of $\mathscr{C}'$. One obtains the exact sequence

$$0\longrightarrow\mathscr{M}'^{v}\longrightarrow\mathscr{M}^{v}\longrightarrow\bigoplus_{1}^{r'-r}\mathcal{O}_{\tilde X}\longrightarrow 0$$

where $\mathscr{M}$ is the indecomposable full sheaf associated to $\mathscr{C}$ and $\mathscr{M}'$ is the full sheaf associated to $\mathscr{C}'$ and to the r' sections. $\mathscr{M}'$ is not necessarily indecomposable and can contain $\mathcal{O}_{\tilde X}$ factors. Write $\mathscr{M}'=\mathscr{N}\oplus\mathcal{O}_{\tilde X}^{s}$, where $\mathscr{N}$ is without $\mathcal{O}_{\tilde X}$ factor. Then $\mathscr{M}$ and $\mathscr{N}$ are not isomorphic as long as $\mathscr{C}$ and $\mathscr{C}'$ are not isomorphic. They have the same first Chern class. If now D is irreducible this gives two non isomorphic indecomposable full sheaves with the same first Chern class (but not with the same rank).

(2. 10). We will study the rank r of a full indecomposable sheaf $\mathscr{M}$. We have seen in (2. 4) that it is exactly the minimal number of generators of the associated $\mathscr{C}$. Let Z be the fundamental cycle of $\tilde X$, that means $\mathcal{O}_{\tilde X}(-Z)=f^*m/\text{torsion}$, where m is the maximal ideal on X. Let μ be the multiplicity of X. Let s be the smallest power of ω_X which has an invertible reflexive hull. Then one has $\omega_{\tilde X}^{s}\cong\mathcal{O}_{\tilde X}(-\Delta)$ where Δ is an effective divisor concentrated on the exceptional locus of f if $\tilde X$ is the *minimal* desingularization.

Proposition (2. 10). *Assume $\tilde X$ to be minimal. Let $\mathscr{M}$, r, Δ, Z, μ be as above. Then one has*

i) $D\cdot Z/(\mu-1)\leqq r\leqq D\cdot Z$,

ii) $\dim R^1f_*\mathscr{M}^v\geqq h^1(\mathcal{O}_Z\otimes\mathscr{M}^v)=D\cdot Z-r$,

$\dim R^1f_*\mathscr{M}^v=h^1(\mathcal{O}_\Delta\otimes\mathscr{M}^v)$.

Proof. Tensorize (2. 3. 1) with $\omega_{\tilde X}$, restrict to Z and apply f_*. This is right exact because of (2. 2) iii). Therefore $D\cdot Z\leqq r\cdot(\mu-1)=r\cdot h^0(\omega_Z(-Z))$. Now restrict simply (2. 3. 1) to Z and apply f_*. One obtains:

$$\dim R^1f_*\mathscr{M}^v\geqq h^1(\mathcal{O}_Z\otimes\mathscr{M}^v)\geqq D\cdot Z-r.$$

Since $\mathcal{O}_{\tilde X}(-Z)$ is generated by global sections, the image $f_*\bigoplus_1^r\mathcal{O}_{\tilde X}(-Z)$ in $f_*\mathcal{O}_D(-Z)$ is isomorphic to $m\cdot\mathscr{C}$. Therefore $\mathscr{C}/m\cdot\mathscr{C}$ lies in $\mathbb{C}^{D\cdot Z}$ and one has $D\cdot Z\geqq r$. Moreover, the surjective map $\bigoplus_1^r\cdot H^0(\mathcal{O}_Z)\to\mathscr{C}/m\cdot\mathscr{C}$ has to be an isomorphism. Therefore one has $h^0(\mathcal{O}_Z\otimes\mathscr{M}^v)=0$ and $h^1(\mathcal{O}_Z\otimes\mathscr{M}^v)=D\cdot Z-r$. Now since $R^1f_*\mathscr{M}^v\otimes\omega_{\tilde X}=0$ and since $\omega_{\tilde X}$ is generated by global sections on the minimal desingularization $\tilde X$, one obtains that $R^1f_*\mathscr{M}^v\otimes\omega_{\tilde X}^k=0$ for all $k\geqq 1$. Therefore one has $R^1f_*\mathscr{M}^v(-\Delta)=0$. This finishs the proof.

§ 3. Reflexive rank one modules an quotient surface singularities

(3. 1). In this chapter we assume X to be a quotient singularity. An invertible sheaf $\mathscr{L}$ on a desingularization $\tilde{X}$ of X is said to be *arithmetically positive* (a.p.) if $\deg_{E_i}\mathscr{L} \geqq 0$ for all exceptional curves E_i. Since X has finitely many rank one reflexive modules [9] some power of $\mathscr{L}$ verifies: $\mathscr{L}^N = \mathcal{O}_{\tilde{X}}(-A)$ where A is an effective divisor concentrated on the exceptional locus of $f: \tilde{X} \to X$. We denote by $\left[\frac{A}{N}\right]$ the integral part of the $\mathbb{Q}$-divisor $\frac{A}{N}$ and by $\left\{\frac{A}{N}\right\}$ the divisor $-\left[-\frac{A}{N}\right]$.

Proposition (3. 2). *Let $\mathscr{L}$ be an a.p. invertible sheaf on $\tilde{X}$. Write $\mathscr{L}^N = \mathcal{O}(-A)$ as in* (3. 1).

i) *$\mathscr{L}$ is full if and only if* $\chi\left(\mathcal{O}_{\left[\frac{A}{N}\right]} \otimes \mathscr{L}\left(\left[\frac{A}{N}\right]\right)\right) = 0.$

ii) $R^1 f_* \mathscr{L}^{-1} = -\chi\left(\mathcal{O}_{\left\{\frac{A}{N}\right\}} \otimes \mathscr{L}^{-1}\right).$

Proof. Let $L^{[i]}$ be the reflexive hull of $f_*\mathscr{L}^i$. The $\mathcal{O}_X$-module $E = \bigoplus_0^{N-1} L^{[i]}$ has an $\mathcal{O}_X$-algebra structure defined by $\mathcal{O}_X \cong L^{[N]}$, which is normal. Therefore $\operatorname{Spec}_{\mathcal{O}_X} E$ has rational singularities (1. 2). The $\mathcal{O}_{\tilde{X}}$-module $\mathscr{E} = \bigoplus_0^{N-1} \mathscr{L}^i\left(\left[\frac{iA}{N}\right]\right)$ has an $\mathcal{O}_X$-algebra structure defined by the inclusion $\mathscr{L}^{-N} \xrightarrow[-A]{} \mathcal{O}_{\tilde{X}}$, which is normal and has only rational singularities ([4], (1. 5) and corresponding references). Therefore the composition of $\operatorname{Spec}_{\mathcal{O}_{\tilde{X}}} \mathscr{E} \to \tilde{X}$ and f factorizes on $\operatorname{Spec}_{\mathcal{O}_X} E$. One obtains

$$R^q f_* \mathscr{L}^i\left(\left[\frac{iA}{N}\right]\right) = \begin{cases} L^{[i]} & q=0 \\ 0 & q=1 \end{cases}$$

for $0 \leqq i \leqq (N-1)$.

i) Take the exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}\left(\left[\frac{A}{N}\right]\right) \longrightarrow \mathcal{O}_{\left[\frac{A}{N}\right]} \otimes \mathscr{L}\left(\left[\frac{A}{N}\right]\right) \longrightarrow 0.$$

If $\mathscr{L}$ is full, then $R^1 f_* \mathscr{L} = 0$ and $f_*\mathscr{L} = f_*\mathscr{L}\left(\left[\frac{A}{N}\right]\right)$. Therefore

$$h^0\left(\mathcal{O}_{\left[\frac{A}{N}\right]} \otimes \mathscr{L}\left(\left[\frac{A}{N}\right]\right)\right) = 0.$$

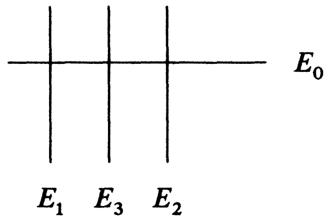
This proves the "only if part". If $\chi\left(\mathcal{O}_{\left[\frac{A}{N}\right]} \otimes \mathscr{L}\left(\left[\frac{A}{N}\right]\right)\right) = 0$ then $f_*\mathscr{L} = f_*\mathscr{L}\left(\left[\frac{A}{N}\right]\right)$ is reflexive and $R^1 f_* \mathscr{L} = 0$. Let $\mathscr{N}$ be the subsheaf of $\mathscr{L}$ generated by global sections. One has $\mathscr{L} = \mathscr{N}(F)$ for an effective vertical divisor F. Then $\mathscr{L}/\mathscr{N} = \mathcal{O}_F \otimes \mathscr{L}$ verifies $\chi(\mathcal{O}_F \otimes \mathscr{L}) = \chi(\mathcal{O}_F) + \mathscr{L} \cdot F = 0$. Since $\mathscr{L} \cdot F \geqq 0$, $\chi(\mathcal{O}_F)$ and thereby F have to be zero.

ii) Write $\mathscr{L}^{-1}=\mathscr{L}^{N-1}(A)$ and take the exact sequence

$$0 \longrightarrow \mathscr{L}^{N-1}\left(\left[\frac{(N-1)\,A}{N}\right]\right) \longrightarrow \mathscr{L}^{-1} \longrightarrow \mathcal{O}_{\{\frac{A}{N}\}} \otimes \mathscr{L}^{-1} \longrightarrow 0.$$

(3. 3). *We now give an example of a quotient singularity for which there are an invertible full sheaf $\mathscr{L}$ and a rank 2 indecomposable full sheaf $\mathscr{M}$ having the same first Chern class.*

Let X be the singularity whose graph of desingularization is given by



$$E_0^2=E_1^2=E_2^2=-2$$
$$E_3^2=-3.$$

This singularity exists and is a quotient singularity, as one can see in the list of Riemenschneider [11]. One can also compute:

$$\omega_{\tilde{X}}^{4}=\mathcal{O}_{\tilde{X}}(-2E_0-E_1-E_2-2E_3)=\mathcal{O}_{\tilde{X}}(-\Delta)$$

and point out that $\left[\dfrac{\Delta}{4}\right]=0$ which is a criterion for X to be quotient ([3], (1. 7)).

(3. 4). Take $\mathscr{L}=\mathcal{O}(D)$, $\deg_{E_0}\mathscr{L}=1$, $\deg_{E_i}\mathscr{L}=0$, $i\geqq 1$. One has (with the notations of (3. 2)): $\left[\dfrac{A}{N}\right]=E_0$. Therefore $\mathcal{O}_{E_0}\otimes\mathscr{L}(E_0)=\mathcal{O}_{E_0}(-1)$ and by (3. 2) i) $\mathscr{L}$ has to be full. One sees immediately that $\mathscr{L}\otimes\omega_{\tilde{X}}=\mathcal{O}_{\tilde{X}}(-Z)$ where Z is the fundamental cycle. Therefore one has $\dim R^1f_*\mathscr{L}^{-1}=h^1(\omega_Z)=1$. One obtains now $\mathscr{M}$ as the full sheaf associated to $\mathscr{C}'=f_*\mathcal{O}_D$ in the construction (2. 9. 3), whereas $\mathscr{L}$ is associated to $\mathscr{C}=\mathcal{O}_{f_*D}$. D being irreducible, $\mathscr{L}$ as well as $\mathscr{M}$ are indecomposable. Moreover one has

$$\operatorname{rank}\mathscr{M}=\text{minimal number of } \mathcal{O}_X \text{ generators of } f_*\mathcal{O}_D=2.$$

From the exact sequence $0\rightarrow\mathscr{L}^{-1}\rightarrow\mathscr{M}^v\rightarrow\mathcal{O}_{\tilde{X}}\rightarrow 0$ one obtains

$$0\rightarrow f_*\mathscr{L}^{-1}\rightarrow f_*\mathscr{M}^v\rightarrow \mathfrak{m}\rightarrow 0$$

since $\dim R^1f_*\mathscr{L}^{-1}=1$. On the other hand $f_*\mathscr{L}^{-1}=\omega_X$ by construction. Therefore $f_*\mathscr{M}^v$ is the unique extension of $\mathfrak{m}$ by ω_X, that means $f_*\mathscr{M}^v$ is the reflexive hull of the holomorphic one forms on the regular locus of X[2]. And $f_*\mathscr{M}$ is the dual of the one forms.

(3. 5) Some remarks and questions. (3. 5. 1) As one has seen in (3. 2) it is easy to compute with invertible sheaves whereas one does not have any numerical criterion for an higher rank sheaf to be full.

(3. 6. 2) Since it is not true that the first Chern class of a full indecomposable sheaf determines it, one might ask whether the Chern polynomial (that means the first Chern class and the rank) determines it. The fact that one can easily produce counterexamples if one allows decomposable sheaves casts some doubts on this.

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