PERIODS FOR IRREGULAR CONNECTIONS ON CURVES.

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Abstract. We define local epsilon factors for periods of meromorphic connections defined over curves over a subfield of the complex numbers, and show a product formula.

1. Introduction

The study of Galois representations in algebraic number theory can crudely be thought of in three parts. At almost all primes, one has the action of the Frobenius automorphism. At a finite set of primes, one has a ramified structure, a Swan conductor, etc. And then finally, one has the "reciprocity" which says that the particular local data comes from a global object.

A geometric model of this theory has developed in recent years, in which Galois representations are replaced by holonomic $\mathcal{D}$-modules on an algebraic curve. The analogue of the punctual frobenius action is the interplay between two rational structures, the algebraic, or de Rham, structure and a choice of transcendental Betti or Stokes structure. Suppose, for example, we are on $\mathbb{A}^1_{\mathbb{Q}}$ and our $\mathcal{D}$-module is the rank 1 connection on $\mathcal{O}_{\mathbb{A}^1}$ given by $\nabla(1) = -d(t^2)$. The fibre at a closed point $x \in \mathbb{A}^1_{\mathbb{Q}}$ has the evident $\mathbb{Q}(x)$-rational structure. Analytically, the horizontal sections define a trivial local system $\mathbb{C} \cdot \exp(t^2)$. If we choose a reduction of structure of this local system to a $\mathbb{Q}$-local system, e.g. by choosing $c \in \mathbb{C}^\times$ and writing $\mathbb{C} \cdot \exp(t^2) = \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q} \cdot c \cdot \exp(t^2)$ our fibre at a closed point x is a collection of $\mathbb{C}$-lines $\ell_\sigma, \sigma : \mathbb{Q}(x) \hookrightarrow \mathbb{C}$. Each ℓ_σ has a $\mathbb{Q}(x)$-structure coming from the algebraic bundle and a $\mathbb{Q}$-structure $\mathbb{Q} \cdot c \cdot \exp(\sigma(x)^2)$.

More generally, when the $\mathcal{D}$-module M restricts to a connection on an open set of the curve X, the reduction of structure is required to be compatible with the Stokes structure at the singular points of M. With this proviso, the de Rham cohomology $H^*_{DR}(X, M)$ inherits two

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reductions of structure. (We adopt the modern convention whereby $H^*_{DR}(X, M)$ sits in degrees $-1, 0, 1$.) Our focus will be on the *global epsilon factor*

$$(1.1)\quad \varepsilon(X, M) := \det H^*_{DR}(X, M) = \det H^0_{DR}(X, M) \otimes \det H^1_{DR}(X, M)^{-1} \otimes \det H^{-1}_{DR}(X, M)^{-1}$$

We view $\varepsilon(X, M)$ as a superline in degree equal the Euler characteristic $\chi(H^*_{DR}(X, M)$. (For more on superlines, see [1].) Techniques for constructing Betti cohomology in this context are discussed in Section 2.3. In the above example, the de Rham cohomology is one dimensional, generated by the class of dt in H^0_{DR}. A generator for the Betti homology is obtained by coupling a horizontal section of the dual connection, $c^{-1}\exp(-t^2)$ to the "rapid decay" path $[-\infty, +\infty]$. The period, or ratio of the DR and Betti structures is in this case

$$(1.2)\qquad \varepsilon_{DR}/\varepsilon_{Betti} = \mathbb{Q}^\times \cdot c^{-1} \int_{-\infty}^{\infty} \exp(-t^2)dt = \mathbb{Q}^\times \cdot c^{-1}\sqrt{\pi}$$

Note that the period depends on the choice of Betti structure on the $\mathcal{D}$-module, i.e. in this case the choice of c.

In general, there will be monodromy, and the Betti structure will only be defined over some finite extension of $\mathbb{Q}$. Consider for example the connection on the trivial bundle on $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$ defined by

$$(1.3)\qquad \nabla(1) = -dt + s\frac{dt}{t}.$$

Here $s \in \mathbb{C} - \mathbb{Z}$. The de Rham structure is defined over $k = \mathbb{Q}(s)$. The horizontal sections are spanned by $\exp(t)t^{-s}$, so the monodromy field is $F = \mathbb{Q}(\exp(2\pi is))$. Again, $H^*_{DR} = H^0_{DR}$ is one dimensional, spanned by dt/t. If one takes as Betti structure on the dual local system the reduction to F given by $F \cdot \exp(-t)t^s$, then $H_{0,Betti}$ is spanned by $\exp(-t)t^s|_\sigma$ where σ is the keyhole shaped path running from $+\infty$ to $\epsilon > 0$, then counterclockwise around a circle of radius ϵ about 0, and finally returning to $+\infty$. We get

$$(1.4)\qquad \varepsilon_{DR}/\varepsilon_{Betti} = k^\times F^\times \int_0^\infty exp(-t)t^s\frac{dt}{t} = k^\times F^\times \Gamma(s).$$

Let k be a field of characteristic 0, and write $K = k((t))$ for the laurent series field. Let E, ∇ be a finite dimensional K-vector space with a k-linear connection. Let $\nu \in \Omega^1_{K/k}$ be a nonzero differential 1-form. The local de Rham ε-line $\varepsilon_{DR}(E, \nu)$ is defined by means of Tate vector spaces and polarized determinants in [1]. When K is the quotient field of the complete local ring $\widehat{\mathcal{O}}_{X,x}$ at a point on a curve and

E (resp. ν) comes by restriction from a holonomic $\mathcal{D}$-module on X (resp. a nonzero meromorphic 1-form on X) we write $\varepsilon_{DR}(E,\nu)_x$ for the local epsilon line to indicate dependence on x.

One of the main results in [1] is a product formula, or reciprocity law identifying (canonically) the global epsilon line with the product of local lines

$$\varepsilon_{DR}(X,E) \cong \bigotimes_{x\in X} \varepsilon_{DR}(E,\nu)_x \tag{1.5}$$

Our main objective will be to Show how the choice of a suitable Betti structure on E will induce Betti structures on the local epsilon factors $\varepsilon_{DR}(E,\nu)_x$. Write $(2\pi i)$ for the superline in degree 0 with de Rham structure $(2\pi i)_{DR} = \mathbb{Q}$ (with a canonical generator) and $(2\pi i)_B = 2\pi i\mathbb{Q} \subset (2\pi i)_{DR} \otimes \mathbb{C}$. Our main result will be the global reciprocity

$$\varepsilon(X,E) \cong (2\pi i)^{(\mathrm{rank}E)(1-g)} \bigotimes_{x\in X} \varepsilon(E,\nu)_x \tag{1.6}$$

compatible with (1.5) and the Betti structures. Here g is the genus of X and rank$E$ refers to the rank at the generic point of X.

This should be compared with the ℓ-adic result (1.7), [7], [12]. Let X be a smooth, complete curve over the finite field $\mathbb{F}_q$. Let E be a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces over X. Epsilon factors in this context are $\overline{\mathbb{Q}}_\ell$-lines with a frobenius action. The global epsilon factor is defined by $\varepsilon_\ell(X,E) := \det(-frob_q | H^*_{et}(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell))$, where $frob_q$ is the geometric frobenius. One has defined local epsilon factors $\varepsilon_\ell(E,\nu)_x$ for $x \in X$ a closed point and ν a nonzero meromorphic 1-form on X. These satisfy the global reciprocity

$$\varepsilon_\ell(X,E) = q^{(1-g)\mathrm{rank}E} \bigotimes_{x\in X} \varepsilon_\ell(E,\nu)_x. \tag{1.7}$$

Here "q" refers to $\mathbb{Q}_\ell$ with frobenius action by multiplication by q.

The content of this paper is as follows. Section 2 is devoted to preliminaries, with subsections on the Levelt-Turritin decomposition, Stokes structures, Betti cohomology for holonomic $\mathcal{D}$-modules on curves, Katz extensions, extending regular singular point connections to the punctured tangent space, local and global classfield theory for holonomic $\mathcal{D}$-modules on curves, and polarized determinants. In section 3 we consider the fourier transform $\mathcal{F}(E)$ for a meromorphic connection E on $\mathbb{P}^1$ which is smooth at infinity. We write $z = 1/t'$, where t' is the dual coordinate, and we identify

$$\mathcal{F}(E) \otimes k((z)) = \oplus_{s\in S} VC(E,s) \tag{1.8}$$

(Corollary 3.8). Here $S \subset \mathbb{A}^1$ is the set of singular points for E and the local vanishing cycles $VC(E,s)$ is a finite dimensional $k((z))$-vector space with connection depending only on the formal connection associated to E at s. We show the determinant connection on $\det VC(E,s)$ has regular singular points upto a simple exponential term which arises from recentering the fourier transform to $s = 0$. (Proposition 3.11). For $\mathcal{V} \subset VC(E,s)$ a suitable $k[[z]]$-lattice,we identify (Proposition 3.9) $\mathcal{V}/z\mathcal{V} = \varepsilon_{DR}(E,dt)$, the local epsilon line as defined in [1].

Sections 4 and 5 apply the method of steepest descent to describe the Betti structure on the local epsilon lines and link it to the global Betti structure on the determinant of cohomology. Again E is a meromorphic connection on $\mathbb{P}^1$ which is smooth at ∞ and we consider the fourier transform $\mathcal{F}(E)$. The decomposition (1.8) is only formal and is false analytically. It does, however, hold asymptotically. We use the Levelt decomposition to study the asymptotic behavior of periods as $z \to 0$.

Section 6 defines the Betti structure on the local epsilon line $\varepsilon(E,dt)$. Here E is a connection over $k((t))$. Because the determinant connection $\det VC(E)$ is regular singular, one gets a rank 1 connection on the punctured tangent line $T - \{0\}$ to $\operatorname{Spec} k[[z]]$ at $z = 0$. By definition, $\varepsilon_B(E,dt)$ is the fibre of this connection at the point d/dz. To understand the Betti structure, imagine extending E to an analytic disk (with arbitrary Stokes structure). A choice of Betti structure on the horizontal sections taken to be compatible with the Stokes structure and defined over some field $M \subset \mathbb{C}$ permits an asymptotic expansion of the period matrix associated with the local fourier transform of E. (See equations (6.6) and (6.7).)

Section 7 gives (cf. (7.1)) the general definition of the local epsilon factors $\varepsilon(E,\nu)$ for $0 \neq \nu \in \Omega^1_{k((t))/k}$. The definition is then extended to the case E a holonomic $\mathcal{D}$-module on $\operatorname{Spec} k[[t]]$. Finally, the projection formula $\varepsilon(E,\pi^*\nu) = \varepsilon(\pi_*E,\nu)$ is proven, where $\pi : \operatorname{Spec} k((t)) \to \operatorname{Spec} k((u))$ is defined by $\pi^*u = t^r$, E is a virtual connection of rank 0 on $\operatorname{Spec} k((t))$, and $0 \neq \nu \in \Omega^1_{k((u))/k}$. The proof given uses the reciprocity law proven in Section 8.

Finally, Section 8 is devoted to the proof of the reciprocity law (product formula) (1.6). The proof is first given when the curve is $\mathbb{P}^1$. that case is used to finish the proof of the projection formula from Section 7, and finally the proof is given in general, using the projection formula.

The Ansatz of the ideas developed here was presented by Pierre Deligne at his seminar at IHES in 1984 [6]. For the rank 1 case with one singularity, it was explained that one should use the steepest descent method to find an asymptotic expansion for computing the determinant of the periods of de Rham cohomology. Our work relies on this idea. Our work also relies very heavily on the ideas of Laumon [12]. In particular, Laumon's insight that Witten's fourier transform method could be applied to the study of epsilon factors is central to this paper. It is really a source of wonder how closely the $\mathcal{D}$-module and ℓ-adic pictures correspond. One need only compare the reciprocity laws (1.6) and (1.7)!

2. Tools and Techniques

In this section we discuss a potpourri of results we will need on connections in dimension 1, including formal, local analytic (punctured disk), and global algebraic techniques. For the most part these are well documented in the literature and we simply give a good reference together with a discussion of how the result will be applied.

2.1. **Levelt-Turrittin.** Let $k \supset \mathbb{Q}$ be a field of characteristic 0. Let (E, ∇_E) be a connection over $F := k((t))$. This means that E is a vector space of dimension $r < \infty$ over F, and that $\nabla_E : E \to E \otimes_F \Omega^1_{F/k}$ is an additive operator which satisfies the Leibniz rule $\nabla(fe) = f\nabla(e) + df \otimes e$ for $e \in E$, $f \in F$. We will say that E is *indecomposable* (resp. *irreducible*) if E cannot be written as a nontrivial direct sum of subconnections $E = \oplus E_i$ (resp. if E does not admit a nontrivial subconnection $E' \subset E$). The connection ∇_E is *regular singular* if there is a *lattice* $\mathcal{V} \subset E$ over $k[[t]]$, that is a free $k[[t]]$-submodule of rank r which spans E, so that $\nabla_E(\mathcal{V}) \subset \frac{dt}{t} \otimes_k \mathcal{V}$. A basic example of a regular singular point connection is the *nilpotent Jordan block* of size n

$$(2.1) \qquad N^{(n)} := F^{\oplus n}; \quad \nabla_{N^{(n)}} = d + \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \frac{dt}{t}.$$

Here the lattice $\mathcal{V}$ is simply $k[[t]]^{\oplus n}$. Let (E_i, ∇_{E_i}) be connections for $i = 1, 2$. Then the tensor product $E_1 \otimes_F E_2$ has a natural connection structure $\nabla(e_1 \otimes e_2) = \nabla_{E_1}(e_1) \otimes e_2 + e_1 \otimes \nabla_{E_2}(e_2)$. If $\pi : \operatorname{Spec} F' \to \operatorname{Spec} F$ is a finite map, and E' is an F'-connection, then $\pi_* E'$ has an F-connection. Indeed, $\Omega^1_{F'/k} = \Omega^1_{F/k} \otimes_F F'$, so by projection formula $\pi_*(E' \otimes \Omega^1_{F'/k}) \cong (\pi_* E') \otimes_F \Omega^1_{F/k}$.

Connections over F admit a Jordan-Hölder type decomposition due to Levelt-Turrittin ([13], Théorème 1.2). Since k is assumed to be algebraically closed in loc. cit., we slightly modify the formulation as in [1], sections 5.9–5.10 to drop this assumption.

Theorem 2.1. *[*Levelt-Turrittin*] Any connection* (E, ∇_E) *over* $F = k((t))$ *with* k *a field of characteristic 0, admits a canonical isotypical decomposition*

$$(E, \nabla_E) = \oplus_{\mathcal{L} \text{ irreducible}}(E^{\mathcal{L}}, \nabla_E^{\mathcal{L}}),$$

where $\mathcal{L} := (L, \nabla_L)$ *is irreducible.* (E^L, ∇_E^L) *is characterized by the property that all its irreducible subquotients are isomorphic to* $\mathcal{L}$*. One has an isomorphism*

$$(E^{\mathcal{L}}, \nabla_E^{\mathcal{L}}) = \oplus_n(\mathcal{L} \otimes (N^{(n)}, \nabla_{N^{(n)}}))$$

for some n *as in* (2.1)*. Furthermore, given* $\mathcal{L}$*, there exists a finite extension* $\pi : \operatorname{Spec} F' \to \operatorname{Spec} F$ *and a* rank 1 *connection* $\mathcal{L}'$ *on* F' *such that*

$$\mathcal{L} \cong \pi_*(\mathcal{L}'); \quad \operatorname{rank}(\mathcal{L}) = [F' : F].$$

Remark 2.2. Usually one formulates the theorem in the way one does for the Jordan-Hölder theorem in linear algebra: given (E, ∇_E), there is a finite extension $\pi : \operatorname{Spec} F' \to \operatorname{Spec} F$ and rank 1 connections $(L_\omega, \nabla_{L_\omega})$ on F' such that

$$(E, \nabla_E) \otimes_F F' \cong \oplus_\omega(L_\omega, \nabla_{L_\omega}) \otimes (\oplus_n(N^{(n)}, \nabla_{N^{(n)}})).$$

2.2. **Stokes Structures.** Suppose now (E, ∇_E) is a meromorphic connection over a punctured disk $D^* \subset \mathbb{C} \setminus \{0\}$ centered at 0. More precisely, let $\mathbb{C}\{z\}$ be the ring of germs of analytic functions at 0 on $\mathbb{C}$, and let $\mathbb{C}\{\{z\}\} = \mathbb{C}\{z\}[z^{-1}]$. Then E is a free $\mathbb{C}\{\{z\}\}$-module of finite rank with a connection $\nabla_E : E \to dz \otimes_{\mathbb{C}} E$. The data (E, ∇_E) is more refined than that of an analytic connection on D^* because ∇_E is assumed to extend to a connection without essential singularities at 0. The classical Riemann-Hilbert correspondence yields an equivalence of tensor categories between analytic connections and local systems on D^*. Deligne's Riemann-Hilbert correspondence yields an equivalence of tensor categories between regular singular connections on D^*, that is those for which a lattice $\mathcal{V} \subset E$ exists over $\mathbb{C}\{z\}$ with $\nabla_E(\mathcal{V}) \subset \frac{dz}{z} \otimes_{\mathbb{C}} \mathcal{V}$, and local systems. Stokes structures extend the Riemann-Hilbert correspondence

(2.2)

$$\begin{array}{ccc}
\{\text{reg. sing. connections}\} & \xrightarrow{\text{equivalence}} & \{\text{local systems}\} \\
\cap\downarrow & & \cap\downarrow \\
\{\text{mero. connections}\} & \xrightarrow{\text{equivalence}} & \{\text{local systems} + \text{Stokes structure}\}
\end{array}$$

We recall the definition of *Stoked structures* which are suitable generalized filtrations on constructible sheaves of $\mathbb{C}$-vector spaces on a circle S. Our reference is [13], chapter IV, and the references cited there. Let $\pi : \widetilde{D} \to D$ be the real blowup of the origin on the disk D. π is an isomorphism outside $\pi^{-1}(0)$, and $\pi^{-1}(0) = S$ is a circle. $\widetilde{D} \cong S \times [0, 1)$ with polar coordinates (r, θ). We have a diagram with evident maps

$$\begin{array}{ccccc} D^* & \xrightarrow{\tilde{j}} & \widetilde{D} & \xleftarrow{i} & S \\ \| & & \downarrow \pi & & \downarrow \\ D^* & \xrightarrow{j} & D & \longleftarrow & \{0\} \end{array} \tag{2.3}$$

Let $\mathcal{E}$ denote the local system on D^* of horizontal sections of E. Let $V = i^*\tilde{j}_*\mathcal{E}$ be the corresponding sheaf on S. It is a local system. We endow V with a sort of a filtration corresponding to the growth of sections as one approaches 0 in a radial direction. More precisely, let

$$\Omega = \{\sum_{-n}^{\infty} a_k z^{\frac{k}{p}} dz\} \Big/ \{\sum_{\frac{k}{p} \geq -1} a_k z^{\frac{k}{p}} dz\} \tag{2.4}$$

where $n, p \in \mathbb{N}$. We view Ω as a local system on S. The generator of $\pi_1(S)$ acts on a_k by multiplication with $\exp(\frac{2\pi k}{p})$. If $\omega, \eta \in \Omega$ and $\theta \in S$, we say $\omega \leq_\theta \eta$ if $\exp(\int^z (\omega - \eta)) = O(|z|^{-N}),\ z = re^{i\tau},\ r \to 0$ for τ in some small sector around θ. This is equivalent to saying that the solution of the rank one connection $(\mathbb{C}\{\{z^{\frac{1}{p}}\}\}, d - (\omega - \eta))$ has moderate growth in some small sector around θ. Note that given $\omega - \eta$, there are finitely many θ's for which neither $\omega \leq_\theta \eta$ nor $\eta \leq_\theta \omega$. Those θ's define lines in D^* which are called *Stokes lines* for $\omega - \eta$. For example, if $\omega = \frac{dz}{z^2}$ and $\eta = 0$, then $\omega \leq_\theta \eta$ for $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, $\eta \leq_\theta \omega$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and there is no inequality defined for $\theta = \pm\frac{\pi}{2}$, thus $\theta = \pm\frac{\pi}{2}$ are the two Stokes lines for $\frac{dz}{z^2}$. Then if θ is a Stokes line for $\omega - \eta$, for θ' closed to the half line defined by θ and on one side of it, one has $\omega \leq_{\theta'} \eta$ and $\omega \neq \eta$ while on the other side, one has $\eta \leq_{\theta'} \omega$ and $\omega \neq \eta$. One writes then *strict inequalities* $\omega <_{\theta'} \eta$ and $\eta <_{\theta'} \omega$.

Classical Levelt-Turrittin theory gives a decomposition

$$\pi^*(E, \nabla_E) = \oplus_\omega (L_\omega \otimes R_\omega) \tag{2.5}$$

where $\pi : D^*_{z^{\frac{1}{p}}} \to D^*_z$ is the ramified cover of degree p for a suitable $p \geq 1$. Here $L_\omega = (\mathbb{C}\{\{z^{\frac{1}{p}}\}\}, d + \omega)$ for ω a 1-form in $z^{\frac{1}{p}}$, and the R_ω are sums of connections as in (2.1). On small sectors, the map π admits sections and we can descend the decomposition (2.5). In this way we can write on stalks

$$V_\theta = \oplus_{\omega\in\Omega} V_{\omega,\theta}. \tag{2.6}$$

The decomposition (2.6) is not canonical, due to the monodromy operation. However, the filtration defined by

$$V_\theta^\eta := \oplus_{\omega\leq_\theta \eta} V_{\omega,\theta} \tag{2.7}$$

is canonical, as the sectors of moderate growth for the solutions of $(\mathbb{C}\{\{z^{\frac{1}{p}}\}\}, d-(\omega-\eta))$ depend only on the most polar coefficient a_n of $(\omega-\eta)$ up to multiplication with $\mathbb{R}^{>0}$. This is not a filtration of the sheaf V in the classical sense because the index set Ω is a local system. In other words, if σ generates the monodromy, then

$$V^{\sigma(\eta)} = \sigma(V^\eta).$$

For V a local system on the circle S, a filtration $\{V^\eta\}_{\eta\in\Omega}$ is a *Stokes filtration* if locally at every $\theta \in S$ there exist $V_{\omega,\theta}$ such that $V_\theta^\eta = \oplus_{\omega\leq_\theta\eta} V_{\omega,\theta}$, where the direct sum is taken over all ω such that $\omega \leq_{\theta'} \eta$ for all θ' in a small sector around θ. We also call this Stokes filtration a *Stokes structure*. Let

$$\Sigma : \{\text{mero. connections on } D^*\} \to \{\text{local systems on } S + \text{Stokes structure}\} \tag{2.8}$$

be the above functor.

One may also talk of a *Stokes gradation* $V = \oplus gr_\omega V$. There is an evident functor gr from Stokes filtrations to Stokes gradations given by

$$gr_\omega V := V^\omega / \sum_{\eta<\omega} V^\eta.$$

It turns out that the composition $gr\circ\Sigma$ factors through the completion functor $\widehat{\ }$: $\{\text{mero. connections on } D^*\} \to \{\mathbb{C}((z))\text{-connections}\}$. The main result in the theory of Stokes structures is

Theorem 2.3. *One has a commutative diagram of functors*

(2.9)

$$\begin{array}{ccc} \{\textit{mero. connections on } D^*\} & \xrightarrow{\Sigma} & \{\textit{loc. syst. on } S + \textit{Stokes str.}\} \\ \big\downarrow \widehat{\ } & & \big\downarrow gr \\ \{\mathbb{C}((z))\textit{-connections}\} & \xrightarrow{\widehat{\Sigma}} & \{\textit{loc. syst. on } S + \textit{Stokes grad.}\} \end{array}$$

The functors Σ and $\widehat{\Sigma}$ are equivalences of category.

Remark 2.4. (i) Rank one meromorphic connections, or regular singular connections have a trival Stokes structure.

(ii) One we have fixed the local parameter z, the functor gr has an evident right inverse, by simply viewing the graded object as a filtered object. It follows from the theorem that there is a preferred lifting of a formal connection to a meromorphic connection on the punctured disk. This is simply the restriction to the punctured disk of the Katz lifting defined by the choice of z of the formal connection to $\mathbb{G}_m$ discussed below.

(iii) Let $K \subset \mathbb{C}$ be a subfield. One says that the (graded or filtered) *Stokes structure is defined over* K if the following conditions are fulfilled.

1) The monodromy is defined over K, that is the image of $\pi_1(D^*)$ via the representation lies in a complex conjugate of $GL(r, K)$.
2) Condition i) implies in particular that the local system V on S is defined over K as well. One requires that for all $\theta \in S$ and all $\eta \in \Omega$ in (2.4), the filtration V_θ^η from (2.7) is a filtration defined over K.

(iv) For rank 1 meromoprhic connections or regular singular connections, the condition for the Stoke's structure to be defined over $K \subset \mathbb{C}$ is reduced to (iii), 1).

2.3. **Cohomology.** Let X be a Riemann surface, and let $T \subset X$ be a finite set of points. Let (E, ∇) be an analytic connection on $X \setminus T$ which is meromorphic at points of T. Let

$$DR((E, \nabla)) := (E \otimes_{\mathcal{O}_X} \Omega_X^*)[1]$$

be the shifted analytic de Rham complex of sheaves for the complex topology for E on X. We write

$$H_{DR}^*(X, (E, \nabla)) := \mathbb{H}^*(X, DR(E)) \tag{2.10}$$

for de Rham cohomology. By a fundamental theorem of Malgrange ([13], Théorème 3.2, p. 61, see Theorem 2.5)), a reduction to a subfield $K \subset \mathbb{C}$ of the Stokes structure induces a K-structure on de Rham cohomology.

On the other hand, if $(X, (E, \nabla))$ is defined over a subfield $k \subset \mathbb{C}$, then de Rham cohomology is a k-vector space.

Theorem 2.5 (Malgrange)**.** *Let X be a Riemann surface, $T \subset X$ be a finite set of points, and (E, ∇) be a meromorphic connection, smooth on $X \setminus T$. Let $\pi : \widetilde{X} \to X$ be the real blowup at points of T, and $\tilde{j} : X \setminus T \hookrightarrow \widetilde{X}$ be the natural inclusion. Let $\mathcal{E}$ be the local system of horizontal sections of (E, ∇) on $X \setminus T$, and let $\mathcal{E}^0 \subset \tilde{j}_*\mathcal{E}$ be the subsheaf of sections with moderate growth at points of $S = \pi^{-1}(T)$.*

(Thus $\mathcal{E}^0 = \tilde{j}_*\mathcal{E}$ off S, and $\mathcal{E}^0|S = (\tilde{j}_*\mathcal{E}|S)^0$, where the superscript 0 on the right is in the sense of the filtration defined in (2.7).) Then there is a natural quasi-isomorphism

$$R\pi_*(\mathcal{E}^0)[1] \approx DR(E).$$

Corollary 2.6. *With notation as above, let $K \subset \mathbb{C}$ be a subfield, and suppose we are given a reduction of structure of the local system $\mathcal{E}$ to K which is compatible with the Stokes structures at the points of T. Then $H^*_{DR}(X, (E, \nabla))$ inherits a natural K-structure, in addition to its k-vector space structure if $((X, (E, \nabla))$ is defined over $k \subset \mathbb{C}$.*

Remark 2.7. One may reformulate Malgrange's theorem in terms of homology. Let $(E^\vee, \nabla^\vee)$ be the dual connection, and let $\mathcal{E}^\vee$ be the dual local system on $X\backslash T$. The homology $H^{rd}_*(X, (E^\vee, \nabla^\vee))$ is defined using chains on $\widetilde{X}$ coupled to sections of $\mathcal{E}^\vee$ with rapid decay on S. For X compact, one has (taking into account the shift in cohomological degree for de Rham cohomology) a perfect pairing

$$H^{rd}_i(X, E^\vee) \times H^{i-1}_{DR}(X, (E, \nabla)) \to \mathbb{C}; \quad (\varepsilon^\vee|\gamma \times \eta) \mapsto \int_\gamma \langle \varepsilon^\vee, \eta\rangle.$$

For details, see [3].

2.4. **Katz extension.** In this section we consider ways to extend an F-connection (E, ∇) to $\mathbb{G}_m/k$, where F is a powerseries field over a field k of characteristic 0.

Theorem 2.8 (Katz extension theorem ([11])). *Fix a local parameter $t \in F$, and use it to identify F as the quotient field of $\widehat{\mathcal{O}}_{\mathbb{A}^1,0}$, where t is the standard coordinate of $\mathbb{A}^1$. Then there exists a unique extension of (E, ∇) to a connection $(\mathbb{E}, \nabla)$ on $\mathbb{G}_m/k$ with the following property. There exists a finite extension k'/k and an integer $N \geq 1$ such that, writing $\iota_N : \mathbb{G}_m \to \mathbb{G}_m$ for the N-th power map, the pullback decomposes*

$$\iota_N^*((\mathbb{E}, \nabla) \otimes k') \cong \oplus_i((L_i, \nabla_{L_i}) \otimes (N^{(n_i)}, \nabla_{N^{(n_i)}})). \tag{2.11}$$

Here the (L_i, ∇_{L_i}) are rank 1 algebraic connections on $\mathbb{G}_m/k'$, without singularities on $\mathbb{G}_m/k'$, with regular singular points at $t = \infty$ and the $(N^{(n)}, \nabla_{N^{(n)}})$ are defined by (2.1) over $\mathbb{G}_m/k$, so without singularities on $\mathbb{G}_m/k$ and regular singular at $t = \infty$ (and of course at $t = 0$ as well).

Remark 2.9. (1) A rank 1 connection (L, ∇) on $\mathbb{G}_m$ is necessarily trivial as a $\mathcal{O}_{\mathbb{G}_m}$-module. Choose a gauge (basis) ℓ. Then

$$\nabla_L(\ell) = \ell \otimes (\sum_{n=-p}^{q} a_n t^n dt).$$

The connection has regular singular points at ∞ if and only if $a_n = 0$ for all $n \geq 0$. Thus, for (E, ∇_E) rank 1 with gauge e and connection $\nabla_E(e) = e \otimes (\sum_{n=-p}^{\infty} a_n t^n dt)$, the existence of a Katz extension follows from the fact that in the gauge $e' = \exp(-\sum_{n\geq 0} \frac{a_n}{n+1} t^{n+1}) \cdot e$ the connection has the form $\nabla(e') = e' \otimes (\sum_{n=-p}^{-1} a_n t^n dt)$.

(2) It follows easily from (2.11) that the restriction of the Katz extension to a punctured disk D^* about 0 coincides with the canonical extension of Malgrange (Remark 2.4 (i)).

2.5. **Extension to punctured tangent space.** Suppose now that (E, ∇) is a regular singular point connection over $F \cong k((t))$. Let T be the tangent space at the origin to the discrete valuation ring R associated to F, and set $T^* := T \setminus \{0\}$. Writing $\mathfrak{m} \subset R$ for the maximal ideal, we have $T = \mathrm{Spec}\,(\oplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1})$. Let $\Xi \subset E$ be a lattice such that $\nabla_E(\Xi) \subset \Xi \frac{dt}{t}$. We have $\nabla_{t\frac{d}{dt}}(t^n \Xi) \subset t^n \Xi$ so $\nabla_{t\frac{d}{dt}}$ induces a regular singular connection on $\mathrm{gr}^\bullet \Xi := \oplus_{n\geq 0} t^n \Xi / t^{n+1} \Xi$.

If e is a k-basis of $\Xi/t\Xi$, we may view e as an $\mathcal{O}_T$-basis of $\mathrm{gr}^\bullet \Xi$. The resulting connection has the form $d + A_0 \frac{d\bar{t}}{\bar{t}}$ with A_0 a constant matrix, where $\bar{t} \in \mathfrak{m}/\mathfrak{m}^2$ is the image of t. The module $\mathrm{gr}^\bullet \Xi$ depends on the choice of lattice Ξ, but the resulting connection $\mathrm{gr}^\bullet \Xi \otimes_{\mathcal{O}_T} \mathcal{O}_{T^*}$ on T^* depends only on (E, ∇_E).

Proposition 2.10. *With notation as above, suppose with respect to a basis e of E, the connection matrix of (E, ∇) has the form $d + (A_0 + A_1 t + \ldots) \frac{dt}{t}$. Let $\Xi \subset E$ be the lattice spanned by the basis. Assume no two eigenvalues of A_0 differ by a nonzero integer. Then the pullback of the graded connection $\mathrm{gr}^\bullet \Xi \otimes_{\mathcal{O}_T} \mathcal{O}_{T^*}$ to the formal powerseries field $k((t))$ at 0 is isomorphic to (E, ∇) and has equation $d + A_0$ in the basis of $\mathrm{gr}^\bullet \Xi \otimes_{\mathcal{O}_T} k((t))$ induced by e.*

Proof. Let $r = \mathrm{rank}\, E$. We will only really use this result in the case $r = 1$, where it is easy. Briefly, the general argument is to look for gauge transformations which kill $A_1, \ldots, A_N$. Assume inductively that $A_1 = \ldots = A_{N-1} = 0$. Write $H = I + H_N t^N$ where H_N is an $r \times r$ matrix, and let $A = A_0 + A_N t^N + \ldots$. We want $HAH^{-1} + tdHH^{-1} = A_0 + B_{N+1} t^{N+1} + \ldots$. This works out to $(ad_{A_0} - N) H_N = A_N$. The eigenvalues of ad_{A_0} are differences of eigenvalues of A_0. If these are distinct from $N \in \mathbb{N} \setminus \{0\}$, we may solve for H_N. $\square$

2.6. **Local Classfield theory.** In this section Λ will be a complete, equicharacteristic 0 discrete valuation ring with quotient field K. If we choose a parameter, we may identify

$$\Lambda = k[[t]], \ K = k((t)).$$

Recall that an *ind-scheme* is a collection $\{X_\alpha, \nu_{\beta\alpha} : X_\alpha \to X_\beta\}$ of schemes with transition morphisms. As a functor on k-algebras, it is defined by $R \mapsto \varinjlim_{\nu_{\alpha\beta}(R)} X_\alpha(R)$, where the right hand term means the inductive limit over the inductive system of points defined by the transition morphisms. Similarly, a *group ind-scheme* is an ind-scheme so that for each α, there is a $\beta \geq \alpha$ and a morphism $X_\alpha \times X_\alpha \xrightarrow{\mu_{\alpha,\beta}} X_\beta$ so that for each R, $\varinjlim_{\nu_{\alpha\beta}(R)} X_\alpha(R)$ becomes a group for

$$\mu : \varinjlim_{\nu_{\alpha\beta}(R)} X_\alpha(R) \times \varinjlim_{\nu_{\alpha\beta}(R)} X_\alpha(R) \to \varinjlim_{\nu_{\alpha\beta}(R)} X_\alpha(R)$$

induced by $\mu_{\alpha,\beta}(R)$.

We write $F^\times$ for the group ind-scheme defined as follows. We consider $\alpha \in \mathbb{N}$, and X_α defined by its points

$$X_\alpha(R) = \sqcup_{0 \leq \beta \leq \alpha} (t^{-\beta} R[[t]])^\times.$$

The transition maps are the obvious inclusions $\nu_{\beta,\alpha}$ on the disjoint union for $\alpha \leq \beta$. In this case, we can easily describe the k-algebra points functor

$$R \mapsto (R \hat{\otimes} K)^\times = R[[t]][t^{-1}]^\times.$$

Recall that a *Picard category* is a tensor category where all objects and all morphisms are invertible. For example, the category of rank 1 bundles with a connection on a variety is a Picard category. Another example is defined by two abelian groups A, B and a homomorphism $d : A \to B$ between them. Then the objects of the category are the elements of B, and tensor product is the group law in B. $\mathrm{Hom}(b, b') = \emptyset$ if $b - b'$ does not lie in the image of d, and if it does, then $\mathrm{Hom}(b, b') = d^{-1}(b - b')$. Another example is given by the category of invariant rank 1 bundles on $F^\times$. A bundle L on the ind-scheme is a collection of bundles L_α on X_α with transition morphisms $\nu_{\beta\alpha}(L) : L_\alpha \to L_\beta$ lifting $\nu_{\alpha\beta}$. It is invariant if

$$\mu^* L \cong p_1^*(L) \boxtimes p_2^*(L)$$

where $p_1 : F^\times \times_k F^\times \to F^\times$ are the two projections, and μ is the multiplication.Such an isomorphism is fixed once one fixes a trivialization of the fibre of L over $1 \in F^\times$.

Proposition 2.11. *There exists a canonical equivalence of categories*

$$a : \mathrm{Pic}^\nabla(K) \to \mathrm{Ext}^\nabla(F^\times, \mathbb{G}_m)$$

between the Picard category of rank 1 *connections on* Spec K *and the Picard category of invariant rank* 1 *connections on* $F^\times$*.*

Proof. Let $\mathfrak{P}$ be the Picard category defined by the two-term complex $K^\times \xrightarrow{d\log} \Omega^1_{K/k}$. The functor

$$\mathfrak{P} \to \mathrm{Pic}^\nabla(K)$$

defined on objects by

$$\alpha \in \Omega^1_{K/k} \mapsto (K, d + \alpha)$$

is an equivalence of category. Indeed choosing a basis e of a K-line $\mathcal{L}$ identifies a connection ∇ on $\mathcal{L}$ with $\nabla(e) \in \Omega^1_{K/k}$ so the functor is essentially surjective. Furthermore, one has

$$\mathrm{Hom}_{\mathfrak{P}}(\alpha, \beta) = \mathrm{Hom}_{\mathrm{Pic}^\nabla(K)}((K, d + \alpha), (K, d + \beta)).$$

Let $\mathfrak{Q}$ be the Picard category defined by the two-term complex $\mathrm{Hom}(F^\times, \mathbb{G}_m) \xrightarrow{d\log} \omega_{F^\times}$ where $\omega_{F^\times}$ is the space of translation-invariant forms and, for $\phi : F^\times \to \mathbb{G}_m$, we define $d\log(\phi) = \phi^*(\frac{dt}{t})$. The functor

$$\mathfrak{Q} \to \mathrm{Ext}^\nabla(F^\times, \mathbb{G}_m)$$

defined on objects by

$$\omega \in \omega_{F^\times} \mapsto (\mathcal{O}_{F^\times}, d + \omega)$$

is an equivalence of categories. Indeed, a translation invariant line bundle on $F^\times$ is trivializable. Choosing a basis e for the line bundle identifies a translation invariant connection ∇ on $\mathcal{O}_{F^\times} \cdot e$ with a translation invariant differential form $\nabla(e) \in \omega_{F^\times}$. Furthermore, one has $\mathrm{Hom}_{\mathfrak{Q}}(\omega, \eta) = \mathrm{Hom}_{\mathrm{Pic}^\nabla(K)}((\mathcal{O}_{F^\times}, d + \omega), (\mathcal{O}_{F^\times}, d + \eta))$.

So we are reduced to defining an equivalence of categories

$$\mathfrak{a} : \mathfrak{P} \to \mathfrak{Q}.$$

This is equivalent to constructing an isomorphism

$$\begin{array}{ccc} K^\times & \xrightarrow{d\log} & \Omega^1_{K/k} \\ {\scriptstyle a^0}\big\downarrow & & {\scriptstyle a^1}\big\downarrow \\ \mathrm{Hom}(F^\times, \mathbb{G}_m) & \xrightarrow{d\log} & \omega_{F^\times} \end{array}$$

between the two complexes defining $\mathfrak{P}$ and $\mathfrak{Q}$ as follows. The Contou-Carrère symbol $\{,\} : F^\times \times F^\times \to \mathbb{G}_m$ is a perfect pairing, so

$$K^\times = F^\times(k) \cong \mathrm{Hom}(F^\times, \mathbb{G}_m).$$

We define $a^0(f) = \{f, *\} \in \mathrm{Hom}(F^\times, \mathbb{G}_m)$. In degree 1, $a^1 : \Omega^1_{K/k} \to \omega_{F^\times}$ is given by

$$a^1(\nu) \in \mathrm{Hom}(F, \mathbb{G}_a) = \mathrm{Hom}(t_{Hom(F^\times,\mathbb{G}_m)}, \mathbb{G}_a) \tag{2.12}$$
$$a^1(\nu)(g) = -\mathrm{Res}(g\nu)$$

Note that $a^1(\frac{dx}{x})(g) = -\mathrm{Res}(g\frac{dx}{x}) = \{x, 1+g\epsilon\}$ so the requisite diagram is commutative. □

We will need a somewhat more elaborate version of the above equivalence of categories, taking into account Betti structures. For this, let $D \subset \mathbb{C}$ be a disk about 0, $D^* = D \setminus \{0\}$, take $k = \mathbb{C}$ and identify K with the formal completion at 0 of the field of meromorphic functions on D. Let R be the ring of germs of meromorphic functions on D^*. Let Ω^1_R be the corresponding module of Kähler differentials. The natural inclusion of complexes $\{R^\times \xrightarrow{d\log} \Omega^1_R\} \hookrightarrow \{K^\times \xrightarrow{d\log} \Omega^1_K\}$ gives an equivalence

$$\mathrm{Pic}^\nabla_{\mathrm{mero}}(R) \cong \mathrm{Pic}^\nabla(K)$$

of Picard categories.

Let $L \in \mathrm{Ob}\ \mathrm{Pic}^\nabla(K)$, and assume we are given $L_{\mathrm{an}} \in \mathrm{Ob}\ \mathrm{Pic}^\nabla_{\mathrm{mero}}(R)$ restricting to L. Fix a gauge e for L_{an} and let $\nabla e = \eta e$ with $\eta \in \Gamma(D^*, \Omega^1_{\mathrm{mero}})$ (differential forms analytic on D^* with meromorphic extension across 0). Let $\mathcal{A} \supset R$ be the ring of germs of functions which are analytic on D^* but with possible essential singularities at 0. Let $\mathcal{A}^\times \subset \mathcal{A}$ be the group of invertible functions. Finally, let $\mathcal{B}^\times \supset \mathcal{A}^\times$ be the larger group of multivalued functions g such that for $z = \rho\exp(2\pi i\theta)$, one has $g(\rho\exp(2\pi i(\theta+1))) = c \cdot g(z)$ for some $c \in \mathbb{C}^\times$. The classical theory of differential equations implies the existence of $b \in \mathcal{B}^\times$ such that $b \cdot e$ is horizontal, or in other words, $\frac{db}{b} = -\eta$.

Let $\mathbb{L} \in \mathrm{Ext}^\nabla(F^\times, \mathbb{G}_m)$ correspond to L as above. Suppose for a moment that the horizontal section $b \cdot e$ above has no monodromy, i.e. $b \in \mathcal{A}^\times$. Then we may view b as a map $K^\times = F^\times(k) \to \mathbb{C}^\times$ by the dilogarithm formula ([8], (2.7.2))

$$(b, f] := \exp\Big(-\frac{1}{2\pi i} \int_{x_0}^{x_0} \log f\ \frac{db}{b}\Big) \cdot b(x_0)^{\mathrm{ord}(f)}. \tag{2.13}$$

(The integration is taken on a circle about 0 in D. The point x_0 lies on the circle, and the integral is independent of the choice of x_0 or the circle. Note that a priori f is a formal power series, but the expression only depends on f modulo degree N for $N >> 0$ so we can make sense of $\log f$ as a function on the circle.) If we pullback the invariant form $\frac{dt}{t}$ on $\mathbb{C}^\times$ by this map, we obtain the translation-invariant 1-form on $F^\times$ given by

$$g \in F \mapsto \exp(\mathrm{Res}(\log(1+g\epsilon)\eta)) = \mathrm{Res}(g\eta)\epsilon. \tag{2.14}$$

It follows from (2.12) that the map $f \mapsto (b, f]$ in (2.13) gives a horizontal section of $\mathbb{L}$. When b is multiple-valued, the recipe is the same. Indeed, (2.14) still makes sense and one still gets a horizontal section of $\mathbb{L}$. We have proven

Proposition 2.12. *The equivalence of categories*

$$a : \mathrm{Pic}^\nabla(K) \to \mathrm{Ext}^\nabla(F^\times, \mathbb{G}_m)$$

in Proposition 2.11 extends to an identification

$$a_{\mathrm{an}} : L^\nabla_{\mathrm{an}} \cong a(L)^\nabla_{\mathrm{an}} \tag{2.15}$$

of horizontal sections over some large ring of multivalued analytic functions $\mathcal{B} \supset K$.

Proposition 2.13. *Let $\phi \in \Lambda\hat{\otimes}\Lambda \cong k[[t,T]]$ be a defining equation for the divisor $\{t = 0\} \setminus \Delta$ in $\mathrm{Spec}\,\Lambda\hat{\otimes}\Lambda$ (Δ the diagonal); i.e. $\phi = u \cdot \frac{t}{t-T}$ for a unit u in $\Lambda\hat{\otimes}\Lambda$. We view ϕ as a map $\mathrm{Spec}\,K \to F^\times$. Then $\phi^* : \mathrm{Ext}^\nabla(F^\times, \mathbb{G}_m) \to \mathrm{Pic}^\nabla(K)$ is an isomorphism. There exists a natural equivalence from the identity functor on $\mathrm{Pic}^\nabla(K)$ to the functor $\phi^* \circ a$ inverse to the isomorphism in Proposition 2.11.*

Proof. The map on two-term complexes is

(2.16)

$$\phi^* : K^\times = \mathrm{Hom}(F^\times, \mathbb{G}_m) \to K^\times; \quad f \mapsto \mathrm{tame}\{f(t), \phi(t,T)\} \in k((T))^\times$$

$$\Omega^1_{K/k} \overset{Res}{\cong} \mathrm{Hom}(F, \mathbb{G}_a) \xrightarrow{\phi^*} \Omega^1_{K/k}; \quad \nu(t) \mapsto \mathrm{Res}_t(\nu(t)\frac{d_T\phi}{\phi})$$

Here tame refers to the tame symbol

$$K_2\Big(k((T))[t, t^{-1}, (t-T)^{-1}]\Big) \to (k((T))^\times$$

We need to show $\phi^* \circ \mathfrak{a} - 1$ is homotopic to 0, where $\mathfrak{a}$ is as in the proof of Proposition 2.11. With u the unit as above, the homotopy is given by

$$\Omega^1_{k((t))/k} \to k((T))^\times : \quad \nu \mapsto \exp(\int res_t(\nu(t)d_T u/u)).$$

□

Suppose now that (E, ∇) is a rank 1 connection on $\operatorname{Spec} k((t))$. Let $\widetilde{F}^\times$ be the group ind-scheme over $\operatorname{Spec} k$ defined by its values $\widetilde{F}^\times(R) = R((T))^\times$ over k-algebras R. We can identify $\widetilde{F}^\times$ with the Picard scheme of line bundles on $\mathbb{P}^1$ with infinite order trivialization at 0 and first order trivialization at ∞. For R a field, for example, we have

$$\operatorname{Pic}(\mathbb{P}^1, \{\infty \cdot 0, \infty\})(R) := \varprojlim_N \operatorname{Pic}(\mathbb{P}^1, \{N \cdot 0, \infty\})(R) = \mathbb{H}^1(\mathbb{P}^1/R, \mathcal{O}^\times \xrightarrow{\text{restriction}} \mathcal{O}^\times_{\{\infty\cdot 0+\infty\}})$$

with an exact sequence

$$0 \to (R[[T]]^\times \oplus R^\times)/R^\times \cong R[[T]]^\times \to \mathbb{H}^1(\mathbb{P}^1/R, \mathcal{O}^\times \xrightarrow{\text{restriction}} \mathcal{O}^\times_{\{\infty\cdot 0+\infty\}}) \to H^1(\mathbb{P}^1/R, \mathcal{O}^\times) = \mathbb{Z} \to 0.$$

This sequence, together with the splitting $N \in \mathbb{Z} \mapsto \mathcal{O}(N \cdot \{1\})$ (endowed with the natural trivialization of infinite order at 0 and ∞) is identified with the exact sequence

$$0 \to R[[T]]^\times \to R((T))^\times \to \mathbb{Z} \to 0$$

and section $N \in \mathbb{Z} \mapsto (\frac{T}{T-1})^N$. (Here T is the standard coordinate on $\mathbb{P}^1$.)

More conceptually the adelic representation yields

$$\operatorname{Pic}(\mathbb{P}^1, \{\infty \cdot 0, \infty\})(R) = \left(R((T))^\times \times \Big(\coprod_{x \in \mathbb{P}^1 \setminus \{0,\infty\}} \mathbb{Z} \Big) \times \big(R((T^{-1}))^\times / (1 + T^{-1} R[[T^{-1}]])^\times\big) \right) \Big/ R(T)^\times \cong R((T))^\times. \tag{2.17}$$

We think of t as being fixed, so we have a point ι_t defined by $T = t$ in $\mathbb{P}^1(k((t)))$. The degree 1 line bundle $\mathcal{O}(\{t\})$ is canonically trivialized at 0 and ∞ and so represents an element in $\widetilde{F}^\times(k((t)))$. In terms of the adelic description (2.17), this class is represented by 1 at the point $T = t$. Multiplying by the global function $\frac{T}{T-t}$ we see

$$[\mathcal{O}(\{t\})] = \frac{T}{T-t} \in \widetilde{F}^\times(k((t))) = k((t))((T))^\times. \tag{2.18}$$

Local classfield theory in this context will associate to (E, ∇) on $\operatorname{Spec} k((t))$ a rank 1 translation-invariant connection (L, ∇) on $\widetilde{F}^\times$ together with an identification $\iota_t^*((L, \nabla)) = (E, \nabla)$. We need to understand this (L, ∇) and its relations with the Katz extension and, in the regular singular case, the extension on the punctured tangent space T^*.

The ind-scheme $\widetilde{F}^\times$ is not reduced, but, since we are interested in connections, we may pass to the reduced subgroup $F^\times := \widetilde{F}^\times_{\text{red}}$. There is a natural homomorphism

$$\text{ord} : F^\times \to \mathbb{Z} = \amalg_{n\in\mathbb{Z}} \text{Spec}\, k$$

defined by the ord at $T = 0$ of the powerseries. We can write

$$F^\times = \amalg F_n^\times; \quad F_n^\times = \text{Spec}\, k[X_0, X_0^{-1}, X_1, X_2, \ldots]. \tag{2.19}$$

The point $X_i = x_i$ corresponds to the powerseries $T^n(x_0 + x_1 T + \ldots)$. For example, $\iota_t \in F_1^\times$ is given by

$$X_i = -\frac{1}{t^{i+1}} \tag{2.20}$$

(because $\frac{T}{T-t} = -t^{-1}T - t^{-2}T^2 - \ldots$). There is an evident action of $F_0^\times$ on $F_n^\times$ with respect to which the invariant 1-forms are the forms ω_r in the series (d_X is the differential $d_X(X_i) = dX_i,\ d_X(T) = 0$)

$$\omega_0 + \omega_1 T + \omega_2 T^2 + \ldots := d_X(X_0 + X_1 T + X_2 T^2 + \ldots)(X_0 + X_1 T + X_2 T^2 + \ldots)^{-1}. \tag{2.21}$$

It follows that

$$\iota_t^*(\omega_0 + \omega_1 T + \omega_2 T^2 + \ldots) = d_t \log(\frac{T}{T-t}) = \frac{dt}{T-t} = -\frac{dt}{t}(1 + \frac{T}{t} + (\frac{T}{t})^2 + \ldots). \tag{2.22}$$

In other words

$$\iota_t^*(\omega_i) = -\frac{dt}{t^{i+1}}. \tag{2.23}$$

Proposition 2.14. *Let (E, ∇) be rank 1 on $\text{Spec}\, k((t))$ as above, with t fixed. Then there exists a unique translation-invariant (L', ∇) on $F^\times$ such that $\iota_t^*((L', \nabla)) \cong (E, \nabla)$.*

Proof. The group of isomorphism classes of such (E, ∇) is identified with

$$k((t))dt/d\log(k((t))^\times) \cong \sum_{n\geq 2} k\frac{dt}{t^n} \oplus (k/\mathbb{Z})\frac{dt}{t}. \tag{2.24}$$

Thus, for a suitable gauge e we can represent the connection on E in the form $\nabla(e) = e \otimes (a_{-n}t^{-n} + \ldots + a_{-1}t^{-1})dt$. The form on the right is unique upto replacing a_{-1} by $a_{-1} + N$ for $N \in \mathbb{Z}$, which corresponds to replacing e by $t^N e$. It follows that $\mathcal{O}_{F^\times}$ with connection $\nabla(1) = -a_{-1}\omega_0 - a_{-2}\omega_1 - \ldots - a_{-n}\omega_n$ on each connected component $F_n^\times$ is a connection $F^\times$ pulling back to E. Note that changing $\nabla(1)$ by an integer multiple of $\omega_0 = \frac{dX_0}{X_0}$ does not change the isomorphism class of (L', ∇). $\square$

We need to rigidify (L', ∇) in Proposition 2.14 in order to make the identification of the pullback with (E, ∇) canonical. It follows from (2.20) that ι_t factors through $\sigma : \operatorname{Spec} k((t)) \to \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, where $\sigma^*(t) = t$:

$$\begin{array}{ccc} \operatorname{Spec} k((t)) & \xrightarrow{\iota_t} & F_1^\times \\ \sigma \downarrow & & \| \\ \mathbb{G}_m & \xrightarrow{\rho_t} & F_1^\times . \end{array} \tag{2.25}$$

Let $(\mathcal{L}, \nabla)$ be the Katz extension of (E, ∇) to $\mathbb{G}_m$, and let $\ell = \mathcal{L}/\mathfrak{m}_1\mathcal{L}$ be the fibre of $\mathcal{L}$ at $t = 1$. A connection on the zero-dimensional scheme $\amalg_{n \in \mathbb{Z}} \operatorname{Spec} k$ (not of finite type) is a locally free sheaf of rank one, thus consists, for each n, of a line ℓ_n defined over k. We have the connection $\ell_{\mathbb{Z}}$ on $\mathbb{Z} = \amalg_{n \in \mathbb{Z}} \operatorname{Spec} k$ defined by $\ell^{\otimes n}$ on $(\operatorname{Spec} k)_n$.

Define

$$(L, \nabla) = (L', \nabla) \otimes \operatorname{ord}^* \ell_{\mathbb{Z}}. \tag{2.26}$$

Proposition 2.15. *The rank* 1 *connection* (L, ∇) *on* $F^\times$ *has the following properties*

(i) (L, ∇) *is translation invariant; i.e. if* $\mu : F^\times \times F^\times \to F^\times$ *is the group law, then* $\mu^*((L, \nabla)) = (L, \nabla) \boxtimes (L, \nabla)$. *(Here "=" means canonical isomorphism.)*

(ii) *The fibre of* L *at any* $k' \supset k$*-rational point of* $F_n^\times$ *is canonically identified with* $\ell^{\otimes n} \otimes_k k'$.

(iii) *The pullback* $\rho_t^*((L, \nabla))$ *is canonically identified with the Katz extension of* (E, ∇) *to* $\mathbb{G}_m$. *As a consequence,* $\iota_t^*(L, \nabla) \cong (E, \nabla)$*, again canonically.*

Proof. Parts (i) and (ii) are immediate. For (iii), note that $\rho_t^*((L, \nabla))$ is an extension of (E, ∇) to $\mathbb{G}_m$, without singularities on $\mathbb{G}_m$, and the connection form has at worst a regular singular point at ∞. It follows that $\rho_t^*((L, \nabla)) \cong (\mathcal{L}, \nabla)$. Once we have fixed such an isomorphism, any other isomorphism will differ from this one by multiplication with an invertible flat section of $\operatorname{End}((\mathcal{L}, \nabla)) = k^\times$. Thus to rigidify the isomorphism, it is sufficient to rigidify it at $t = \{1\}$. Note that $\rho_t(1) = \frac{T}{T-1} \in F_1^\times(k)$. By construction, the fibre of L at this point coincides with the fibre of $\mathcal{L}$. Therefore, there is a unique isomorphism $\rho_t^*(L, \nabla) \xrightarrow{\text{iso}} (\mathcal{L}, \nabla)$ with $\rho_t^*(L)|_1 \xrightarrow{\text{iso}|_1 = \text{identity}} \mathcal{L}|_1$. $\square$

2.7. **Global Classfield Theory.** Let X be a smooth, projective curve over a field k of characteristic 0. Let (L, ∇_L) be a rank 1 meromorphic connection on X. Let $S \subset X$ be a finite set of closed points such that (L, ∇_L) is smooth on $X \setminus S$. Then there exists a *modulus*, i.e. an effective divisor $\mathfrak{m} := \sum_{s\in S} m_s s$ and a translation-invariant rank 1 connection $(\mathcal{L}, \nabla_{\mathcal{L}})$ on the generalized Picard scheme $J^* = \coprod_{n\in\mathbb{Z}} J^n$ parametrized line bundles on X trivialized along $\mathfrak{m}$ together with a canonical identification of (L, ∇_L) with the pullback of $(\mathcal{L}, \nabla_{\mathcal{L}})$ under the natural map

$$X \setminus S \to J^1$$
$$x \mapsto (\mathcal{O}_X(x), \mathcal{O}_X|_{\mathfrak{m}} \xrightarrow{\text{incl. defined by } x} \mathcal{O}_X(x)|_{\mathfrak{m}}).$$

For details on this construction, cf. [2], section 2.

To relate the local and global classfield constructions, let $\mathcal{P}$ be the Poincaré bundle on $X \times J^1$. Let $\phi : X\setminus S \hookrightarrow J^1$ be the natural map. We identify $(1_X \times \phi)^*\mathcal{P} \cong \mathcal{O}(\Delta)$, the line bundle associated to the diagonal restricted to $X \times (X \setminus S)$. Let $s \in S$ and suppose m_s is the multiplicity of s in $\mathfrak{m}$. Define $F^\times = \coprod F_n^\times$ to be the reduced group ind-scheme associated to the units in quotient field K_s of the complete local ring $\widehat{\mathcal{O}}_{X,s}$ as in subsection 2.6. Let $U^{(n)} \subset F_0^\times$ be the congruence subgroup modulo t^n where t is a local parameter at s. Let $\pi_s : F_1^\times/U^{(m_s)} \hookrightarrow J^1$ be the natural map. To an element $f \in F_1^\times(k(s))$, π_s associates the line bundle on $X \times_k k(s)$ defined as follows. We cover with $X \setminus \{s\}$ and $\operatorname{Spec} k(s)[[t]]$ where t is a local parameter at s. We take the trivial line bundle on both these schemes and glue over $\operatorname{Spec} K_s = \operatorname{Spec} k(s)((t))$ using f. We take the trivialization to order m_s at s given by the local patch $\mathcal{O}_{k(s)[[t]]}$.

Lemma 2.16. *The diagram below commutes (here j_s is the natural map centered at s)*

$$\begin{array}{ccc} X \setminus S & \xrightarrow{\phi} & J^1 \\ {\scriptstyle j_s}\uparrow & & {\scriptstyle \pi}\uparrow \\ \operatorname{Spec} K_s & \xrightarrow{\iota_t} & F_1^\times \end{array} \tag{2.27}$$

Proof. We have to identify $\phi \circ j_s = \pi \circ \iota_t : \operatorname{Spec} K_s \to J^1$. For this it suffices to identify the two pullbacks $(1_X \times \phi \circ j_s)^*\mathcal{P} \cong (1_X \times \pi \circ \iota_t)^*\mathcal{P}$ as line bundles on X_{K_s} with modulus $\mathfrak{m}$. $(1_X \times \pi \circ \iota_t)^*\mathcal{P}$ is represented by $\mathcal{O}$ on $(X \setminus \{s\})_{K_s}$ and on $\operatorname{Spec} K_s[[T]]$, glued on $\operatorname{Spec} K_s((T))$ by $\frac{T}{T-t}$. Here T is the coordinate on X at s and $K_s = k(s)((t))$. On the other hand, $(1_X \times \phi)^*\mathcal{P} = \mathcal{O}(\Delta)$ on $X \times (X \setminus S)$. The Cartier divisor

$(1_X \times j_s)^*\Delta$ on X_{K_s} is defined by $\frac{T-t}{T}$ on a punctured neighborhood of s in $(X \setminus \{s\})_{K_s}$ and 1 on $\mathrm{Spec}\, K_s[[T]]$ (a formal neighborhood of $\{s\} \times \mathrm{Spec}\, K_s$). The ratio of these is again $\frac{T}{T-t}$, proving the lemma. □

In Section 6 we will define a local symbol (E, f) associated to a rank 1 meromorphic connection E on $k((t))$ and $f \in k((t))^\times$. (E, f) will be the fibre over $f \in F^\times(k)$ of the associated invariant connection L as in Proposition 2.15. When E comes from a global meromorphic connection on a curve X, and $k((t))$ is the formal power series field at a point $x \in X$, we write $(E, f)_x$ for the local symbol.

Proposition 2.17 (Artin reciprocity)**.** *Let E be a rank 1 meromorphic connection on a curve X, and let $f \in k(X)^\times$ be a nonzero meromorphic function. Then for almost all $x \in X$ closed point, the line $(E, f)_x$ is canonically trivialized. Further one has a canonical trivialization*

$$\mathbf{1} = \bigotimes_{x \in X} (E, f)_x \tag{2.28}$$

Proof. E pulls back from a rank 1 invariant connection $\mathbf{L}$ on a suitable generalized jacobian J^* associated to X and some modulus as above (sf. [2]). For $x \in X$ a closed point, let K_x be the field of formal power series at x. We have $K_x^\times \to J^*(k)$, and for almost all x, the units $\mathcal{O}_x^\times \subset K_x^\times$ map to 0. In particular, for almost all x, $f \in K_x^\times$ maps to the origin in J^*. It follows from Lemma 2.16 above, that the line $(E, f)_x$ is identified with the fibre of $\mathbf{L}$ over the origin for almost all x. This proves the first assertion because the fibre over the origin of a rank 1 translation -invariant connection is canonically trivialized. For the second assertion, it suffices to note the restricted product $\prod'_{x \in X} K_x^\times$ maps to J^*, and the global functions $k(X)^\times$ lie in the kernel. □

2.8. **The DR-Line.** In this section we recall the theory of the de Rham epsilon line. This theory is developed in some detail in [1], and we will omit proofs when they are available there. We restrict to the case of a base field k which we will assume of characteristic 0.

Let V, W be k-vector spaces, possibly infinite dimensional. Let $\mathrm{Hom}_f(V, W) \subset \mathrm{Hom}_{k-vec}(V, W)$ be the space of k-linear maps of finite rank. Define

$$\mathrm{Hom}^\infty(V, W) := \mathrm{Hom}_{k-vec}(V, W)/\mathrm{Hom}_f(V, W). \tag{2.29}$$

The category $Vect^\infty(k)$ has as objects k-vector spaces and as morphisms Hom^∞. There is an evident functor $Vect(k) \to Vect^\infty(k)$. A linear map $f : V \to W$ is said to be *Fredholm* if f^∞ is invertible.

Consider triples (W, V, f^∞) where V, W are k-vector spaces, and $f^\infty : V \to W$ is an isomorphism in $Vect^\infty$. Let $f : V \to W$ be a lifting of

f^∞ to $Vect$. Let $W_1 \subset W$ be a finite dimensional subspace such that $W = f(V) + W_1$. Define

$$\det(W, V, f^\infty) := (\det W_1) \otimes (\det f^{-1}(W_1))^{-1}. \tag{2.30}$$

We may view this as a superline (op. cit.) placed in degree

$$\mathrm{index}(f^\infty) := \dim W_1 - \dim f^{-1}(W_1). \tag{2.31}$$

As such it is well-defined and independent of the choice of W_1, upto canonical isomorphism. Given $U^\infty \xrightarrow{g^\infty} V^\infty \xrightarrow{f^\infty} W^\infty$, one has a canonical identification

$$\det(W, V, f^\infty) \otimes \det(V, U, g^\infty) = \det(W, U, (f \circ g)^\infty). \tag{2.32}$$

Note that when $f^\infty : V \cong W$ in $Vect^\infty$ can be lifted to an isomorphism $f : V \cong W$ in $Vect$, then the choice of the lifting f determines a trivialization of $\det(W, V, f^\infty)$.

Related to, but deeper than, the above notion of infinite determinant is the concept of *polarized determinant*. We work with vector spaces V of the form $V = k((t))^{\oplus n}$. Such a vector space has a natural k-linear topology, with a basis of open sets given by $k[[t]]$-lattices $\mathcal{V} \subset V$. A c-lattice $L \subset V$ is a k-vector subspace commensurable with a $k[[t]]$-lattice; $\mathcal{V} \subset L \subset t^{-N}\mathcal{V}$. (For the more general notion of *Tate* vector space which includes c-Tate vector spaces like $k[[t]]^{\oplus n}$ and discrete Tate vector spaces like $(k((t))/k[[t]])^{\oplus n}$, see (op. cit.).) The quotient

$$V^\infty := V/L \in Ob(Vect_k^\infty) \tag{2.33}$$

is well defined (independent of the choice of c-lattice $L \subset V$) upto canonical isomorphism, so we get a functor

$$\text{Tate vector spaces}/k \to Vect_k^\infty; \quad V \mapsto V^\infty \tag{2.34}$$

On the subcategory $Vect_k = \{\text{discrete Tate vector spaces}\}$ the functor is the identity on objects, while on the category of all Tate vector spaces it is not. This can lead to confusion.

The space V^* of continuous k-linear functionals $V \to k$ on a Tate vector space V has a natural structure of Tate vector space, and one gets a contravariant functor $V \mapsto V^*$. For example, if $V \cong k((t))^{\oplus n}$, then one identifies $V^* = \mathrm{Hom}_{k((t))}(V, k((t))) \otimes \Omega^1_{k((t))}$ via residues.

A k-linear map $f : V \to W$ is *Fredholm* if both f^∞ and $f^{*\infty}$ are isomorphisms. Equivalently, f is Fredholm if there exist c-lattices $\mathcal{V} \subset V$ and $\mathcal{W} \subset W$ such that $f(V) + \mathcal{W} = W$, $\mathcal{V} \cap \ker f = (0)$, and $f^{-1}(\mathcal{W})$, $f(\mathcal{V})$ are again c-lattices. In this context, we may define the determinant line as in (2.30). We take $\mathcal{W}$ sufficiently large so

$f(\mathcal{V}) \subset \mathcal{W}$. Using that $f : \mathcal{V} \cong f(\mathcal{V})$, one identifies

$$\det(W, V, f^\infty) \cong \det(\mathcal{W}/f(\mathcal{V})) \otimes \det(f^{-1}(\mathcal{W}/\mathcal{V})^{-1}. \tag{2.35}$$

Let $V \cong k((t))^{\oplus n}$ be as above, and suppose now that $f : V \to V$ is a Fredholm endomorphism. Let $\mathcal{V} \subset V$ be a c-lattice such that $\mathcal{V} + f(V) = V$. The *polarized determinant* λ_f is given by

$$\lambda_f := \det(\mathcal{V} : f^{-1}(\mathcal{V})) \tag{2.36}$$

Here, for commensurable subspaces $A, B \subset V$ one defines

$$\det(A : B) := \det((A + B)/B) \otimes \det((A + B)/A)^{-1}. \tag{2.37}$$

It is straightforward to check that (2.36) is well-defined independent (upto canonical isomorphism) of the choice of $\mathcal{V}$. Said another way, the association $f \mapsto \lambda_f$ depends only on $f^\infty \in \mathrm{Aut}(V^\infty)$. An important application of this principle occurs when f stabilizes a d-lattice. A d-lattice $M \subset V$ is simply a subvector space which is complimentary to some c-lattice: $V = M \oplus \mathcal{V}$. Let $f : V \to V$ be Fredholm, and assume $f(M) \subset M$. Then $f|M : M \to M$ is Fredholm in the discrete sense, and one has canonically

$$\lambda_f \cong \det(M, M, f^\infty) \tag{2.38}$$

where the right hand side is defined in (2.30). Indeed, in the ∞-category one has $M^\infty \xrightarrow{\cong} V^\infty$, and this isomorphism commutes with f^∞.

A basic example (cf. op. cit. 3.4) of (2.38) is the *Weil reciprocity law.* Let X be a smooth, projective curve over k, and let $S \subset X$ be a finite subset defined over k. Write $K_S := \prod_{s \in S} K_s$ where $K_s \cong k(s)((t_s))$ is the complete discrete valuation field at $s \in S$. Let $M = \Gamma(X - S, \mathcal{O}_X)$. Then K_S is a Tate vector space, and $M \subset K_S$ is a d-lattice. Let $f \in k(X)^\times$, and assume (f) is supported on S, so f is invertible on $X - S$. Then f_M^∞ lifts to an automorphism $f : M \cong M$, so $\det(M, M, f^\infty) = 1$ is a canonically trivialized line. It follows from (2.38) that the polarized determinant $\lambda(f : K_S \to K_S)$ is also canonically trivialized. This is the Weil reciprocity law.

Using duality, the polarized determinant can be related to the discrete infinite determinant on a Tate vector space as follows. Given $f : V \to V$ Fredholm, Write $f^* : V^* \to V^*$ for the dual Fredholm map.Then a small computation (cf. op. cit. (2.13.6)) exhibits a canonical isomorphism

$$\det(V, V, f^\infty) \cong \lambda_f \otimes \lambda_{f^*}. \tag{2.39}$$

2.9. Let V be a connection on $\operatorname{Spec} k((t))$, and let $0 \neq \nu \in \Omega^1_{k((t))}$ be a non-zero meromorphic 1-form. Define $\nabla_{\nu^{-1}} : V \to V$ to be the endomorphism

$$V \xrightarrow{\nabla} V \otimes \Omega^1_{k((t))} \xrightarrow{1\otimes\nu^{-1}} V. \tag{2.40}$$

Then $\nabla_{\nu^{-1}}$ is Fredholm.

Definition 2.18. With notation as above, the local de Rham epsilon line is defined to be the polarized determinant $\varepsilon_{DR}(V,\nu) := \lambda_{\nabla_{\nu^{-1}}}$.

More generally, a connection $V = \prod V_i$ over $K = \prod k_i((t_i))$ a finite product with $[k_i : k] < \infty$, is a Tate vector space over k. One can define $\varepsilon_{DR}(V,\nu)$ to be the polarized determinant. One has

$$\varepsilon_{DR}(V,\nu) = \bigotimes_i \varepsilon_{DR}(V_i,\nu) \tag{2.41}$$

If k' is an extension field of k containing the k_i

$$\varepsilon_{DR}(V,\nu) \otimes_k k' \cong \bigotimes_{k_i \hookrightarrow k'} \varepsilon_{DR}(V_i \otimes_{k_i} k', \nu_i). \tag{2.42}$$

Let X be a smooth, projective curve over k, and let $S \subset X$ be a finite subset. We assume that S is defined over k, but points of S need not be. Write $K_S := \prod_{s\in S} K_s$ as above. Let E, ∇ be a meromorphic connection on X, smooth over $X - S$. Let $\nu \in \Gamma(X-S, \Omega^1_{X-S})$ be a 1-form. We assume that ν generates the module of 1-forms at every point of $X - S$, i.e. that the divisor of ν is supported on S. Let E be a meromorphic connection on X which is smooth on $X - S$. Using the diagram

$$\begin{array}{ccc} \Gamma(X-S,E) & \xrightarrow[\nabla_E]{} & \Gamma(X-S, E\otimes\Omega^1_X) \\ \Big\| & & \cong\Big\downarrow \nu^{-1} \\ \Gamma(X-S,E) & \xrightarrow[\nu^{-1}\circ\nabla_E]{} & \Gamma(X-S,E) \end{array} \tag{2.43}$$

we define the global DR-epsilon line

$$\varepsilon_{DR}(X-S,E) := \det\Big(\Gamma(X-S, E\otimes\Omega^1_X), \Gamma(X-S,E), \nabla_E^\infty\Big). \tag{2.44}$$

It follows from (2.32) that this line is canonically identified (independent of the choice of ν) as indicated

$$\varepsilon_{DR}(X-S,E) = \det\Big(\Gamma(X-S,E), \Gamma(X-S,E), (\nu^{-1}\circ\nabla_E)^\infty\Big). \tag{2.45}$$

Proposition 2.19 (DR Reciprocity law)**.** *Let E, ∇_E, X, S, ν be as above, so ν generates $\Gamma(X-S, \Omega^1_X)$. For $s \in S$ a closed point, write $\varepsilon_{DR}(V,\nu)_s$ for the local epsilon factor as in definition 2.18. One has a canonical identification of DR-epsilon lines*

$$\varepsilon_{DR}(X-S, E) = \bigotimes_{s \in S} \varepsilon(E, \nu)_s \tag{2.46}$$

Proof. (cf. op. cit. prop. 4.10) This follows from (2.38), applied to the Fredholm endomorphism $\nu^{-1} \circ \nabla_E$ of $E \otimes K_S$ where $K_S = \prod K_s$ is the product of the local fields at points of S. Indeed, $\Gamma(X-S, E) \subset E \otimes K_S$ is a d-lattice stabilized by $\nu^{-1} \circ \nabla_E$, and one uses (2.45). □

3. Fourier transform and Vanishing cycles for irregular connections

The purpose of this section is to discuss the Fourier transform and vanishing cycles.

3.1. Good lattice pairs. Let R be a complete, discrete valuation ring of equicharacteristic 0, and let $K \supset R$ be its quotient field. By a wellknown theorem of Cohen ([16], Vol. II, p. 304) R admits a field k of representatives, so we may write $R = k[[t]]$ and $K = k((t))$. Let E be a finite dimensional K-vector space, and let $\nabla_E : E \to dt \otimes_k E$ be a k-linear connection. In our applications, R will be the completion of a local ring on a curve over some field k_0 and E will arise by restricting to the formal neighbourhood of some point a meromorphic k_0-linear connection on the curve. In this situation, k/k_0 is finite, and (E, ∇_E) is obtained by base change from k_0 to k.

Definition 3.1. Let $\mathcal{V} \subset \mathcal{W}$ be R-lattices in E. We will say that $(\mathcal{V}, \mathcal{W})$ is a *good lattice pair* for the connection (E, ∇_E) if the following conditions hold.

(i) $\nabla : \mathcal{V} \to \frac{dt}{t} \otimes \mathcal{W}$.
(ii) For any $n \geq 0$, the natural inclusion of complexes $(\mathcal{V} \to \frac{dt}{t} \otimes \mathcal{W}) \hookrightarrow (t^{-n}\mathcal{V} \to \frac{dt}{t} \otimes t^{-n}\mathcal{W})$ is a quasi-isomorphism.

Remark 3.2. (i) When rank $E = 1$ the pair $(\mathcal{V}, \mathcal{W})$ is a good lattice pair for any lattice $\mathcal{V}$ and $\mathcal{W} \supset \mathcal{V}$ minimum such that $\nabla(\mathcal{V}) \subset \frac{dt}{t} \otimes \mathcal{W}$.
(ii) By a theorem of Deligne, good lattice pairs always exist. One construction ([5], pp. 110-112) yields $\mathcal{W} \subset t^{-n}\mathcal{V}$, $\mathcal{W} \not\subset t^{-n+1}\mathcal{V}$, where $n = \dim_k(\mathcal{W}/\mathcal{V})$. There is no uniqueness, however, and one can frequently find good lattice pairs with $\mathcal{W} \subset t^{-m}\mathcal{V}$ for

$m < n$. A discussion on the construction of other lattice pairs can be found in [4].

(iii) When (E, ∇_E) is regular singular, one can find lattices $\mathcal{V}$ such that $(\mathcal{V}, \mathcal{V})$ is a good lattice pair. Note that this assertion has two implications. There exist lattices with respect to which the connection has logarithmic poles, and moreover their de Rham cohomology is stable under adding more poles.

(iv) If $(\mathcal{V}, \mathcal{W})$ is a good lattice pair, then

$$\frac{dt}{t} \otimes \mathcal{W} = \frac{dt}{t} \otimes \mathcal{V} + \nabla(\mathcal{V}) \subset \frac{dt}{t} \otimes E.$$

Indeed, given $w \in \mathcal{W}$ and $n \geq 1$ we can by (ii) in the definition find $v \in \mathcal{V}$ and $w' \in \mathcal{W}$ such that $\nabla(t^{-n}v) = \frac{dt}{t} \otimes t^{-n}w + \frac{dt}{t} \otimes w'$. It follows that $\nabla(v) = \frac{dt}{t} \otimes w + \frac{dt}{t} \otimes t^n w' + \frac{dt}{t} \otimes nv$. Thus we just have to take n large enough so that $t^n \mathcal{W} \subset \mathcal{V}$.

(v) It follows from (ii) in the definition that for $(\mathcal{V}, \mathcal{W})$ a good lattice pair, the inclusion $(\mathcal{V} \to \frac{dt}{t} \otimes \mathcal{W}) \hookrightarrow (E \to dt \otimes E)$ is a quasi-isomorphism.

(vi) The integer $i = \dim(\mathcal{W}/\mathcal{V})$ is called the *irregularity*. It depends only on (E, ∇_E) and not on the particular choice of a good lattice pair. To see this, it is convenient to globalize the notion of good lattice pair.

Definition 3.3. Let (E, ∇_E) be a meromorphic connection on a smooth curve X. Let $\mathcal{V} \subset \mathcal{W} \subset E$ be lattices. The pair $(\mathcal{V}, \mathcal{W})$ is a *good lattice pair* for (E, ∇_E) if, writing $S \subset X$ for the finite set of points where ∇_E has poles, we have

(i) $\nabla(\mathcal{V}) \subset \omega_X(S) \otimes \mathcal{W}$.

(ii) For any $n \geq 0$ the inclusion of complexes of sheaves on X

$$\Big(\mathcal{V} \xrightarrow{\nabla} \omega_X(S) \otimes \mathcal{W}\Big) \hookrightarrow \Big(\mathcal{V}(nS) \xrightarrow{\nabla} \omega_X(S) \otimes \mathcal{W}(nS)\Big)$$

is a quasi-isomorphism.

Remark 3.4. (i) There exists a $1-1$ correspondence between lattices $\mathcal{V} \subset E$ and collections $\{\widehat{\mathcal{V}}_s\}_{s \in S}$ of lattices inside the formal completions of E at points of S. A global lattice pair $(\mathcal{V}, \mathcal{W})$ is good if and only if each $(\widehat{\mathcal{V}}_s, \widehat{\mathcal{W}}_s)$ is good in the sense of Definition 3.1. Indeed, the correspondence between global and formal lattices is standard. To check that a lattice pair is good, one has to show that various quotient complexes $\mathcal{V}(nS)/\mathcal{V} \to \omega(S) \otimes \mathcal{W}(nS)/\mathcal{W}$ are acyclic. But these complexes are the same if one works with collections of formal connections at points of S.

(ii) We can now establish the invariance of the irregularity $\dim_k \widehat{\mathcal{W}}/\widehat{\mathcal{V}}$ for a good lattice pair associated to a formal local connection (Remark (3.2), (vi)). Let $(\mathbb{E}, \nabla)$ be the Katz extension of the formal connection to a meromorphic connection on $\mathbb{P}^1$. Let $(\mathcal{V}, \mathcal{W})$ be the good global lattice pair for $(\mathbb{E}, \nabla)$ associated to the given good formal pair $\widehat{\mathcal{V}} \subset \widehat{\mathcal{W}}$. Note $(\mathbb{E}, \nabla)$ has a regular singular point at infinity, so can take $\mathcal{V} = \mathcal{W}$ away from 0. Then the Euler-Poincaré characteristic

$$\chi((\mathbb{E}, \nabla)) = \chi(\mathcal{V}) - \chi(\omega(\{0, \infty\}) \otimes \mathcal{W}) = \chi(\mathcal{V}) - \chi(\mathcal{W}) = \\ - \chi(\mathcal{W}/\mathcal{V}) = -\dim(\mathcal{W}/\mathcal{V}).$$

This is independent of the choice of lattice.

3.2. Fourier. Let (E, ∇_E) be a meromorphic connection on $\mathbb{A}^1_t = \operatorname{Spec} k[t]$. Let $S \subset \mathbb{A}^1$ be the finite set of closed points where (E, ∇_E) has singularities. We will assume in what follows that (E, ∇) extends to a connection on $\mathbb{P}^1$ with at worst a regular singular point at ∞. We are interested in the Fourier transform of (E, ∇_E), which is defined on the dual affine line $\mathbb{A}^1_{t'}$ as the cokernel of the relative connection $\Phi := \nabla_E + t'dt$

$$\mathcal{F}(E) := \operatorname{Coker}\Big(E[t'] \xrightarrow{\Phi} E[t']dt\Big). \tag{3.1}$$

The operator Φ lifts to an integrable connection $\nabla_E + d(t't)$ in two variables t and t', so the Gauß-Manin mechanism yields a connection in t' on $\mathcal{F}(E)$. We are interested in the behaviour in a formal neighborhood of $t' = \infty$. Let $z := t'^{-1}$. We wish to compute $\mathcal{F}(E) \otimes_{k[t']} k((z))$. Let $\mathcal{V} \subset \mathcal{W} \subset E$ be a global good lattice pair for (E, ∇_E) on $\mathbb{P}^1$.

Lemma 3.5. *The lattice pair $\mathcal{V} \subset \mathcal{W}(\infty)$ is good for the $k((z))$-connection $\Phi = \nabla_E + \frac{dt}{z}$.*

Proof. The only issue is at $t = \infty$. We need to check that the induced map

$$\mathcal{V}(n\infty)/\mathcal{V}((n-1)\infty) \otimes k((z)) \to \omega(\infty) \otimes [\mathcal{W}((n+1)\infty)/\mathcal{W}(n\infty)] \otimes k((z))$$

is an isomorphism. But this map is multiplication by dt/z which has a pole of order 2 at infinity, so the assertion is clear. □

Replacing $(\mathcal{V}, \mathcal{W})$ by $(\mathcal{V}(n\infty), \mathcal{W}(n\infty))$ for n large, we may assume these coherent sheaves on $\mathbb{P}^1$ have no H^1. It follows that

$$\mathcal{F}(E) \otimes k((z)) \cong \mathbb{H}^1\Big(\mathbb{P}^1_{k((z))}, \mathcal{V}_{k((z))} \xrightarrow{\Phi} \omega(S + \infty) \otimes \mathcal{W}(\infty)_{k((z))}\Big). \tag{3.2}$$

We can define a z-lattice in $\mathcal{F}(E) \otimes k((z))$

$$\Psi(\mathcal{V}, \mathcal{W}) := \mathbb{H}^1\Big(\mathbb{P}^1_{k[[z]]}, \mathcal{V}[[z]] \xrightarrow{z\Phi} \omega(S+\infty) \otimes \mathcal{W}(\infty)[[z]]\Big), \tag{3.3}$$

where the differential $z\Phi = z\nabla_E + dt$ is z-linear. It is easy to check that $\Psi(\mathcal{V}, \mathcal{W})$ has no z-torsion, and $\Psi \otimes k((z)) \cong \mathcal{F}(E) \otimes k((z))$.

Let $P/\mathrm{Spf}\ k[[z]]$ be the formal scheme associated to $\mathbb{P}^1_{k[[z]]}$. By standard Grothendieck formal scheme theory ([9] III, th. 4.1.5), $(\mathcal{V}[[z]], \mathcal{W}[[z]])$ give rise to coherent sheaves $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}})$ on P, and (3.3) carries over:

$$\Psi(\mathcal{V}, \mathcal{W}) := \mathbb{H}^1\Big(P, \widetilde{\mathcal{V}} \xrightarrow{z\Phi} \omega(S+\infty) \otimes \widetilde{\mathcal{W}}(\infty)\Big). \tag{3.4}$$

It is also clear that $\Psi(\mathcal{V}, \mathcal{W})/z^n\Psi(\mathcal{V}, \mathcal{W})$ is the hypercohomology of this complex pulled back to $P_n := P \times \mathrm{Spec}\, k[z]/(z^n)$.

Write $\mathcal{H} = \mathcal{H}(\mathcal{V}, \mathcal{W})$ for the (Zariski) sheaf cokernel of $z\Phi$. It is easy to check that this map of sheaves is injective, so we conclude

$$\Psi(\mathcal{V}, \mathcal{W}) \cong \Gamma(P, \mathcal{H}). \tag{3.5}$$

Proposition 3.6. *The Zariski sheaf $\mathcal{H}$ on the formal scheme P is a skyscraper sheaf supported at the points $S \subset \mathbb{A}^1$ where (E, ∇_E) is singular. (Note, as a topological space, $P = \mathbb{P}^1_k$.) For each $s \in S$, let $M_s \subset (\mathcal{W} \otimes \omega(S))_s$ be a splitting (on the level of k-vector spaces) of the natural surjection on the stalk*

$$(\mathcal{W} \otimes \omega(S))_s \twoheadrightarrow (\omega(S) \otimes \mathcal{W})_s / dt \otimes \mathcal{V}_s.$$

Then $\dim M_s = \mathrm{rank}\ E + i_s(E)$ *where i_s is the irregularity. The stalks of $\mathcal{H}$ are given by*

$$\mathcal{H}_x = \begin{cases} 0 & x \in \mathbb{P}^1 \setminus S \\ M_s[[z]] & x = s \in S. \end{cases} \tag{3.6}$$

Proof. For $x = \infty$, let $u = 1/t$. Then $\mathcal{H}_\infty$ is identified with the cokernel

$$\begin{gathered} \mathcal{V}_\infty[[z]] \to \omega(2\infty) \otimes \mathcal{V}_\infty[[z]] \\ \sum_{\ell \geq 0} z^\ell v_\ell \mapsto -v_0 du/u^2 + \sum_{\ell \geq 1} z^\ell(\nabla_E(v_{\ell-1}) - v_\ell du/u^2) \end{gathered} \tag{3.7}$$

Given $\sum z^\ell w_\ell du/u^2 \in \omega(2\infty) \otimes \mathcal{V}_\infty[[z]]$, one solves inductively for v_ℓ, starting with $v_0 = -w_0$, so $\mathcal{H}_\infty = (0)$.

For $x \in \mathbb{A}^1 \setminus S$, triviality of $\mathcal{H}_x$ follows from the contraction mapping theorem. Indeed, multiplication by dt is an isomorphism $\mathcal{V}_x[[z]] \to \omega \otimes \mathcal{V}_x[[z]]$, and $z\nabla_E$ is contracting in the z-adic topology.

The analogue of (3.7) for $x = s \in S$ is

$$\text{(3.8)} \qquad \mathcal{V}_s[[z]] \to \omega(s) \otimes \mathcal{W}_s[[z]]$$
$$\sum_{\ell \geq 0} z^\ell v_\ell \mapsto v_0 dt + \sum_{\ell \geq 1} z^\ell (\nabla(v_{\ell-1}) + v_\ell dt).$$

Again, inductively, an element $\sum_{\ell \geq 0} z^\ell w_\ell dt/t$ is hit if and only if

$$\text{(3.9)} \quad v_0 = w_0/t \in \mathcal{V}_s, \ v_1 = w_1/t - \nabla_{\frac{d}{dt}}(v_0), \ldots, v_i = w_i/t - \nabla_{\frac{d}{dt}}(v_{i-1}).$$

The obstructions to solving at each stage are the images of $w_i/t - \nabla_{\frac{d}{dt}}(v_{i-1}) \in \mathcal{W}_s(s)/\mathcal{V}_s$. The assertion of the proposition follows. □

Corollary 3.7. *The stalks $\mathcal{H}_x$ depend only on the restriction of E to the power series field at x.*

Proof. This is clear for $x \notin S$. for $x \in S$, the discussion in (3.8) and (3.9) is exactly the same if $(\mathcal{V}_s, \mathcal{W}_s)$ are replaced by their formal completions $(\widehat{\mathcal{V}}_s, \widehat{\mathcal{W}}_s)$ in t. □

We define the *vanishing cycles* for (E, ∇_E) and the good lattice pair at $s \in S$

$$\text{(3.10)} \qquad VC(s; \mathcal{V}, \mathcal{W}) := \mathcal{H}(\mathcal{V}, \mathcal{W})_s$$
$$VC(s) = VC(s, E) := \mathcal{H}(\mathcal{V}, \mathcal{W})_s \otimes_{k[[z]]} k((z))$$

Corollary 3.8. *The $VC(s; \mathcal{V}, \mathcal{W})$ are free, finitely generated $k[[z]]$-modules. The $VC(s, E)$ are finite $k((z))$-vector spaces which are independent of the choice of a good lattice pair. We have*

$$\text{(3.11)} \qquad \mathcal{F}(E) \otimes k((z)) \cong \oplus_{s \in S} VC(s, E).$$

Proof. The finiteness statements follow from the finiteness for $\Psi(\mathcal{V}, \mathcal{W})$, (3.4). The direct sum decomposition (3.11) follows from (3.3)-(3.5). Independence of the lattice pair follows from (3.11) because, given two good lattice pairs $(\mathcal{V}, \mathcal{W})$ and $(\mathcal{V}', \mathcal{W}')$ we can define $(\mathcal{V}'', \mathcal{W}'' = \mathcal{V}'(N \cdot S), \mathcal{W}'(N \cdot S))$ for $N >> 0$ so both pairs are included in a third pair. The projections in (3.11) are functorial with respect to the inclusion of one good lattice pair in another. Since the left side is independent of the lattice pair, independence on the right follows. □

An examination of the argument in Proposition 3.6 shows that the identification $(\omega(s) \otimes \mathcal{W}_s/dt \otimes \mathcal{V}_s)[[z]] \cong M_s[[z]] \cong VC(s; \mathcal{V}, \mathcal{W})$ is canonical mod z:

$$\text{(3.12)} \qquad \omega(S) \otimes \mathcal{W}_s/dt \otimes \mathcal{V}_s = VC(s; \mathcal{V}, \mathcal{W})/zVC(s; \mathcal{V}, \mathcal{W}).$$

(Here “canonical” means independent of the choice of the splitting M_s.) Given a finite dimensional $k((t))$-vector space V with a connection

∇ and a non-zero 1-form $\nu = f dt$, we introduce in [1], Section 4, an epsilon factor $\varepsilon(V,\nabla)_\nu$ defined as the polarized determinant of the endomorphism $\nu^{-1}\circ\nabla : V \to V$.

Proposition 3.9. *Let $r = \operatorname{rank} E$ and let i_s be the irregularity at s. Then we have a canonical identification of k-lines (depending on the choice of dt)*

$$\det_k \Big(VC(s;\mathcal{V},\mathcal{W})/zVC(s;\mathcal{V},\mathcal{W})\Big) = \varepsilon(E_s, dt), \tag{3.13}$$

where $\varepsilon(E_s, dt)$ is the local epsilon factor defined in Definition 2.18. In particular, the left hand determinant is independent of the choice of good lattice pair $(\mathcal{V},\mathcal{W})$.

Proof. By definition of good lattice pair, we have that $\nabla_{d/dt}$ induces an isomorphism $\widehat{E}_s/\mathcal{V}_s \cong \widehat{E}_s/\mathcal{W}_s t^{-1}$. This amounts to saying

$$(\nabla_{d/dt})^{-1}(\mathcal{W}_s t^{-1}) = \mathcal{V}_s \text{ and } (\nabla_{d/dt})(\widehat{E}_s) + \mathcal{W}_s t^{-1} = \widehat{E}_s.$$

From this it follows ([1], (2.13.2)) that $\varepsilon(\widehat{E}_s,\nabla_E)_{dt} = \det \mathcal{W}_s t^{-1}/\mathcal{V}_s$. Multiply numerator and denominator by dt and compare (3.12). □

3.3. Connections. We continue to assume that (E,∇_E) is a meromorphic connection on $\mathbb{A}^1_k$ with at worst a regular singular point at $t=\infty$.

Proposition 3.10. *The Fourier transform $\mathcal{F}(E)$ is smooth for $t' \neq 0,\infty$ and has at worst a regular singular point at $t' = 0$.*

Proof. Let $(\mathcal{V}\subset\mathcal{W})$ be a good lattice pair for E on $\mathbb{P}^1_t$. Twisting enough, we may assume these sheaves have vanishing H^1. Define

$$L_1 := \operatorname{coker}\Big(\Gamma(\mathbb{P}^1,\mathcal{V})[t'] \xrightarrow{\nabla_E + t'dt} \Gamma(\mathbb{P}^1,\omega(S+2\infty)\otimes\mathcal{W})[t']\Big). \tag{3.14}$$

Note L_1 is a finitely generated $k[t']$-module, and by Lemma 3.5, we have $L_1\otimes k(t') \cong \mathcal{F}(E)\otimes k(t')$. Define $L\subset\mathcal{F}(E)$ to be the image of L_1. We claim that L is stable under $t'd/dt'$. We know that d/dt' acts by multiplication by t (or, more precisely, the diagram

$$\begin{array}{ccccc} dt\otimes E[t'] & \to & \mathcal{F}(E) & \to & 0 \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle d/dt'} & & \\ dt\otimes E[t'] & \to & \mathcal{F}(E) & \to & 0 \end{array} \tag{3.15}$$

commutes.) For $w\in\Gamma(\mathbb{P}^1,\omega(S+2\infty)\otimes\mathcal{W})$ we want to show that $t'tw$ is congruent to an element in $\Gamma(\mathbb{P}^1,\omega(S+2\infty)\otimes\mathcal{W})$ modulo $(\nabla_E +$

$t'dt)E[t']$. Consider the exact diagram (write $\mathcal{W}' = \omega(S+2\infty)\otimes\mathcal{W}$ to shorten)

$$
\begin{array}{ccccccccc}
0 & \to & \Gamma(\mathcal{V})[t'] & \to & \Gamma(\mathcal{V}(\infty))[t'] & \to & \mathcal{V}(\infty)/\mathcal{V}[t'] & \to & 0 \\
 & & \downarrow{\scriptstyle \nabla_E+t'dt} & & \downarrow{\scriptstyle \nabla_E+t'dt} & & \downarrow{\scriptstyle t'dt} & & \\
0 & \to & \Gamma(\mathcal{W}')[t'] & \to & \Gamma(\mathcal{W}'(\infty))[t'] & \to & \mathcal{W}'(\infty)/\mathcal{W}'[t'] & \to & 0.
\end{array}
\tag{3.16}
$$

It follows by a diagram chase that the element $tt'w \in t'\Gamma(\mathcal{W}'(\infty))[t']$ is congruent modulo the image of $\nabla_E + t'dt$ to an element in $\Gamma(\mathcal{W}')[t']$. □

We consider now the case at $t' = \infty$, i.e. the z-connections on the vanishing cycles $VC(s, E)$. It is in general not true that these connections are regular singular. However, their determinants are regular singular up to a simple exponential factor. In what follows, we work with the connection (E, ∇_E) over the power series field at s. As remarked at the beginning of section 3, the powerseries ring will contain a field of representatives, and our connections will be linar over this field. We may therefore assume $k = k(s)$. Write $EXP(s/z)$ for the rank 1 connection $d + sdz/z^2 = d - d(s/z)$, and write $\tau = t - s$. The absolute Fourier transform connection on (E, ∇_E) looks like $\nabla_E + d\tau/z - (\tau + s)dz/z^2$. Let $r = \text{rank } E$ and write n for the irregularity of E as s.

Proposition 3.11. *The connection on*

$$
EXP(s/z)^{\otimes r+n} \det_{k((z))} VC(s, E)
\tag{3.17}
$$

is regular singular. The two connections $\nabla_{\det VC(s,E)}$ and $\nabla_{\det E_s}$ have the same residue in $k/\mathbb{Z}$.

Proof. Since $r + n = \text{rank } VC(s, E)$ (Proposition 3.6), (3.17) coincides with $\det_{k((z))} EXP(s/z) \otimes VC(s, E)$. We may therefore assume that the absolute connection has the form $\nabla_E + dt/z - tdz/z^2$, i.e. that $s = 0$. The connection arises from the complex (to simplify we drop the subscript s)

$$
\begin{array}{ccccc}
 & & \mathcal{V}((z)) & = & \mathcal{V}((z)) \\
 & & {\scriptstyle \nabla_E+dt/z-tdz/z^2}\downarrow & & {\scriptstyle \nabla_E+dt/z}\downarrow \\
\mathcal{V}((z))dz & \to & \mathcal{W}((z))dt/t + \mathcal{V}((z))dz & \to & \mathcal{W}((z))dt/t \\
\downarrow & & {\scriptstyle \nabla_E+dt/z-tdz/z^2}\downarrow & & \\
\mathcal{W}((z))dt/t\wedge dz & = & \mathcal{W}((z))dt/t\wedge dz & &
\end{array}
\tag{3.18}
$$

by taking the coboundary from the cokernel on the right to the cokernel on the left. Choose $w_1, \dots, w_r \in \mathcal{W}$ and integers $n_i \geq 0$ in such a way that w_j (resp. $v_j := t^{n_j}w_j$) form a $k[[t]]$-basis for $\mathcal{W}$ (resp. for $\mathcal{V}$). Using Proposition 3.6, we see that the elements $t^i w_j dt$, $-1 \leq i < n_j, 1 \leq j \leq r$ form a k-basis of $VC(s)$. It follows from diagram (3.18) that

$$\nabla_{VC}(t^i w_j dt) = \pm t^{i+1} w_j dt \wedge dz/z^2. \tag{3.19}$$

Define $\lambda = \bigwedge_{i,j} t^i w_j dt$, so $\det VC(s) = k((z))\lambda$. Differentiating λ factor by factor, the only factors that contribute are the factors $t^{n_j-1}w_j dt$. But

$$\begin{aligned}\nabla_{VC}(t^{n_j-1}w_j dt) = v_j dt \wedge dz/z^2 \equiv \nabla_E(v_j) \wedge dz/z \equiv \\ \sum a_{jkl}t^k w_l dt \wedge dz/z + v' dt \wedge dz/z \equiv \\ \sum a_{jkl}t^k w_l dt \wedge dz/z \pm \nabla(v') \wedge dz \equiv \dots\end{aligned} \tag{3.20}$$

Every time this process is iterated, one gets an extra factor of z. It follows that we get z-adic convergence, so the lattice $k[[z]]\lambda$ is stabilized by $\nabla_{\det VC, zd/dz}$.

Note in (3.20) the only term which contributes to $\nabla_{\det VC}(\lambda)$ is

$$a_{j,n_j-1,j}t^{n_j-1}w_j dt \wedge dz/z = a_{j,n_j-1,j}t^{-1}v_j dt \wedge dz/z. \tag{3.21}$$

If we drop the dz/z, this is precisely the term in $t^{-1}v_j$ arising in $\nabla_E(v_j)$. It follows that the two rank 1 connections $\nabla_{\det VC(s,E)}$ and $\nabla_{\det E_s}$ have the same residue in $k/\mathbb{Z}$. □

The following corollary is precisely analogous to Laumon's result in the ℓ-adic situation, [12], th. 3.4.2. It is this which enables us to pass from the fibre of $\det \mathcal{F}(E)$ at $t' = 0$ which contains the global period determinant of E to the fibre at $t' = \infty$ which can be evaluated using vanishing cycles.

Corollary 3.12. *The rank* 1 *meromorphic connection*

$$\mathcal{L} := \det(\mathcal{F}(E)) \otimes \bigotimes_{s \in S} EXP((r + n_s)st')$$

on $\mathbb{P}^1_{t'}$ *has residue at* $t' = \infty$ *equal to* $-\mathrm{res}_{t=\infty} \det E$. *If* $\det E$ *extends smoothly across* ∞, *then* $\mathcal{L}$ *is the trivial connection.*

Proof. By (3.11) we have $\det \mathcal{F}(E) \otimes k((z)) = \otimes_{s \in S} \det VC(s, E)$. It follows from Proposition 3.11 that $\mathcal{L}$ has a regular singular point at $t' = \infty$ with residue equal to the sum of the residues of $\det E$ at points of S. Since the global sum of the residues of $(\det E, \det \nabla_E)$ is 0 in $k/\mathbb{Z}$, this proves the first part of the corollary. Note that $\mathcal{L}$ is smooth away from $t' = 0, \infty$ and, by Proposition 3.10, $\mathcal{L}$ has a regular singular point at $t' = 0$. If the residue at infinity of $\mathcal{L}$ is trivial, it follows that the residue at zero is trivial as well, and $\mathcal{L}$ is globally trivial on $\mathbb{P}^1$. □

The rest of this section is devoted to the identification of the connection on the ε-line as defined in [1], when k is a field extension of the field k_0 and ∇ is an integrable connection with respect to k_0, inducing the connection $\nabla_{/k}$ relative to k with respect to which we have constructed the vanishing cycles. It won't be used in the rest of this article and is just meant to shed a slightly different light on the basic construction of [1].

Let us recall first the construction of the connection on the ε-line as identified in Proposition 3.9. The parameter t on $\mathbb{A}^1_k$ yields a splitting

$$T_{\mathbb{A}^1/k_0} = T_{\mathbb{A}^1_k/k} \oplus T_{k/k_0} \otimes \mathcal{O}_{\mathbb{A}^1_k}. \tag{3.22}$$

Furthermore, via this splitting one has

$$[T_{\mathbb{A}^1/k}, T_{k/k_0}] = 0. \tag{3.23}$$

Let $s \in k$ be transcendent over k_0 and define $\theta := s\partial_s \in T_{k/k_0}$.

Lemma 3.13. *There is a cyclic vector $e \in E \otimes k[[\tau]]$ with respect to $\tau\partial_\tau$ such that $\theta(e)$ is a cyclic vector with respect to $\tau\partial_\tau$ as well.*

Proof. If e is a cyclic vector with respect to $\tau\partial_\tau$, then

$$\theta((\tau\partial_\tau)^i s^M e) = s^M (M + \theta)((\tau\partial_\tau)^i e). \tag{3.24}$$

Since $M + \theta$ is invertible for M large, the vectors $\theta((\tau\partial_\tau)^i s^M e) = (\tau\partial_\tau)^i \theta(s^M e)$ are independent for $i = 0, \ldots, r-1$. Thus $s^M e$ and $\theta(s^M e)$ are both cyclic vectors for M large. □

Now recall from [5], p.110-112 that a pair of good lattices is obtained, starting from a cyclic vector e with respect to $\tau\partial_\tau$, by considering

$$\mathcal{V} = < \frac{e}{\tau^N}, \ldots, \frac{(\tau\partial_\tau)^{r-1} e}{\tau^N} >$$

for N large enough. Then, assuming that $H^0(\nabla) = 0$, which is permissible to prove Theorem 3.14, one has $\mathcal{W} = \tau\partial_\tau \circ \nabla(\mathcal{V})$. We denote by $(\theta(\mathcal{V}), \theta(\mathcal{W}))$ a pair of good lattices obtained by replacing a cyclic vector e by $\theta(e)$ such that $\theta(e)$ is cyclic as well, and taking N large enough so that it defines a good $\mathcal{V}$ both for $\frac{e}{\tau^N}$ and for $\frac{\theta(e)}{\tau^N}$. One has a commutative diagramm

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\partial_\tau} & \frac{1}{\tau}\mathcal{W} \\ {\scriptstyle \theta\circ\nabla}\downarrow & & \downarrow{\scriptstyle \theta\circ\nabla} \\ \theta(\mathcal{V}) & \xrightarrow[\partial_\tau]{} & \frac{1}{\tau}\theta(\mathcal{W}) \end{array} \tag{3.25}$$

Functoriality of the construction of the ϵ line for the fixed differential form dt defines the ϵ connection with values in Ω^1_{k/k_0} on it as follows. The vertical maps induce

$$\det(\frac{1}{\tau}\mathcal{W}/\mathcal{V}) \xrightarrow{\theta} \det(\frac{1}{\tau}\theta(\mathcal{W})/\theta(\mathcal{V})). \tag{3.26}$$

By definition, θ is the k/k_0 connection on the k ϵ-line associated to dt.

On the other hand, the connection $\nabla + d(\frac{t}{z})$ is an absolute connection as well. Thus the $\Omega^1_{k((z))/k}$-valued Gauß-Manin connection studied before is indeed the z-part of the Gauß-Manin connection

$$VC(s) \xrightarrow{GM} \Omega^1_{k((z))/k_0} \otimes VC(s). \tag{3.27}$$

Since

$$[\theta, \partial_z] = 0, [\theta, \partial_t] = 0, \tag{3.28}$$

one has

$$\theta \circ (\nabla + d(\frac{t}{z})) = \theta \circ \nabla. \tag{3.29}$$

Thus the commutative diagram

$$\begin{array}{ccc} \mathcal{V} \otimes k[[z]] & \xrightarrow{z\partial_\tau + 1} & \frac{1}{\tau}\mathcal{W} \otimes k[[z]] \\ {\scriptstyle \theta\circ\nabla}\downarrow & & \downarrow{\scriptstyle \theta\circ\nabla} \\ \theta(\mathcal{V}) \otimes k[[z]] & \xrightarrow[z\partial_\tau + 1]{} & \frac{1}{\tau}\theta(\mathcal{W}) \otimes k[[z]] \end{array} \tag{3.30}$$

yields the expression of the Gauß-Manin connection on lattices

$$VC(s, \mathcal{V}, \mathcal{W}) \xrightarrow{\theta\circ(\nabla + d(\frac{t}{z}))} VC(s, \theta(\mathcal{V}, \mathcal{W})) \tag{3.31}$$

as the map induced from the vertical map of (3.30) on the cohomology of the horizontal maps. Since the whole diagram (3.30) commutes with z, it induces the commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\text{identity}} & \frac{1}{\tau}\mathcal{W} \\ {\scriptstyle \theta\circ\nabla}\downarrow & & \downarrow{\scriptstyle \theta\circ\nabla} \\ \theta(\mathcal{V}) & \xrightarrow[\text{identity}]{} & \frac{1}{\tau}\theta(\mathcal{W}) \end{array} \tag{3.32}$$

The vertical map of (3.32) induces the map (3.26), which is the ϵ connection. In conclusion, we have proven

Theorem 3.14. *Under the assumption that k contains a subfield k_0 and that ∇ is an integrable connection with respect to k_0, the identification of the ϵ-line with respect to the differential form dt as the determinant of the vanishing cycles in Proposition 3.9 carries over to* *the* Ω^1_{k/k_0}*-valued* ϵ *connection. It is the restriction to* $z = 0$ *of the Gauß-Manin connection defined on* $\det VC(s)$*, coming from the absolute connection* $\nabla + d(\frac{t}{z})$.

4. Local Asymptotic Study of Periods

In this section we begin the study of the Fourier period matrix as a function of z near $z = 0$. We have already seen (Corollary 3.8) that $\mathcal{F}(E) \otimes k((z)) \cong \oplus_{s\in S} VC(s, E)$. Note this decomposition occurs only on the formal level. We cannot expect a simple decomposition of the period matrix into local pieces viewed as analytic functions of z. The analogue of the formal decomposition of the connection will be a decomposition of the matrix of asymptotic expansions (as $z = t'^{-1} \to 0$) of the Fourier periods.

We begin by recalling the basic theory of asymptotic expansions (cf. [13], pp. 53-55). Let $D \subset \mathbb{C}$ be a disk about 0 with coordinate z, and let $\pi : \widetilde{D} \to D$ be the real blowup of D at 0. (Cf. subsection 2.2). With reference to the diagram (2.3), we define a subsheaf $\mathcal{A} \subset \tilde{j}_*\mathcal{O}_{D^*}$ on $\widetilde{D}$ as follows. Away from the exceptional divisor $S = \pi^{-1}(0)$ take $\mathcal{A} = \mathcal{O}_{D^*}$. For $\theta \in S$ we can think of neighborhoods of θ as corresponding to small sectors $\mathcal{S} = \{re^{i\tau} \mid \theta - \epsilon < \tau < \theta + \epsilon,\ 0 < r << 1\}$ around 0 on D. Let $f \in (\tilde{j}_*\mathcal{O}_{D^*})_\theta$ live on such a sector. Let $g(z) = \sum_{n\geq -N} a_n z^n \in \mathbb{C}((z))$. We say that g is the *asymptotic expansion* of f as $z \to 0$ if for a sufficiently small sector $\mathcal{S}$ there exist constants $C_p > 0$ for all $p >> 0$ such that

$$|f(z) - \sum_{n\leq p} a_n z^n| \leq C_p |z|^{p+1},\ z \in \mathcal{S} \tag{4.1}$$

Define $\mathcal{A}_\theta \subset (\tilde{j}_*\mathcal{O}_{D^*})_\theta$ to be the collection of those f which admit asymptotic expansions in $\mathbb{C}((z))$. Define $\mathcal{A}^{<0} \subset \mathcal{A}$ to be the subsheaf of functions whose asymptotic expansion is identically 0. For example the function $e^{1/z} \in \mathcal{A}^{<0}_\theta$ (resp. $e^{1/z} \notin \mathcal{A}_\theta$) if $\cos\theta < 0$ (resp. $\cos\theta \geq 0$). The basic result of Borel-Ritt (op. cit.) is that the asymptotic expansion map is surjective. We get an exact sequence of sheaves

$$0 \to \mathcal{A}^{<0} \to \mathcal{A} \xrightarrow{A.E.} \mathbb{C}((z))_S \to 0 \tag{4.2}$$

(Here $\mathbb{C}((z))_S$ is the constant sheaf on S with stalk $\mathbb{C}((z))$.) Note that $A.E.$ is a homomorphism of sheaves of rings. Using the Cauchy inequalities, one shows that $A.E.$ commutes with the action of d/dz.

The classical theory of steepest descent will apply to show that our fourier period integrals have the form $e^{Q(z)} \cdot f(z)$ where Q is some polynomial in $1/z$ and $f \in \mathcal{A}$.

Let $\pi : \operatorname{Spec} k((u)) \to \operatorname{Spec} k((t))$ be the ramified cover $\pi^* t = u^p$. Let (L, ∇) be a rank 1 connection on $\operatorname{Spec} k((u))$, and let $\mathfrak{Four} = (k((t)), \nabla_{\mathfrak{Four}}(1) = d(\frac{t}{z}))$ be the Fourier connection. We want to investigate asymptotic periods for the connection $\pi_* L \otimes \mathfrak{Four}$.

For a suitable gauge on L, we have

$$L = k((u)); \quad \nabla_L(1) = (b_{-n-1}u^{-n-1} + \ldots + b_{-1}u^{-1})du. \tag{4.3}$$

We think of these connections as being defined on small analytic disks in the respective coordinate planes. (Note that choosing a gauge for which the connection form has no non-negative powers of u amounts to taking the Katz extension to $\mathbb{G}_m$.) Let $\epsilon^\vee$ be a (possibly multi-valued) solution (recall *solution* means horizontal section for the dual connection) of $\pi_* L$. Then $\epsilon^\vee \exp(\frac{t}{z})$ is a solution for $\pi_* L \otimes \mathfrak{Four}$.

Let $\tilde{\sigma}$ be a path through a point $u_0 \neq 0$ on the u-disk, and write $\tilde{\gamma} = \pi_* \tilde{\sigma}$. (In what follows, $\tilde{\sigma}$ and $\tilde{\gamma}$ will denote germs of paths on disks. σ and γ will be global paths.) The solution $\epsilon^\vee$ along $\tilde{\gamma}$ lifts to a solution $\mu^\vee$ along $\tilde{\sigma}$ for L. For a section η of $\pi_* L \otimes \mathfrak{Four} \otimes \Omega^1_t(*)$ (where $\Omega^1_t(*)$ is the space of 1-forms on the t-disk meromorphic at $t = 0$) we can form

$$\int_{\tilde{\gamma}} \langle \epsilon^\vee \exp(\frac{t}{z}), \eta \rangle = \int_{\tilde{\sigma}} \mu^\vee \exp(\frac{u^p}{z}) \pi^* \eta. \tag{4.4}$$

Here $\pi^*\eta$ is viewed as a section of $L \otimes \pi^* \mathfrak{Four} \otimes \Omega^1_u(*)$. Note that, having trivialized L, we can write $\pi_* L \otimes \Omega^1_t(*) = \sum_{i=0}^{p-1} u^i k((t)) \frac{dt}{t}$, and $u^i \pi^* t^j \frac{dt}{t} = p u^{jp+i} \frac{du}{u}$. When L has been trivialized, we write $\pi^* \eta = f(u)du$. With ∇_L as in (4.3), the integral on the right becomes for some $c \in \mathbb{C}^\times$

$$\int_{\tilde{\sigma}} c u^{b_{-1}} \exp(a_{-n}u^{-n} + \ldots + a_{-1}u^{-1} + \frac{u^p}{z}) f(u) du; \quad a_{-j} = -\frac{b_{-j-1}}{j}. \tag{4.5}$$

We assume now that the connection is irregular, i.e. $a_{-n} \neq 0$. (For the regular singular case, cf. remark 4.3 below.) We will give a steepest descent asymptotic expansion of this integral as a function in z as $z \to 0$ when $\tilde{\sigma}$ is a suitable path through a critical point (in u) of the function

$$h(u, z) := a_{-n}u^{-n} + \ldots + a_{-1}u^{-1} + \frac{u^p}{z}. \tag{4.6}$$

In general, the path $\tilde{\sigma}$ depends on z. The resulting asymptotic expansion in z will have the shape

$$A.E. \int_{\tilde{\gamma}} \langle \epsilon^\vee \exp(\frac{t}{z}), \eta \rangle = \delta(\epsilon^\vee|_{\tilde{\gamma}}) \cdot q(\epsilon^\vee|_{\tilde{\gamma}}, \eta); \quad z \to 0 \tag{4.7}$$

Here δ is an exponential term which does not depend on η, and $q \in \mathbb{C}((z^{1/(n+p)}))$. The determinant of a matrix with such entries will be the crucial ingredient in the definition of the local epsilon factors.

We take $0 < z << 1$ and consider the harmonic function (in u) Re $h(u,z)$ which is well-defined on $\mathbb{P}^1 \setminus \{0,\infty\}$. The critical points of h (i.e. zeroes of $\frac{\partial h}{\partial u}$) are easily computed to be of the form

(4.8)

$$\zeta_\ell(z) = z^{\frac{1}{n+p}}\kappa_\ell; \quad \kappa_\ell = \kappa_\ell(z) := \exp(\frac{2\pi i\ell}{n+p})\Big(\frac{na_{-n}}{p} + O(z^{\frac{1}{n+p}})\Big)^{\frac{1}{n+p}}, \\ 0 \le \ell < n+p.$$

Here the $(n+p)$-th root of the factor $\frac{na_{-n}}{p} + \dots$ on the right is fixed. We define K_ℓ and M_ℓ by

$$\text{(4.9)} \qquad h(\zeta_\ell, z) = z^{\frac{-n}{n+p}} a_{-n}^{\frac{p}{n+p}} e^{\frac{2\pi i\ell p}{n+p}} (\frac{n}{p})^{\frac{p}{n+p}} (\frac{n+p}{p} + O(z^{\frac{1}{n+p}})) \\ =: z^{\frac{-n}{n+p}} K_\ell$$

$$\text{(4.10)} \quad h''(\zeta_\ell, z) = z^{-1+\frac{p-2}{n+p}} a_{-n}^{\frac{p-2}{n+p}} e^{\frac{2\pi i\ell(p-2)}{n+p}} p(n+p)\Big((\frac{n}{p})^{\frac{p-2}{n+p}} + O(z^{\frac{1}{n+p}})\Big) \\ =: z^{-1+\frac{p-2}{n+p}} M_\ell = z^{\frac{-n-2}{n+p}} M_\ell.$$

The second derivative h'' is taken with respect to the variable u and is non-zero for $z << 1$, so these critical points are non-degenerate.

In general, at a non-degenerate critical point we can write $h = h_0(z) + c(z)u^2 + O(u^3)$, and writing $u = x+iy$, we have Re $h(x,y;z) =$ Re $h_0(z) + Q(x,y) + O((x^2+y^2)^{3/2})$ where Q is a quadratic form with one negative and one positive eigenvalue. We take our path $\tilde\sigma(s)$, $-\epsilon \le s \le \epsilon$ such that $\tilde\sigma(0) = \zeta_\ell(z)$ and such that $\tilde\sigma$ is a straight line in the direction of the negative eigenvalue. This means that Re h has an absolute maximum along $\tilde\sigma$ at the point $u = \zeta_\ell(z)$. We refer to $\tilde\sigma$ as a local steepest descent path at $\zeta_\ell(z)$.

It remains to explicit the asymptotics of $\int_{\tilde\sigma} u^{b-1}\exp(h(u,z))f(u)du$ (4.5). In $v = z^{\frac{-1}{n+p}}u$-coordinates, the path $\tilde\sigma$ is constant in z up to $o(1)$ as $z \to 0$. A painful, but standard, calculation (cf. [14], pp. 486-491) using the coordinates

$$\text{(4.11)} \qquad w := \sqrt{\frac{-h''(\zeta_\ell,z)}{2!}}(u-\zeta_\ell); \quad u = \sqrt{\frac{2!}{-h''(\zeta_\ell,z)}}w + \zeta_\ell.$$

yields an asymptotic expansion (with $\kappa_\ell, K_\ell, M_\ell$ as in (4.9) and (4.10))

(4.12)

$$\int_{\tilde\sigma} u^{b-1}\exp(h(u,z))f(u)du \sim \exp(h(\zeta_\ell,z))\zeta_\ell^{b-1}f(\zeta_\ell)\sqrt{\frac{2\pi}{-h''(\zeta_\ell,z)}} \sim$$

$$\exp(z^{\frac{-n}{n+p}}K_\ell)(z^{\frac{1}{n+p}}\kappa_\ell)^{b-1}f(z^{\frac{1}{n+p}}\kappa_\ell)z^{\frac{n+2}{2(n+p)}}\sqrt{\frac{2\pi}{-M_\ell}};\quad z\to 0.$$

In the notation of (4.7), $\delta(\epsilon^\vee|_{\tilde\gamma}) = \exp(h(\zeta_\ell,z))$ and $q(\epsilon^\vee|_{\tilde\gamma},\eta)$ is the product of the remaining factors in the asymptotic expansion.

Proposition 4.1. *Let notation be as above, but write $\tilde\sigma_\ell$, $0\le \ell < n+p$ for the local rapid decay path through ζ_ℓ. Let $\eta_m = p^{-1}u^{m-p[\frac{m}{p}]+1}t^{[\frac{m}{p}]}\frac{dt}{t}$ be the local section of $\pi_*L\otimes\mathfrak{Four}\otimes\Omega^1_t(*)$ with $\pi^*\eta_m = u^m du$, $0\le m < n+p$. Let $\tilde\gamma_l = \pi_*\tilde\sigma_\ell$. Choose a non-trivial solution $\mu^\vee$ for L on a punctured disk in the u-plane about $u=0$. Let $0 < z << 1$ (it is convenient but not necessary to take z real at this point). Then the paths $\tilde\sigma_\ell$ will lie in the u-disk, and we can define solutions $\epsilon^\vee_\ell|_{\tilde\gamma_\ell}$ along $\tilde\gamma_\ell$ by pushing forward $\mu^\vee$. Then the matrix of asymptotic expansions for $z\to 0$:*

$$\Big(A.E.\int_{\tilde\gamma_\ell}\langle\epsilon^\vee_\ell\exp(\frac{t}{z}),\eta_m\rangle\Big)_{0\le\ell,m<n+p} \tag{4.13}$$

has non-trivial determinant.

Proof. The entries in this matrix have the form (4.12). Note that in the righthand expression in (4.12), the only term depending on the differential form is the term $f(\zeta_\ell)$. This means that the matrix (4.13) can be written in the form MD, where D is diagonal with nonzero entries coming from the remaining factors on the right in (4.12), and

$$M = ((z^{\frac{1}{n+p}}\kappa_\ell)^m)_{0\le\ell,m<n+p} \tag{4.14}$$

is a Vandermonde matrix. □

Remark 4.2. The result above extends easily to connections of the form $\pi_*L\otimes V$ for V a nilpotent connection. Indeed, we can choose a basis of solutions for V of the form $\nu_i^\vee = v_i^\vee + \sum_{j>i}a_{ij}(t)v_j^\vee$, where the $v_i^\vee$ are a $k((t))$-basis of the dual connection $V^\vee$. The determinant will not see the off-diagonal terms $\sum_{j>i}a_{ij}(t)v_j^\vee$.

Remark 4.3. The regular singular term $b_{-1}u^{-1}du$ in the connection (4.3) contributes the factor $(z^{\frac{1}{n+p}}\kappa_\ell)^{b-1}$ to the asymptotic expansion (4.12). Thus, inclusion of a regular singular term in the connection has the effect of multiplying the asymptotic period matrix (4.13) by a diagonal matrix. The determinant is then multiplied by $\prod_\ell (z^{\frac{1}{n+p}}\kappa_\ell)^{b-1}$.

Finally, we consider the case when the connection is regular singular. In this case, we need only take $p = 1$, so $h(t,z) = t/z$. Instead of applying steepest descent, we simply change variables, taking $v = t/z$. In place of (4.5) we have

$$cz^{b-1}\int_{z^{-1}\sigma} v^{b-1}\exp(v)f(zv)zdv \tag{4.15}$$

Here the path is global. Because the integrand has rapid decay at the endpoints, the integral will not see the deformation of the path. In the case $f(t) = t^q$ the result corresponds exactly to the right hand side of (4.12) taking $n = 0,\ p = 1$. The exponential term is replaced with a constant Γ-factor, and the square root is omitted.

5. Global Asymptotic Study of Periods

We return to the global picture, with (E,∇) a meromorphic connection on $\mathbb{P}^1\setminus S$ defined over a field $k\subset\mathbb{C}$, where $S\subset\mathbb{A}^1$ is a finite set of closed points. Our objective is to endow the local and global de Rham ε-lines with Betti structures. Since the de Rham k-lines exist already ([1], as recalled in section 2.8) there is no loss of generality in assuming the in S are k-rational. We may further assume that the formal connections $\widehat{E}_s$ at points of S have a Levelt decomposition defined over k (see (5.4) below).

The standard parameter on $\mathbb{P}^1$ will be denoted t. The Fourier sheaf is by definition the connection

$$\mathfrak{Four} := \big(\mathcal{O}_{\mathbb{A}^1}, \nabla_{\mathfrak{Four}}(1) = d(\frac{t}{z})\big),$$

where z is a variable. Note this is an integrable connection with respect to k, but we will also consider it as a relative connection with respect to $k((z))$. We have from Corollary 3.8 (for the definition of Φ, see (3.1))

$$H^1(\Phi)\otimes_{k(z)} k((z)) = H^1_{DR/k((z))}(\mathbb{A}^1\setminus S, E\otimes\mathfrak{Four}) \cong \bigoplus_{s\in S} VC(s). \tag{5.1}$$

We can also treat $H^1(\Phi)$ as a meromorphic connection on a z-disk, though this meromorphic structure is not compatible with the formal direct sum decomposition into vanishing cycle groups.

Let $\mathcal{E} = E_{\rm an}^{\nabla}$ be the local system on $\mathbb{P}^1\setminus S$ of flat sections of E, and let $\mathcal{E}^\vee$ be the dual local system, called the local system of solutions for E. Let γ be a 1-chain on $\mathbb{P}^1_{\mathbb{C}}$ with $\partial\gamma$ supported on S. Let $\epsilon^\vee$ be a section of $\mathcal{E}^\vee$ along γ. The exponential $\exp(\frac{t}{z})$ is a solution for $\mathfrak{Four}$, so we may interpret $\epsilon^\vee \exp(\frac{t}{z})$ as a solution for the connection $E \otimes \mathfrak{Four}$ along γ. Here z is assumed to lie in a small sector about the positive real axis on a punctured disk around $z = 0$. We assume this solution has rapid decay in the sense of [13], [3] at points of $S \cup \{\infty\}$ lying in the support of γ. Then for any section η of $E \otimes \mathfrak{Four} \otimes \Omega^1_{\mathbb{P}^1}(*S + *\infty)$ which is analytic on an open in $\mathbb{P}^1$ containing the support of γ except for possible meromorphic poles at points of S and ∞, the *period integral*

$$\int_\gamma \langle \epsilon^\vee \exp(\frac{t}{z}), \eta\rangle \tag{5.2}$$

is defined.

More generally, one can choose bases $\{\epsilon_j^\vee \exp(\frac{t}{z})|_{\gamma_j}\}$ for the rapid decay homology [3] and η_k for $H^1(\Phi)$ (5.1) and consider the period matrix

$$\mathcal{P}\mathrm{er}(z) := \Big(\int_{\gamma_j} \langle \epsilon_j^\vee \exp(\frac{t}{z}), \eta_k\rangle\Big)_{j,k}. \tag{5.3}$$

Note that the paths γ_j may depend on z in the above, provided they vary continuously with endpoints fixed. Indeed, a variant of Stokes theorem taking into account the rapid decay at the endpoints will imply independence of the integral on continuous variation of the path. We will say that a path γ is associated to a point $s \in S$ if the support of γ meets no other point in S. We allow γ to contain ∞ as well.

Lemma 5.1. *The rapid decay homology $H_1^{\mathrm{rd}}(\mathbb{A}^1 \setminus S, E^\vee \otimes \mathfrak{Four}^\vee)$ is generated by elements $\epsilon^\vee \exp(\frac{t}{z})|_\gamma$ such that γ is associated to a point of S, i.e. γ avoids all but one point of S.*

Proof. This is straightforward. Given a path γ, one may cut it into pieces, $\gamma = \sum \gamma_i$ such that each γ_i contains at most one point of S. If $p_i \in \gamma_i \cap \gamma_{i+1}$ is a cut point, one joins p_i to ∞ along a path σ_i such that $\exp(\frac{t}{z})$ has rapid decay along σ_i for z in our fixed sector. Note that by assumption (E, ∇_E) is smooth at ∞ so the $\epsilon^\vee$ approach a finite limit. One then replaces γ_i by $\gamma_i + \sigma_i$ and γ_{i+1} by $-\sigma_i + \gamma_{i+1}$. $\square$

We now fix $s \in S$ and consider rapid decay homology cycles with paths associated to s. To simplify notation, we assume $s = 0$. We want to further classify paths associated to 0. Replacing k with a finite extension, we have as in Theorem 2.1

$$E \otimes k((t)) \cong \bigoplus_i (\pi_{i*} L_i \otimes V_i). \tag{5.4}$$

Here for a given i (we drop i from the notation to simplify)

$$\pi : \mathrm{Spec}\, k((u)) \to \mathrm{Spec}\, k((t))$$

is the ramified cover given by $t \mapsto u^p$. (We will also write $\pi : \mathbb{P}^1_u \to \mathbb{P}^1_t$ for the corresponding global map $\pi^*(t) = u^p$.) L is the rank 1 connection on $k((u))$

$$L = k((u)); \quad \nabla_L(1) = (b_{-n-1}u^{-n-1} + \ldots + b_{-1}u^{-1})du, \tag{5.5}$$

and the connection on V is nilpotent; i.e. V admits a flat filtration such that the connection on the graded pieces is trivial.

The formal decomposition (5.4) extends to an analytic decomposition in a small sector in t about any given angle.

In the previous section (cf. (4.4) and the discussion following) we have analysed the asymptotic behaviour of an integral like (5.2) when the connection has the form π_*L and the (global) path γ is replaced with a germ $\tilde{\gamma}$ of a rapid decay path through a suitable critical point. Our objective now is to show that this asymptotic analysis is valid globally. We learned the following lemma from C. Skiadas.

Lemma 5.2 (Skiadas)**.** *Let h (resp. $\zeta(z)$) be as in* (4.6) *(resp.* (4.8)*), and consider the path $\tilde{\sigma}$ as in Proposition 4.1. We view h as defined and meromorphic on the global u-plane $\mathbb{P}^1_u$. Let $C \in \mathbb{R}$ be such that*

$$\text{Re } h(\sigma(\pm\epsilon)) < C < \text{Re } h(\zeta).$$

Then there exists a path $\sigma : [-1, 1] \to u$-plane such that

1. $\sigma = \tilde{\sigma}$ *on* $[-\epsilon, \epsilon]$.
2. *Re* $h(\sigma(s)) < C$ *for* $|s| \geq \epsilon$.
3. *The* 0*-chain* $\partial\sigma$ *is supported on* $\{0, \infty\}$.
4. *The path σ can be taken to avoid any given finite set of points on $\mathbb{P}^1$ distinct from* $0, \infty, \zeta$.

Proof. Let $U \subset \mathbb{P}^1 \setminus \{0, \infty\}$ be a connected component of the set $\{\text{Re } h < C\}$ containing one of the points Re $h(\pm\epsilon)$. Let $\overline{U}$ be the closure of U in $\mathbb{P}^1$. We claim $\overline{U}$ necessarily contains at least one of the points $0, \infty$. Indeed, if $\overline{U} \subset \mathbb{P}^1 \setminus \{0, \infty\}$, the function Re h is everywhere defined on this compact set and so takes on a maximum and a minimum. By the maximum principle, these must fall on $\overline{U} \setminus U$. But Re $h \equiv C$ on $\overline{U} \setminus U$, so the function is necessarily constant, a contradiction.

As a consequence, the points $\tilde{\sigma}(\pm\epsilon)$ can be joined to the points $\{0, \infty\}$ by paths lying wholly in the set $\{\text{Re } h < C\} \cup \{0, \infty\}$. Shrinking ϵ and modifying the paths on $\{\text{Re } h < C\}$, one can avoid any set of points as required in the 4-th condition. □

Definition 5.3. Paths σ and (somewhat abusively) $\gamma := \pi_*\sigma \subset \mathbb{P}^1_t$ above will be referred to as *rapid descent paths*. Thus, associated to a rapid descent path one has singularities $(0, \infty)$ containing the support of the boundary of the path, and a unique critical point ζ lying on the path. One considers period integrals which are functions of z asymptotically supported on a small interval of the path through ζ as $z \to 0$. If $n = 0$, so $p = 1$ and the Levelt summand is regular singular, this is an abuse of language. Indeed, the path one has to take starts at ∞, comes to the singularity along a rapid decay line for $\exp(\frac{t}{z})$, winds around the singularity and comes back to ∞ again along a rapid decay line for $\exp(\frac{t}{z})$.

Lemma 5.4. *We continue to study the connection $\pi_* L$, where π : $\operatorname{Spec} k((u)) \to \operatorname{Spec} k((t))$ is given by $\pi^* t = u^p$. We assume $n \geq 1$. We have the asymptotic expansion as $z \to 0$ (cf. (4.4), (4.12). We write $\pi^*\eta = f(u)du$.) If $n \geq 1$, then*

$$\int_\gamma \langle \epsilon^\vee \exp(\frac{t}{z}), \eta\rangle \sim \int_{\tilde{\gamma}} \langle \epsilon^\vee \exp(\frac{t}{z}), \eta\rangle \sim c\exp(z^{\frac{-n}{n+p}} K_\ell)(z^{\frac{1}{n+p}}\kappa_\ell)^{b-1} f(z^{\frac{1}{n+p}}\kappa_\ell) z^{\frac{n+2}{2(n+p)}} \sqrt{\frac{2\pi}{-M_\ell}}; \quad z \to 0. \tag{5.6}$$

Proof. We assume to fix ideas that $0 < z << 1$. With notation as in (4.5), we need an asymptotic expansion for the integral

$$\int_\sigma c u^{b-1} \exp(h(u,z)) f(u) du.$$

(Note the branch of u^{b-1} along σ is part of the data, i.e. is determined by $\mu^\vee$.) If we make the change of coordinates $u = z^{\frac{1}{n+p}} v$, then $h(u,z) = z^{\frac{-n}{n+p}}(a_{-n} v^{-n} + v^p + O(z^{\frac{1}{n+p}}))$. In v-coordinates, the point $\zeta_\ell = \exp(\frac{2\pi i \ell}{n+p})\kappa_\ell$ (cf. (4.11)). We define the path $\tilde{\sigma}$ on a small interval $[-\epsilon, \epsilon]$ to be centered at the point ζ_ℓ and to follow the path of steepest descent. Then for some $C <$ Re $z^{\frac{n}{n+p}} h(\zeta, z)$ and independent of z, we can arrange by Lemma 5.2 (applied using v coordinates) that $|\exp(h(u,z))| < \exp(C z^{\frac{-n}{n+p}})$ on σ away from $\tilde{\sigma}$. It follows that the two "tails" of the integrals are $O(\exp(C z^{\frac{-n}{n+p}}))$:

$$\Big|\int_{\sigma - \tilde{\sigma}}\Big| \leq \exp(C z^{\frac{-n}{n+p}}) \Big|\int_{\sigma-\tilde{\sigma}} \exp(-C z^{\frac{-n}{n+p}}) \exp(h(u,z)) f(u) du\Big|. \tag{5.7}$$

Notice $\exp(-C z^{\frac{-n}{n+p}})\exp(h(u,z))$ has absolute value ≤ 1 over the path of integration and has rapid decay at $u = 0, \infty$. By assumption $f(u)$ is analytic except for possible meromorphic poles at $0, \infty$. It follows that the integral on the right is $O(1)$, so the right hand side is $O(\exp(Cz^{\frac{-n}{n+p}}))$. The integral over $\tilde{\sigma}$ thus dominates the two tails, proving the lemma. □

Remark 5.5. As in Remark 4.2, replacing π_*L with $\pi_*L \otimes V$ for V nilpotent doesn't substantially affect the above analysis. The asymptotic expansions of certain periods may involve polynomials in $\log z$, but the determinant of the matrix of periods will be simply raised to the rank V power.

We want to show now that the *formal* vanishing cycle decomposition (5.1) is compatible with the asymptotic period matrix, i.e. that the latter can be written in block diagonal form with one block for each $s \in S$, where $S \subset \mathbb{A}^1$ are the singularities of the connection. Note first that computing asymptotic expansions permits us to tensor the space of differential forms η with $k((z))$. That is, an asymptotic expansion like (5.6) makes sense for η a linear combination of forms with coefficients in $k((z))$. In particular, with reference to (5.1), we can assume the forms η run through a basis of $H^1(\Phi) \otimes k((z))$ compatible with the decomposition $\oplus_S VC(s)$.

Lemma 5.6. *We assume $0 \in S$. Let η represent a class in $H^1(\Phi) \otimes k((z))$ (5.1) which dies under the projection to $VC(0)$. Let $\epsilon^v \otimes \nu \exp(\frac{t}{z})|_\gamma$ be a rapid decay chain (see Definition 5.3) for some Levelt factor $\pi_*L \otimes V$ associated to $0 \in S$, and let $\exp(z^{\frac{-n}{n+p}} K_\ell)$ be the exponential factor associated to this chain as in (4.9). Then $\exp(-z^{\frac{-n}{n+p}} K_\ell) \int_\gamma \langle \epsilon^\vee \otimes \nu \exp(\frac{t}{z}), \eta \rangle$ has rapid decay as $z \to 0$ is a small sector around the positive real axis.*

Proof. The solution ν for V plays no role, so we drop it. We assume first $n \geq 1$. By Lemma 5.4, we may replace γ by a small interval $\tilde{\gamma}$ through the critical point. We need to evaluate an integral

$$(5.8) \qquad (*) := \int_{\tilde{\gamma}} \exp(h(u,z)) f(u,z) du$$

where h is as in (4.6). Suppose given a critical point $\zeta(z)$ for h, i.e. $\frac{dh}{dt}(z, \zeta(z)) = 0$. Assume ζ is a Laurent power series in some fractional power of z. Write $u = U + \zeta(z)$, so

$$(5.9) \quad h(u,z) = h(U + \zeta(z), z) = h(\zeta(z), z) + a_2(z)U^2 + a_3(z)U^3 + \ldots$$

Define

$$(5.10) \qquad y := \sqrt{-a_2(z)U^2 - \ldots}$$

and treat y, z as local coordinates, so

$$h(u,z) = h(\zeta(z),z) - y^2 \tag{5.11}$$

$$(*) = \exp(h(\zeta(z),z)) \int \exp(-y^2) f(u(z,y),z) \frac{du}{dy} dy \tag{5.12}$$

The path of integration for $(*)$ will be a small interval from $U = -\epsilon(z)$ to $U = +\epsilon(z)$ for a suitable $\epsilon(z)$. In the y coordinate, this becomes $[-X(z), Y(z)]$ where $\mathbb{R} \ni X, Y \to \infty$ as $z \to 0$. Indeed, in the v coordinates, ϵ and ζ_ℓ have the form const.$+o(1)$ in z. Hence, in the $u = z^{\frac{1}{n+p}} v$-coordinate, $\epsilon(z) \sim C z^{\frac{1}{n+p}}$; $z \to 0$. Also $\sqrt{-a_2(z)} \sim C' z^{\frac{-n-2}{2(n+p)}}$ by (5.10) and (4.10), so

$$U = u - \zeta_\ell(z) \sim C' z^{\frac{n+2}{2(n+p)}} y, \quad X(z), Y(z) \sim C'' z^{\frac{-n}{2(n+p)}}, \ z \to 0. \tag{5.13}$$

To compute the asymptotic expansion of (5.12), expand

$$f(u(z,y),z)\frac{dt}{dy} = b_0(z) + b_1(z)y + \dots. \tag{5.14}$$

As f is meromorphic, (5.13) implies $\mathrm{ord}_z b_i(z) \to \infty$ as $i \to \infty$. Define constants α_k by

$$\alpha_k := \int_{-\infty}^{\infty} \exp(-y^2) y^k dy = \begin{cases} \frac{(2m)!\sqrt{\pi}}{m! 2^{2m}} & k = 2m \text{ even} \\ 0 & k \text{ odd.} \end{cases} \tag{5.15}$$

substituting (5.14) into (5.12) and applying (5.15) term by term leads to the expansion

$$(*) = \exp(h(\zeta(z),z)) \int_{-X(z)}^{Y(z)} \exp(-y^2) f(u(z,y),z) \frac{du}{dy} dy \sim \exp(h(\zeta(z),z)) \sum_{k \geq 0} \alpha_k b_k(z); \ z \to 0. \tag{5.16}$$

It is straightforward to justify this asymptotic expansion assuming $X(z)$ and $Y(z)$ grow more rapidly than z^c for some $c < 0$ (which they do in our case) as $z \to 0$.

Let $g(y)$ be a power series which is meromorphic in a neighborhood of the real axis. Then

$$0 = \int_{-\infty}^{\infty} d_y(\exp(-y^2) g(y)) = \int_{-\infty}^{\infty} \exp(-y^2)(-2yg(y) + g'(y)) dy. \tag{5.17}$$

Take u, z, y as above, and let $g(u,z)$ be a power series in z with coefficients $g_i(u)$ which are meromorphic along the path γ. In the following, the path of integration is a steepest descent path in the u-plane of the sort denoted $\tilde{\sigma}$ in section 4. (See the discussion below (5.12).) The crucial point is that the path of integration in y coordinates approaches the real axis $[-\infty, \infty]$ as $z \to 0$.

$$(5.18) \quad \int d_u(\exp(h(u,z))g(u,z)) = \int \exp(h)(\frac{dh}{du}g + \frac{dg}{du})du =$$

$$\int \exp(h(u(y),z))\Big(\frac{dh}{du}(u(y),z)g(u(y),z) + \frac{dg}{du}\Big)\frac{du}{dy}dy =$$

$$\int \exp(h(u(y),z)\Big(\frac{dh(u(y),z)}{dy}g(u(y),z) + \frac{dg(u(y),z)}{dy}\Big)dy =$$

$$\exp(h(\zeta(z),z)\int \exp(-y^2)(-2yg + \frac{dg}{dy})dy =$$

$$\exp(z^{\frac{-n}{n+p}}K_\ell)\Big(\int \sim 0\Big); \quad z \to 0.$$

For the lemma, note the condition that the form η correspond to a function of the form $\frac{dh}{du}g + \frac{dg}{du}$ is precisely the condition that $\eta \mapsto 0 \in VC(0)$.

Finally if $n = 0$, then η has log poles at some $0 \neq s \in S$, and the integral is of the shape $\int_\gamma ct^{b-1}\exp(\frac{t}{z})\frac{dt}{\prod_{s\neq 0}(t-s)}$, where γ is as in Definition 5.3 and $b_{-1} \notin \mathbb{Z}, \mathrm{Re}(b_{-1}) > 0$. The integral along the rapid decay piece decays rapidly, while the integral on a circle around 0 is of the shape $z^{b_{-1}+1}\int_{S^1} u^b \exp(u)du$. This converges to 0 when $z \to 0$ on the real axis.

□

We now consider the global period matrix, involving all the points of $S \subset \mathbb{A}^1$. Because $\exp(\frac{t}{z}) = \exp(\frac{s}{z})\exp(\frac{t-s}{z})$, the asymptotic formulae such as (5.6) inherit an extra factor of $\exp(\frac{s}{z})$. We index the columns of our period matrix by the homology chains, which we group together according to the different $s \in S$. The rows correspond to a basis of $H^1(\Phi) \otimes_{k(z)} k((z))$ which we take to be compatible with the vanishing cycle direct sum decomposition (5.1). The resulting period matrix $\mathcal{P}\mathrm{er(z)}$ (5.3) can be written in the form $\mathcal{P}\mathrm{er(z)} = \mathrm{QD}$, where D is the diagonal matrix with entries of the form $\exp(\frac{s}{z} + b_{-1}\log z + z^{\frac{-n}{n+p}}K_\ell)$ as in (5.6), and the entries of Q all admit asymptotic expansions in $\mathbb{C}((z^{1/N}))[\log z]$ for some N as $z \to 0$. Indeed, Q can be written in block matrix form

$$(5.19) \qquad Q = (Q_{ss'})_{s,s' \in S}$$

where Q_{ss} is a square matrix of size $\mathrm{rank}E + \mathrm{irregularity}(E,s)$ (Proposition 4.1), and, by Lemma 5.6, the entries of $Q_{ss'}$ have rapid decay for $s \neq s'$. Note the presence of $\log z$ in these entries comes from the nilpotent connections V_i, (5.4). These play no role in the determinant of the asymptotic period matrix, which is our main interest. Indeed, by Remark 4.2, for suitable bases, the log terms will occur only above the diagonal.

Proposition 5.7. *Let $\mathcal{A}$ be the ring of germs of analytic functions in z on our sector about $z > 0$ admitting an asymptotic expansion in $\mathbb{C}((z^{1/N}))$ at the origin for some N (cf. Section 4). With notation as above (in particular E is the original meromorphic connection on $\mathbb{P}^1$), define*

$$\mathfrak{s} = \sum_{s\in S} (\mathrm{rank} E + \mathrm{irregularity}(E, s)) \cdot s.$$

Then $\exp(-\frac{\mathfrak{s}}{z}) \det D$ *and* $\det Q$ *lie in* $z^{\mathbb{C}} \cdot \mathcal{A}$*, so they have asymptotic expansions, which we denote by* $A.E(*)$*, which are lying in* $z^{\mathbb{C}} \cdot \mathbb{C}((z^{1/N}))$*. Moreover* $\exp(-\frac{\mathfrak{s}}{z}) \det \mathcal{P}\mathrm{er(z)}$ *lies* $\in \mathbb{C}((z))$*. We have product formulae indexed by* $s \in S$*:*

$$\det D = \prod_S D_s$$

$$A.E. \det Q = \prod_S A.E. \det Q_{ss}$$

$$\exp(-\frac{\mathfrak{s}}{z}) \det \mathcal{P}\mathrm{er(z)} = \mathrm{A.E.}(\exp(-\frac{\mathfrak{s}}{\mathrm{z}}) \det \mathrm{D}) \prod_{\mathrm{S}} \mathrm{A.E.} \det \mathrm{Q_{ss}}$$

Proof. We know from Proposition 3.11 that the Gauß-Manin connection on $\det H^1(\Phi)$ has the form $\nabla(1) = -d(\frac{\mathfrak{s}}{z})$ for a suitable gauge. Since $\det \mathcal{P}\mathrm{er(z)}$ is horizontal for this connection, it follows that $\exp(-\frac{\mathfrak{s}}{z}) \det \mathcal{P}\mathrm{er(z)}$ is horizontal for a trivializable rank 1 formal connection over $\mathbb{C}((z))$. In particular, this quantity lies in $\mathbb{C}((z))$. The other assertions follow from this, together with Proposition 4.1 and Lemma 5.6. As remarked above, $\log z$ does not occur in the determinant because the period matrices associated to the nilpotent connections V_i are upper triangular. The factors $z^{\mathbb{C}}$ come from the regular singular point contribution. □

Corollary 5.8. *There is a product expression*

$$\exp(-\frac{\mathfrak{s}}{z}) \det \mathcal{P}\mathrm{er(z)} = \prod_{\mathrm{s}\in\mathrm{S}} \xi_{\mathrm{s}} \tag{5.20}$$

$$\xi_s = \exp\Big(- (\mathrm{rank} E + \mathrm{irregularity}(E, s)) \cdot s/z\Big) D_s \cdot A.E. \det Q_{ss}$$

Remark 5.9. The definition of the ξ_s are purely local, depending only on the connection (E, ∇_E) over the formal powerseries field at $s \in \mathbb{P}^1_t$. They will play the central role in the definition of local epsilon factors. Notice ξ_s is a product of an exponential factor times an element in $\mathbb{C}((z^{1/N}))^\times$ for some N. To be precise, we should think of

$$\xi_s \in \Big(K^{\rm ess} \otimes_{K^{\rm mero}} \mathbb{C}((z^{1/N}))\Big)^\times$$

where $K^{\rm ess}$ (resp. $K^{\rm mero}$) denotes the field of germs of functions in $z^{1/N}$ with possible essential (resp. meromorphic) singularities at the origin. It may be interpreted as a solution in that ring of a certain rank 1 connection.

Remark 5.10. The reader may inquire why we have never dealt with the possibility that our global connection E has nontrivial Stokes structures at points of S. The reason is that changing Stokes structures amounts to twisting by a cocycle in the sheaf $End(E)$ (on the real blowup at points of S) of the form $I + M$ where $M \in End(E)^{<0}$ has rapid decay. Such a twist will disappear when we take the asymptotic expansion. Indeed, let $e_1, \dots, e_r$ be a local basis in a sector such that (say) $\nabla_E(e_i) = e_i \otimes \sigma_i dt$. Suppose $e_i{}' = e_i + \sum_j g_{ij} e_j$ where the $e_i{}'$ form a local basis of E on an analytic neighborhood of $z = 0$. Assume the g_{ij} have rapid decay and the matrix $h := (\delta_{ij} + g_{ij})$ is flat as an endomorphism of $E = \oplus \mathcal{O} e_i$. Let $\epsilon_i^\vee = \phi_i e_i^\vee$ be flat sections of E. Then de Rham cohomology classes can be represented in the form $e_j{}' \otimes f_{jk}(t)dt$ for $f_{jk}(t)$ meromorphic at 0. Our period integrals look locally like

$$\int_{\gamma_i} \langle \epsilon_i^\vee exp(t/z), e_j{}' \otimes f_{jk}(t)dt \rangle = \int_{\gamma_i} exp(t/z)\phi_i f_{jk} \cdot (\delta_{ij} + g_{ji})dt. \tag{5.21}$$

But the right hand integral has the same asymptotic expansion for $z \to 0$ as the integral

$$\int_{\gamma_i} exp(t/z)\phi_i \delta_{ij} f_{jk} dt = \int_{\gamma_i} \langle \epsilon_i^\vee exp(t/z), e_j \otimes \delta_{ij} f_{jk}(t)dt \rangle. \tag{5.22}$$

To see this, note that in (4.12), the term $f(z^{\frac{1}{n+p}}\kappa_\ell)$ which appears on the right involves a positive power of z. Replacing f by $f + g$, where g has rapid decay, does not change the asymptotic expansion.

6. Epsilon Lines

Let $k, M \subset \mathbb{C}$ be subfields of the complex numbers.

Definition 6.1. The category line(k, M $\subset \mathbb{C}$) is the groupoid with objects the category of triples $\ell_{\mathbb{C}}, \ell_k, \ell_M$ of lines over the indicated field together with inclusions $\ell_k \subset \ell_{\mathbb{C}}$, $\ell_M \subset \ell_{\mathbb{C}}$. A morphism $\{\ell_k, \ell_M, \ell_{\mathbb{C}}\} \to \{L_k, L_M, L_{\mathbb{C}}\}$ is a $\mathbb{C}$-linear isomorphism $\ell_{\mathbb{C}} \to L_{\mathbb{C}}$ which carries $\ell_F \to L_F$ for $F = k, M$. The corresponding category of superlines, $\mathrm{superline(k, M \subset \mathbb{C})}$, is defined analogously. We will occasionally refer to the line ℓ_k as the DR-line and the line ℓ_M as the Betti line.

Example 6.2. (i) Let E be a rank 1 connection on a smooth variety X defined over k, and let $x \in X(k)$ be a k-point. Suppose given an embedding $k \hookrightarrow \mathbb{C}$, and write $\mathcal{E} = E_{\mathbb{C}}^{an,\nabla=0}$ for the local system of horizontal sections. Take $\ell_k := E_x$ to be the fibre at x and take $\ell_{\mathbb{C}} = \mathcal{E}_x$ to be the fibre of the local system. One has a canonical identification $E_k \otimes_k \mathbb{C} = \mathcal{E}_x$ and hence an inclusion $\ell_k \subset \ell_{\mathbb{C}}$. If further we are given a reduction of structure for $\mathcal{E}$ to a local system $\mathcal{E}_M$ of M-vector spaces for some subfield $M \subset \mathbb{C}$, then we may define an object in $\mathrm{line(k, M \subset \mathbb{C})}$ by taking $\ell_M = \mathcal{E}_{M,x}$. More generally, if E has arbitrary rank, we may do this construction on $\det E$.
(ii) For $n \in \mathbb{Z}$, we write $(2\pi i)^n \in \mathrm{line}(\mathbb{Q}, \mathbb{Q} \subset \mathbb{C})$ for the line $\ell_k = \mathbb{Q}$, $\ell_{\mathbb{C}} = \mathbb{C}$, $\ell_M = (2\pi i)^{-n}\mathbb{Q}$. As a superline, $(2\pi i)^n$ has degree 0. Writing $\ell^\vee$ for the dual line, we get $\langle \ell_k, \ell_M^\vee \rangle = (2\pi i)^n\mathbb{Q}$. Thus, $(2\pi i) = (H^1_{DR}(\mathbb{G}_m, \mathbb{Q}), H^1_B(\mathbb{G}_m, \mathbb{Q}))$.
(iii) The categories $\mathrm{line(k, M \subset \mathbb{C})}$ have evident tensor structures which we will typically denote by juxtaposition, $\ell\ell'$ rather than $\ell \otimes \ell'$. More generally, if $k' \subset k$ and $M' \subset M$ then one has a tensor operation $\mathrm{line(k', M' \subset \mathbb{C})} \times \mathrm{line(k, M \subset \mathbb{C})} \to \mathrm{line(k, M \subset \mathbb{C})}$. Thus, e.g. $(2\pi i)^n\ell$ is always defined.
(iv) Let E be a rank 1 connection on $\mathrm{Spec}\, K$, where K is a powerseries field in one variable over $k \subset \mathbb{C}$. As in subsection 2.6, we associate to E a rank 1 translation invariant connection $\mathbb{E}$ on $F^\times$. Suppose further E, ∇_E is identified as coming by pullback from an analytic connection E_{an}, ∇_{an} on a punctured disk. We have seen in this case (Proposition 2.12) that the horizontal sections of E_{an} are identified with the horizontal sections of $\mathbb{E}_{an}$. If we are given a *monodromy* field $M \subset \mathbb{C}$ and a reduction of structure of the local system $E_{an}^{\nabla=0}$ to M, then for each $f \in F^\times(k) = K^\times$, the fibre of $\mathbb{E}$ over f defines an object in $\mathrm{line(k, M \subset \mathbb{C})}$ which we denote (E, f). Translation-invariance implies

$$(E, f_1f_2) = (E, f_1)(E, f_2); \quad (E, 1) = \mathbf{1} \tag{6.1}$$

where $\mathbf{1}$ is the line $(k, M, k \otimes_k \mathbb{C} = M \otimes_M \mathbb{C})$.
(v) Our most important example of an object of $\mathrm{superline(k, M \subset \mathbb{C})}$ is the *epsilon line* $\varepsilon(E, \nu)$ associated to a connection E over a powerseries field in 1 variable and a non-zero 1-form ν. The remainder of this section is devoted to the definition and properties of this line.

6.1. Let k be a field of characteristic 0, and let K be the powerseries field in one variable over k. Let E be a finite dimensional K vector space endowed with a k-linear connection $\nabla : E \to E \otimes \Omega^1_{K/k} = Edt$ for a choice of a parameter $t \in K \cong k((t))$. (Note of course the Kähler differentials are taken to be dual to the continuous derivations so $\Omega^1_{K/k}$ is a rank 1 K-vector space.) Let $\nu \in \Omega^1_{K/k}$ be a non-zero 1-form. Recall we have defined (Definition 2.18) we have defined the de Rham *epsilon (super)line* $\varepsilon_{DR}(E, \nu)$ to be the polarized determinant of the endomorphism $\nu^{-1} \circ \nabla : E \to E$. Briefly, if $L \subset M \subset E$ are lattices such that $\nu^{-1} \circ \nabla$ induces an isomorphism $E/L \cong E/M$, then $\varepsilon_{DR}(E, \nabla, \nu) := \det_k(M/L)$.

Recall further in Proposition 3.9 above for $\nu = dt$ we reinterpreted this de Rham line in terms of a canonical isomorphism with the vanishing cycles:

$$\varepsilon_{DR}(E, dt) = \det_k \Big(VC(E; \mathcal{V}, \mathcal{W})/zVC(E; \mathcal{V}, \mathcal{W})\Big). \tag{6.2}$$

The connection $\det VC(E)$ had a regular singular point (Proposition 3.11) and the lattice $\Xi := \det VC(E; \mathcal{V}, \mathcal{W}) \subset \det VC(E)$ was stable under the operator $\nabla_{z\frac{d}{dz}}$. (At the moment we work locally, so $s = 0$ and the exponential factor $EXP((r + n)s/z)$ one sees in Proposition 3.11 is not present.) As in Subsection 2.5 there is an induced connection on $gr\Xi = \oplus_{i\geq 0} z^i\Xi/z^{i+1}\Xi$ which we identify as a rank 1 regular singular point connection on the tangent space T to $\operatorname{Spec} k[[z]]$ at the origin. For any point $\zeta \in T$, we have therefore canonical identifications

$$\varepsilon_{DR}(E, dt) = \Xi/z\Xi \xrightarrow{\cong} gr\Xi|_\zeta. \tag{6.3}$$

6.2. Let $\iota : k \hookrightarrow \mathbb{C}$ be a fixed embedding. The rank 1 connection $gr\Xi$ gives rise to an analytic local system $(gr\Xi \otimes_k \mathbb{C})^{an,\nabla=0}$ on the punctured tangent space $T^* := T - \{0\}$. By definition, the *Betti* epsilon line is the $\mathbb{C}$-line

$$\varepsilon_B(E, dt) := (gr\Xi \otimes_k \mathbb{C})^{an,\nabla=0}|_{\frac{d}{dz}} \tag{6.4}$$

where we identify d/dz as a point in T^*. From (6.3) we get a canonical identification

$$\varepsilon_B(E, dt) = \varepsilon_{DR}(E, dt) \otimes_k \mathbb{C}. \tag{6.5}$$

Suppose now that we are given a meromorphic connection E_{an}, ∇_{an} on a punctured disk D^*. Suppose further our powerseries field K is identified with the completion of the field of meromorphic functions at the origin of the disk in such a way that E, ∇ is the base extension of E_{an}, ∇_{an}. Let $M \subset \mathbb{C}$ be a subfield. In Remark 2.4(iii) we defined the notion of reduction to M for the Stokes structure of E_{an}. We want to show how a reduction to M of the Stokes structure of E_{an} determines an M-structure on the Betti line $\varepsilon_B(E, dt)$. Let $\mathbb{E}$ be the Katz extension of E to $\mathbb{P}^1$ associated to the parameter t as in Subsection 2.4. $\mathbb{E}$ is identified with E at the origin and has regular singular points at ∞. We want to apply the asymptotic analysis from Sections 3.2, 4, and 5 to the fourier transform of the meromorphic connection $\mathbb{E} \otimes_k \mathbb{C}$. For k'/k an extension field, we may extend ι to $\iota' : k' \hookrightarrow \mathbb{C}$. Since $\mathbb{E} \otimes_k \mathbb{C} \cong E \otimes_k k' \otimes_{k'} \mathbb{C}$, we may replace E and $\mathbb{E}$ in our discussion by $E \otimes_k k'$ and $\mathbb{E} \otimes_k k'$ and assume that E admits a Levelt decomposition as in Subsection 2.1. We choose suitable paths γ_j with endpoints in $\{0, \infty\}$ and suitable sections $\epsilon_j^\vee \in \mathcal{E}^\vee|_{\gamma_j}$ where $\mathcal{E}$ is the local system of horizontal sections of $\mathbb{E}$. Our choice of a M-Stokes structure determines such a structure on $\mathcal{E}^\vee$ and we choose the $\epsilon_j^\vee$ to be defined over M.

We now analyse the asymptotic behavior as $z \to 0$ of the determinant of the period matrix (5.3)

$$\det \mathcal{P}er(z) := \det \Big(\int_{\gamma_j} \langle \epsilon_j^\vee \exp\big(\frac{t}{z}\big), \eta_k \rangle \Big)_{j,k} \tag{6.6}$$

where η_k are a basis for $VC(E)$. As a function of $\bigwedge_k \eta_k \in \det VC(E)$, this is horizontal with respect to the regular singular point (Prop. 3.11) connection $\nabla_{\det VC}$. We know from the results in Section 5 that the period matrix has the form $D \cdot Q$ where the entries of Q admit asymptotic expansions in $z^{\mathbb{C}} \cdot \mathbb{C}((z))[\log z]$ and D is diagonal with entries $\exp(f_j)$ where f_j is meromorphic and independent of η_k. Write $f = \sum_j f_j$. Because the determinant connection is regular singular, the function f must be regular at $z = 0$. Because this connection has rank 1 there will be no logarithm term in the expansion. We take the η_k to be a $k[[z]]$-basis for the lattice $VC(E; \mathcal{V}, \mathcal{W})$ (Corollary 3.8 and write $\det \mathcal{P}er(z) = z^\alpha g(z)$ where $g(0)$ is defined. The M-line

$$M \cdot g(0) z^\alpha \subset (gr\Xi)_{an}^{\vee, \nabla=0} \tag{6.7}$$

determines a (dual) M-structure on $gr\ \Xi_{an}^{\nabla=0} = gr \det VC(E; \mathcal{V}, \mathcal{W})_{an}$ and hence a M-structure on $\varepsilon_B(E, dt) = gr\ \Xi_{an}^{\nabla=0}|_{d/dz}$. Note that the k-line $\varepsilon_{DR}(E, dt)$ does not depend on fixing $d/dz \in T^*$, but the Betti line does.

Remark 6.3. The choice of $M \subset \mathbb{C}$ depends on the Stokes structure, but (cf. Remark 5.10) the asymptotic expansion of (6.6) does not. this is why were are free to use the Katz extension (which corresponds to the trivial Stokes structure) to calculate our epsilon line.

6.3. Example. Suppose E has rank r and has regular singular points. If we permit ourselves to extend the field of definition k, we may assume there exists a gauge with respect to which the connection matrix has the form $d + A\frac{dt}{t}$ where the matrix A is constant with entries in k. (Of course, extending k loses information on $\varepsilon_{DR}(E,\nu)$ but it doesn't affect the calculation of ε_B which is the crucial point.) Let $\{v_i\}$ be the corresponding basis of E. The Katz extension $\mathbb{E}$ is the meromorphic connection on $\mathbb{P}^1$ with trivial rank r bundle and the same connection matrix. The Fourier twist has connection form $d+A\frac{dt}{t}+\frac{dt}{z}$. A basis for H^1 of this connection is represented by the elements $v_i \otimes \frac{dt}{t}$, $1 \leq i \leq r$. A basis for solutions is given by the columns of the matrix

$$\exp(A\log t + \frac{t}{z}I) \tag{6.8}$$

Let γ be the path in the t-plane which follows the $\mathbb{R}^+$ axis from $+\infty$ to some point $0 < \delta << 1$, winds counter-clockwise about 0, and then returns to $+\infty$. For convenience we take $0 < z << 1$. Write $z\gamma$ for the scaled path. The period matrix can be written

$$\Big(\int_{z\gamma} \exp(A\log t + \frac{t}{z})_{ij}\frac{dt}{t}\Big)_{1\leq i,j\leq r} \tag{6.9}$$

Write $s = t/z$. The period matrix becomes

$$\Big(\int_{\gamma} \exp(A\log s + A\log z + s)_{ij}\frac{ds}{s}\Big)_{1\leq i,j\leq r} \tag{6.10}$$

Conjugating A by a suitable constant matrix, we may assume A is upper triangular with diagonal entries $a_1, \ldots, a_r \in \mathbb{C}$. The determinant of (6.10) is then

$$z^{\mathrm{Tr}(A)}\prod_i \int_\gamma s^{a_i}e^s\frac{ds}{s} \tag{6.11}$$

The vanishing cycle determinant connection is then

$$d_z + \mathrm{Tr}(A)\frac{dz}{z}. \tag{6.12}$$

The corresponding graded connection on the tangent space to Spec $k[[z]]$ at $z=0$ is the same, so $\varepsilon_{DR}(E,dt)$ is the fibre at $z=1$ of this connection. We identify $\varepsilon_B = \varepsilon_{DR}\otimes_k \mathbb{C}$ with the fibre at $z=1$

Let $M \subset \mathbb{C}$ be a field containing the $\exp(2\pi i a_j)$, $1 \leq j \leq r$. A horizontal section of $\det VC$ is given by $z^{-\mathrm{Tr}(A)}$. The M-structure we want on the fibre at $z=1$ is given by

$$M\cdot\Big(\prod_i\int_\gamma s^{a_i}e^s\frac{ds}{s}\Big)^{-1}z^{-\mathrm{Tr}(A)}|_{z=1} \tag{6.13}$$

Note that this is well-defined because $\exp(2\pi i \mathrm{Tr}(A)) \in M$.

It is worth remarking here that we have made the choice of a M-Betti structure on E for which a basis of solutions is given by the columns of $\exp(A\log t)$. A different choice would have as basis the columns of $B \cdot \exp(A \log t)$ for an invertible matrix B with entries in $\mathbb{C}$. The new M-structure would be calculated from the period determinant

$$\det\Big(\int_{z\gamma}\Big(B\cdot\exp(A\log t+\frac{t}{z})\Big)_{ij}\frac{dt}{t}\Big)_{1\le i,j\le r} = \det\Big(\int_{z\gamma}\sum_k B_{ik}\exp(A\log t+\frac{t}{z})_{kj}\frac{dt}{t}\Big)_{1\le i,j\le r} = \det B\cdot z^{\mathrm{Tr}(A)}\prod_i\int_\gamma s^{a_i}e^s\frac{ds}{s}. \tag{6.14}$$

Thus the M-structure on ε_B is multiplied by $\det B^{-1}$.

7. Projection Formula

7.1. Change of Coordinate. Let E be a connection over $k((t))$, and let $\nu = f dt \in k((t))dt$ be a nonzero 1-form. We want to justify the following formula-definition (cf. Example 6.2(iv))

$$\varepsilon(E,\nu) := (2\pi i)^{ord(f)\cdot\mathrm{rank}(E)}(\det E, f)\varepsilon(E, dt). \tag{7.1}$$

This is to be viewed as an identity between objects in the category superline(k, M ⊂ ℂ), subject to the choice of $M \subset \mathbb{C}$ and a M-Stokes structure on E, (see example 6.2). We must show the right hand side is independent of the choice of parameter t. Let $\sigma \in k((t))$ be another parameter, so $d\sigma = \frac{d\sigma}{dt}dt$ with $\frac{d\sigma}{dt} \in k[[t]]^\times$, and $f dt = f(\frac{d\sigma}{dt})^{-1}d\sigma$. Bearing in mind multiplicativity of the symbol $(\det E, f)$ in f (6.1), independence of t follows from

Theorem 7.1. *With notation as above, we have a canonical isomorphism*

$$(\det E, \frac{d\sigma}{dt})\varepsilon(E, dt) = \varepsilon(E, d\sigma). \tag{7.2}$$

Proof. Let $U^\times = \operatorname{Spec} k[X_0, X_0^{-1}, X_1, X_2, \dots]$ be the scheme of units in $k[[t]]$, and let $\omega_0^\times$ be the scheme of 1-forms generating $\omega^1_{k[[t]]/k}$ over $k[[t]]$. We have a natural torsor structure

$$U^\times \times \omega_0^\times \to \omega_0^\times;\quad f, \eta \mapsto f\eta. \tag{7.3}$$

On $U^\times \times \operatorname{Spec} k((t))$ (resp. on $\omega_0^\times \times \operatorname{Spec} k((t))$ we have a canonical unit u (resp. a canonical 1-form $d_t\tau$)

$$u := X_0 + X_1 t + X_2 t^2 + \dots \tag{7.4}$$

$$d_t\tau := u dt \tag{7.5}$$

(The notation $d\tau$ without the subscript t will refer to the differential relative to k). By classfield theory, (Subsection 2.6) we have associated to the rank 1 connection $\det E$ on $\operatorname{Spec} k((t))$ (together with a choice of gauge e) a translation-invariant form η on $U^\times$.

Lemma 7.2. *Write $\nabla_{\det E}(e) = a(t)dt \cdot e$. Then*

$$\eta = -Res_{t=0} a(t)dt \wedge du/u.$$

Here du refers to the differential relative to k, so $dX_i \neq 0$.

proof of lemma. Recall η is defined to be the unique invariant 1-form which pulls back to $a(t)dt$ under the mapping $X_i \mapsto -t^{-i-1}$. (This comes from identifying $U^\times$ with a subscheme of the Picard scheme of $\mathbb{P}^1$ with level 1 at ∞ and infinite level at 0, and then identifying $\operatorname{Spec} k((t))$ with the point $T = t$ on $\mathbb{P}^1$.) Now $dt \wedge du/u = dt \wedge d_U u/u$ where $d_U u = dX_0 + t dX_1 + \dots$, and $d_U u/u = \eta_0 + \eta_1 t + \dots$, where the η_i are the invariant 1-froms on $U^\times$. To distinguish between the different roles of t, write $v = X_0 + X_1 T + X_2 T^2 + \dots$. Under the substitution $X_i \mapsto -t^{-i-1}$, we have $v \mapsto \frac{1}{T-t}$, and $d_U v/v \mapsto d_t \log \frac{1}{T-t} = -\sum_{i=0}^\infty \frac{T^i dt}{t^{i+1}}$. That is $\eta_i \mapsto \frac{-dt}{t^{i+1}}$.

Suppose $\det E$ has irregularity n, so $a(t)dt = (a_{-n-1}t^{-n-1} + \dots)dt$. Then

$$\operatorname{Res}_{t=0}(a(t)dt \wedge d_U u/u) = a_{-n-1}\eta_n + a_{-n}\eta_{n-1} + \dots \mapsto -\sum_{p<0} a_p t^p dt. \tag{7.6}$$

□

We return to the proof of the theorem. Let $V \subset W$ be a good lattice pair for E as in Subsection 3.1. Let $r = \operatorname{rank} E$ and let $w_1, \dots, w_r$ be a $k[[t]]$-basis for W such that for suitable $n_j \geq 0$ the elements $e_j := t^{n_j} w_j$ form a basis for V. We take as gauge in $\det E$ the element $e = e_1 \wedge \dots \wedge e_r \in \det E$.

Our strategy will be extend the vanishing cycles $VC(E)$ to a connection over $\mathcal{O}_{\omega_0^\times}((z))$ by replacing dt by $d_t\tau$. That is, we consider the family of vanishing cycles, defined by the z-linear sequence

$$V((z)) \xrightarrow{\nabla_E + \frac{d_t\tau}{z}} W((z))\frac{dt}{t} \to VC \to 0. \tag{7.7}$$

(To simplify notation, we omit writing completed tensor products $\widehat{\otimes}\mathcal{O}_{\omega_0^\times}$ everywhere.)

We define a z-lattice fil^0VC in VC to be spanned over $\mathcal{O}_{\omega_0^\times}[[z]]$ by the image in VC of $W\frac{dt}{t}$. Define $fil^iVC = z^i fil^0VC,\ i \in \mathbb{Z}$. Arguing as in section 2 of the ms, we know that VC is free of rank $n+r$ over $\mathcal{O}_{\omega_0^\times}((z))$ where n is now the irregularity of E. As a base we can take $\{t^iw_jdt \mid -1 \le i < n_j\}$. The Gauß-Manin connection on VC is defined as usual by the diagram

$$\begin{array}{ccccc}
 & & V((z)) & = & V((z)) \\
 & & \Big\downarrow{\scriptstyle \widetilde{\nabla}_E + \frac{d\tau}{z}} & & \Big\downarrow{\scriptstyle \nabla_E + \frac{d_t\tau}{z}} \\
V((z)) \otimes \Omega^1_{\omega_0^\times} & \longrightarrow & V((z)) \otimes \Omega^1_{\omega_0^\times} + W((z))\frac{dt}{t} & \longrightarrow & W((z))\frac{dt}{t} \\
\Big\downarrow{\scriptstyle (\nabla_E + \frac{d_t\tau}{z})\otimes 1} & & \Big\downarrow{\scriptstyle \widetilde{\nabla}_E + \frac{d\tau}{z}} & & \\
W((z))\frac{dt}{t} \otimes \Omega^1_{\omega_0^\times} & = & W((z))\frac{dt}{t} \otimes \Omega^1_{\omega_0^\times} & &
\end{array} \tag{7.8}$$

Here all maps are z-linear, ∇_E is $\mathcal{O}_{\omega_0^\times}$-linear as well, while $\widetilde{\nabla}_E$ differentiates on $\mathcal{O}_{\omega_0^\times}$. (In the bottom line, we kill $\Omega^2_{\omega_0^\times}$.)

As a consequence of (7.8) we have a z-linear connection $\nabla_{VC} : VC \to VC \otimes \Omega^1_{\omega_0^\times}$, and the corresponding determinant connection $\nabla_{\det VC} : \det VC \to \det VC \otimes \Omega^1_{\omega_0^\times}$. Here the exterior power involved in defining the determinant is taken over $\mathcal{O}_{\omega_0^\times}((z))$. In particular, the z-lattice structure fil^0VC induces a z-lattice structure $fil^0 \det VC$. Concretely, one has

$$\nabla_{VC}(t^iw_jdt) = t^iw_jdt \wedge \frac{d\tau}{z} = t^iw_jdt \wedge \frac{d_U\tau}{z}. \tag{7.9}$$

Define a basis

$$\lambda := \bigwedge_{i,j}(t^iw_jdt) \in \det VC; \quad 1 \le j \le r,\ -1 \le i < n_j. \tag{7.10}$$

(The wedge is not t-linear.)

Lemma 7.3. *The connection $\nabla_{\det VC}$ preserves the filtration $fil^i \det VC$.*

Proof. The connection is z-linear so it suffices to show this for $i = 0$. Write $d_U\tau = \sum_i (d_U\tau)_i t^i$, and $(d_U\tau)_{<m} = \sum_{i<m}(d_U\tau)_i t^i$ (similarly

$(d_U\tau)_{\geq m}$, etc.). We have

$$(7.11)\quad \nabla_{\det VC}\lambda = \sum_{i,j} \pm\nabla_{VC}(t^i w_j dt) \wedge \bigwedge_{(k,\ell)\neq(i,j)} (t^k w_\ell dt) = \sum_{i,j} \pm t^i w_j dt \wedge \frac{d_U\tau}{z} \wedge \bigwedge_{(k,\ell)\neq(i,j)} (t^k w_\ell dt) = \sum_{i,j} \pm t^i w_j dt \wedge \Big(\frac{d_U\tau}{z}\Big)_{\geq n_j - i} \wedge \bigwedge_{(k,\ell)\neq(i,j)} (t^k w_\ell dt).$$

Note in the last line, the terms in $(d_U\tau)_{<n_j-i}$ give rise to terms $t^k w_j dt$ which appear already in the wedge on the right. These terms die. In $VC \otimes \Omega^1_{\omega_0^\times}$, we have (note $d_t\tau = udt$)

$$(7.12)\quad t^i w_j dt \wedge \Big(\frac{d_U\tau}{z}\Big)_{\geq n_j-i} = t^{n_j} w_j dt \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}z} = e_j dt \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}z} = \pm e_j \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}u} \wedge \frac{d_t\tau}{z} \equiv \pm \nabla_E\Big(e_j \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}u}\Big).$$

Note that the RHS lies in $fil^0 VC$, proving the lemma. □

Next we want to compute $gr_z\nabla_{\det VC}$. Let $A = (a_{ij})$ be the connection matrix for E in the basis $\{e_j\}$, so $\nabla_E = d + Adt$. Since $\nabla_E(V) \subset W\frac{dt}{t}$, we have $a_{jk} = 0,\ k < -n_j - 1$. In VC, for $p > 0$

$$(7.13)\quad \nabla_E(t^p e_j) = t^p\nabla_E(e_j) + e_j d(t^p) \equiv t^p\nabla_E(e_j) + pz\nabla_E(t^{p-1}u^{-1}e_j).$$

Combining (7.11)-(7.13), we see that to compute $gr_z\nabla_{\det VC}$, it suffices to compute

$$(7.14)\qquad \nabla_E(e_j)\frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}u} = \sum_\ell a_{j\ell}e_\ell dt \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}u}.$$

Consider on the RHS terms with $\ell \neq j$. Each $a_{j\ell}e_\ell \in t^{-1}W$, and the factor to the right of the wedge is a power series in non-negative powers of t. Since for $p \geq 0$, $t^p e_\ell \equiv 0 \mod (z)$ as in (7.13), and since terms $t^p e_\ell \in t^{-1}W$ will appear already in the right-hand wedge in (7.11), all terms with $\ell \neq j$ can be dropped. thus we need only consider

$$(7.15)\qquad a_{jj}e_j dt \wedge \frac{(d_U\tau)_{\geq n_j-i}}{t^{n_j-i}u} = t^i w_j dt \wedge \Big(\frac{a_{jj}(d_U\tau)_{\geq n_j-i}}{u}\Big)$$

Arguing as above, we need only consider the term in t^0 in the factor to the right of the wedge on the RHS. In other words, the contribution for i, j fixed is (note $t^{n_j+1}a_{jj}$ has no pole)

$$\text{(7.16)} \qquad \mathrm{Res}_{t=0} a_{jj} \frac{dt}{t} \wedge \frac{(d_U\tau)_{\geq n_j - i}}{u} = \mathrm{Res}_{t=0} a_{jj} \frac{dt}{t} \wedge \sum_{\ell=n_j-i}^{n_j+1} \frac{(d_U\tau)_\ell t^\ell}{u}.$$

Summing over i, j

(7.17)

$$\sum_j \sum_{i=-1}^{n_j-1} \mathrm{Res}_{t=0} a_{jj} dt \wedge \sum_{\ell=1}^{n_j+1} \frac{\ell (d_U\tau)_\ell t^{\ell-1}}{u} = \sum_j \mathrm{Res}_{t=0} a_{jj} dt \wedge \frac{d_U u}{u}.$$

Since, with reference to lemma 7.2, $a(t) = \sum_j a_{jj}$, we conclude from (7.17) that the connection on $U^\times$ associated to $\det E$ with gauge $e = e_1 \wedge \ldots \wedge e_r$ by classfield theory coincides with the connection

$$\text{(7.18)} \qquad gr^0_z \det VC$$

with gauge λ as in (7.10) on $\omega_0^\times$. Note the assertion here is that two translation invariant forms are equal. Since $\omega_0^\times$ is a torsor for $U^\times$, the spaces of translation-invariant forms are identified, so "equality" makes sense.

Another way of saying this is that under the torsor map $\pi : U^\times \times \omega_0^\times \to \omega_0^\times$ we have

$$\text{(7.19)} \qquad \pi^* gr^0_z \det VC = [\det E] \boxtimes gr^0_z \det VC,$$

where $[\det E]$ is the rank 1 invariant connection from classfield theory. Indeed, the fact that the connection form on $gr^0_z \det VC$ is translation invariant means that the pullback will be such an exterior tensor product. To identify the two factors one restricts to $1 \in U^\times(k)$ and $dt \in \omega_0^\times(k)$.

To get what we want from (7.19), note that the composition

$$\text{(7.20)} \qquad gr^0_z \det VC \hookrightarrow \oplus_{i\geq 0} gr^i_z \det VC \twoheadrightarrow (\oplus_{i\geq 0} gr^i_z \det VC)/(z-1)$$

is an isomorphism compatible with the connection. The RHS is the family over $\omega_0^\times$ of fibres at $z = 1$ which is, by definition, the family of epsilon lines.

To complete the proof of theorem 7.1, we have to check that the M-Betti structure is flat for the $\omega_0^\times$-connection, and that in (7.19) that the M-structure on $[\det E]$ coincides with the M-structure coming from classfield theory (Subsection 2.6). For flatness, consider the period determinant

$$\text{(7.21)} \qquad \det\Big(\int_{\gamma_i} \langle \varepsilon_i^\vee \exp(\tau/z), \eta_j \rangle \Big)$$

obtained by replacing t in our previous calculations from Section 5 with $\tau = X_0 t + X_1 t^2/2 + \dots$ such that $d_t\tau = udt$ as above. Locally over $\omega_0^\times$ we may take the η_j independent of the X_i. It follows that the determinant (7.21) will be a solution for the connection along $\omega_0^\times$. It follows that the M-structure, which is specified on the graded regular singular point connection in z associated to (7.21) is horizontal with respect to the connection on $\omega_0^\times$. Finally, we remark that, given the M-Betti structure on $gr_z^0 \det VC$, (7.19) determines a M-Betti structure on $[\det E]$ over $U^\times$. It suffices to verify at one point of $U^\times$ that this is the M-structure coming from classfield theory. But at the point $1 \in U^\times$, (7.19) forces this M-structure to coincide with the canonical trivialization of the translation-invariant connection over the identity. This completes the proof of theorem 7.1. □

7.2. Local ε-factors for holonomic $\mathbb{D}$-modules. In order for our local ε-lines to be multiplicative in exact sequences, it is necessary to extend our construction to holonomic $\mathcal{D}$-modules. We consider holonomic $\mathcal{D}$-modules on $\operatorname{Spec} k[[t]]$. Let $j : \operatorname{Spec} k((t)) \to \operatorname{Spec} k[[t]]$.

Lemma 7.4. *Let M be a holonomic $\mathcal{D}$-module on* $\operatorname{Spec} k[[t]]$*. Then we have a canonical decomposition*

$$M = M_r \oplus M_i \tag{7.22}$$

where j^*M_r (resp. j^*M_i) is a regular singular point connection (resp. a purely irregular connection, i.e. a connection with all slopes > 0 ([13], chap. III).). Further, $M_i \cong j_*j^*M_i \cong j_!j^*M_i$. This decomposition is unique and functorial.

Proof. For $k = \overline{k}$, this is op. cit. pp. 51-52. In general, we write $M \otimes \overline{k} = (M \otimes \overline{k})_r \oplus (M \otimes \overline{k})_i$ and note that because $\operatorname{Hom}_{\mathcal{D}}(M_i, M_r) = \operatorname{Hom}_{\mathcal{D}}(M_r, M_i) = (0)$, the two factors are galois stable so they descend. □

Definition 7.5. Let $M = M_i$ be a purely irregular holonomic $\mathcal{D}$-module as above, and let $K \subset \mathbb{C}$ be a subfield. Write $j^*M_{\mathbb{C}} := j^*M \otimes_{k((t))} \mathbb{C}((t))$. Let $A \supset \mathbb{C}((t))$ be an algebra such that $j^*M_{\mathbb{C}} \otimes A$ is spanned by its horizontal sections. A K-Betti structure on M is a K-structure on the space of horizontal sections of $j^*M_{\mathbb{C}} \otimes A$ which is compatible with the Stokes grading in the sense of op. cit. IV, 2. Said another way, $j^*M_{\mathbb{C}}$ may be extended to a meromorphic connection on a small punctured disk, and one may look at the local system of horizontal sections for the extended analytic connection. This extension is not well defined, but the graded for the Stokes filtration is canonically defined. We assume our K-structure is compatible with this grading (cf. Remark 2.4(iii)). In practical terms, the Levelt decomposition enables us to write

$$j^* M_{\mathbb{C}} = \oplus \pi_{i*} L_i \otimes U_i \tag{7.23}$$

where the π_i are ramified projections defined by $t = u_i^{n_i}$, the L_i are distinct irregular rank 1 connections on $\mathbb{C}((u_i))$, and the U_i are nilpotent regular singular connections. Our K-Betti structure is compatible with this grading.

Definition 7.6. Let $M = M_r$ be a regular holonomic $\mathcal{D}$-module as above, and let $K \subset \mathbb{C}$ be a subfield. Our M extends canonically to an analytic holonomic $\mathcal{D}$-module M_{an} on a disk about the origin (op. cit. th. 5.3, p. 38). A K-Betti structure on M is a descent to the derived category of constructible sheaves of K vector spaces on the disk for the de Rham complex $DR(M_{an})$.

Note, $j^* DR(M_{an})$ is just the local system of horizontal sections of $j^* M_{an}$.

Definition 7.7. Let $M = M_r \oplus M_i$ be a holonomic $\mathcal{D}$-module as above, and let $K \subset \mathbb{C}$ be a subfield. Then a K-Betti structure on M is a choice of K-Betti structures on M_i and M_r. Of course the K-Betti structure on M_i is required to be compatible with the Stokes structure as in Subsection 2.2.

Example 7.8. Let M be a holonomic $\mathcal{D}$-module with punctual support on $\operatorname{Spec} k[[t]]$. Then $H^{-1}_{DR}(M) = (0)$ and a K-Betti structure on M is a K-structure on $H^0_{DR}(M)$. By Kashiwara's theorem, $M = i_* H^0_{DR}(M)$, where $i : \operatorname{Spec} k \to \operatorname{Spec} k[[t]]$ is the closed point, and $H^0_{DR}(M)$ is viewed as a $\mathcal{D}$-module on $\operatorname{Spec} k$. Recall, for V a k-vector space, we have $i_* V := V \otimes_k \mathcal{D}/t\mathcal{D} \otimes_{k[[t]]} \omega^{-1}$ where $\mathcal{D} := k[[t, \partial_t]$ and $\omega := k[[t]]dt$. Then

$$H^0_{DR}(i_* V) = i_* V \otimes_{\mathcal{D}} \omega \cong V \otimes_k (\omega / t\omega) \otimes_{k[[t]]} \omega^{-1} = V \tag{7.24}$$

Thus a K-Betti structure on $i_* V$ is simply a K-structure on $V \otimes_k \mathbb{C}$.

Definition 7.9. Let M be a holonomic $\mathcal{D}$-module on $\operatorname{Spec} k[[t]]$ as above, and let $\nu \in k((t))dt$ be a nonzero meromorphic 1-form. Define

$$\varepsilon(M, \nu) := \varepsilon(j^* M, \nu) \det H^0_{DR}(A) \det H^0_{DR}(B)^{-1} \tag{7.25}$$

where A and B are punctual $\mathcal{D}$-modules defined by the exact sequence

$$0 \to A \to M \to j_* j^* M \to B \to 0. \tag{7.26}$$

Here $j^* M$ is a meromorphic connection over $k((t))$ and $\varepsilon(j^* M, \nu)$ is defined in (7.1).

Proposition 7.10. *A K-Betti structure on M in the sense of definition 7.7 induces a K-Betti structure on $\varepsilon(M,\nu)$.*

Proof. When $M = M_i$ we have $A = B = (0)$, so we may assume $M = M_r$. We must show that a K-Betti structure on M in the sense of definition 7.6 gives rise to a K-structure on $\det H^0_{DR}(A) \otimes (\det H^0_{DR}(B))^{-1} \otimes \mathbb{C}$. This line is identified with $\det H^*_{DR}(M) \otimes \det H^*_{DR}(j_*j^*M)^{-1}$. By assumption, the analytic de Rham cohomology of M_{an} is endowed with a K-structure in the derived category. Thus, $H^*_{DR}(M)$, which is the cohomology of this complex, gets a K-structure as does $H^*_{DR}(j^*M)$ which is the cohomology of the pullback complex. Since j_* is exact on $\mathcal{D}$-modules, we have $H^*_{DR}(j^*M) = H^*_{DR}(j_*j^*M)$. □

Proposition 7.11. *Let ν be as above.*
(i) Let E be a meromorphic connection. Assume E extends to a smooth connection $\mathcal{E}$ over $k[[t]]$. Let $\mathcal{E}_0$ be the central fibre. In this situation, $\mathcal{E}$ is trivial as a connection and a $K \subset \mathbb{C}$ Betti structure is simply a choice of K-structure on the horizontal sections. Such a choice defines an object (of degree $\mathrm{rank}E$*)* $\det \mathcal{E}_0 \in \mathrm{superline}(\mathrm{k}, \mathrm{K} \subset \mathbb{C})$*, and*

$$\varepsilon(\mathcal{E},\nu) = (2\pi i)^{ord(\nu)\mathrm{rank}(E)}(\det \mathcal{E}_0)^{ord(\nu)}.$$

As a superline, $\varepsilon(\mathcal{E},\nu)$ has degree $(\mathrm{rank}E)(ord\nu)$*.*
(ii) For an exact sequence of holonomic $\mathcal{D}$-modules $0 \to M' \to M \to M'' \to 0$ with compatible K-Betti structure, we have

$$\varepsilon(M,\nu) = \varepsilon(M',\nu)\varepsilon(M'',\nu).$$

Proof. (i) When E extends smoothly across $\mathrm{Spec}\, k[[t]]$ the vanishing cycles are trivial, so $\varepsilon(\mathcal{E}, dt) = \mathbf{1}$ is the trivial line given by $k, K \subset \mathbb{C}$. The fibre at 1 of the Katz extension $\mathbb{E}$ of E is $\mathcal{E}_0$. (Indeed $\mathbb{E}$ is trivial as a connection so the fibres at 1 and 0 are identified.) It follows that the translation invariant connection on $F^\times$ associated to $\det E$ is the trivial connection with fibre $(\det \mathcal{E}_0)^{\otimes n}$ on $F^\times_n$. It follows that $(\det E, f) = (\det \mathcal{E}_0)^{\otimes ord(f)}$.

The proof of (ii) is immediate because the right hand side in (7.25) is multiplicative in exact sequences. □

Remark 7.12. With hypotheses as in (i) above, it is important to distinguish between $\varepsilon(\mathcal{E},\nu)$ and $\varepsilon(j_*E,\nu)$, where $j : \mathrm{Spec}\, k((t)) \to \mathrm{Spec}\, k[[t]]$. Indeed, the exact sequence $0 \to \mathcal{E} \to j_*E \to j_*E/\mathcal{E} \to 0$ leads to the formula

$$\varepsilon(j_*E,\nu) = (2\pi i)^{ord(\nu)\mathrm{rank}(E)}(\det \mathcal{E}_0)^{ord(\nu)+1} \tag{7.27}$$

As a superline, $\varepsilon(j_*E,\nu)$ has degree $(\mathrm{rank}E)(\mathrm{ord}(\nu)+1)$.

7.3. Projection Formula. Let S be a circle, and let $\pi : S \to S$ be the map $z \mapsto z^m$. For $\mathcal{F}$ a sheaf on S and $s \in S$, we have $(\pi_*\mathcal{F})_s = \oplus_{s' \mapsto s} \mathcal{F}_{s'}$. If $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces with a Stokes structure (resp. a graded Stokes structure) then $\pi_*\mathcal{F}$ inherits a Stokes (resp. graded Stokes) structure where the filtration (resp. grading) on the stalks $(\pi_*\mathcal{F})_s$ is simply the direct sum of the corresponding structures on the $\mathcal{F}_{s'}$.

Proposition 7.13 (Projection Formula)**.** *Let*

$$\pi : \operatorname{Spec} k((t)) \to \operatorname{Spec} k((u)); \quad \pi^* u = t^m, \ m \geq 1.$$

Let E be a virtual connection with K-Stokes structure on $\operatorname{Spec} k((t))$*. Assume* $\operatorname{rank} E = 0$*, and let $\pi_* E$ be the direct image on* $\operatorname{Spec} k((u))$*. We view $\pi_* E$ as a virtual connection of* rank 0 *and endow it with a K-Stokes structure as above. Let ν be a nonzero* 1*-form on* $\operatorname{Spec} k((u))$*. Then we have a canonical identification of objects in* superline(k, K $\subset$ $\mathbb{C}$)

$$\varepsilon(\pi_* E, \nu) = \varepsilon(E, \pi^* \nu). \tag{7.28}$$

Proof. We first show that (7.28) for a particular ν implies the identity for any other $\nu' = f(u)\nu$. Since E has rank 0, (7.1) implies

$$\begin{aligned} \varepsilon(\pi_* E, f\nu) &= (\det \pi_* E, f)\varepsilon(\pi_* E, \nu) \\ \varepsilon(E, \pi^*(f\nu)) &= (\det E, \pi^* f)\varepsilon(E, \pi^* \nu). \end{aligned} \tag{7.29}$$

We must therefore verify

$$(\det \pi_* E, f) = (\det E, \pi^* f). \tag{7.30}$$

We may assume $E = V - \mathcal{O}^r$ where V is a rank r connection on $\operatorname{Spec} k((t))$ and $\mathcal{O}$ denotes the trivial connection. Let $v_1, \ldots, v_r$ be a basis of V, and let $A(t)\frac{dt}{t}$ be the connection matrix. Write $A(t) = A_0(u) + A_1(u)t + \ldots + A_{m-1}(u)t^{m-1}$. the connection on $\pi_* V$ with respect to the basis $t^j v_i, \ 0 \leq j < m$ then becomes

$$\frac{1}{m} \begin{pmatrix} A_0(u) & A_1(u) & \ldots & A_{m-1}(u) \\ uA_{m-1}(u) & A_0(u) + 1 & \ldots & A_{m-2}(u) \\ \vdots & \vdots & \ldots & \vdots \\ uA_1(u) & uA_2(u) & \ldots & A_0(u) + m - 1 \end{pmatrix} \frac{du}{u} \tag{7.31}$$

It follows that $\det \pi_* V$ has connection form $(\mathrm{Tr} A_0(u) + \frac{r(m-1)}{2})\frac{du}{u}$, and $\det \pi_* V \otimes (\det \pi_* \mathcal{O})^{-r}$ has connection form $\mathrm{Tr} A_0(u)\frac{du}{u}$. This coincides with the connection form for $N_\pi \det V$, where N_π is the norm map associating to a rank 1 connection with connection form $\eta(t)$ the rank 1-connection with form $\mathrm{Tr}_{k((t))/k((u))}\eta$. The identification for connections of virtual rank 0 can be written

$$N_\pi \det E = \det \pi_* E. \tag{7.32}$$

It is compatible with a choice of K-Betti structure for some subfield $K \subset \mathbb{C}$.

Let $[\det E]$ be the rank 1 invariant connection on $F_t^\times$ associated to $\det E$ as in Subsection 2.6. Write

$$F_t^\times = \coprod \operatorname{Spec} k[Y_0, Y_0^{-1}, Y_1, Y_2, \dots]$$
$$F_u^\times = \coprod \operatorname{Spec} k[X_0, X_0^{-1}, X_1, X_2, \dots]$$

The pullback map $\pi^* : F_u^\times \to F_t^\times$ is defined on coordinates by

$$(\pi^*)^*(Y_i) = \begin{cases} 0 & m \not| i \\ X_{i/m} & m|i \end{cases} \tag{7.33}$$

together with the above discussion, this shows in the rank 0 case

$$(\pi^*)^*[\det E] = [\det \pi_* E] \tag{7.34}$$

To check compatibility on the fibre, identify points in $F_u^\times$ (resp. $F_t^\times$) with power series in U (resp. T). The fibre of $(\pi^*)^*[\det E]$ at $\frac{U}{U-1}$ coincides with the fibre of $[\det E]$ at $\frac{T^m}{T^m-1}$. By translation invariance, this is the tensor of the fibres of $[\det E]$ at the points $\frac{T}{T-\zeta}$ with $\zeta^m = 1$, or the fibre at ζ of the Katz extension of $\det E$. This in turn coincides with $\det(\oplus_\zeta E_\zeta) = (\det \pi_* E)_1$. Note the fact that $\text{rank} E = 0$ makes this identification canonical, i.e. independent of an ordering of the m-th roots of 1 in the determinant. The desired formula (7.30) now follows from (7.34).

The next step in the proof of Proposition 7.13 is to reduce to the case where E is a linear combination of connections with regular singular points. Let $\mathbb{E}$ be the Katz extension to $\mathbb{P}_t^1$. $\mathbb{E}$ is a linear combination of connections which are smooth on $\mathbb{G}_m$ and have regular singular points at ∞. We have an evident diagram of schemes

$$\begin{array}{ccc} \operatorname{Spec} k((t)) & \to & \mathbb{P}_t^1 \\ \downarrow \pi & & \downarrow \pi \\ \operatorname{Spec} k((u)) & \to & \mathbb{P}_u^1. \end{array} \tag{7.35}$$

In the next section we will define a global epsilon line for a meromorphic connection on a curve. Our main result will be a product formula expressing this line as a product of local epsilon lines. Applying this formula to the Katz extensions $\mathbb{E}$ and $\pi_*\mathbb{E}$ yields

$$(7.36)\qquad \prod_{z\in\mathbb{P}^1_u}\varepsilon_z(\pi_*\mathbb{E},\eta)=\varepsilon(\pi_*\mathbb{E})=\varepsilon(\mathbb{E})=\prod_{w\in\mathbb{P}^1_t}\varepsilon_w(\mathbb{E},\pi^*\eta)$$

Here η is a nonzero meromorphic 1-form on $\mathbb{P}^1_u$. (Because $\mathbb{E}$ and $\pi_*\mathbb{E}$ have rank 0, there are no factors of $2\pi i$ in this equation.) The global epsilon factors $\varepsilon(\mathbb{E})$ and $\varepsilon(\pi_*\mathbb{E})$ come from the determinant of cohomology for $\mathbb{E}|_{\mathbb{G}_m}$ and $\pi_*\mathbb{E}|_{\mathbb{G}_m}$. They agree since these two cohomologies agree. We have written ε_z and ε_w for the local epsilon lines at the closed points z and w. Let $y=\frac{u}{u-1}$ and take $\eta=dy$. We apply proposition 7.11(i) repeatedly to the factors in (7.36) to deduce

$$(7.37)\qquad \varepsilon_0(E,\pi^*dy)\varepsilon_\infty(\mathbb{E},\pi^*dy)\prod_{\zeta^m=1}(\det E_\zeta)^{-2}=\\ \varepsilon_0(\pi_*E,dy)\varepsilon_\infty(\pi_*\mathbb{E},dy)(\det(\pi_*\mathbb{E})_1)^{-2}.$$

The identification $\det(\pi_*\mathbb{E})_1=\prod_{\zeta^m=1}\det E_\zeta$ follows as above, so we conclude

$$(7.38)\qquad \varepsilon(E,\pi^*dy)=\varepsilon(\pi_*E,dy)\Leftrightarrow\varepsilon_\infty(\mathbb{E},\pi^*dy)=\varepsilon_\infty(\pi_*\mathbb{E},dy).$$

In this way we are reduced to proving the projection formula in the case of a regular singular point and we may assume $\nu=du$.

For the de Rham line, the projection formula follows from Definition 2.18 because the polarized determinant for $(du)^{-1}\circ\nabla_{\pi_*E}$ equals the polarized determinant for $\pi^*(du)^{-1}\circ\nabla_E$ in the sense that the infinite dimensional k-vector spaces E and π_*E coincide, and this coincidence identifies the two operators. It remains to investigate the K-Stokes structures. Once the de Rham lines are identified, the Stokes structures are insensitive to extending the field k, so we may assume k is "sufficiently large". As in Example 6.3, we may assume $E=V-\mathcal{O}^{\mathrm{rank}V}$, where V admits a gauge with connection of the form $d+A\frac{dt}{t}$ and A constant. Conjugating by a constant matrix, we may assume A is upper triangular. We may further assume that none of the eigenvalues of A are integers.

We now argue as in Example 6.3. We give V the Stokes structure determined by a basis of solutions corresponding to columns of the matrix

$$(7.39)\qquad M\exp(A\log t),$$

where M is an invertible matrix with $\mathbb{C}$-coefficients. (Note that since V is regular singular, a K-Stokes structure is determined by a reduction to K of the local system of solutions on a circle about the origin.) We interpret ε_{DR} and ε_B as the fibres at the point $z=1$ (cf. (6.4)) for the connection on $k((z))$ with $\nabla(1) = \mathrm{Tr}A\frac{dz}{z}$ and K-structure on the horizontal sections determined by the K-span of the columns of $\exp(A\log t)$.

We need to identify

$$\varepsilon_*(\pi_*V - \pi_*\mathcal{O}^r, du) \overset{?}{=} \varepsilon_*(V - \mathcal{O}^r, mt^{m-1}dt) = (\det V, mt^{m-1})_* \otimes \varepsilon_*(V - \mathcal{O}^r, dt), \tag{7.40}$$

where the subscript $*$ is either DR or B. (We take the right hand identity as true by definion of ε.)

Recall the line $(\det V, mt^{m-1})_*$ is the fibre over $mT^{m-1} \in F^\times(k)$ of the translation-invariant rank 1 connection L on $F^\times$ associated to $\det V$. The DR structure is given by the trivial bundle on $F^\times$ with connection $\nabla(1) = \mathrm{Tr}A\frac{dX_0}{X_0}$, where X_0 is the coordinate function on $F^\times$ defined by $X_0(f) =$ leading coefficient of f. The K-Betti structure is

$$K \cdot \exp\Big(\frac{1}{2\pi i}\int_{x_0}^{x_0} \log(mt^{m-1})\mathrm{Tr}A\frac{dt}{t}\Big)(x_0)^{-(m-1)\mathrm{Tr}A}(\det M)^{m-1}. \tag{7.41}$$

By assumption the monodromy is defined over the field K whence $\exp(2\pi i\mathrm{Tr}A) \in K$. We may therefore take $x_0 = 1$ and ignore the last factor. Also

$$\frac{1}{2\pi i}\int_1^1 \log(t)\mathrm{Tr}A\frac{dt}{t} = \frac{\mathrm{Tr}A}{4\pi i}(\log^2 t)\Big|_1^1 = \pi i\mathrm{Tr}A \tag{7.42}$$

Since we have taken the Betti structure on $\det V$ to be given by $K \cdot \det M \cdot t^{\mathrm{Tr}A}$, (7.40), it follows that $(\det V, mt^{m-1})$ is the trivial DR line with Betti fibre $K \cdot (\det M)^{m-1}((-1)^{m-1}m)^{\mathrm{Tr}A}$.

On the right hand side of (7.40), the K-Betti structure now becomes

$$((-1)^{m-1}m)^{\mathrm{Tr}A} \cdot (\det M)^m\Big(\prod_i \int_\gamma s^{a_i}e^s\frac{ds}{s}\Big)^{-1}. \tag{7.43}$$

(There is a confusing point here. The DR line for $\det V$ is trivialized, generated say by 1. The Betti line is generated by $z^{-\mathrm{Tr}A} \cdot 1$. But, by assumption, the monodromy local system reduces to K, so $\exp(2\pi i\mathrm{Tr}A) \in K$ and the Betti and DR fibres at 1 have a common trivialization: $K \cdot z^{-\mathrm{Tr}A}|_{z=1} = K \cdot 1$. Thus we may simply omit $z^{-\mathrm{Tr}A}|_{z=1}$ in the Betti line.)

Let B be the $m \times m$ diagonal matrix with entries $0, \frac{1}{m}, \ldots, \frac{m-1}{m}$. $\varepsilon(\pi_*V - \pi_*\mathcal{O}^r, du)$ is the epsilon line corresponding to the difference of two regular singular point connections

$$\Big(d + (\frac{1}{m}A \otimes I_m + I_r \otimes B)\frac{du}{u}\Big) - \Big(d + I_r \otimes B\frac{du}{u}\Big) \tag{7.44}$$

Note however that the natural K-Betti structure as discussed in subsection 7.3 is not obtained by upper-triagularizing the connection matrix in (7.44). Indeed, if $\{v_j\}$ is a basis of V such that the monodromy maps $v_j \mapsto \exp(2\pi i a_j)v_j$, then a basis of $\pi_* V$ adapted to upper-triangularizing the connection matrix is $\{t^p v_j\}_{0\le p<m}$. This leads to a K-structure on $\varepsilon(\pi_* V - \pi_*\mathcal{O}^r, du)$ of the form

$$K\cdot(\det M)^m\cdot\Big(\prod_{j,p}\int_\gamma s^{(a_j+p)/m}e^s\frac{ds}{s}\Big)^{-1}\times \Big(\prod_{p=1}^{m-1}\int_\gamma s^{p/m}e^s\frac{ds}{s}\Big)^r. \tag{7.45}$$

The actual basis we want is

$$w_{s,j} := \sum_{q=0}^{m-1}\exp(2\pi i s(a+q)/m)t^q v_j;\quad 0\le s\le m-1. \tag{7.46}$$

The monodromy map $t\mapsto \exp(2\pi i)t$ then maps $w_{0,j}\mapsto w_{1,j}\mapsto \ldots \mapsto w_{m-1,j}\mapsto \exp(2\pi i a)w_{0,j}$. The change of basis $t^s v_j\mapsto w_{s,j}$ is block-diagonal with has blocks

$$\begin{pmatrix} 1 & 1 & \ldots & 1\\ \exp(2\pi i a_j/m) & \exp(2\pi i(a_j+1)/m) & \ldots & \exp(2\pi i(a_j+m-1)/m)\\ \vdots & \vdots & \ldots & \vdots\\ \exp(2\pi i(m-1)a_j/m) & \exp(2\pi i(m-1)(a_j+1)/m) & \ldots & \exp(2\pi i(m-1)(a_j+m-1)/m)\end{pmatrix} \tag{7.47}$$

This is a Vandermonde matrix with determinant

$$\exp(\pi i(m-1)a_j)\prod_{0\le j<\ell<m}(\zeta^\ell-\zeta^j);\quad \zeta=\exp(2\pi i/m). \tag{7.48}$$

Taking the product of these for the eigenvalues a_j, $1\le j\le r$ from the connection matrix A, and then dividing by the r-th power of the corresponding determinant with $a=0$ (coming from $\pi_*\mathcal{O}^r$), we get that the K-structure (7.45) has to be multiplied by

$$\exp(\pi i(m-1)\mathrm{Tr}A) = (-1)^{(m-1)\mathrm{Tr}A}. \tag{7.49}$$

One has $\int_\gamma s^a e^s ds/s = (\exp(2\pi i a)-1)\Gamma(a)$. By assumption, none of the a_j are integers, so the terms $\exp(2\pi i a_j)-1\in K^\times$ and can be ignored.

The distribution relation for the Γ-function reads

$$\Gamma(x) = (2\pi)^{(1-m)/2}m^{x-\frac{1}{2}}\prod_{p=0}^{m-1}\Gamma(\frac{x+p}{m}) \tag{7.50}$$

Apply this with $x = 1$ to get

$$1 = (2\pi)^{(1-m)/2} m^{\frac{1}{2}} \prod_{p=1}^{m-1} \Gamma(\frac{p}{m}). \tag{7.51}$$

(Note we can omit $\Gamma(1)$ in the product.) Substituting in (7.45) and including the factor (7.49), we find that the K-Betti structure associated to $\varepsilon(\pi_* V - \pi_* \mathcal{O}^r, du)$ is

$$K \cdot ((-1)^{m-1} m)^{\mathrm{Tr} A} \Gamma(\mathrm{Tr} A)^{-1} (\det M)^r \tag{7.52}$$

This coincides with the K-Betti structure from (7.43), so the projection formula (proposition 7.13) is proven, modulo the global product formula to be proven in the next section. □

Remark 7.14. It is straightforward to extend the projection formula to virtual holonomic $\mathcal{D}$-modules of generic rank 0.

8. The Reciprocity Law

Our objective in this section is

Theorem 8.1 (Reciprocity)**.** *Let X be a smooth, projective curve of genus g over a field $k \subset \mathbb{C}$. Let E be a holonomic $\mathcal{D}$-module on X, and fix a Stokes structure on E defined over some $K \subset \mathbb{C}$. Let $\nu \neq 0$ be a meromorphic 1-form on X.*
(i) Let $X \subset X$ be a finite set of closed points such that $E|_{X-S}$ is a locally free $\mathcal{O}_X$-module of finite rank and $\Omega^1_{X-S} = \mathcal{O}_{X-S} \cdot \nu$. For any $x \in X - S$ the local ε-factor $\varepsilon_x(E, \nu)$ is canonically trivialized

$$\varepsilon_x(M, \nu) = \mathbf{1}. \tag{8.1}$$

(ii) One has canonically

$$\varepsilon(X, E) = (2\pi i)^{\mathrm{rank} E(1-g)} \bigotimes_{x \in X} \varepsilon_x(E, \nu). \tag{8.2}$$

Proof. Part (i) follows from Proposition 7.11. For (ii), suppose first that E has punctual support. Both sides of (8.2) are multiplicative, so By Kashiwara's theorem we reduce to the case $E = i_* V$ where V is a vector space and i is the inclusion of a point in X. By (7.24) and (7.25) we see that both sides are identified with $\det V$.

The argument is thus reduced to the case $E = j_* \mathcal{E}$, where $\mathcal{E}$ is a meromorphic connection on $X - S$ and $j : X - S \hookrightarrow X$.

We will first consider the case $X = \mathbb{P}^1$. For a suitable choice of coordinate, we may suppose that E is smooth at ∞. The following proposition will be proven later.

Proposition 8.2. *Let E be a meromorphic connection on $\mathbb{P}^1$ which extends smoothly across ∞ with fibre E_∞. Let $\mathcal{F}(E)$ be the Fourier transform of $E|_{\mathbb{A}^1}$. Write $\mathcal{L} = \det \mathcal{F}(E)$, viewed as a meromorphic rank 1 connection on $\mathbb{P}^1$ with coordinate t'. Then $\mathcal{L}$ extends smoothly across $t' = 0$ (corollary 3.12), and we have a canonical identification of superlines for the fibre at 0*

$$\mathcal{L}_0 = \det(E_\infty) \otimes \det(H^*_{DR}(\mathbb{A}^1, E|_{\mathbb{A}^1})). \tag{8.3}$$

This identification is compatible with both the DR and Betti structures. (The superline associated to a determinant is taken to have degree the rank of the vector space.)

Returning to the proof of the theorem in the case $X = \mathbb{P}^1$, note that by (7.1), replacing ν by $f \cdot \nu$ multiplies on the right by a factor

$$\prod_{x \in \mathbb{P}^1} (\det E, f)_x \tag{8.4}$$

By the reciprocity law in classfield theory (Proposition 2.17), this product is trivialized, both in the DR and Betti realizations. Thus, we may assume $\nu = dt$.

Let $r = \mathrm{rank} E$, and let $S \subset \mathbb{P}^1$ be the set of all singular points of E. For $s \in S$, let $i(s)$ denote the irregularity of E at s. Define $\mathfrak{s} = \sum_{s \in S}(r+i(s))s \in k$ and define $\mathcal{K}$ to be the meromorphic connection on the trivial bundle on $\mathbb{P}^1_{t'}$ defined by $\nabla_{\mathcal{K}}(1) = -\mathfrak{s} dt'$.

By Proposition 3.10, the rank 1 meromorphic connection $\mathcal{L} \otimes \mathcal{K}$ on $\mathbb{P}^1$ is regular singular (at worst) at ∞ and smooth away from ∞. It follows that this connection is trivial. We give $\mathcal{K}$ the $\mathbb{Q}$-Betti structure which is generated by the horizontal section $\exp(\mathfrak{s}t')$. Note that $\mathcal{K}_0 \cong \mathbf{1}$ (canonically). Thus, by Proposition 8.2, we have

$$\det H^*_{DR}(\mathbb{A}^1, E|_{\mathbb{A}^1}) \otimes \det E_\infty = \mathcal{L}_0 = \mathcal{L}_0 \otimes \mathcal{K}_0 = (\mathcal{L} \otimes \mathcal{K})_\infty. \tag{8.5}$$

The connections $EXP((r + n(s))s/z) \otimes \det VC(s, E)$ are regular singular (Proposition 3.11), and

$$\bigotimes_{s \in S} EXP((r + n(s))s/z) \otimes \det VC(s, E) = (\mathcal{K} \otimes \mathcal{L}) \otimes k((z)). \tag{8.6}$$

We may choose our lattices $\Xi_s \subset EXP((r + n(s))s/z) \otimes \det VC(s, E)$ (cf. (6.2), (6.3), and the discussion at that point) in such a way that $\otimes_S \Xi_s \subset (\mathcal{K} \otimes \mathcal{L}) \otimes k((z))$ is the $k[[z]]$-lattice spanned by a horizontal section of this trivial connection. It then follows that the Betti structures on the local lines $\Xi_s / z\Xi_s$ tensor together to give the Betti structure on $(\mathcal{K}\otimes\mathcal{L})_\infty$. We obtain in this way an identification of lines

$$\det H^*_{DR}(\mathbb{A}^1, E|_{\mathbb{A}^1}) \otimes \det E_\infty = \bigotimes_{s\in S} \varepsilon_s(E, dt). \tag{8.7}$$

Let $\mu : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and $i : \{\infty\} \to \mathbb{P}^1$ be the inclusions. Since E extends smoothly across ∞, we have a distinguished triangle

$$E \to R\mu_*\mu^* E \to i_* E_\infty[-1](-1) \tag{8.8}$$

where (-1) means that the Betti structure has to be multiplied by $2\pi i$. This yields for determinant lines

$$\det H^*_{DR}(\mathbb{A}^1, E|_{\mathbb{A}^1}) = \det H^*_{DR}(\mathbb{P}^1, E) \otimes \det(E_\infty) \otimes (2\pi i)^{\mathrm{rank} E} \tag{8.9}$$

Combining (8.7) and (8.9),

$$\det H^*_{DR}(\mathbb{P}^1, E) = (\det E_\infty)^{-2} \otimes (2\pi i)^{-\mathrm{rank} E} \otimes \bigotimes_{s\in S} \varepsilon_s(E, dt). \tag{8.10}$$

From Proposition 7.11(i), we get $\varepsilon_\infty(E, dt) = (2\pi i)^{-2\mathrm{rank} E} \otimes (\det E_\infty)^{-2}$. With this substitution, the desired formula

$$\varepsilon(\mathbb{P}^1, E) = (2\pi i)^{\mathrm{rank} E} \otimes \bigotimes_{x\in \mathbb{P}^1} \varepsilon_x(E, dt) \tag{8.11}$$

emerges.

Finally, we prove Theorem 8.1 for X of genus $g > 0$. Since the theorem is now proven for genus 0, we may use the local projection formula Proposition 7.13. We already have by Proposition 2.19 an identification of DR-lines

$$\varepsilon(X, E)_{DR} = \bigotimes_{x\in X} \varepsilon_x(E, \nu), \tag{8.12}$$

and it remains to show compatibility with the Betti structures. Suppose first $E = \mathcal{O}_X$. Note in this case, both sides in (8.12) are canonically trivialized. On the left, one has

$$\begin{aligned} \det H^1_{DR}(X) &= H^{2g}_{DR}(Jac(X)) = k \cdot [pt] \\ H^2_{DR}(X) &= k \cdot [pt]; \quad H^0_{DR}(X) = k \cdot [X] \end{aligned} \tag{8.13}$$

On the right, the local epsilon factors are trivialized using Proposition 7.11. Presumably, these trivializations coincide. In any case, for $X/\mathbb{Q}$ the ratio of the trivializations lies in $\mathbb{Q}^\times$. One can use the fact ([1], Proposition 4.10) that (8.12) is compatible in a suitable sense with the Gauß-Manin connection in families to argue that this ratio is constant over moduli, and hence always in $\mathbb{Q}^\times$.

We can take our monodromy field to be $\mathbb{Q}$. Our Stokes structure is simply the choice of a $\mathbb{Q}$-structure on the constant local system $\mathbb{C}_X$.

Assume first we take the obvious one, $\mathbb{C}_X = \mathbb{Q}_X \otimes \mathbb{C}$. We have to evaluate the ratio of DR and Betti structures on $\det H^*_B(X, \mathbb{C})$. We know that $H^2_B(X, \mathbb{Q}(1))$ contains a DR-class (the cycle class of a point). Also $(\det H^1_B(X, \mathbb{Q}))(g) \cong H^{2g}_B(J(X), \mathbb{Q}(g))$ contains the class of a point on the jacobian $J(X)$. It follows that, writing $e \in \varepsilon_{DR}(X, \mathcal{O}_X)$ for the above trivialization, we have $\mathbb{Q} \cdot e = (2\pi i)^{g-1} \det H^*_B(X, \mathbb{Q})$. On the right hand side, the divisor (ν) contributes $(2\pi i)^{2g-2}$ (cf. Proposition 7.11) and the reciprocity formula (8.2) has an initial factor in this case of $(2\pi i)^{1-g}$. Taken together, this shows that the Betti structures coincide. Note that if we scale the Betti structure on $\mathbb{C}_X$ by $c \in \mathbb{C}^\times$, the same scale factor c^{2g-2} appears on left and right.

We now are reduced to proving the theorem for E on X a virtual holonomic $\mathcal{D}$-module of rank 0. We may therefore use the local projection formula (Proposition 7.13). We choose a finite projection $p : X \to \mathbb{P}^1$. Let $S \subset X$ be a finite set containing the ramification locus and the set of nonsmooth points for E, and let $j_X : X - S \hookrightarrow X$, $j_{\mathbb{P}^1} : \mathbb{P}^1 - p(S) \hookrightarrow \mathbb{P}^1$ be the inclusions. Using the definition of epsilon factors for holonomic $\mathcal{D}$-modules with punctual support (Proposition 7.10), we can reduce to the case $E = j_{X*}E_0$ where E_0 is smooth, rank 0 on $X - S$. Then $p_*E = j_{\mathbb{P}^1*}p_{0*}E_0$, where $p_0 : X - S \to \mathbb{P}^1 - p(S)$. We have in this case

$$\det H^*_{DR}(X, E) \cong \det H^*_{DR}(\mathbb{P}^1 - p(S), p_{0*}E_0) \tag{8.14}$$

Using the reciprocity law in classfield theory (Proposition 2.17) and arguing as in (8.4), we reduce further to the case $\nu = p^*\mu$ for μ meromorphic on $\mathbb{P}^1$. Now the local projection formula (Proposition 7.13) identifies the product of the epsilon factors for $j_{X*}E_0$ with the corresponding product for $j_{\mathbb{P}^1*}p_{0*}E_0$, completing the proof of the theorem. □

proof of Proposition 8.2. More precisely, as in Corollary 2.6, a choice of a reduction to $K \subset \mathbb{C}$ of the local system $\mathcal{E}$ of horizontal sections of E which is compatible with Stokes structures at points of S determines a K-structure on $H^*_{DR}(\mathbb{A}^1, E) \otimes_k \mathbb{C}$. By assumption, E extends smoothly across infinity, so $\mathcal{E}_\infty \cong E_\infty \otimes_k \mathbb{C}$ inherits a K-structure as well. Finally, the Fourier sheaf has an evident K-structure given by $K \cdot \exp(-tt')$, so $\mathcal{F}(E)$ and $\mathcal{L}$ inherit a K-structure as well. The assertion is that the identification (8.3) is compatible with these K-structures.

Both sides of (8.3) are multiplicative in exact sequences. From the sequence

$$0 \to A \to E \to j_*j^*E \to B \to 0 \tag{8.15}$$

where $j : \mathbb{A}^1 - S \hookrightarrow \mathbb{A}^1$ and A, B are punctually supported, it suffices to verify the assertion in the two cases E supported on S and $E = j_*E'$.

When E has punctual support, $\det E_\infty = k$ (with canonical generator), and

$$\mathcal{L} = \det(\mathcal{O} \otimes_k H^0_{DR}(E)); \quad H^{-1}_{DR}(E) = (0), \tag{8.16}$$

so the assertion is correct in that case. We are reduced to the case E a direct image of a smooth connection on $\mathbb{A}^1 - S$. When $E = \mathcal{O}_{\mathbb{A}^1 - S}$, the sequence

$$0 \to \mathcal{O}_{\mathbb{A}^1} \to E \to \delta_S \to 0 \tag{8.17}$$

reduces us to considering $E \cong \mathcal{O}_{\mathbb{A}^1}$. Let σ be a nontrivial global flat section. Then $E_\infty = k \cdot \sigma$, and $\mathcal{F}(E)$ is supported at $t' = 0$ so $\mathcal{L}$ is canonically the trivial connection. Finally $H^*_{DR}(E) = H^{-1}_{DR}(E) = k \cdot \sigma$. Note in (8.3), the dependence on σ in $\det H^*_{DR}(E)$ will be inverted because it appears in degree 1. Put together, both sides of (8.3) are canonically trivialized in this case.

Finally we must consider the case where E is the direct image of a smooth connection on $\mathbb{A}^1 - S$ and $H^{-1}_{DR}(\mathbb{A}^1, E) = (0)$. Write $U := \mathbb{P}^1 \backslash S$. We have the following diagram of de Rham cohomologies, where all groups are tensored over k with $k((t'))$:

$$\begin{array}{ccccccc} 0 \to & \Gamma(U,E) & \xrightarrow{\nabla} & \Gamma(U, E \otimes \Omega^1(\infty)) \to & H^1_{DR}(\mathbb{A}^1 \setminus S, E) & \to 0 \\ & & & \| & & \\ 0 \to & \Gamma(U, E(-\infty)) & \xrightarrow{\nabla + t'dt} & \Gamma(U, E \otimes \Omega^1(\infty)) \to & H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four}) & \to 0 \end{array} \tag{8.18}$$

These spaces have a natural $k[[t']]$-structure. Let $h = \dim H^1_{DR}(\mathbb{A}^1 \setminus S, E)$. Then

$$\dim H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four}) = h + \mathrm{rank} E. \tag{8.19}$$

Let $\eta_1, \ldots, \eta_h \in \Gamma(U, E \otimes \Omega^1(\infty))$ induce a $k[[t']]$-basis in $H^1_{DR}(\mathbb{A}^1 \backslash S, E)$ and be linearly independent and saturated in $H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four})$. (The existence of such elements simply boils down to finding a subspace of dimension h in $\Gamma(U, E \otimes \Omega^1(\infty))$ meeting the two subspaces of finite codimension $\nabla(\Gamma(U,E))$ and $(\nabla + t'dt)(\Gamma(U, E(-\infty)))$ properly. Note, by a snake lemma argument applied to the horizontal sequences in (8.18), the $k[[t']]$-modules $H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four})$ and $H^1_{DR}(\mathbb{A}^1 \setminus S, E)$ have no t'-torsion.) We may assume

$$\Gamma(U, E \otimes \Omega^1(\infty)) = \nabla(\Gamma(U,E)) \oplus \bigoplus_i k[[t']] \cdot \eta_i \tag{8.20}$$

as $k[[t']]$-module. It follows that there exist $\tau_1, \ldots, \tau_{\mathrm{rank} E} \in \Gamma(U,E)$ such that

$$\eta_1, \ldots, \eta_h, \nabla\tau_1, \ldots, \nabla\tau_{\mathrm{rank} E} \tag{8.21}$$

induce a $k[[t']]$-basis of $H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four})$. Notice that if $\tau \in \Gamma(U, E(-\infty))$, then $\nabla\tau \equiv -t'\tau dt \in t' H^1_{DR}(\mathbb{A}^1 \setminus S, E \otimes \mathfrak{Four})$. It follows that necessarily the $\tau_i|_\infty$ form a basis of the fibre $E|_\infty$.

Write $V := U \setminus \{\infty\} = \mathbb{P}^1 \setminus T$, where $T := S \cup \{\infty\}$. We have exact sequences of homology (cf. (2.3) in [3])

$$(8.22)\quad 0 \to H_1(V, \mathcal{E}^\vee) \to H_1(\mathbb{P}^1, T; E^\vee, \nabla^\vee) \to \oplus_S H_1(\Delta_s, \delta_s \cup \{s\}; E^\vee, \nabla^\vee) \to H_0(V, \mathcal{E}^\vee) \to 0$$

and

$$(8.23)\quad 0 \to H_1(V, \mathcal{E}^\vee \otimes \mathcal{F}^\vee) \to H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \to \oplus_{\tau \in T} H_1(\Delta_\tau, \delta_\tau \cup \{\tau\}; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \to H_0(V, \mathcal{E}^\vee \otimes \mathcal{F}^\vee) \to 0.$$

Here the superscript $\vee$ denotes dual, $\mathcal{E}$ and $\mathcal{F}$ are analytic local systems of horizontal sections for E and $\mathfrak{Four}$, and Δ_s is a disk about s with boundary δ_s. The homology in the second and third terms of these sequences is with rapid decay. For an example, classes in $H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee)$ are represented by expressions $\epsilon^\vee \exp(tt')|_\gamma$ where $\epsilon^\vee \exp(tt')$ is a horizontal section of $E^\vee \otimes \mathfrak{Four}^\vee$ along the path γ. this horizontal section is assumed to have rapid decay as γ approaches points of T. (It is technically convenient to take γ to be a path on the *real blowup* of $\mathbb{P}^1_t$ at points $\tau \in T$.)

In particular, in the first sequence we can take the $\oplus_S$ even though we have also removed ∞ because there are no rapid decay horizontal sections at infinity. Finally, the sequences are shorter by one term than the sequences in the cited paper because by assumption $H^0_{DR}(V; E, \nabla) = H^0_{DR}(V; E \otimes \mathfrak{Four}, \nabla + t'dt) = (0)$, and these DR-cohomology groups are dual to the missing terms.

We have $\mathcal{F}^\vee = \mathbb{C} \cdot \exp(t't)$. Multiplication by $\exp(t't)$ gives a commutative diagram

(8.24)

$$\begin{array}{ccccc}
\oplus_S H_1(\Delta_s, \delta_s \cup \{s\}; E^\vee, \nabla^\vee) & \xrightarrow{a} & H_0(V, \mathcal{E}^\vee) & \to 0 \\
\downarrow \cdot \exp(t't) & & \cong \downarrow \cdot \exp(t't) & \\
\oplus_{\tau \in T} H_1(\Delta_\tau, \delta_\tau \cup \{\tau\}; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) & \xrightarrow{b} & H_0(V, \mathcal{E}^\vee \otimes \mathcal{F}^\vee) & \to 0.
\end{array}$$

This yields an exact sequence

$$(8.25)\quad 0 \to \ker a \to \ker b \to H_1(\Delta_\infty, \delta_\infty \cup \{\infty\}; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \to 0$$

We also get $\exp(t't) : H_1(V, \mathcal{E}^\vee) \cong H_1(V, \mathcal{E}^\vee \otimes \mathcal{F}^\vee)$ and $\exp(t't) : H_1(\mathbb{P}^1, T; E^\vee, \nabla^\vee) \hookrightarrow H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee)$. Putting all this together, we get an exact sequence

$$(8.26) \quad 0 \to H_1(\mathbb{P}^1, T; E^\vee, \nabla^\vee) \xrightarrow{\exp(t't)} H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \xrightarrow{exc} H_1(\Delta_\infty, \delta_\infty \cup \{\infty\}; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \to 0$$

Here chains for the $E^\vee$-homology avoid ∞, so multiplication by $\exp(t't)$ makes sense. The map *exc* is excision on the disk Δ_∞ at ∞.

We view (8.26) as defining a filtration on $H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee)$. We want to compute the determinant (for a basis on homology compatible with (8.26) and the basis (8.21) on cohomology) of the pairing

$$(8.27) \qquad H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee) \times H^1_{DR}(V, E \otimes \mathfrak{Four}) \to \mathbb{C}$$

This determinant will be a function of t', and we are interested in the limit as $t' \to 0$. Let ω be one of the 1-forms (8.21), and let μ be a 1-cycle lying in the image of $\exp(t't)$ in (8.26). We find

$$(8.28) \qquad \langle \mu, \omega \rangle = \int_\gamma \langle \epsilon, \omega \rangle \exp(t't).$$

Here $\epsilon|_\gamma$ represents a class in $H_1(\mathbb{P}^1, T; E^\vee, \nabla^\vee)$. In particular, γ is a path away from ∞ and ϵ is a section of $\mathcal{E}^\vee$ along γ. Since γ is compact and away from ∞, we have

$$(8.29) \qquad \lim_{t' \to 0} \langle \mu, \omega \rangle = \int_\gamma \langle \epsilon, \omega \rangle = \langle \bar{\mu}, \bar{\omega} \rangle,$$

where the bar indicates the corresponding class in $H_1(\mathbb{P}^1, T; E^\vee, \nabla^\vee)$ and $H^1_{DR}(V, E)$. In particular, this is zero if $\omega = \nabla \tau_i$. Since (8.28) is holomorphic in t', we can write $\langle \mu, \nabla \tau_i \rangle = t' f_i(t')$ with f_i analytic at $t' = 0$.

Now let $\sigma \in H_1(\mathbb{P}^1, T; E^\vee \otimes \mathfrak{Four}^\vee, \nabla^\vee)$ and $\omega \in \{\eta_i, \nabla \tau_j\}$ be arbitrary.

Lemma 8.3. $\lim_{t' \to 0} t' \langle \sigma, \omega \rangle = 0$.

proof of lemma. Represent $\sigma = \rho \exp(t't)|_c$ where c is a path through ∞ and $\rho \exp(t't)$ is a horizontal section of $E^\vee \otimes \mathfrak{Four}^\vee$ along c. Write $c = c_0 + c_\infty$ where c_0 is away from ∞ and c_∞ is a radius at infinity where $\exp(t't)$ has rapid decay. As in (8.28), the integral over c_0 is bounded in t', so $t' \int_{c_0} \to 0$. By assumption, ω has at worst a log pole at $t = \infty$. Writing $u = t^{-1}$, we reduce to calculating a limit of the form

$$(8.30) \qquad \lim_{t' \to 0^-} t' \int_0^1 f(u) \exp(t'/u) \frac{du}{u}$$

where f is holomorphic at $u = 0$. We write this as $t'(\int_0^{-t'} + \int_{-t'}^1)$. On $[-t', 1]$, we get a bound like $|t' \log |t'|| \to 0$. On $[0, -t']$ we take $v = -u/t'$. The integral becomes

$$t' \int_0^1 f(-t'v) \exp(-1/v) \frac{dv}{v} \to 0; \quad t' \to 0. \tag{8.31}$$

□

As a consequence, we deduce that for a basis $\epsilon_r \exp(t't), \sigma_s$ of $H_1(\mathbb{P}^1, T; E^\vee \otimes F^\vee, \nabla^\vee)$ compatible with (8.26), we have

$$\lim_{t'\to 0} \det((8.27)) = \lim_{t'\to 0} \det\langle(\epsilon_r \exp(t't), \eta_i); (\sigma_s, \nabla\tau_j)\rangle = \det\langle \epsilon_r, \eta_i\rangle \cdot \lim_{t'\to 0} \det\langle \sigma_s, \nabla\tau_j\rangle. \tag{8.32}$$

Lemma 8.4. *Write $\sigma_s = \rho_s \exp(t't)|_{c_s}$, where c_s is a path through ∞ and ρ_s is a horizontal section of $\mathcal{E}^\vee$ along c_s. Then*

$$\lim_{t'\to 0} \det\langle \sigma_s, \nabla\tau_j\rangle = \pm \det\langle \rho_s|_\infty, \tau_j|_\infty\rangle \tag{8.33}$$

proof of lemma. We have

$$\langle \sigma_s, \nabla\tau_j\rangle = \int_{c_s} \langle \rho_s, \nabla\tau_j\rangle \exp(t't) = -t' \int_{c_s} \langle \rho_s, \tau_j\rangle \exp(t't)dt. \tag{8.34}$$

As before, to compute the limit as $t' \to 0$, we can ignore the part of the path away from $t = \infty$ and take c_s to be a short radius at ∞. By assumption also, $\langle \rho_s, \tau_j\rangle = \langle \rho_s|_\infty, \tau_j|_\infty\rangle + O(t^{-1})$. We know by (8.30) that $\lim_{t'\to 0} t' \int_N^\infty \exp(t't)\frac{dt}{t} = 0$. (By assumption, the radial path is such that $\exp(t't)$ has rapid decay at ∞.) Thus, we are reduced to calculating

$$-\langle \rho_s|_\infty, \tau_j|_\infty\rangle t' \int_N^\infty \exp(t't)dt = \langle \rho_s|_\infty, \tau_j|_\infty\rangle \exp(t'N) \to \langle \rho_s|_\infty, \tau_j|_\infty\rangle \tag{8.35}$$

as $t' \to 0$. □

We have remarked that the $\tau_j|_\infty$ form a basis of the fibre of E, and by the surjectivity of the excision map *exc* in (8.26), the $\rho_s|_\infty$ form a basis for $E^\vee|_\infty$. The assertion of Proposition 8.2 follows from (8.33), (8.32), and (8.29). □

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