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Variation on Artin's vanishing theorem☆

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À Michael Artin, à l'occasion de son 70-ième anniversaire

Abstract

We give a proof of Artin's vanishing theorem in characteristic zero, based on Deligne's Riemann–Hilbert correspondence.

Keywords: Vanishing theorem; Riemann–Hilbert correspondence

1. Introduction

In [1, Corollaire 3.5], Artin shows that if X is a affine variety over a separably closed field k, and if $\mathcal{F}$ is a constructible sheaf on it, then étale cohomology $H^m_{\text{ét}}(X, \mathcal{F})$ is vanishing for $m \geqslant \dim(X) + 1$. He reduces the proof to dimension 1 by applying base change for proper morphisms. The purpose of this note is to give an analytic proof of Artin's famous vanishing theorem in the analytic category, that is when k is the field of complex numbers, $\mathcal{F}$ is a constructible sheaf of complex vector spaces with possibly infinite monodromies on strata, and the cohomology is the analytic one. Surely, this is not necessary to have an analytic proof to apply Artin's vanishing theorem analytically,

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but this is perhaps of interest to know it exists. Indeed, we give two proofs in the framework of Deligne's Riemann–Hilbert correspondence [3]. The first one is based on a splitting property proven in [4, Proposition 1.2]. The second one is a slightly different form of this and is due to Pierre Deligne. He did not write it down in his Lectures Notes [3], but communicated to us in a discussion on the first proof. It is based on [2, Proposition 5, p. 412]. In both cases, it allows to compute the analytic cohomology of the extension by 0 of a local system on a Zariski open set as the hypercohomology of a suitable algebraic de Rham complex.

2. Artin's vanishing in complex geometry

Theorem 1. *Let X be an affine variety defined over the field of complex numbers, and $\mathcal{F}$ be a constructible sheaf. Then $H^m(X_{\mathrm{an}}, \mathcal{F}) = 0$ for $m > \dim(X)$.*

Proof. If $Y \subset X$ is the support of $\mathcal{F}$, then $H^m(X_{\mathrm{an}}, \mathcal{F}) = H^m(Y_{\mathrm{an}}, \mathcal{F})$ and Y is affine. Thus we may assume that the support of $\mathcal{F}$ is X. Let $\pi : X \to \mathbb{A}^n$ be a Noether normalization, with $n = \dim(X)$. Then $H^m(X_{\mathrm{an}}, \mathcal{F}) = H^m(\mathbb{A}^n_{\mathrm{an}}, \pi_*\mathcal{F})$, and $\pi_*\mathcal{F}$ is a constructible sheaf with support $\mathbb{A}^n$. Thus we may assume $X = \mathbb{A}^n$. Let $j : \emptyset \neq U \hookrightarrow X$ be a Zariski open set so that $\mathcal{F}|_{U_{\mathrm{an}}} = \mathcal{V}$ is a local system. Thus $j_!\mathcal{V} \subset \mathcal{F}$ and the quotient $\mathcal{F}/j_!\mathcal{V}$ is a constructible sheaf with support in dimension $< n$. Thus inducting on $\dim(X)$, we may assume $\mathcal{F} = j_!\mathcal{V}$, as for $\dim(X) = 0$ the theorem is trivial. Let $Y := X \setminus U$. If Y has codimension $\geqslant 2$, then $\mathcal{V}$ is the constant sheaf $\mathbb{C}^r$, where r is the rank of $\mathcal{V}$. From the exact sequence $0 \to j_!\mathbb{C}^r \to \mathbb{C}^r \to \mathbb{C}^r|_Y \to 0$, and the induction, we reduce the theorem to the vanishing of $H^m(\mathbb{A}^n_{\mathrm{an}}, \mathbb{C}) = 0$ for $m > n$, which is trivial. Else Y is a divisor. Then there is an algebraic bundle V on X (which has to be trivial since $X = \mathbb{A}^n$) with an algebraic connection $\nabla : V \to \Omega^1_X(mY) \otimes V$ with meromorphic poles along Y, such that $(V_{\mathrm{an}}|_U)^\nabla = \mathcal{V}$. We choose for (V, ∇) any extension to X of the unique regular singular connection on $X \setminus Y$ [3, Théorème 5.9]. In particular, it is regular singular "at the ∞ of X". Let $\mathcal{I}$ be the ideal sheaf of Y. For $N \in \mathbb{N}, N > n$, one defines the complex

$$\begin{aligned}\mathcal{K}_N : \mathcal{I}^N \otimes V \to \mathcal{I}^{N-1} \otimes V \otimes \Omega^1_X(mY) \to \cdots \\ \cdots \to \mathcal{I}^{N-n} \otimes V \otimes \Omega^n_X(nmY).\end{aligned} \tag{2.1}$$

Lemma 2. *For $N \in \mathbb{N}$ large, $H^m(X_{\mathrm{an}}, j_!\mathcal{V})$ is a direct summand of $\mathbb{H}^m(X, \mathcal{K}_N)$.*

Proof (*See Esnault [4, Proposition 1.2]*). Let $\pi : \mathbb{P} \to \mathbb{P}^n$ be a birational projective morphism, with $\mathbb{P}$ smooth, so that $\bar{Z} + H$ is a normal crossing divisor. Here we denote by Z the inverse image $\pi^{-1}(Y)$, by $\bar{Z}$ its Zariski closure in $\mathbb{P}$ and by H the inverse image $\pi^{-1}(\mathbb{P}^n \setminus \mathbb{A}^n)$. We also set $\lambda : \mathbb{A}^n \to \mathbb{P}^n, \tilde{\lambda} : \pi^{-1}(\mathbb{A}^n) = \mathbb{P} \setminus H \to \mathbb{P}, j : U \to \mathbb{A}^n, \tilde{j} : U \to \mathbb{P} \setminus H$. Let $(\bar{V}, \bar{\nabla})$ be an extension to $\mathbb{P}$ of (V, ∇) on U, so that $\bar{V}$ is locally free and $\bar{\nabla}$ has logarithmic poles along

$\bar{Z}+H$. Then for M large enough, $R\tilde{\lambda}_*\tilde{j}_!V \to (\Omega^\bullet_{\mathbb{P}}(\log(\bar{Z}+H))(-M\bar{Z})\otimes \bar{V})_{\mathrm{an}} \to (\Omega^\bullet_{\mathbb{P}}(\log \bar{Z})(*H)(-M\bar{Z})\otimes \bar{V})_{\mathrm{an}}$ are quasi-isomorphisms ([3, Corollaire 6.10], and via duality [5, (2.9),(2.11)]), thus by GAGA [6], it induces an isomorphism $H^m(\mathbb{P}\setminus H, \tilde{j}_!\mathcal{V})(= H^m(\mathbb{A}^n, j_!\mathcal{V})) \xrightarrow{\cong} \mathbb{H}^m(\mathbb{P}\setminus H, \Omega^\bullet_{\mathbb{P}\setminus H}(\log Z)(-MZ)\otimes \bar{V})$. Taking now any coherent extension V' to $\mathbb{P}^n$ of V, with $\mathcal{I}'$ being the ideal sheaf of the Zariski closure Y' of Y in $\mathbb{P}^n$, one has $j_*\mathcal{K}_N = (\mathcal{I}')^{N-\bullet}\otimes V'\otimes \Omega^\bullet_{\mathbb{P}^n}(\bullet mY')(*(\mathbb{P}^n\setminus\mathbb{A}^n))$. By GAGA again, one has $H^m(\mathbb{A}^n,\mathcal{K}_N) = \mathbb{H}^m(\mathbb{P}^n, (\mathcal{I}')^{N-\bullet}\otimes V'\otimes \Omega^\bullet_{\mathbb{P}^n}(\bullet mY')(*(\mathbb{P}^n\setminus\mathbb{A}^n))) = \mathbb{H}^m(\mathbb{P}^n, ((\mathcal{I}')^{N-\bullet}\otimes V'\otimes \Omega^\bullet_{\mathbb{P}^n}(\bullet mY')(*(\mathbb{P}^n\setminus\mathbb{A}^n)))_{\mathrm{an}})$. The latter group receives $H^m(\mathbb{A}^n_{\mathrm{an}}, j_!\mathcal{V})$ while functoriality implies the existence of

$$\pi^{-1}:\mathcal{K}_N \to R\pi_*(\Omega^\bullet_{\mathbb{P}\setminus H}(\log Z)(-MZ)\otimes \bar{V}),$$

which is an isomorphism on U. This finishes the proof. □

As X is affine, one has $\mathbb{H}^m(X,\mathcal{K}_N) = H^m(\Gamma(X,\mathcal{K}_N)) = 0$ for $m > \dim(X)$. Using Lemma 2 this concludes the proof. □

Remark 3. Of course in Lemma 2, we could replace $X=\mathbb{A}^n$ by any smooth variety X.

We now reproduce a communication by Deligne, which completes Lemma 2. Instead of considering (2.1) for a fixed N, which leads to a splitting of $j_!$, we consider the inverse system of such. More generally, let $\mathcal{V}$ be a local system on a smooth Zariski open $\emptyset \neq U$ of a complex variety X, and let $\mathcal{K}$ be a bounded complex of coherent sheaves, extending $V_0\otimes\Omega^\bullet_U$ to X, where (V_0,∇_0) is the unique algebraic bundle with a regular singular connection on U with underlying $\mathcal{V}$ [3, Corollaire 6.10]. Let $\mathcal{I}$ be a sheaf of ideals with supports $Y := X\setminus U$, and let $(\mathcal{L}_N)_N$ be the projective system of complexes $\mathcal{L}_N$ defined by

$$(\mathcal{L}_N)^p = \mathcal{I}^{N-p}\mathcal{K}^p := \mathrm{Im}(\mathcal{I}^{N-p}\otimes_{\mathcal{O}_X}\mathcal{K}^p \to \mathcal{K}^p). \tag{2.2}$$

For example, for $\mathcal{K} = \Omega^\bullet_X(\bullet mY)\otimes V$ as in the proof of Theorem 1, then $\mathcal{L}_N$ is nearly equal to $\mathcal{K}_N$ of (2.1) (nearly as we have not assumed V to be locally free, thus the tensor product is not necessarily equal to the product in (2.2)). We denote by $j: U\to X$ the open embedding.

Proposition 4 (*Deligne*). *One has*

$$H^m(X_{\mathrm{an}}, j_!\mathcal{V}) = \varprojlim_N \mathbb{H}^m(X,\mathcal{L}_N).$$

Proof (*Deligne*). Let $\sigma:\tilde{X}\to X$ be a projective birational morphism with $\sigma|_U = \mathrm{id}$, with $\tilde{X}$ smooth and $\sigma^{-1}(Y) =: Z$ a normal crossing divisor. Let us set $\mathcal{J} = \sigma^*\mathcal{I}$ and choose (V,∇) an extension to $\tilde{X}$ of (V_0,∇_0) on U. By [2, Proposition 5], the projective system $R^a\sigma_*(\mathcal{J}^N \otimes V \otimes \Omega^p_{\tilde{X}}(\log Z))$ is essentially constant of value 0 for $a > 0$. This means that for $N \geqslant 0$ given, there is a $N' > N$ so that $R^a\sigma_*(\mathcal{J}^{N'} \otimes V \otimes \Omega^p_{\tilde{X}}(\log Z)) \xrightarrow{0} R^a\sigma_*(\mathcal{J}^N \otimes V \otimes \Omega^p_{\tilde{X}}(\log Z))$. Moreover, for all N, there are $N_i, i = 1, 2$ with $(\mathcal{L}_{N_2})^p \to \sigma_*(V \otimes \mathcal{J}^{N_1} \otimes \Omega^p_{\tilde{X}}(\log Z)) \to (\mathcal{L}_N)^p$. This shows that in the proposition, we may assume that X is smooth, $Z = X \setminus U$ is a normal crossing divisor, $\mathcal{K}^\bullet = V \otimes \Omega^\bullet_X(\log Z)$, where (V, ∇) extends (V_0, ∇_0) with V locally free and ∇ with logarithmic poles. Again for all N, there there are $N_i, i = 1, 2$ with $\mathcal{L}_{N_2} \to V \otimes \mathcal{O}_X(-N_1 Z) \otimes \Omega^\bullet_X(\log Z) \to \mathcal{L}_N$. Thus $\varprojlim_N \mathbb{H}^m(X, \mathcal{L}_N) = \varprojlim_N \mathbb{H}^m(X, V \otimes \mathcal{O}_X(-NZ) \otimes \Omega^\bullet_X(\log Z))$. Taking $\bar{X} \supset X$ a good compactification with $\infty = \bar{X} \setminus X$ a normal crossing divisor, and with $(\bar{V}, \bar{\nabla})$ an extension to $\bar{X}$ of (V, ∇) with $\bar{V}$ locally free and $\bar{\nabla}$ with logarithmic poles, one has $\mathbb{H}^m(X, V \otimes \mathcal{O}_X(-NZ) \otimes \Omega^\bullet_X(\log Z)) = \mathbb{H}^m(\bar{X}, \bar{V} \otimes \mathcal{O}_{\bar{X}}(-N\bar{Z}) \otimes \Omega^\bullet_{\bar{X}}(\log \bar{Z})(*\infty))$. Here $\bar{Z} \subset \bar{X}$ is the Zariski closure of $Z \subset X$. Again by GAGA [6],

$$\begin{aligned}&\mathbb{H}^m(\bar{X}, \bar{V} \otimes \mathcal{O}_{\bar{X}}(-N\bar{Z}) \otimes \Omega^\bullet_{\bar{X}}(\log \bar{Z})(*\infty))\\&\quad = \mathbb{H}^m(X_{\mathrm{an}}, (V \otimes \mathcal{O}_X(-NZ) \otimes \Omega^\bullet_X(\log Z))_{\mathrm{an}})\end{aligned}$$

and the latter for N large enough is equal to $H^m(X_{\mathrm{an}}, j_!\mathcal{V})$ by [3, Corollaire 6.10], and via duality [5, (2.9),(2.11)]. $\square$

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