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Inventiones mathematicae

Dyson's Lemma for polynomials in several variables (and the Theorem of Roth)

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Liouville's theorem asserts that for every algebraic number α of degree $d \geqq 2$ and every $\varepsilon > 0$ there exists a constant $c(\alpha, \varepsilon)$ such that

$$\left|\alpha - \frac{p}{q}\right| > q^{-d-\varepsilon} \quad \text{for all } \frac{p}{q} \in \mathbb{Q} \quad \text{with } q \geqq c(\alpha, \varepsilon).$$

Whereas this result was obtained by considering the value of the irreducible equation of α at a point $\frac{p}{q}$, one had to consider two approximations $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$ and auxiliary polynomials in two variables in order to replace the exponent d by

$$\frac{d}{2} + 1 \qquad \text{(Thue)}$$

$$\operatorname{Min}\left\{\frac{d}{s+1} + s;\ s = 0, \ldots, d-1\right\} \qquad \text{(Siegel)}$$

$$\sqrt{2 \cdot d} \qquad \text{(Dyson and Gelfond).}$$

Finally Roth, [9], obtained

$$\left|\alpha - \frac{p}{q}\right| > q^{-2-\varepsilon} \quad \text{for all } \frac{p}{q} \in \mathbb{Q} \quad \text{with } q \geqq c(\alpha, \varepsilon),$$

by considering auxiliary polynomials in several variables, having a zero of high order at $(\alpha, \ldots, \alpha)$ and a zero of low order at $\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)$. Unfortunately the constant $c(\alpha, \varepsilon)$ is not effectively computable, except in Liouville's inequality.

In order to obtain effective bounds for approximations by rational numbers of generators of certain number fields, Bombieri, [1], reconsidered and general-

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ized Dyson's approach in handling auxiliary polynomials in two variables. In this article he posed the problem of the generalization of Dyson's Lemma to the several-variable case. More precisely, in a latter dated December 2, 1981, Bombieri formulated it in the following way:

(0.1) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial of multidegree $\underline{d} = (d_1, \ldots, d_n)$, i.e.: $\deg_{x_i}(f) \leqq d_i$. In order to measure how badly a given point $\zeta \in \mathbb{C}^n$ occurs as a zero of f, we define for $t \in \mathbb{R}_{\geqq 0}$ and $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geqq 0}$:

(0.2) *Definition.* f has a *zero of type* $(\underline{a}, t)$ at ζ if $\frac{\partial^{\underline{i}} f}{\partial x_1^{i_1} \cdot \ldots \cdot \partial x_n^{i_n}}(\zeta) = 0$ for every n-tuple $\underline{i} = (i_1, \ldots, i_n)$ of natural numbers with $\sum_{\nu=1}^{n} i_\nu \cdot a_\nu \leqq t$.

Of course, it is enough to consider only the n-tuples $\underline{i}$ with $i_\nu \leqq d_\nu$, $\nu = 1, \ldots, n$. Hence the number of equations we have to look for in (0.2) is approximately

$$d_1 \cdot \ldots \cdot d_n \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t)) = d_1 \cdot \ldots \cdot d_n \cdot \int_{I(\underline{d}, \underline{a}, t)} d\xi_1 \wedge \ldots \wedge d\xi_n$$

where $I^n = \{(\xi_\nu) \in \mathbb{R}^n;\ 0 \leqq \xi_\nu \leqq 1\}$ and

$$I(\underline{d}, \underline{a}, t) = \left\{(\xi_\nu) \in I^n;\ \sum_{\nu=1}^{n} d_\nu \cdot \xi_\nu \cdot a_\nu \leqq t\right\}.$$

(0.3) **Bombieri's problem.** *Assume that there exists* $f \neq 0$ *of multidegree* $\underline{d}$ *such that* f *has a zero of type* $(\underline{a}^{(\mu)}, t_\mu)$ *at the point* $\zeta_\mu \in \mathbb{C}^n$, *for* $\mu = 1, \ldots, M$. *Under which conditions on* $\{\zeta_\mu;\ \mu = 1, \ldots, M\}$ *and* $\{(\underline{a}^{(\mu)}, t_\mu);\ \mu = 1, \ldots, M\}$ *can one find an inequality*

$$\sum_{\mu=1}^{M} \mathrm{Vol}(I(\underline{d}, \underline{a}^{(\mu)}, t_\mu)) \leqq 1 + \varepsilon(d)$$

such that $\varepsilon(d)$ *is small for* $d_1 \gg d_2 \gg \ldots \gg d_n > 0$.

The main result of this article is:

(0.4) **Theorem.** *Assume that*

a) $\zeta_\mu = (\zeta_{\mu,1}, \ldots, \zeta_{\mu,n})$, *for* $\mu = 1, \ldots, M$, *are* M *points in* $\mathbb{C}^n$ *such that* $\zeta_{\mu,\nu} \neq \zeta_{\gamma,\nu}$ *for* $\mu \neq \gamma$ *and* $\nu = 1, \ldots, n$.

b) $\underline{a} \in \mathbb{R}^n_{\geqq 0}$ *(independently of* μ*) and* $t_\mu \in \mathbb{R}_{\geqq 0}$ *for* $\mu = 1, \ldots, M$.

c) $\underline{d} \in \mathbb{N}^n$ *such that* $d_1 \geqq d_2 \geqq \ldots \geqq d_n > 0$.

Then, if there exists $f \in \mathbb{C}[x_1, \ldots, x_n]$ *of multidegree* $\underline{d}$ *having a zero of type* $(\underline{a}^{(\mu)}, t_\mu)$ *at* ζ_μ *for* $\mu = 1, \ldots, M$, *one has the inequality*

$$\sum_{\mu=1}^{M} \mathrm{Vol}(I(\underline{d}, \underline{a}, t_\mu)) \leqq \prod_{j=1}^{n} \left(1 + (M' - 2) \cdot \sum_{i=j+1}^{n} \frac{d_i}{d_j}\right)$$

where $M' = \mathrm{Max}\{M, 2\}$.

Without assumption a) or b) the inequality fails to be true (see the end of § 10 for more remarks concerning this point).

If we consider (0.4) for $n=2$ we find on the right hand side

$$1+(M'-2)\cdot d_2\cdot d_1^{-1}.$$

In fact, one is able to improve the inequality (see § 10) to obtain

$$\text{(0.5)} \qquad \sum_{\mu=1}^{M} \mathrm{Vol}(I(\underline{d}, \underline{a}, t_\mu)) \leqq 1+\frac{(M'-2)\cdot d_2}{2\cdot d_1}.$$

This is finally the inequality known as "Dyson's Lemma", [2], (under the additional assumption that $0<t\leqq \mathrm{Min}\{d_1\cdot a_1,\ d_2\cdot a_2\}$) and it was obtained in the form stated above by Bombieri, [1], using Wronskian-determinants. Viola ([14]) found the first proof of (0.5) avoiding the use of Wronskian-determinants. His arguments are based on a careful local and global analysis of the singularities of reducible algebraic curves. In some way the proof of theorem 0.4 – using methods from the complex projective geometry – is close in spirit to Viola's approach. In fact, the first tool, the positivity results for direct images of dualizing sheaves (see § 6), were developed in order to be applied in the classification theory of higher dimensional varieties ([3], [7], [12], [13]). The second tool is the generalized "Kodaira-Vanishing-Theorem" for integral parts of divisors with coefficients in $\mathbb{Q}$ (see (4.6), [8], [10]) which we already applied to "zeros of polynomials" in [5].

We are grateful to Enrico Bombieri who suggested this problem to us in a letter and who explained to us, as non-specialists, a lot about problem (0.3) and its applications in the theory of approximation of algebraic numbers.

(0.6) The first two sections give the translation of (0.2) in terms of ideals of $\mathbb{C}[x_1, \ldots, x_n]$ (§ 1) and of ideal-sheaves on $(\mathbb{P}^1)^n$ (§ 2). We study the powers of these ideals (1.9) and some combinatorial conditions (2.7), (2.9) on the $(2\cdot n+1)$-tuples $(\underline{d}, \underline{a}, t_\mu)$ imposed by the existence of the polynomial f of (0.4). It is there, where we need the assumptions a) and b) made in (0.4). In § 3 we study the behaviour of the ideal-sheaves under "blowing-ups". To this end we consider (locally) certain coverings of $(\mathbb{P}^1)^n$, a construction which is used again in § 7. § 4 starts with a crash-course on weakly positive coherent sheaves. This notation is just made to study direct images of certain sheaves in § 6, but mostly used in this paper in the case of invertible sheaves, where it serves as a convenient notation avoiding to many "limit-processes". At the end of this section we present the first tool the proof of (0.4) is based on, the generalized "Kodaira-Vanishing-Theorem" for integral parts of divisors with coefficients in $\mathbb{Q}$. This tool was already used in [5] in order to study zeros of polynomials. In the next section we formulate the Main Lemma (5.3) and we use it together with (2.9) to prove that a certain sheaf is arithmetically positive (5.4). At the end of § 5 we show how this implies Theorem (0.4). It seems, that (5.4) is just the "*sheaf-theoretic version of Dylon's Lemma*". In § 6 we discuss the second tool, the weak positivity for the direct images of certain sheaves (6.2). This theory is based on Kawamata's results [7], applied to certain coverings (see [3], [4], [11] and [12] for a general discussion of cyclic coverings). Whereas the first tool, the Vanishing-Theorems, can be obtained by interpretating the symmetry of the Hodge-numbers of projective varieties and the closedness of global logarithmic differential forms (due to Deligne), the proof of Kawamata's Main Theorem is based on Griffith's theory of variations of Hodge-structures and the Nilpotent-Orbit-Theorem (due to W. Schmid). Somehow the analytic theory replacing the Wronskian determinants used in [1] is hidden behind (6.1) and (6.2). At the end of § 6 we reformulate (6.2) using (4.5) in a quite technical lemma, needed in § 8. The following section reformulates the Main Lemma in a slightly more general set-up and in form of an induction step. This finally is proved in § 8, using again a covering construction (made at the end of § 7). Even if it is well known to the specialists, we indicate in § 9 how to obtain the theorem of Roth using (0.4) (for $a_i = d_i^{-1}$). § 10 contains a discussion of possible ameliorations of the inequality of (0.4). For example, if one knows the decomposition of the polynomial f into irreducible factors, one can improve the inequality. In the general case one can replace the constant $(M'-2)$ by a slightly smaller constant. We explain this only in the two-variable case and prove the inequality (0.5).

(0.7) *Notations.* In this paper we are using the "standard notations" of algebraic geometry, as they can be found for example in [6]. Special notations are:

– If D is a Cartier-divisor on an algebraic variety X, we write $\mathcal{O}_X(D)$ to be the associated invertible sheaf.

– If $\mathcal{F}$ is any sheaf on X, we write $\mathcal{F}^a = \mathcal{F}^{\otimes a}$ for the tensorproduct and $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$. In order to create confusion: $\mathcal{F}^a(D) = \mathcal{F}^a \otimes \mathcal{O}_X(D)$ but $\mathcal{F}(D)^a = (\mathcal{F} \otimes \mathcal{O}_X(D))^a$.

– The invertible sheaf of degree one on $\mathbb{P}^1$ is denoted by $\mathcal{O}_{\mathbb{P}^1}(1)$ and the sheaf $\mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \dots, d_n)$ on $(\mathbb{P}^1)^n$ is $\bigotimes_{\nu=1}^{n} \mathrm{pr}_\nu^* \mathcal{O}_{\mathbb{P}^1}(1)^{d_\nu}$, where pr_ν denotes the projection on the ν-th factor.

– If $f: X \to (\mathbb{P}^1)^n$ is any morphism and $\mathcal{F}$ a sheaf on X, then

$$\mathcal{F}(d_1, \dots, d_n) = \mathcal{F} \otimes f^* \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \dots, d_n).$$

Again $\mathcal{F}^a(d_1, \dots, d_n) = \mathcal{F}^a \otimes f^* \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \dots, d_n)$ and

$$\mathcal{F}(d_1, \dots, d_n)^a = (\mathcal{F}(d_1, \dots, d_n))^a.$$

– We write $h^i(X, \mathcal{F}) = \dim_{\mathbb{C}}(H^i(X, \mathcal{F}))$ and for an invertible sheaf $\mathcal{L}$ we define the "$\mathcal{L}$ dimension"

$$\kappa(X, \mathcal{L}) = \kappa(\mathcal{L}) = \begin{cases} \mathrm{tr.dg}\left(\bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)\right) - 1 & \text{if } H^0(X, \mathcal{L}^i) \neq 0 \text{ for some } i > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

– All varieties are supposed to be nonsingular, irreducible, projective and defined over the field of complex numbers $\mathbb{C}$, if not explicitely other properties are given. An open subset is an open subvariety, which is always supposed to be not empty (even if we sometimes forget to mention it).

– The canonical sheaf of a nonsingular variety X is written $\omega_X = \bigwedge^{\dim(X)} \Omega^1_X$ and if $f: X \to Y$ is a morphism between nonsingular varieties we write $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$.

§ 1. The ideals of polynomials with zeros of type $(\underline{a}, t)$

Using the notation introduced in (0.1) we consider a polynomial $f \in \mathbb{C}(x_1, \ldots, x_n]$ of multidegree $\underline{d}$, having a zero of type $(\underline{a}, t)$ in $\zeta \in \mathbb{C}^n$. The set of $\underline{i}$ with $\sum_{\nu=1}^{n} a_\nu \cdot i_\nu \leqq t$ remains the same after replacing $\underline{a}$ by $\underline{a}'$ and t by t' as long as $0 \leqq a_\nu - a'_\nu \leqq \varepsilon$ *and* $0 \leqq t' - t \leqq \varepsilon$ for a very small perturbation $\varepsilon > 0$. Since $I(\underline{d}, \underline{a}, t) \subseteq I(\underline{d}, \underline{a}', t')$, one can assume that $(\underline{a}, t)$ is a $(n+1)$-tuple of non negative rational numbers, or, multiplying by the same integer, that it is a $(n+1)$-tuple of non negative integers.

(1.1) *In the rest of this paper, one has* $(\underline{a}, t) \in \mathbb{N}^{n+1}$ *for the type of all zeros considered.*

(1.2) *Definition.* For a given type $(\underline{a}, t)$ and a given point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ we define $\mathfrak{m}_\zeta^{(\underline{a}, t)}$ to be the ideal generated in $\mathbb{C}[x_1, \ldots, x_n]$ by all the monomials $(x_1 - \zeta_1)^{i_1} \ldots (x_n - \zeta_n)^{i_n}$ such that $a_1 \cdot i_1 + \ldots + a_n \cdot i_n \geqq t$.

(1.3) If f has a zero of type $(\underline{a}, t)$ in ζ, then $f \in \mathfrak{m}_\zeta^{(\underline{a}, t)}$. Even if the converse is not true, because we allowed in (1.2) monomials at the boundary (for which $a_1 \cdot i_1 + \ldots + a_n \cdot i_n = t$), we will say "*$f$ has a zero of type $(\underline{a}, t)$*" instead of saying $f \in \mathfrak{m}_\zeta^{(\underline{a}, t)}$.

Moreover, if $f = \sum_{\underline{i} \in \mathbb{N}^n}^{n} \alpha_{\underline{i}} \cdot (x_1 - \zeta_1)^{i_1} \cdot \ldots \cdot (x_n - \zeta_n)^{i_n}$ is in $\mathfrak{m}_\zeta^{(\underline{a}, t)}$, all the monomials with $\alpha_{\underline{i}} \neq 0$ are in the ideal.

If a given $\underline{d}$ is to small, $\mathfrak{m}_\zeta^{(\underline{a}, t)}$ can not be generated by monomials of multidegree $\underline{d}$.

(1.4) *Definition.* For a given type $(\underline{a}, t)$ and a given point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ we define $\ell_\zeta^{(\underline{d}, \underline{a}, t)}$ to be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by those monomials $(x_1 - \zeta_1)^{i_1} \cdot \ldots \cdot (x_n - \zeta_n)^{i_n}$ out of $\mathfrak{m}_\zeta^{(\underline{a}, t)}$ for which $0 \leqq i_\nu \leqq d_\nu$ for all ν.

(1.5) For $N \in \mathbb{N}$, one has the inclusions

$$\mathfrak{m}_\zeta^{(\underline{a}, t)^N} \to \mathfrak{m}_\zeta^{(\underline{a}, N \cdot t)}$$

$$\ell_\zeta^{(\underline{d}, \underline{a}, t)^N} \to \ell_\zeta^{(N \cdot \underline{d}, \underline{a}, N \cdot t)}.$$

In order to have equalities we need some additional assumptions which are always fulfilled after replacing $(\underline{d}, t)$ by $(s \cdot \underline{d}, s \cdot t)$, i.e. after replacing the polynomial f of (0.4) by f^s.

(1.6) *Assumption on* $(\underline{a}, t)$: For all ν, $n \cdot a_\nu$ divides t.

In other words, the edge points of $\left\{(\xi_\nu) \in \mathbb{R}^n_{\geqq 0};\ \sum_{\nu=1}^{n} a_\nu \cdot \xi_\nu = t\right\}$ have coordinates in $n \cdot \mathbb{N}$.

(1.7) Consider the $(n-1)$-dimensional polygon

$$\Gamma(\underline{d},\underline{a},t)=\left\{(\xi_\nu)\in\mathbb{R}^n;\ 0\leqq\xi_\nu\leqq d_\nu \text{ and } \sum_{\nu=1}^{n} a_\nu\cdot\xi_\nu=t\right\}.$$

Let $E_1,\dots,E_r$ be the edge points of $\Gamma(\underline{d},\underline{a},t)$ (see figure (1.12)). The coordinates of an edge point are all 0 or d_ν but one and satisfy the equality $\sum_{\nu=1}^{n} a_\nu\cdot\xi_\nu=t$. Of course, they are all rational numbers. The edge points of the polygon $\Gamma(s\cdot\underline{d},\underline{a},s\cdot t)$ for a given positive integer s are multiplied by s. Hence we may assume (1.8).

(1.8) *Assumption on* $(\underline{d},\underline{a},t)\in\mathbb{N}^{2n+1}$: the edge points $E_1,\dots,E_r$ have coordinates in $r\cdot\mathbb{N}$.

(1.9) **Lemma.**

i) *Assume that* $(\underline{a},t)$ *satisfies* (1.6). *Then one has* $\mathcal{m}_\zeta^{(\underline{a},t)^N}=\mathcal{m}_\zeta^{(\underline{a},N\cdot t)}$ *for all* $N\in\mathbb{N}$.

ii) *Assume that* $(\underline{d},\underline{a},t)$ *satisfies* (1.8). *Then one has* $\ell_\zeta^{(\underline{d},\underline{a},t)^N}=\ell_\zeta^{(N\cdot\underline{d},\underline{a},N\cdot t)}$ *for all* $N\in\mathbb{N}$.

Proof. Of course $\mathcal{m}_\zeta^{(\underline{a},t)}$ is of the form $\ell_\zeta^{(\underline{d},\underline{a},t)}$ for some big $\underline{d}$. In this case, the n edge points of the polygon corresponding to $\mathcal{m}_\zeta^{(\underline{a},t)}$ have coordinates in $n\cdot\mathbb{N}$ by assumption (1.6). So i) is a special case of ii).

We write for simplicity $\zeta=(0,\dots,0)$. Let $x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}$ be any generator of $\ell_\zeta^{(N\cdot\underline{d},a,N\cdot t)}$, i.e. $0\leqq i_\nu\leqq N\cdot d_\nu$ for all ν and $\sum_{\nu=1}^{n} a_\nu\cdot i_\nu=(N+\varepsilon)\cdot t$ for some $\varepsilon\geqq 0$. We have to present this monomial as a product of N elements of $\ell_\zeta^{(\underline{d},\underline{a},t)}$. By induction on N and the definition of the ideals it is sufficient to find n-tuples

$$(j_\nu)\in\Gamma((N-1)\cdot\underline{d},\underline{a},(N-1)\cdot t)\cap\mathbb{Z}^n$$

and

$$(k_\nu)\in\Gamma((1+\varepsilon)\cdot\underline{d},\underline{a},(1+\varepsilon)\cdot t)\cap\mathbb{Z}^n$$

such that $j_\nu+k_\nu=i_\nu$ for $\nu=1,\dots,n$.

Let $E_1,\dots,E_r$ be the edge points of $\Gamma(\underline{d},\underline{a},t)$, as in (1.7). The point (i_ν) can be written as $(i_\nu)=\sum_{s=1}^{r}\alpha_s\cdot E_s$ for $(\alpha_1,\dots,\alpha_r)\in[0,N+\varepsilon]$ and $\sum_{s=1}^{r}\alpha_s=N+\varepsilon$.

Claim. There exist natural numbers β_s such that

$$\alpha_s-(1+\varepsilon)\leqq\frac{\beta_s}{r}\leqq\operatorname{Min}\{\alpha_s,N-1\}\tag{1.10}$$

and

$$\sum_{s=1}^{r}\frac{\beta_s}{r}=N-1.\tag{1.11}$$

Proof. If [ ] denotes the integral part of a real number, we have the inequalities

$$N+\varepsilon\leqq\sum_{s=1}^{r}\left(\frac{[\alpha_s\cdot r]}{r}+\frac{1}{r}\right)$$

and

$$N-1 \leqq \sum_{s=1}^{r} \operatorname{Min}\left\{\frac{[\alpha_s \cdot r]}{r}, N-1\right\} \leqq N+\varepsilon.$$

For $\beta'_s = \operatorname{Min}\{[\alpha_s \cdot r], (N-1)\cdot r\}$ the condition (1.10) is fulfilled. In fact the second inequality follows from the choice of β'_s and the first from $\alpha_s \leqq N-1+1+\varepsilon$ and $\alpha_s \leqq [\alpha_s \cdot r]\cdot r^{-1}+1$. Assume that (1.11) is not true. Then

$$\sum_{s=1}^{r} \frac{\beta'_s}{r} \geqq (N-1)+\frac{1}{r}$$

and

$$\sum_{s=1}^{r} \left(\alpha_s - \frac{\beta'_s}{r}\right) \leqq 1+\varepsilon-\frac{1}{r}.$$

We can find some index s_0 such that $\beta'_{s_0}>0$ and $\alpha_{s_0}-\frac{\beta'_{s_0}}{r} \leqq 1+\varepsilon-\frac{1}{r}$.

If we replace β'_{s_0} by $\beta'_{s_0}-1$, we have another solution of (1.10) such that $\sum_{s=1}^{r} \frac{\beta'_s}{r} \geqq N-1$.

We are allowed to repeat this step until we have found the r-tuple β_s satisfying both conditions, (1.10) and (1.11).

(1.10) guarantees that $\frac{\beta_s}{r} \in [0, N-1]$ and $\alpha_s - \frac{\beta_s}{r} \in [0, 1+\varepsilon]$. Moreover, from (1.11) we have in addition that $\sum_{s=1}^{r} \left(\alpha_s - \frac{\beta_s}{r}\right) = 1+\varepsilon$.

Hence

$$(j_v) = \sum_{s=1}^{r} \frac{\beta_s}{r} \cdot E_s \in \Gamma((N-1)\cdot \underline{d}, \underline{a}, (N-1)\cdot t)$$

and

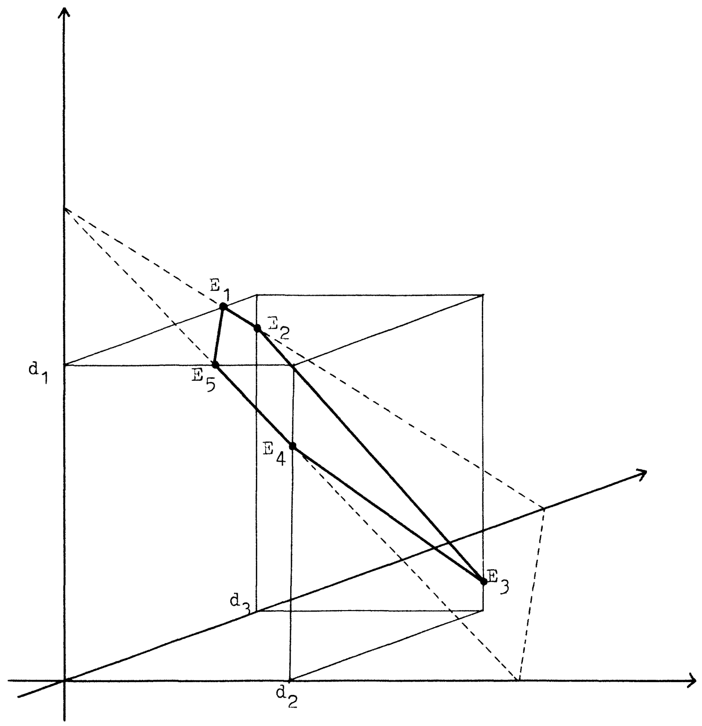
$$(k_v) = \sum_{s=1}^{r} \left(\alpha_s - \frac{\beta_s}{r}\right) \cdot E_s \in \Gamma((1+\varepsilon)\cdot \underline{d}, \underline{a}, (1+\varepsilon)\cdot t).$$

From (1.8) we know that (j_v) is a point with integer coordinates, and hence (k_v) too. By construction $j_v + k_v = i_v$.

(1.12) In the following figure we consider

$$a_1 = \frac{2\cdot t}{3\cdot d_1}, \quad a_2 = \frac{t}{2\cdot d_2} \quad \text{and} \quad a_3 = \frac{2\cdot t}{5\cdot d_3}.$$

The monomials generating $\ell_\zeta^{(\underline{d},\underline{a},t)}$ are corresponding to integer points inside of the box, but over the dotted hypersurface. In other words, we show the image of $I(\underline{d}, \underline{a}, t)$ under multiplication with (d_1, d_2, d_3).



§ 2. Homogeneous polynomials and some combinatorial conditions

(2.1) Let i: $\mathbb{C}^n \to (\mathbb{P}^1)^n$ be the embedding of $\mathbb{C}^n$ into the multiprojective space such that $i(\beta_1, \ldots, \beta_n) = ((\beta_1, 1), \ldots, (\beta_n, 1))$.

Any polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of multidegree $\underline{d}$ gives rise to a polynomial $F(X_1, Y_1, \ldots, X_n, Y_n)$, homogeneous of degree d_ν in X_ν, Y_ν for all ν, such that $F(x_1, 1, \ldots, x_n, 1) = f(x_1, \ldots, x_n)$. We may as well consider F as an element of $H^0((\mathbb{P}^1)^n, \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n))$ and in the sequel *we don't distinguish polynomials and sections.*

(2.2) Let ζ be a point of $\mathbb{C}^n$ (where we denote $i(\zeta)$ again by ζ). We consider $\mathscr{m}_\zeta^{(\underline{a},t)}$ and $\ell_\zeta^{(\underline{d},\underline{a},t)}$ (see (1.2) and (1.4)) as ideal sheaves on $\mathbb{C}^n$ and we denote $i_* \mathscr{m}_\zeta^{(\underline{a},t)} \cap \mathcal{O}_{(\mathbb{P}^1)^n}$ (and $i_* \ell_\zeta^{(\underline{d},\underline{a},t)} \cap \mathcal{O}_{(\mathbb{P}^1)^n}$) again by $\mathscr{m}_\zeta^{(\underline{a},t)}$ (and $\ell_\zeta^{(\underline{d},\underline{a},t)}$ respectively).

(2.3) *Remark.*

i) $\mathscr{m}_\zeta^{(\underline{a},t)}$ can also be defined by the following property: Let $\zeta = ((\lambda_1, \eta_1), \ldots, (\lambda_n, \eta_n))$ and choose a second point $\zeta' = ((\lambda'_1, \eta'_1), \ldots, (\lambda'_n, \eta'_n))$ such that $(\lambda_\nu, \eta_\nu) \neq (\lambda'_\nu, \eta'_\nu)$ for $\nu = 1, \ldots, n$. Then for all n-tuple $\underline{d} = (d_1, \ldots, d_n)$ of natural numbers the vector-space

$$H^0((\mathbb{P}^1)^n, \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n) \otimes \mathscr{m}_\zeta^{(\underline{a},t)})$$

has a basis given by the monomials

$$\prod_{\nu=1}^{n} (\eta_\nu \cdot X_\nu - \lambda_\nu \cdot Y_\nu)^{i_\nu} \cdot (\eta'_\nu \cdot X_\nu - \lambda'_\nu \cdot Y_\nu)^{d_\nu - i_\nu}$$

such that $\sum_{\nu=1}^{n} i_\nu \cdot a_\nu \geqq t$ and $0 \leqq i_\nu \leqq d_\nu$ for all ν.

ii) The description given above shows that

$$\begin{aligned} &H^0((\mathbb{P}^1)^n, \ell_\zeta^{(\underline{d},\underline{a},t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n)) \\ &\quad = H^0((\mathbb{P}^1)^n, m_\zeta^{(\underline{a},t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n)). \end{aligned}$$

Moreover, $\ell_\zeta^{(\underline{d},\underline{a},t)}$ is the uniquely determined subsheaf of $m_\zeta^{(\underline{a},t)}$ such that $\ell_\zeta^{(\underline{d},\underline{a},t)}) \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n)$ is generated by its global sections. In fact, if we choose $\zeta' = ((1,0), \ldots, (1,0))$, the basis given in i) is just consisting of the multihomogeneous monomials corresponding to the generators of $\ell_\zeta^{(\underline{d},\underline{a},t)}$ in (1.4).

(2.4) The volume introduced in the introduction can now be described in the following way:

i) For $N \in \mathbb{N} - \{0\}$, $\mathrm{Vol}(I(\underline{d}, \underline{a}, t)) = \mathrm{Vol}(I(N \cdot \underline{d}, \underline{a}, N \cdot t))$

ii) $\mathrm{Vol}(I(\underline{d}, \underline{a}, t)) = \lim_{N \to \infty} (N^n \cdot d_1 \cdot \ldots \cdot d_n)^{-1}$

$$\cdot \left| \left(I(\underline{d}, \underline{a}, t) \cap \frac{1}{d_1 \cdot N} \cdot \mathbb{Z} \oplus \frac{1}{d_2 \cdot N} \cdot \mathbb{Z} \oplus \ldots \oplus \frac{1}{d_n \cdot N} \cdot \mathbb{Z} \right) \right|,$$

where $|\;|$ denotes the number of elements.

In other terms,

$$\begin{aligned} &d_1 \cdot \ldots \cdot d_n \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t)) \\ &= \lim_{N \to \infty} N^{-n} \cdot \left| \left\{ (\xi_\nu) \in \mathbb{Z}^n;\; 0 \leqq \xi_\nu \leqq N \cdot d_\nu \text{ and } \sum_{\nu=1}^{n} a_\nu \cdot \xi_\nu < N \cdot t \right\} \right| \\ &= \lim_{N \to \infty} N^{-n} \cdot ((N \cdot d_1 + 1) \cdot \ldots \cdot (N \cdot d_n + 1) \\ &\qquad - h^0((\mathbb{P}^1)^n, m_\zeta^{(\underline{a}, N \cdot t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(N \cdot d_1, \ldots, N \cdot d_n))) \\ &= d_1 \cdot \ldots \cdot d_n - \lim_{N \to \infty} N^{-n} \cdot h^0((\mathbb{P}^1)^n, \ell_\zeta^{(N \cdot \underline{d}, \underline{a}, N \cdot t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(N \cdot d_1, \ldots, N \cdot d_n)). \end{aligned}$$

iii) If $\underline{d}'$ is a second n-tuple of natural numbers and $d'_1 \geqq d_1$, $d'_2 \geqq d_2, \ldots, d'_n \geqq d_n$, then the first equality shows that

$$d'_1 \cdot \ldots \cdot d'_n \cdot \mathrm{Vol}(I(\underline{d}', \underline{a}, t)) \geqq d_1 \cdot \ldots \cdot d_n \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t)).$$

The zero-set of the ideal $\ell_\zeta^{(\underline{d},\underline{a},t)}$ is in general not concentrated in the point ζ. For example in the situation described in figure (1.12) in § 1, this zero-set contains $V(X_1)$ and $V(X_2, X_3)$.

(2.5) **Lemma.** *Let* $C_\zeta^{(\underline{d},\underline{a},t)} = \operatorname{coker}(\ell_\zeta^{(\underline{d},\underline{a},t)} \to \mathcal{O}_{(\mathbb{P}^1)^n})$, *then the support of* $C_\zeta^{(\underline{d},\underline{a},t)}$ *is exactly* $\bigcup_{I \subseteq \{1,\ldots,n\}} S_I$ *for*

$$S_I = \begin{cases} \emptyset & \text{if } \sum_{v\in I} a_v \cdot d_v \geqq t \\ V(\eta_v \cdot X_v - \lambda_v \cdot Y_v;\ v \notin I) & \text{if } \sum_{v\in I} a_v \cdot d_v < t. \end{cases}$$

Proof. We may assume that $\zeta = ((0,1), \ldots, (0,1))$. From (2.3, ii)) we know that (for $\zeta' = ((1,0), \ldots, (1,0)))$ the sheaf $\ell_\zeta^{(\underline{d},\underline{a},t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1, \ldots, d_n)$ is generated by $\prod_{v=1}^n X_v^{i_v} \cdot Y_v^{d_v - i_v}$ for all $\underline{i}$ with $\sum_{v=1}^n a_v \cdot i_v \geqq t$ and $0 \leqq i_v \leqq d_v$ for $v = 1, \ldots, n$.

If for some $I \subseteq \{1, \ldots, n\}$ one has $\sum_{v\in I} a_v \cdot d_v < t$, then every n-tuple $\underline{i}$ with $\sum_{v=1}^n a_v \cdot i_v \geqq t$ must have $i_v \neq 0$ for at least one $v \notin I$. Hence $\ell_\zeta^{(\underline{d},\underline{a},t)}$ is contained in $\langle X_v, v \notin I \rangle$ or $S_I \subset C_\zeta^{(\underline{d},\underline{a},t)}$.

On the other hand, if P is any point in the support of $C_\zeta^{(\underline{d},\underline{a},t)}$, we can choose $J \subseteq \{1, \ldots, n\}$ to be the set of all v such that $P \in V(X_v)$. Then the monomial $\prod_{v\notin J} X_v^{d_v} \cdot \prod_{v\in J} Y_v^{d_v}$, being non-zero at P, can not be one of the generating monomials and $\sum_{v\notin J} a_v \cdot d_v < t$.

(2.6) For the rest of this section we consider for $\mu = 1, \ldots. M$ the points $\zeta_\mu = (\zeta_{\mu,1}, \ldots, \zeta_{\mu,n}) \in (\mathbb{P}^1)^n$ where $\zeta_{\mu,v} = (\lambda_{\mu,v}, \eta_{\mu,v}) \in \mathbb{P}^1$. We fix $t_\mu \in \mathbb{N}$, $\underline{a}^{(\mu)} \in \mathbb{N}^n$, for $\mu = 1, \ldots, M$, and we write

$$\mathcal{M}' = \bigcap_{\mu=1}^M m_{\zeta_\mu}^{(\underline{a}^{(\mu)}, t_\mu)} \qquad \mathcal{L}'^{(\underline{d})} = \bigcap_{\mu=1}^M \ell_{\zeta_\mu}^{(\underline{d}, \underline{a}^{(\mu)}, t_\mu)}.$$

For simplicity, if $\underline{d}$ is fix, we write $\mathcal{L}' = \mathcal{L}'^{(\underline{d})}$. As in (2.3, ii)) we have $H^0((\mathbb{P}^1)^n, \mathcal{M}'(d_1, \ldots, d_n)) = H^0((\mathbb{P}^1)^n, \mathcal{L}'(d_1, \ldots, d_n))$, but for $M \geqq 2$ it is no longer true that $\mathcal{L}'(d_1, \ldots, d_n)$ is generated by its global sections. We assume in the sequel that

$$H^0((\mathbb{P}^1)^n, \mathcal{M}'(d_1, \ldots, d_n)) \neq 0$$

which (see (1.3) and (2.2)) just means that there exists a non-zero polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of multidegree $\underline{d}$ having a zero of type $(\underline{a}^{(\mu)}, t_\mu)$ at ζ_μ for all μ. *We also have to assume that* $\underline{a}^{(\mu)} = \underline{a}$, *independent of* μ.

(2.7) **Lemma.** *Let* $1 \leqq \mu$, $\gamma \leqq M$ *and* $\mu \neq \gamma$. *Assume that* $\zeta_{\mu,v} \neq \zeta_{\gamma,v}$ *for* $v = 1, \ldots, n$. *Then*

$$\sum_{v=1}^n a_v \cdot d_v \geqq t_\mu + t_\gamma.$$

Proof. Of course, we may assume that

$$\zeta_\mu = ((0,1), \ldots, (0,1)) \quad \text{and} \quad \zeta_\gamma = ((1,0), \ldots, (1,0)).$$

If $F=\sum_{\underline{i}}\alpha_{\underline{i}}\cdot\prod_{\nu=1}^{n}X_\nu^{i_\nu}\cdot Y_\nu^{d_\nu-i_\nu}$ is a non-zero section of $\mathcal{M}'(d_1,\ldots,d_n)$ and if $\underline{i}$ is any n-tuple with $\alpha_{\underline{i}}\neq 0$, the monomial $\prod_{\nu=1}^{n}X_\nu^{i_\nu}\cdot Y_\nu^{d_\nu-i_\nu}$ must be one of the generators given in (2.3, i)) for $H^0((\mathbb{P}^1)^n, m_{\zeta_\mu}^{(\underline{a},t_\mu)}\otimes\mathcal{O}_{(\mathbb{P}^1)^n}(d_1,\ldots,d_n))$. Hence $\sum_{\nu=1}^{n}a_\nu\cdot i_\nu\geqq t_\mu$ and (exchanging μ and γ) $\sum_{\nu=1}^{n}a_\nu\cdot(d_\nu-i_\nu)\geqq t_\gamma$. Adding up both inequalities we find (2.7).

(2.8) **Lemma** (see [1], Lemma 4). *Under the assumption of* (2.7)

$$\mathrm{Supp}(C_{\zeta_\mu}^{(\underline{d},\underline{a},t_\mu)})\cap\mathrm{Supp}(C_{\zeta_\gamma}^{(\underline{d},\underline{a},t_\gamma)})=\emptyset.$$

Proof. Again we assume that

$$\zeta_\mu=((0,1),\ldots,(0,1))\quad\text{and}\quad\zeta_\gamma=((1,0),\ldots,(1,0)).$$

If (2.8) were wrong, (2.5) would show the existence of subsets I_μ and I_γ of $\{1,\ldots,n\}$ such that

$$\sum_{\nu\in I_\mu}a_\nu\cdot d_\nu<t_\mu\quad\text{and}\quad\sum_{\nu\in I_\gamma}a_\nu\cdot d_\nu<t_\gamma$$

and such that $V(X_\nu;\,\nu\notin I_\mu)\cap V(Y_\nu;\,\nu\notin I_\gamma)\neq\emptyset$. Of course, for any ν_0 we have $V(X_{\nu_0})\cap V(Y_{\nu_0})=\emptyset$ and therefore $I_\mu\cup I_\gamma=\{1,\ldots,n\}$. Then

$$\sum_{\nu=1}^{n}a_\nu\cdot d_\nu\leqq\sum_{\nu\in I_\mu}a_\nu\cdot d_\nu+\sum_{\nu\in I_\gamma}a_\nu\cdot d_\nu<t_\mu+t_\gamma$$

in contradiction to (2.7).

As we mentioned already in (2.6) the sheaf $\mathcal{L}'(d_1,\ldots,d_n)$ is in general not generated by its global sections. We can not even exclude the case, that all sections are zero along $(\mathbb{P}^1)^{n-1}$ regarded as a subvariety of $(\mathbb{P}^1)^n$ by the inclusion j, given by

$$j(P_1,\ldots,P_{n-1})=(P_1,\ldots,P_{k-1},\zeta_{s,k},P_k,\ldots,P_{n-1})$$

for a fixed pair s,k. If the subvariety $j((\mathbb{P}^1)^{n-1})$ does not meet $C_{\zeta_\mu}^{(\underline{d},\underline{a},t_\mu)}$, for $\mu\neq s$, we can forget about the other points and use (2.3, ii)) to produce sections of $j^*\mathcal{L}'(d_1,\ldots,d_n)$.

(2.9) **Lemma.** *Using the notations introduced in* (2.6) *we assume that all the* $(2\cdot n+1)$*-tuples* $(\underline{d},\underline{a},t_\mu)$ *satisfy* (1.8) *and that* $\zeta_{\mu,\nu}\neq\zeta_{\gamma,\nu}$ *for* $\mu\neq\gamma$ *and* $\nu=1,\ldots,n$. *Let* $k\in\{1,\ldots,n\}$ *and* $s\in\{1,\ldots,M\}$ *be fixed numbers such that* $V(\eta_{s,k}\cdot X_k-\lambda_{s,k}\cdot Y_k)$ *is not contained in the support of* $C_{\zeta_s}^{(\underline{d},\underline{a},t_s)}$. *We write* $\underline{\delta}=(d_1,\ldots,d_{k-1},d_{k+1},\ldots,d_n)$, $\underline{\alpha}=(a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n)$, $\tau_s=t_s$ *and* $\tau_\mu=\mathrm{Max}\{0,t_\mu-a_k\cdot d_k\}$ *for* $\mu\neq s$.

On $(\mathbb{P}^1)^{n-1}$ *we consider* $\mathcal{L}''=\bigcap_{\mu=1}^{M}\ell_{\zeta'_\mu}^{(\underline{\delta},\underline{\alpha},\tau_\mu)}$ *for*

$$\zeta'_\mu=(\zeta_{\mu,1},\ldots,\zeta_{\mu,k-1},\zeta_{\mu,k+1},\ldots,\zeta_{\mu,n}).$$

Then

i) $\mathscr{L}''=\mathrm{Im}(j^*\mathscr{L}'\to\mathcal{O}_{(\mathbb{P}^1)^{n-1}})$

ii) $(\mathscr{L}''\otimes\mathcal{O}_{(\mathbb{P}^1)^{n-1}}(d_1,\dots,d_{k-1},d_{k+1},\dots,d_n))^N$ *has a non trivial section for some N big enough.*

Proof. We may write $k=1$ and $s=1$ and $\zeta_1=((0,1),\dots,(0,1))$. From (2.8) we know that the supports of $C_{\zeta_\mu}^{(\underline{d},\underline{a},t_\mu)}$ are disjoint for different μ. Hence in order to prove i) we can consider each point separately.

$j^*(\ell_{\zeta_1}^{(\underline{d},\underline{a},t_1)}\otimes\mathcal{O}_{(\mathbb{P}^1)^n}(d_1,\dots,d_n))$ is generated by the monomials

$$X_2^{i_2}\cdot\ldots\cdot X_n^{i_n}\cdot Y_2^{d_2-i_2}\cdot\ldots\cdot Y_n^{d_n-i_n}$$

with $\sum_{\nu=2}^{n}a_\nu\cdot i_\nu\geqq t_1$. In fact, j^* is right exact, and all the other generators written down in (2.3, ii)) are mapped to zero under j^*. If ζ_μ is another point, let's say $\mu=2$ and $\zeta_2=((1,0),\dots,(1,0))$, then $j^*(\ell_{\zeta_2}^{(\underline{d},\underline{a},t_2)}\otimes\mathcal{O}_{(\mathbb{P}^1)^n}(d_1,\dots,d_n))$ is generated by the monomials $X_2^{d_2-i_2}\cdot\ldots\cdot X_n^{d_n-i_n}\cdot Y_1^{d_1}\cdot Y_2^{i_2}\cdot\ldots\cdot Y_n^{i_n}$ with $a_1\cdot d_1+\sum_{\nu=2}^{n}a_\nu\cdot i_\nu\geqq t_2$, using the same argument, and in both cases we verified i).

Renumbering the points we may assume that $t_2\geqq t_3\geqq\ldots\geqq t_M$. If $a_1\cdot d_1\geqq t_2$, then $\tau_\mu=0$ for $\mu=2,\dots,M$ and $\mathscr{L}''=\ell_{\zeta_1'}^{(\underline{\delta},\underline{\alpha},\tau_1)}$. The non trivial section exists by (2.3, ii)). Therefore, we may assume that $\tau_2=t_2-a_1\cdot d_1>0$. Let F be a general section of $\mathscr{L}'(d_1,\dots,d_n)$ and c the largest number such that X_1^c divides F.

Consider the polynomial $G=j^*(F\cdot X_1^{-c})$.

G has a zero of type $(\underline{\alpha},\tau_1-c\cdot a_1)$ at ζ_1' and a zero of type $(\underline{\alpha},\tau_\mu+c\cdot a_1)$ at ζ_μ' for $\mu=2,\dots,M$.

For example, if $\zeta_\mu=((1,0),\dots,(1,0))$ and

$$F\cdot X_1^{-c}=\sum_{\underline{i}}\beta_{\underline{i}}\cdot X_1^{i_1}\cdot\ldots\cdot X_n^{i_n}\cdot Y_1^{d_1-i_1}\cdot\ldots\cdot Y_n^{d_n-i_n},$$

then

$$G=\sum\beta_{\underline{i}}\cdot X_2^{i_2}\cdot\ldots\cdot X_n^{i_n}\cdot Y_2^{d_2-i_2}\cdot\ldots\cdot Y_n^{d_n-i_n}$$

where the second sum is taken over all $\underline{i}$ with $i_1=0$. For those $\underline{i}$ the coefficient $\beta_{\underline{i}}$ can only be non-zero if $a_1\cdot c+\sum_{\nu=2}^{n}a_\nu\cdot i_\nu\geqq t_1$ and $a_1\cdot(d_1-c)+\sum_{\nu=2}^{n}a_\nu\cdot(d_\nu-i_\nu)\geqq t_\mu$. For $w=\sum_{\nu=2}^{n}a_\nu\cdot d_\nu$ we know from (2.7) that $w-t_1\geqq t_2-a_1\cdot d_1=\tau_2>0$.

Consider $H=G^{w-t_1}\cdot\prod_{\nu=2}^{n}X_\nu^{c\cdot a_1\cdot d_\nu}$. At ζ_1', H has a zero of type $(\underline{\alpha},(w-t_1)\cdot(\tau_1-c\cdot a_1)+c\cdot a_1\cdot w)$ and at ζ_μ' for $\mu\neq1$ a zero of type $(\underline{\alpha},(w-t_1)\cdot(\tau_\mu+c\cdot a_1))$.

The multidegree of H is $(w-t_1+c\cdot a_1)\cdot\underline{\delta}$.

We have $(w-t_1)\cdot(\tau_1-c\cdot a_1)+c\cdot a_1\cdot w=(w-t_1+c\cdot a_1)\cdot\tau_1$ since $t_1=\tau_1$, and $(w-t_1)\cdot(\tau_\mu+c\cdot a_1)=(w-t_1)\cdot\tau_\mu+(w-t_1)\cdot c\cdot a_1$ and this is bigger than or equal to $(w-t_1+c\cdot a_1)\cdot\tau_\mu$. Hence for $N=w-t_1+c\cdot a_1$ we found the section needed in ii).

§ 3. A covering construction

In this § we want to study the behaviour of the ideals $\ell_\zeta^{(\underline{d},\underline{a},t)}$ and $m_\zeta^{(\underline{a},t)}$ under certain blowing up's. To this end we consider certain coverings. This construction is going to appear again in §7. Some general remarks about cyclic coverings can be found at the end of the §.

(3.1) *Generalities on blowing up's*. Let X be a variety and $\mathcal{J}$ a sheaf of ideals. The blowing up of $\mathcal{J}$ is a morphism $\pi: X' \to X$ such that $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X'} = \mathrm{Im}(\pi^* J \to \mathcal{O}_{X'})$ becomes invertible and such that π is universal for this property ([6], p. 164). The variety X' is nothing but $\mathrm{Proj}(\bigoplus_{d \geqq 0} \mathcal{J}^d)$. Moreover, π is birational and is an isomorphism on the open variety on which $\mathcal{J}$ is invertible. The blowing up of $\mathcal{J}^l$ is the same as the blowing up of $\mathcal{J}$ for all integers $l > 0$.

(3.2) We define E to be the effective divisor such that $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$. The divisor $-E$ is relatively ample and – by the theorems of Serre ([6], p. 228) – one has $R^i \pi_* \mathcal{O}_{X'}(-l \cdot E) = 0$ for all $i > 0$ and all $l > 0$ big enough and the natural inclusion $\mathcal{J}^l \to \pi_* \mathcal{O}_{X'}(-l \cdot E)$ becomes an equality for l big enough.

We take a disingularization $\sigma: Y \to X'$ (possible after Hironaka's construction, see [6] p. 391 for references) and set $\tau = \pi \circ \sigma: Y \to X$.

(3.3) *Definition*. The sheaf of ideals $\mathcal{J}$ is said to be *full* if the natural inclusion

$$\mathcal{J} \to \tau_* \sigma^* \mathcal{O}_{X'}(-E) = \tau_* \mathrm{Im}(\tau^* \mathcal{J} \to \mathcal{O}_Y)$$

is an isomorphism.

Using the projection formula ([6], p. 124) one sees that this definition is independent of the chosen desingularization Y. Coming from the possible non normality of X', (3.2) does *not* imply that $\mathcal{J}^l$ is full for l big enough.

In order to prove that our ideals $\mathcal{m}_\zeta^{(\underline{a},t)}$ and $\ell_\zeta^{(\underline{d},\underline{a},t)}$ are full, we use the following numerical conditions.

(3.4) i) *Assumption on* $(\underline{a}, t) \in \mathbb{N}^{n+1}$: a_ν divides t.

ii) *Assumption on* $(\underline{d}, \underline{a}, t)$: Let $(\xi_\nu) \in \mathbb{R}^n_{\geqq 0}$ be any solution of the equation $\sum_{\nu=1}^n a_\nu \cdot \xi_\nu = t$ such that ξ_ν is either 0 or d_ν for all ν but one. Then (ξ_ν) is a point of $\mathbb{N}^n$.

(3.5) **Proposition.** *Under the assumptions* (3.4)

i) $\mathcal{m}_\zeta^{(\underline{a},t)}$ is full,

ii) $\ell_\zeta^{(\underline{d},\underline{a},t)}$ is full.

Of course, i) is a special case of ii). We prove here that i) implies ii) and prove i) in (3.6).

Choose $\tau: Y \to (\mathbb{P}^1)^n$ "big enough" such that both sheaves $\mathcal{m} = \tau^{-1} \mathcal{m}_\zeta^{(\underline{a},t)} \cdot \mathcal{O}_Y$ and $\ell = \tau^{-1} \ell_\zeta^{(\underline{d},\underline{a},t)} \cdot \mathcal{O}_Y$ are invertible. One has the inclusions (3.5 i))

$$\ell_\zeta^{(\underline{d},\underline{a},t)} \to \tau_* \ell \to \tau_* \mathcal{m} = \mathcal{m}_\zeta^{(\underline{a},t)}.$$

If the first inclusion is not an equality, we choose an embedding $\mathbb{C}^n \to (\mathbb{P}^1)^n$ for which the restriction of this inclusion is not an equality. In other words, there is a polynomial f which is in $\tau_* \ell|_{\mathbb{C}^n}$ but not in $\ell_\zeta^{(\underline{d},\underline{a},t)}|_{\mathbb{C}^n}$. One can assume that $\zeta = (0, \ldots, 0) \in \mathbb{C}^n$. Every monomial $x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ appearing in f must satisfy the inequality $a_1 \cdot i_i + \ldots + a_n \cdot i_n \geqq t$. After permuting the variables, one can assume that there is one monomial such that for some $k \geqq 1$

$$i_1 > d_1, \ldots, i_k > d_k, \quad i_{k+1} \leqq d_{k+1}, \ldots, i_n \leqq d_n$$

and

$$x_{1_1}^{d} \cdot \ldots \cdot x_k^{d_k} \cdot x_{k+1}^{i_{k+1}} \cdot \ldots \cdot x_n^{i_n} \notin \ell_\zeta^{(\underline{d}, \underline{a}, t)}|_{\mathbb{C}^n}.$$

In other words, one has $\sum_{\nu=1}^{k} a_\nu \cdot d_\nu + \sum_{\nu=k+1}^{n} a_\nu \cdot i_\nu < t$.

Let j: $\mathbb{C}^{n-k} \to \mathbb{C}^n$ be the inclusion given by

$$j(\alpha_{k+1}, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_k, \alpha_{k+1}, \ldots, \alpha_n)$$

for a point $(\beta_1, \ldots, \beta_k)$ in general position. One has

$$j^* \ell_\zeta^{(\underline{d}, \underline{a}, t)} = \ell_{\zeta'}^{(\underline{d}', \underline{a}', t')} \quad \text{for } \zeta' = (0, \ldots, 0),$$

$\underline{a}' = (a_{k+1}, \ldots, a_n)$, $\underline{d}' = (d_{k+1}, \ldots, d_n)$ and $t' = t - \sum_{\nu=1}^{k} a_\nu \cdot d_\nu$.

From (3.5, i)) one has again $j^*(f) \in j^* \ell \hookrightarrow m_{\zeta'}^{(\underline{a}', t')}$.

But (3.4. ii)) is just formulated to make (3.4, i)) true for $(\underline{a}', t')$. However, $j^*(f)$ contains the monomial $\prod_{\nu=k+1}^{n} x_\nu^{i_\nu}$ whereas $\sum_{\nu=k+1}^{n} i_\nu \cdot a_\nu < t - \sum_{\nu=1}^{k} d_\nu \cdot a_\nu$.

(3.6) *The proof of* (3.5, i)): If $m_\zeta^{(\underline{a}, t)}$ is a power of the maximal ideal of the point ζ, i.e. if $a_\nu = a$ for all ν, then it is well known. So the idea is to come back to this situation. Since $m_\zeta^{(\underline{a}, t)}$ is invertible outside of ζ, we may assume that the situation is local and consider $m_\zeta^{(\underline{a}, t)}$ in the affine space where $\zeta = (0, \ldots, 0)$.

(3.7) Let ρ: $X \to \mathbb{C}^n$ be any finite covering, such that X is smooth and the discriminant $\Delta(X/\mathbb{C}^n)$ is contained in the coordinate-axes $V(x_1 \cdot \ldots \cdot x_n)$. Assume that for every point $\eta \in \rho^{-1}(\zeta)$ one can find a local parameter system $(u_1, \ldots, u_n)$ such that $u_\nu^{\alpha_\nu} \cdot f_\nu = x_\nu$ where $\alpha_\nu \in \mathbb{N}$ and f_ν is a unit at η. Define n_η to be the ideal generated by $\prod_{\nu=1}^{n} u_\nu^{j_\nu}$ such that $\sum_{\nu=1}^{n} j_\nu \cdot \frac{a_\nu}{\alpha_\nu} \geqq t$.

(3.8) **Lemma.** *One has the inclusion* $\rho^* m_\zeta^{(\underline{a}, t)} \hookrightarrow \bigcap_{\rho^{-1}(\zeta)} n_\zeta = \mathcal{N}'$ *and a commutative diagram*

$$\begin{array}{ccccc} m_\zeta^{(\underline{a}, t)} & \hookrightarrow & \rho_* \mathcal{N}' & \to & m_\zeta^{(\underline{a}, t)} \\ \cup & & \cup & & \cup \\ \mathcal{O}_{\mathbb{C}^n} & \hookrightarrow & \rho_* \mathcal{O}_X & \to & \mathcal{O}_{\mathbb{C}^n} \end{array}$$

where the second row consists of the natural splitting given by the natural inclusion and the trace map.

Proof. Near η, $\rho^* m_\zeta^{(\underline{a}, t)}$ is the ideal generated by $\prod_{\nu=1}^{n} u_\nu^{\alpha_\nu \cdot i_\nu} \cdot f_\nu^{i_\nu}$ such that $\sum_{\nu=1}^{n} i_\nu \cdot \alpha_\nu \cdot \frac{a_\nu}{\alpha_\nu} \geqq t$. Hence the first inclusion is obvious.

We have the natural inclusion $\mathscr{m}_\zeta^{(a,t)} \to \rho_* \mathscr{N}'$, compatible with $\mathcal{O}_{\mathbb{C}^n} \to \rho_* \mathcal{O}_X$. So we have only to show that the image of $\rho_* \mathscr{N}'$ under the trace map is contained in $\mathscr{m}_\zeta^{(a,t)}$. To this end, one can assume that $\rho^{-1}(\tau) = \eta$ and suppose $\mathcal{O}_{\mathbb{C}^n}$ and $\mathcal{O}_X$ to be complete. Write $X = \operatorname{Spec} \mathbb{C}[\![u_1, \ldots, u_n]\!]$. Then $f_\nu = 1$ and the trace map is just the sum of the conjugates under the operation of the Galois group $G = \mathbb{Z}/\alpha_1 \times \ldots \times \mathbb{Z}/\alpha_n$. The ideal $\mathscr{N}'$ is invariant under G. So the image is generated by the G-invariant elements of $\mathscr{N}'$, this means by the G-invariant monomials $\prod_{\nu=1}^{n} u_\nu^{l_\nu}$. This implies that $l_\nu = \alpha_\nu \cdot m_\nu$ for $\nu = 1, \ldots, n$ and that

$$\sum_{\nu=1}^{n} l_\nu \cdot \frac{a_\nu}{\alpha_\nu} = \sum_{\nu=1}^{n} m_\nu \cdot a_\nu \geqq t.$$

(3.9) Now let N be any positive integer divisible by the smallest common multiple of the edge points t/a_ν (which are integers by (3.4, i)). For $\alpha_\nu = \dfrac{N \cdot a_\nu}{t}$ and $X = \operatorname{Spec} \mathbb{C}[u_1, \ldots, u_n] \to \mathbb{C}^n$ for $u_\nu^{\alpha_\nu} = x_\nu$, the ideal $\mathscr{n}_\eta$ is nothing but the N-th power of the maximal ideal of the point $\eta = 0$ in X.

We know that $\mathscr{n}_\eta$ is full. Choose a diagram of non singular varieties

$$\begin{array}{ccc} Z & \xrightarrow{\ \tau'\ } & \mathbb{C}^n \\ {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \rho} \\ Y & \xrightarrow[\ \tau\ ]{} & \mathbb{C}^n \end{array}$$

such that τ, τ' are birational, $\tau'^{-1} \mathscr{n}_\eta \cdot \mathcal{O}_Z = \mathscr{N}$ and $\tau^{-1} \mathscr{m}_\zeta^{(a,t)} \cdot \mathcal{O}_Y = \mathscr{M}$ are invertible.

We have the natural inclusions

$$\rho'^* \mathscr{M} = \tau'^{-1}(\rho^* \mathscr{m}_\zeta^{(a,t)}) \cdot \mathcal{O}_Z \to \mathscr{N}$$

and

$$\mathscr{M} \hookrightarrow \rho'_* \rho'^* \mathscr{M} \hookrightarrow \rho'_* \mathscr{N}.$$

Hence, $\tau_* \mathscr{M}$ is contained in $\rho_* \tau'_* \mathscr{N} = \rho_* \mathscr{n}_\eta$. Moreover the image is G-invariant and from (3.8) we find

$$\tau_* \mathscr{M} \hookrightarrow \mathscr{m}_\zeta^{(a,t)} \hookrightarrow \tau_* \mathscr{M}.$$

(3.10) Let D be any effective divisor on the non singular quasiprojective variety Y and $D = \sum \gamma_j \cdot E_j$ the decomposition into prime components. For all integers $i \geqq 0$ and $N > 0$ we write

$$\left[\frac{i \cdot D}{N}\right] = \sum \left[\frac{i \cdot \gamma_j}{N}\right] \cdot E_j$$

where [ ] denotes the integral part of a real number.

(3.11) *Notation.* D is called a *normal crossing divisor* if the E_j are non singular and intersect transversally. This means that for all points of Y one can find a local parameter system $(u_1, \ldots, u_n)$ such that near the point the equation of D_{red} is $u_1 \cdot \ldots \cdot u_s = 0$.

The integral parts of divisors are compatible with blowing up's (simple exercise, or see [10], 2.3 and 2.4):

(3.12) **Lemma.** *Let $\tau\colon Z \to Y$ be a birational morphism of non singular varieties such that D and $D' = \tau^* D$ are both normal crossing divisors. Then one has*

$$\tau_* \mathcal{O}_Z\left(\left[\frac{i \cdot D'}{N}\right]\right) = \mathcal{O}_Y\left(\left[\frac{i \cdot D}{N}\right]\right)$$

and

$$\tau_*\left(\omega_Z \otimes \mathcal{O}_Z\left(-\left[\frac{i \cdot D'}{N}\right]\right)\right) = \omega_Y \otimes \mathcal{O}_Y\left(-\left[\frac{i \cdot D}{N}\right]\right).$$

(3.13) The divisors $\left[\frac{i \cdot D}{N}\right]$ occur in a natural way in the following construction. If $i \geqq N$, we replace D by $r \cdot D$ and N by $r \cdot N$ for r big enough and we keep the same i. So we assume $i < N$. Assume that there exists an invertible sheaf $\mathscr{L}$ such that $\mathscr{L}^N = \mathcal{O}(D)$, where D is a normal crossing divisor. The section of $\mathscr{L}^N$, whose zero-set is D, defines on the $\mathcal{O}_Y$-module $\mathscr{A} = \bigoplus_{i=0}^{N-1} \mathscr{L}^{-i}$ an $\mathcal{O}_Y$-algebra structure. Let T be any desingularization of $Spec\,\mathscr{A}$ and $\rho\colon T \to Y$ the corresponding morphism. Then one has

$$\rho_* \mathcal{O}_T = \bigoplus_{i=0}^{N-1} \mathscr{L}^{-i}\left(\left[\frac{i \cdot D}{N}\right]\right)$$

and

$$\rho_* \omega_T = \bigoplus_{i=0}^{N-1} \omega_Y \otimes \mathscr{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right).$$

For the proof and some applications of this construction, see [4].

§ 4. Positivity and vanishing theorems

In this section we introduce and discuss the notations of weakly positive sheaves and arithmetically positive invertible sheaves, and we formulate the generalized "Kodaira-Vanishing-Theorem".

(4.1) *Notation.* Let $\mathscr{F}$ be any coherent, torsion-free sheaf on a nonsingular quasi-projective variety Y.

i) Let U be an open subvariety of Y. We say that $\mathscr{F}$ is *generated by its global sections over* U, if there is a map $\bigoplus \mathcal{O}_Y \to \mathscr{F}$, surjective over U. This is equivalent to the fact, that for all $y \in U$ we can find elements in $H^0(Y, \mathscr{F})$ which generate $\mathscr{F}$ near y.

ii) Let $i: V \to Y$ be the biggest open subvariety of Y such that $i^* \mathscr{F}$ is locally free. Then we define for $\beta > 0$ $\hat{S}^\beta(\mathscr{F}) = i_* S^\beta(i^* \mathscr{F})$, where S^β denotes the usual symmetric product.

(4.2) *Definition.* Let $\mathscr{F}$ be a coherent torsion-free sheaf on Y and $\mathscr{U}$ a set of open subvarieties of Y, closed under intersections (i.e. $U_1, U_2 \in \mathscr{U}$ implies $U_1 \cap U_2 \in \mathscr{U}$).

i) $\mathscr{F}$ is called *weakly positive with respect to* $\mathscr{U}$ if

a) $\mathscr{F}|_U$ is locally free for some $U \in \mathscr{U}$.

b) For all ample invertible sheaves $\mathscr{H}$ on Y and all $\alpha > 0$ there exist some $\beta > 0$ and some $U' \in \mathscr{U}$ such that $\hat{S}^{\alpha \cdot \beta}(\mathscr{F}) \otimes \mathscr{H}^\beta$ is generated by its global sections over U'. (In a) and b), we assume – of course – U and U' to be non-empty.)

ii) If $\mathscr{U}$ is the set of all open subvarieties of Y, we say $\mathscr{F}$ to be *weakly positive* instead of "weakly positive with respect to $\mathscr{U}$".

iii) If $\mathscr{U} = \{U\}$, we just say that $\mathscr{F}$ is *weakly positive over U.*

The notation "weakly positive" was introduced to study direct images of certain sheaves under surjective morphisms (see §6). In §7 and §8 we apply this notation to invertible sheaves and to a quite special class of open subvarieties. This is nothing but a convenient way to avoid too many "limit-processes". General properties of weakly positive sheaves are discussed in [11], [12] and [13]. The most important are:

(4.3) *Properties of weakly positive sheaves.* Let X, Y and Z be nonsingular quasiprojective varieties, $\mathscr{U}$ a set of open subvarieties of Y, closed under intersections, and $\mathscr{F}$ and $\mathscr{G}$ coherent, torsion-free sheaves on Y, locally free over some $U \in \mathscr{U}$.

1) If $\mathscr{F}$ is weakly positive with respect to $\mathscr{U}$ and $\mathscr{F} \to \mathscr{G}$ a map, surjective over some $U \in \mathscr{U}$, then $\mathscr{G}$ is weakly positive with respect to $\mathscr{U}$.

2) Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be any invertible sheaves on Y. Assume that for all $\gamma > 0$ there is some $\mu > 0$ such that $\hat{S}^{\gamma \cdot \mu}(\mathscr{F}) \otimes \mathscr{L}_1^\mu \otimes \mathscr{L}_2$ is weakly positive with respect to $\mathscr{U}$. Then $\mathscr{F}$ is weakly positive with respect to $\mathscr{U}$.

3) If $\mathscr{F}$ and $\mathscr{G}$ are weakly positive with respect to $\mathscr{U}$, then $\mathscr{F} \otimes \mathscr{G}$, $\det(\mathscr{F})$ and $\hat{S}^\gamma(\mathscr{F})$ (for all $\gamma > 0$) are weakly positive with respect to $\mathscr{U}$.

4) Let $\tau: Z \to Y$ be any morphism and $\mathscr{F}$ be weakly positive with respect to $\mathscr{U}$. If either τ is flat or $\mathscr{F}$ locally free, then $\tau^* \mathscr{F}$ is weakly positive with respect to $\tau^{-1}(\mathscr{U}) = \{\tau^{-1}(U);\ U \in \mathscr{U}\}$.

Moreover, if τ is a finite covering (and hence flat), then: $\tau^* \mathscr{F}$ is weakly positive with respect to $\tau^{-1}(\mathscr{U})$ if and only if $\mathscr{F}$ is weakly positive with respect to $\mathscr{U}$.

5) Let $\tau': Y \to X$ be a birational morphism and $\mathscr{F}$ weakly positive with respect to $\mathscr{U}$. Then $\tau'_* \mathscr{F}$ is weakly positive with respect to $\{\tau'(U) \cap V;\ U \in \mathscr{U}\}$ where V is the biggest open subvariety of X such that $\tau'|_{\tau'^{-1}(V)}$ is an isomorphism. More precisely we can say that for all ample invertible sheaves $\mathscr{H}$ on X and all $\alpha > 0$ there is some $\beta > 0$ such that $\hat{S}^{\alpha \cdot \beta}(\mathscr{F}) \otimes \tau'^*(\mathscr{H}^\beta)$ is generated by its global sections over $\tau'^{-1}(V) \cap U$ (or equivalently that $\tau'_* \hat{S}^{\alpha \cdot \beta}(\mathscr{F}) \otimes \mathscr{H}^\beta$ is generated by its global sections over $V \cap \tau'(U)$) for some $U \in \mathscr{U}$.

6) Assume that $\mathscr{F}$ is locally free and that Y is projective. Then $\mathscr{F}$ is weakly positive over Y, if and only if for all curves C, for all morphisms $j: C \to Y$ and for all invertible quotient sheaves $\mathscr{L}$ of $j^*\mathscr{F}$ one has $\deg_C(\mathscr{L}) \geqq 0$.

Proof. 1) follows immediately from the definition and 3) is proven in [13], 3.2. If $\mathscr{F}$ is an invertible sheaf (and this is the only situation where we need 3)) then 3) is obvious. 6) can be found in [11], 1, 10.

2) Let $\mathscr{H}$ be any ample invertible sheaf on Y and $\alpha > 0$. For δ big enough both $\mathscr{H}^\delta \otimes \mathscr{L}_2^{-1}$ and $\mathscr{H}^\delta \otimes \mathscr{L}_2^{-1}$ are ample. We can find $\mu > 0$, by our assumption, such that $\hat{S}^{2\cdot\delta\cdot\alpha\cdot\mu}(\mathscr{F}) \otimes \mathscr{L}_1^\mu \otimes \mathscr{L}_2$ is weakly positive with respect to $\mathscr{U}$. For some $\beta > 0$ and $U \in \mathscr{U}$, $\mathscr{F}|_U$ is locally free and

$$\hat{S}^\beta(\hat{S}^{2\cdot\delta\cdot\alpha\cdot\mu}(\mathscr{F})) \otimes \mathscr{L}_1^{\mu\cdot\beta} \otimes \mathscr{L}_2^\beta \otimes (\mathscr{H}^{\delta\cdot\mu} \otimes \mathscr{L}_1^{-\mu} \otimes \mathscr{H}^\delta \otimes \mathscr{L}_2^{-1})^\beta$$

is generated by its global sections over some $U \in \mathscr{U}$. Then

$$\hat{S}^{2\cdot\beta\cdot\delta\cdot\alpha\cdot\mu}(\mathscr{F}) \otimes \mathscr{H}^{\beta\cdot\delta\cdot\mu+\delta\cdot\beta} \quad \text{and} \quad \hat{S}^{2\cdot\beta\cdot\delta\cdot\mu\cdot\alpha}(\mathscr{F}) \otimes \mathscr{H}^{2\cdot\beta\cdot\delta\cdot\mu}$$

as well are generated by their global sections over U.

4) For $\mathscr{H}$ ample on Y and $\gamma > 0$ we can find some $\mu > 0$ that $\hat{S}^{\gamma\cdot\mu}(\mathscr{F}) \otimes \mathscr{H}^\mu$ is generated by its global sections over some $U \in \mathscr{U}$. We may assume that $\mathscr{F}$ is locally free over U and each of the assumptions gives an inclusion of $\tau^*(\hat{S}^{\gamma\cdot\mu}(\mathscr{F}) \otimes \mathscr{H}^\mu)$ in $\hat{S}^{\gamma\cdot\mu}(\tau^*\mathscr{F}) \otimes \tau^*\mathscr{H}^\mu$ and hence a map $\oplus \mathcal{O}_Z \to \hat{S}^{\gamma\cdot\mu}(\tau^*\mathscr{F}) \otimes \tau^*\mathscr{H}^\mu$, both surjective over $\tau^{-1}(U)$. The sheaf $\oplus \mathcal{O}_Z$ is weakly positive over Z and from 1) and 2) we obtain the weak positivity of $\tau^*\mathscr{F}$. The other direction – if τ is finite – is in [11], 1.7: $\tau^*\mathscr{H}$ is ample and if $\tau^*\mathscr{F}$ is weakly positive we can find for given α some $\beta > 0$ such that $\hat{S}^{2\cdot\alpha\cdot\beta}(\tau^*\mathscr{F}) \otimes \tau^*\mathscr{H}^\beta$ is generated by its global sections over $\tau^{-1}(U)$ for some $U \in \mathscr{U}$. We hence have maps, surjective over U,

$$\oplus \mathscr{H}^\beta \otimes \tau_*\mathcal{O}_Z \to \hat{S}^{2\cdot\alpha\cdot\beta}(\mathscr{F}) \otimes \mathscr{H}^{2\cdot\beta} \otimes \tau_*\mathcal{O}_Z \to \hat{S}^{2\cdot\alpha\cdot\beta}(\mathscr{F}) \otimes \mathscr{H}^{2\cdot\beta}.$$

For β big enough, the first sheaf is generated by its global sections.

5) For given $\mathscr{H}$ and α we have to find $U \in \mathscr{U}$ and $\beta > 0$ such that $\tau'_*\hat{S}^{\alpha\cdot\beta}(\mathscr{F}) \otimes \mathscr{H}^\beta$ is generated by its global sections over $V \cap \tau'(U)$. This sheaf does not change, if we replace Y by its biggest open subvariety on which $\mathscr{F}$ is locally free. By 4) we may replace Y by any birational $Y' \to Y$ and assume that τ' is just a sequence of blowing-ups (see for example [6], V.5.6.1). Then – as in (3.2) – there is an effective divisor E with support in the exceptional locus of τ', such that $\mathcal{O}_Y(-E)$ is relatively ample. In other words, for some $\gamma > 0$ the sheaf $\tau'^*\mathscr{H}^\gamma(-E)$ is ample.

Hence for β' big enough and some $U \in \mathscr{U}$ $\hat{S}^{\alpha\cdot\gamma\cdot\beta'}(\mathscr{F}) \otimes (\tau'^*\mathscr{H}^\gamma(-E))^{\beta'}$ is generated by its global sections over U. This sheaf is included in $\hat{S}^{\alpha\cdot\gamma\cdot\beta'}(\mathscr{F}) \otimes \tau'^*\mathscr{H}^{\gamma\cdot\beta'}$, isomorphic over V, and for $\beta = \gamma \cdot \beta'$ we get 5).

(4.4) *Definition.* An invertible sheaf $\mathscr{L}$ over a projective nonsingular variety Y is called *arithmetically positive* if one of the following equivalent conditions is fulfilled:

a) $\mathscr{L}$ is weakly positive over Y.

b) For all curves C in Y we have $\deg_C(\mathscr{L}|_C) \geqq 0$.

In fact, if $j: C \to Y$ is any non trivial morphism and s the degree of C over $j(C)$, one has $\deg_C(j^*\mathscr{L}) = s \cdot \deg_{j(C)}(\mathscr{L}|_{j(C)})$ and the equivalence of a) and b) is (4.3, 6)).

For arithmetically positive invertible sheaves one has the following vanishing theorem – using the notation introduced in (3.10).

(4.5) **Theorem.** *Let Y be a nonsingular projective variety, D a normal crossing divisor and $\mathscr{L}$ an invertible sheaf on Y. We assume that for some $N > 0$ the sheaf $\mathscr{L}^N(-D)$ is arithmetically positive, and we fix $i > 0$ and $p > 0$. Then*

$$H^p\left(Y, \mathscr{L}^i \otimes \mathcal{O}_Y\left(-\left[\frac{i \cdot D}{N}\right]\right) \otimes \omega_Y\right) = 0$$

if one of the following conditions if fulfilled:

i) *The selfintersection number $c_1(\mathscr{L}^N(-D))^{\dim(Y)} > 0$.*

ii) *The "$\mathscr{L}$-dimension" $\kappa(\mathscr{L}^N(-D)) = \dim(Y)$.*

iii) *$N > i$ and the "$\mathscr{L}$-dimension" $\kappa\left(\mathscr{L}^{N-i}\left(-\left[\frac{N-i}{N} \cdot D\right]\right)\right) = \dim(Y)$.*

This theorem was proven by Kawamata in [8] and independently in [10] and [4]. The conditions i) and ii) are equivalent and – replacing N by $l \cdot N$ and D by $l \cdot D$, if necessary – they imply iii) (see [10], 2.2, 3.1 and 3.2). One possible proof of (4.5) uses the covering-construction indicated in (3.13), the symmetry of Hodge-numbers of projective varieties and the closedness of global logarithmic differential forms.

(4.6) **Corollary.** *Let Y and X be nonsingular projective varieties and $\tau: Y \to X$ a birational morphism. For a normal crossing divisor D and an invertible sheaf $\mathscr{L}$ on Y we assume that $\mathscr{L}^N(-D)$ is arithmetically positive for some $N > 0$. Then for all $i > 0$ and $q > 0$*

$$R^q \tau_*\left(\omega_{Y/X} \otimes \mathscr{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)\right) = 0.$$

Moreover, if one of the conditions of (4.5) *is fulfilled,*

$$H^p\left(X, \tau_*\left(\omega_{Y/X} \otimes \mathscr{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)\right) \otimes \omega_X\right) = 0$$

for $p > 0$.

Proof. The second statement follows from the first one and (4.5). In fact, the Leray-spectral-sequence gives

$$H^p\left(Y, \mathscr{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right) \otimes \omega_Y\right) = H^p\left(X, \tau_*\left(\omega_{Y/X} \otimes \mathscr{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right) \otimes \omega_X\right).$$

In order to show the vanishing of the higher direct images of τ we can use the projection formula ([6], p. 253) and replace $\mathscr{L}$ by $\mathscr{L}\otimes\tau^*\mathscr{H}$ for any ample invertible sheaf $\mathscr{H}$ on X. Especially we may assume that $\kappa(\mathscr{L}^N(-D))=\dim(Y)$, that

$$R^q\tau_*\left(\omega_{Y/X}\otimes\mathscr{L}^i\left(-\left[\frac{i\cdot D}{N}\right]\right)\right)\otimes\omega_X$$

is generated by its global sections and that

$$H^p\left(X, R^q\tau_*\left(\omega_{Y/X}\otimes\mathscr{L}^i\left(-\left[\frac{i\cdot D}{N}\right]\right)\otimes\omega_X\right)=0$$

for all $p>0$ and $q\geqq 0$. In this situation the Leray-spectral-sequence gives

$$H^p\left(Y, \mathscr{L}^i\left(-\left[\frac{i\cdot D}{N}\right]\right)\otimes\omega_Y\right)=H^0\left(X, R^p\tau_*\left(\omega_{Y/X}\otimes\mathscr{L}^i\left(-\left[\frac{i\cdot D}{N}\right]\right)\otimes\omega_X\right)\right).$$

Hence from (4.5) both sides must be zero and we obtain (4.6).

(4.7) **Corollary.** *Using the notations from* (4.6) *we assume* $\mathscr{L}$ *to be arithmetically positive itself. Then there exists a polynomial* $P(l)$ *of degree at most* $\dim(Y)-1$ *such that for all* $l\geqq 0$

$$h^1(X, \tau_*\mathscr{L}^l)\leqq P(l).$$

Proof. Let H be a very ample divisor on Y such that $\mathcal{O}_Y(H)\otimes\omega_Y^{-1}$ is ample. From (4.5) (or from the usual "Kodaira-Vanishing-Theorem") we know that $H^q(Y, \mathscr{L}^l(H))$ is zero for $q>0$. $h^0(H, \mathscr{L}^l(H)|_H)$ is bounded by a polynomial of degree $\dim(H)=\dim(Y)-1$ and using the exact sequence

$$0\to\mathscr{L}^l\to\mathscr{L}^l(H)-\mathscr{L}^l(H)|_H\to 0$$

we find $h^1(Y, \mathscr{L}^l)$ to be bounded by the same polynomial. The Leray-spectral-sequence gives an inclusion $H^1(X, \tau_*\mathscr{L}^l)\to H^1(Y, \mathscr{L}^l)$.

§5. The Main Lemma and the proof of (0.4)

(5.1) Let $\underline{d}$ be a n-tuple of natural numbers satisfying $d_1\geqq d_2\geqq\ldots\geqq d_n$. Using the notations and assumptions made in (2.6) we choose a birational morphism $\tau: \mathbb{P}\to(\mathbb{P}^1)^n$ such that $\mathbb{P}$ is nonsingular and projective and such that $\mathscr{M}=\operatorname{Im}(\tau^*\mathscr{M}'\to\mathcal{O}_{\mathbb{P}})$ is invertible (see (3.2)). Of course we can choose τ to be an isomorphism on $\tau^{-1}((\mathbb{P}^1)^n-\{\zeta_1, \ldots, \zeta_M\})$.

Again in this section we have to assume that $\underline{a}^{(\mu)}=\underline{a}$ for all μ and we want $(\underline{a}, t_\mu)$ to satisfy (1.6) and (3.4, i)).

In order to distinguish the different factors of the $(\mathbb{P}^1)^n$ we number them: $(\mathbb{P}^1)^n=\mathbb{P}_1^1\times\mathbb{P}_2^1\times\ldots\times\mathbb{P}_n^1$. For every subset $I=\{i_1, \ldots, i_s\}\subseteq\{1, \ldots, n\}$ we write

$$\pi_I': \mathbb{P}_1^1\times\mathbb{P}_2^1\times\ldots\times\mathbb{P}_n^1\to\mathbb{P}_{i_1}^1\times\mathbb{P}_{i_2}^1\times\ldots\times\mathbb{P}_{i_s}^1=(\mathbb{P}^1)^I$$

for the projection, $\pi_I=\pi_I'\cdot\tau$, $\pi_k'=\pi_{\{k\}}'$ and $\pi_k=\pi_{\{k\}}$.

(5.2) *Notation.* An open subvariety of $\mathbb{P}$ is called a *product open set* if it is of the form $\tau^{-1}(U_1 \times U_2 \times \ldots \times U_n)$ for non-empty open subvarieties $U_\nu \subseteq \mathbb{P}^1_\nu$.

Instead of saying that some sheaf is weakly positive with respect to the set of product open sets, we just say that the sheaf is *weakly positive over some product open set.*

In §7 and §8 we are going to prove (remember that we assume $\mathcal{M}(d_1, \ldots, d_n)$ to have a non trivial section!):

(5.3) **Main Lemma.** *In addition to* (5.1) *we assume that for* $\gamma \neq \mu$ *and* $\nu = 1, \ldots, n$ *we have* $\zeta_{\gamma,\nu} \neq \zeta_{\mu,\nu}$. *Let* $M' = \operatorname{Max}\{M, 2\}$ *and*

$$d_i' \geqq d_i + (M'-2) \cdot \sum_{j=i+1}^{n} d_j.$$

Then the invertible sheaf $\mathcal{M}(d_1', \ldots, d_n')$ *is weakly positive over some product open set.*

(5.4) **Corollary.** *Under the assumptions of* (5.3) *we assume that* $(\underline{d}', \underline{a}, t_\mu)$ *satisfies* (1.8) *and* (3.4, ii)) *for all* μ.

Let $\mathcal{L}'^{(d')}$ *be the sheaf introduced in* (2.6) *and* $\tau' \colon \mathbb{P}' \to (\mathbb{P}^1)^n$ *be a birational morphism of nonsingular projective varieties, such that* τ' *is an isomorphism over some product open set, the fibres of* $\pi'_\nu \cdot \tau'$ *are normal crossing divisors for all* ν *and* $\mathcal{L} = \operatorname{Im}(\tau'^* \mathcal{L}'^{(d)} \to \mathcal{O}_{\mathbb{P}'})$ *is invertible. Then* $\mathcal{L}(d_1', \ldots, d_n')$ *is arithmetically positive.*

We start the proof of (5.4) with:

(5.5) *Claim.* $\mathcal{L}(d_1', \ldots, d_n')$ is weakly positive over some product open set.

Proof. If one of the $t_\mu = 0$ we can leave away the point ζ_μ and replace M by $M-1$. Therefore we can assume that

$$0 < t = \operatorname{Min}\{t_1, \ldots, t_M\}$$

and that there exists some $\gamma > 0$ such that $\gamma \cdot t \geqq t_\mu$ for $\mu = 1, \ldots, M$. (2.7) implies

$$\sum_{\nu=1}^{n} a_\nu \cdot d_\nu' \geqq \sum_{\nu=1}^{n} a_\nu \cdot d_\nu \geqq t_\mu + t \qquad \text{for all } \mu.$$

The polynomial $H = \prod_{\mu=1}^{M} \prod_{\nu=1}^{n} (\eta_{\mu,\nu} \cdot X_\nu - \lambda_{\mu,\nu} \cdot Y_\nu)^{d'_\nu}$ has multidegree $M \cdot \underline{d}'$ and a zero of type $(\underline{a}, t_\mu + t)$ at ζ_μ. Moreover, in ζ_μ, H is in the ideal $\ell_{\zeta_\mu}^{(\underline{d}', \underline{a}, t_\mu + t)}$.

For $\alpha > 0$ given, (4.3, 5)) applied to (5.3) shows the existence of some $\beta > 0$ such that

$$\mathcal{M}'(d_1', \ldots, d_n')^{\alpha \cdot \beta} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1', \ldots, d_n')^{\beta} = \tau_*(\mathcal{M}(d_1', \ldots, d_n')^{\alpha \cdot \beta} \otimes \mathcal{O}_{\mathbb{P}}(d_1', \ldots, d_n')^{\beta})$$

is generated by its global sections over a product open set.

These sections of multidegree $(\alpha \cdot \beta + \beta) \cdot \underline{d}'$ have zeros of type $(\underline{a}, \alpha \cdot \beta \cdot t_\mu)$ at ζ_μ. Multiplying them with the fixed polynomial $H^{\gamma \cdot \beta}$, we get sections, generating the sheaf over some product open set, of

$$\bigcap_{\mu=1}^{M} \ell_{\zeta_\mu}^{((\alpha+\gamma+1)\cdot\beta\cdot\underline{d}',\, \underline{a},\, (\alpha+\gamma)\cdot\beta\cdot t_\mu+\gamma\cdot\beta\cdot t)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)^{(\alpha+M\cdot\gamma+1)\cdot\beta}.$$

This is contained – isomorphic over a product open set – in

$$\mathcal{L}'^{(\underline{d}')}(d'_1, \ldots, d'_n)^{\alpha\cdot\beta} \otimes \mathcal{F}^\beta$$

for

$$\mathcal{F} = (\mathcal{L}'^{(\underline{d}')})^{1+\gamma} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)^{1+M\cdot\gamma}.$$

τ'^* being right exact, we find that $\mathcal{L}(d'_1, \ldots, d'_n)^{\alpha\cdot\beta} \otimes (\tau'^* \mathcal{F})^\beta$ is generated by its global sections over a product open set, and hence it is weakly positive over this set. (5.5) now follows from (4.3, 2)).

(5.6) *Claim.* If $M=1$, then (5.4) is true.

Proof. From (2.3, ii)) we know that in this situation

$$\mathcal{L}'^{(\underline{d}')}(d'_1, \ldots, d'_n) = \ell_{\zeta_1}^{(\underline{d}', \underline{a}, t_1)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)$$

is generated by its global sections and hence the same is true for $\mathcal{L}(d'_1, \ldots, d'_n)$.

In the general case we prove (5.4) by induction on n. For $n=1$ there is nothing to prove. For simplicity we write $\tau=\tau'$, $\mathbb{P}=\mathbb{P}'$ and $\pi_k=\pi'_k\cdot\tau$. Let C be any curve in $\mathbb{P}$. We have to show that

$$\deg_C(\mathcal{L}(d'_1, \ldots, d'_n)|_C) \geqq 0. \tag{5.7}$$

Case (5.8). $\pi_\nu(C)=\mathbb{P}^1_\nu$ for all ν.

This just means that C meets every product open set U. If not, C would be contained in one of the irreducible components of $\mathbb{P}-U$, which are components of $\pi_\nu^{-1}(P)$ for some $P\in\mathbb{P}^1_\nu$. From (4.3, 6)) applied to (5.5) we find for all $\alpha>0$ some $\beta>0$ such that

$$\mathcal{L}(d'_1, \ldots, d'_n)^{\alpha\cdot\beta} \otimes \mathcal{O}_{\mathbb{P}}(\beta, \ldots, \beta)$$

has a section, which is non-zero in some points of C. Hence

$$\alpha\cdot\beta\cdot\deg_C(\mathcal{L}(d'_1, \ldots, d'_n)|_C) \geqq -\beta\cdot\deg_C(\mathcal{O}_C(1, \ldots, 1))$$

for all $\alpha>0$ and we find (5.7).

Case (5.9). Using the notation introduced in (2.5) assume that $\tau(C)$ is contained in the support of $C_{\zeta_s}^{(\underline{d}', \underline{a}, t_s)}$ for some s.

(2.8) guarantees that $\tau(C)$ does not meet the support of $C_{\zeta_\mu}^{(\underline{d}', \underline{a}, t_\mu)}$ for $\mu \neq s$ and

$$\deg_C(\mathcal{L}(d'_1, \ldots, d'_n)|_C) = \deg_C(\operatorname{Im}(\tau^* \ell_{\zeta_s}^{(\underline{d}', \underline{a}, t_s)} \to \mathcal{O}_{\mathbb{P}}) \otimes \mathcal{O}_C(d'_1, \ldots, d'_n)).$$

We get (5.7) directly from (5.6).

Case (5.10). For some $k \in \{1, \ldots, n\}$ one has $\pi_k(C) = P \in \mathbb{P}^1_k$ and

$$\tau(C) \not\subset \operatorname{Supp}(C_{\zeta_\mu}^{(\underline{d}', \underline{a}, t_\mu)})$$

for any μ.

τ induces a morphism $C \to (\mathbb{P}^1)^n$ which factors over

$$j\colon C \to \pi_k'^{-1}(P) \simeq (\mathbb{P}^1)^{n-1}.$$

Over some open subset of C

$$j^* \mathscr{L}'^{(\underline{d}')}(d_1', \ldots, d_n')|_{\pi_k'^{-1}(P)} \to \mathcal{O}_C(d_1', \ldots, d_n')$$

is surjective and the image is nothing but $\mathscr{L}(d_1', \ldots, d_n')|_C$. As we have seen in (2.9) the sheaf $\mathscr{L}'^{(\underline{d})}|_{\pi_k^{-1}(P)}$ is again of the form $\bigcap_{\mu=1}^{M} \mathcal{I}_{\zeta_\mu'}^{(\underline{\delta}, \underline{a}, \tau_\mu)}$ (on $(\mathbb{P}^1)^{n-1}$) and $(\mathscr{L}'^{(\underline{d})}(d_1, \ldots, d_n)|_{\pi_k'^{-1}(P)})^N$ has a nontrivial section. Replacing $\mathscr{L}(d_1', \ldots, d_n')$ by some power, we may assume that $(\underline{\alpha}, \tau_\mu)$ and $(\underline{\delta}, \underline{\alpha}, \tau_\mu)$ satisfy the numerical conditions (1.6), (1.8) and (3.4) and that $N=1$.

Let Y be the proper transform of $\pi_k'^{-1}(P)$ in $\mathbb{P}$.

By the induction hypothesis we know that $\mathscr{L}(d_1', \ldots, d_n')|_Y$ is arithmetically positive, and we get

$$\deg_C((\mathscr{L}(d_1', \ldots, d_n')|_Y)|_C) = \deg_C(\mathscr{L}(d_1', \ldots, d_n')|_C) \geqq 0.$$

(5.11) *The proof that* (5.3) *implies Theorem* (0.4).

As we have seen in §1 and §2, the existence of f in (0.4) guarantees that $H^0(\mathbb{P}, \mathscr{M}(d_1, \ldots, d_n)) \neq 0$. The conclusion of (0.4) remains the same if we replace $\underline{d}$ by $N \cdot \underline{d}$ and t_μ by $N \cdot t_\mu$ (see (2.4, i)) and we can assume that (1.6), (1.8) and (3.4) are fulfilled. Choosing $\tau'\colon \mathbb{P}' \to (\mathbb{P}^1)^n$ as in (5.4) – which is possible by (2.5) – we find that $\mathscr{L}(d_1', \ldots, d_n')$ is arithmetically positive for

$$d_i' = d_i + \sum_{j=i+1}^{n} (M'-2) \cdot d_j.$$

By (3.5) and (4.7) we know that for some polynomial $P'(N)$ of degree $n-1$

$$h^1((\mathbb{P}^1)^n, \mathscr{L}'^{(\underline{d}')}(d_1', \ldots, d_n')^N) \leqq P'(N) \quad \text{for all} \quad N > 0.$$

Let $C^{(N)} = \bigoplus_{\mu=1}^{M} C_{\zeta_\mu}^{(N \cdot \underline{d}', \underline{a}, N \cdot t_\mu)}$. From (2.8) we get the exact sequence

$$0 \to \mathscr{L}'^{(\underline{d}')}(d_1', \ldots, d_n')^N \to \mathcal{O}_{(\mathbb{P}^1)^n}(d_1', \ldots, d_n')^N \to C^{(N)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1', \ldots, d_n')^N \to 0.$$

(5.12) We find

$$\begin{aligned}
&(N \cdot d_1' + 1) \cdot \ldots \cdot (N \cdot d_n' + 1)\\
&\quad \leqq h^0((\mathbb{P}^1)^n, C^{(N)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d_1', \ldots, d_n')^N) + h^0((\mathbb{P}^1)^n, \mathscr{L}'^{(\underline{d}')}(d_1', \ldots, d_n')^N)\\
&\quad \leqq (N \cdot d_1' + 1) \cdot \ldots \cdot (N \cdot d_n' + 1) + P'(N).
\end{aligned}$$

For some $s \in \{1, \ldots, M\}$ we forget the points ζ_μ for $\mu \neq s$ i.e. we replace t_μ by 0 for $\mu \neq s$ and find using (5.12) that

$$h^0((\mathbb{P}^1)^n, C_{\zeta_s}^{(N \cdot \underline{d}', \underline{a}, t_s \cdot N)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)^N)$$

is bigger than

$$-h^0((\mathbb{P}^1)^n, \ell_{\zeta_s}^{(N \cdot \underline{d}', \underline{a}, N \cdot t_s)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)^N) + (N \cdot d'_1 + 1) \cdot (N \cdot d'_2 + 1) \cdot \ldots \cdot (N \cdot d'_n + 1).$$

Altogether we obtain for some $P(N)$ of degree $n-1$

$$N^n \cdot d'_1 \cdot \ldots \cdot d'_n \geqq \sum_{\mu=1}^{M} (d'_1 \cdot \ldots \cdot d'_n \cdot N^n - h^0((\mathbb{P}^1)^n, \ell_{\zeta_\mu}^{(N \cdot \underline{d}', \underline{a}, N \cdot t_\mu)} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(d'_1, \ldots, d'_n)^N)) + h^0((\mathbb{P}^1)^n, \mathcal{L}'^{(\underline{d}')}(d'_1, \ldots, d'_n)^N) - P(N).$$

Using (2.4) we get slightly more than requested in (0.4):

(5.13) *Under the assumptions made in* (0.4) *one has*

$$\sum_{\mu=1}^{M} d_1 \cdot \ldots \cdot d_n \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t_\mu)) \leqq \sum_{\mu=1}^{M} d'_1 \cdot \ldots \cdot d'_n \cdot \mathrm{Vol}(I(\underline{d}', \underline{a}, t)) \leqq d'_1 \cdot \ldots \cdot d'_n - \lim_{N \to \infty} N^{-n} \cdot h^0((\mathbb{P}^1)^n, \mathcal{M}'(d'_1, \ldots, d'_n)^N) \leqq d'_1 \cdot \ldots \cdot d'_n.$$

§ 6. Weak positivity for direct images of certain sheaves

(6.1) **Theorem.** *Let V and W be nonsingular quasi-projective varieties and $f: V \to W$ a surjective projective morphism. Assume that for some open subvariety W_0 of W*

a) *The restriction of f to $f^{-1}(W_0)$ is smooth.*

Then $f_* \omega_{V/W}$ is weakly positive over W_0. Moreover, if

b) *$W - W_0$ is a normal crossing divisor.*

Then $f_* \omega_{V/W}$ is locally free.

Let $r = \dim(V) - \dim(W)$. Studying the variation of Hodge-structures on $R^r f_* \mathbb{C}|_{f^{-1}(W_0)}$ and its degeneration along $W - W_0$, Kawamata proved (6.1) in [7], Theorem 5, under the additional conditions:

c) $f^{-1}(w)$ is connected for $w \in W$,

d) V and W are projective,

e) the local monodromies of $R^r f_* \mathbb{C}|_{f^{-1}(W_0)}$ around the components of $W - W_0$ are unipotent.

In fact, he obtained the weak positivity of $f_* \omega_{V/W}$ over W itself (using (4.3, 6)). Choosing good compactifications, we may always assume that d) is satisfied. Replacing W by a finite covering and V by the normalization of the fibre-product we may assume that each component of V satisfies c) and e) (see

[7] and [12], §4). (6.1) follows from a careful analysis of the behaviour of $f_* \omega_{V/W}$ under fibre-product and normalization ([3], Lemme 13 and [12], 3.2) and the properties (4.3, 4) and 5)) of weakly positive sheaves. The details can be found in [12], §4.

(6.2) **Corollary** (*see also* [12], 5.1). *Let $f: V \to W$ be a surjective projective morphism of nonsingular quasi-projective varieties, D an effective normal crossing divisor on V. Assume that for some invertible sheaf $\mathcal{L}$ and some $N > 0$ one has an inclusion $\mathcal{O}_V(D) \to \mathcal{L}^N$, surjective over $f^{-1}(U)$ for some non-trivial open subvariety U of W. Then, for all $i > 0$,*

$$f_*\left(\omega_{V/W} \otimes \mathcal{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)\right)$$

is weakly positive over an open subvariety Y of W.

Proof. If $i \geqq N$ we replace D by $v \cdot D$ and N by $v \cdot N$ for some $v \gg 0$, and hence we may assume that $i < N$. By (3.12) we are allowed to replace V by any "blowing up", as long as D remains a normal crossing divisor. In this way we can assume that $\mathcal{L}^N = \mathcal{O}_V(D')$ for a normal crossing divisor $D' \geqq D$. The natural inclusion

$$f_*\left(\omega_{V/W} \otimes \mathcal{L}^i\left(-\left[\frac{i \cdot D'}{N}\right]\right)\right) \to f_*\left(\omega_{V/W} \otimes \mathcal{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)\right)$$

is surjective over U and using (4.3, 1)) we may assume that $D' = D$. As we have seen in (3.13) we can find $\rho: T \to V$ such that T is nonsingular and such that $\rho_* \omega_T = \bigoplus_{i=0}^{N-1} \omega_V \otimes \mathcal{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)$. (6.1) applied to $f \cdot \rho$ gives the weak positivity of $f_* \rho_* \omega_{T/W}$ and using (4.3, 1)) the weak positivity of the direct summand $f_*\left(\omega_{V/W} \otimes \mathcal{L}^i\left(-\left[\frac{i \cdot D}{N}\right]\right)\right)$ over an open subvariety of W.

(6.3) We want to lift the positivity statement of (6.2) from the base to the total space, using (4.5), in the following situation:

Let $p: C \to T$ be a surjective, flat morphism of nonsingular projective varieties, $k = \dim(C) - \dim(T)$. Let $\mathcal{I}$ be an ideal-sheaf and $\mathcal{L}$ and $\mathcal{K}$ invertible sheaves on C. We fix open subvarieties X of C and Y of T and some $N > 0$. We assume:

a) $(\mathcal{L}^N \otimes \mathcal{I})^{\mu_1}$ is generated by its global sections over $X \cap p^{-1}(Y)$ for some $\mu_1 \gg 0$.

b) $\mathcal{K}^{\mu_2}$ is generated by its global sections for some $\mu_2 \gg 0$ and the "$\mathcal{L}$ dimension" $\kappa(\mathcal{K}|_{p^{-1}(t)}) = k$ for a point $t \in T$ in general position.

c) For $t \in T$ in general position let x and x' be two different points in $p^{-1}(t) \cap X$. The there exists an effective divisor A on $p^{-1}(t)$ such that $x \in A$, $x' \notin A$ and such that $\mathcal{O}_{p^{-1}(t)}(A)$ is numerically equivalent to $\mathcal{K}$.

d) Let $\tau': V \to C$ be any birational morphism such that

$$\tau'^{-1} \mathcal{I} \cdot \mathcal{O}_V = \mathcal{O}_V(-B)$$

for a normal crossing divisor B. Then $\tau'_* \omega_{V/C}\left(-\left[\frac{B}{N}\right]\right) \to \mathcal{O}_C$ is an isomorphism over X.

(6.4) **Proposition.** *Under the assumptions made in* (6.3), *there exists an open subvariety* $Y' \subseteq T$ *such that* $\mathcal{L} \otimes \mathcal{K}^{k+1} \otimes \omega_{C/T}$ *is weakly positive over* $p^{-1}(Y') \cap X$.

Proof. It is enough to find $Y' \subseteq Y$ and making X smaller if necessary we may assume that $X \subseteq p^{-1}(Y)$. Further replacing $\mathcal{I}$ by $\mathcal{I}^{\mu_1 \cdot \mu_2}$ and N by $\mu_1 \cdot \mu_2 \cdot N$, we may assume that $\mu_1 = 1$ and that μ_2 divides N.

By a) we can find a subsheaf $\mathcal{I}'$ of $\mathcal{I}$, such that $\mathcal{I}'|_X = \mathcal{I}|_X$ and such that $\mathcal{L}^N \otimes \mathcal{I}'$ is generated by its global sections on C. Since d) is independent of the blowing up chosen (3.2), one can define τ' such that $\tau'^{-1} \mathcal{I}' \cdot \mathcal{O}_V = \mathcal{O}(-B')$ for a divisor B' containing B. Then $\tau'^* \mathcal{L}^N \otimes \mathcal{O}_V(-B')$ is generated by its global sections over V and by d) one has $\tau'_* \omega_V\left(-\left[\frac{B'}{N}\right]\right) = \omega_C$ over X.

Let s be a general section of $\tau'^* \mathcal{L}^N \otimes \tau'^* \mathcal{K}^{N \cdot (k+1)} \otimes \mathcal{O}_V(-B')$. $V(s)$ is reduced, intersects B' properly and $V(s) \cup B'$ is a normal crossing divisor (use the theorem of Bertini [6], III 10.9). We have therefore $\left[\frac{B' + V(s)}{N}\right] = \left[\frac{B'}{N}\right]$ and from (6.2)

$$p_* \tau'_* \left(\tau'^* \mathcal{L} \otimes \mathcal{K}^{k+1}) \otimes \mathcal{O}_V\left(-\left[\frac{B'}{N}\right]\right) \otimes \omega_{V/T}\right)$$

$$= p_*(\mathcal{L} \otimes \mathcal{K}^{k+1}) \otimes \omega_{C/T} \otimes \tau'_* \left(\omega_{V/C}\left(-\left[\frac{B'}{N}\right]\right)\right)$$

is weakly positive over some $Y' \subseteq Y \subseteq T$. We have natural maps

$$p^* p_*(\mathcal{L} \otimes \mathcal{K}^{k+1}) \otimes \omega_{C/T} \otimes \tau'_* \left(\omega_{V/C}\left(-\left[\frac{B'}{N}\right]\right)\right) \xrightarrow{\alpha_1} \mathcal{L} \otimes \mathcal{K}^{k+1} \otimes \omega_{C/T}$$

$$\otimes \tau'_* \left(\omega_{V/C}\left(-\left[\frac{B'}{N}\right]\right)\right) \xrightarrow{\alpha_2} \mathcal{L} \otimes \mathcal{K}^{k+1} \otimes \omega_{C/T}.$$

From (4.3, 4)) we get the weak positivity of the image of $\alpha = \alpha_2 \cdot \alpha_1$ over $p^{-1}(Y')$. α_2 being an isomorphism over X (6.4) follows from

(6.5) *Claim.* Making Y' smaller – if necessary – α_1 is surjective over $p^{-1}(Y') \cap X$.

Proof. (6.5) is just saying that for $t \in Y'$ in general position the sheaf

$$\mathcal{L}_t \otimes \mathcal{K}_t^{k+1} \otimes \omega_{C_t} \otimes \tau'_{t*} \left(\omega_{V_t/C_t}\left(-\left[\frac{B'_t}{N}\right]\right)\right)$$

is generated by its global sections over X_t, where the index "t" just denotes the restriction to the fibre $C_t = p^{-1}(t)$ and $V_t = \tau'^{-1}(C_t)$.

Let $x \in X_t$ be a point and $\rho: C'_t \to C_t$ the birational morphism obtained by blowing up the maximal ideal of x (3.1). Let E_x be the reduced exceptional divisor of ρ. From [6], II, Ex. 8.5 we know that $\omega_{C'_t/C_t} = \mathcal{O}_{C'_t}((k-1) \cdot E_x)$.

(6.6) *Claim.* $\rho^* \mathcal{K}_t^{k+1}(-k \cdot E_x)$ is arithmetically positive and

$$\kappa(\rho^* \mathcal{K}_t^{k+1}(-k \cdot E_x)) = k.$$

Proof. Since $\kappa(\mathcal{K}_t) = \kappa(\rho^* \mathcal{K}_t) = k$, it is enough to show that $\rho^* \mathcal{K}_t(-E_x)$ is arithmetically positive.

In fact, $\kappa(\rho^* \mathcal{K}_t) = k$ is equivalent to the fact that $\rho^* \mathcal{K}_t^\beta$ contains an ample subsheaf for some $\beta \gg 0$ (see for example [12], 6.3). Hence

$$\rho^* \mathcal{K}_t^\rho \otimes (\rho^* \mathcal{K}_t^k(-k \cdot E_x))^\rho$$

contains an ample subsheaf. Now let Γ be any curve in C'_t. If $\rho(\Gamma) = x$, then

$$\deg_\Gamma(\rho^* \mathcal{K}_t(-E_x)) = \deg_\Gamma(\mathcal{O}_{C'_t}(-E_x)) > 0.$$

If $\rho(\Gamma) \neq x$, then there exists by c) an effective divisor A_t, numerically equivalent to $\mathcal{K}_t$ such that $\rho(\Gamma) \nsubseteq A_t$ and $x \in A_t$. Hence $\deg_\Gamma(\rho^* \mathcal{K}_t(-E_x)) = \Gamma \cdot (\rho^* A_t - E_x) \geqq 0$ since $\rho^* A_t - E_x$ is an effective divisor not containing Γ.

In order to finish the proof of (6.5) we may assume that V_t factors over $\eta: V_t \to C'_t$, i.e. $\tau'_t = \rho \cdot \eta$. Write $F_x = \eta^* E_x$. The sheaf

$$\tau'^*_t(\mathcal{L}_t \otimes \mathcal{K}_t^{k+1})^N \otimes \mathcal{O}_{V_t}(-(N \cdot k) \cdot F_x - B'_t)$$

is arithmetically positive and of "$\mathcal{L}$ dimension" k. We hence can apply (4.6) and we get a surjection

$$H^0(C_t, \mathcal{L}_t \otimes \mathcal{K}_t^{k+1} \otimes \omega_{C_t}) \to H^0(C_t, \mathcal{F})$$

where

$$\mathcal{F} = \operatorname{Coker}\left(\tau'_{t*}\left(\omega_{V_t/C_t}\left(-k \cdot F_x - \left[\frac{B'_t}{N}\right]\right)\right) \to \mathcal{O}_{C_t}\right) \otimes \mathcal{L}_t \otimes \mathcal{K}_t^{k+1} \otimes \omega_{C_t}.$$

By the assumption d), $\mathcal{F}|_X$ has its support in the point x. If $\mathcal{F}|_X$ were zero,

$$\omega_{V_t/C_t}\left(-k \cdot F_x - \left[\frac{B'_t}{N}\right]\right)\bigg|_{\tau_t'^{-1}(X_t)} = \mathcal{O}_{\tau_t'^{-1}(X_t)}(E)$$

for an effective divisor E with $\tau'_t(E) = x$.

We would obtain a relative canonical divisor of the form

$$E + k \cdot F_x + \left[\frac{B'_t}{N}\right]$$

which contains the proper transform of E_x under η with the multiplicity at least k, contradicting the description of $\omega_{C'_t/C_t}$ given above.

Hence for some non-trivial sky-scraper sheaf $\mathcal{F}_x$, concentrated in x we get a surjection

$$H^0(C_t, \mathcal{L}_t \otimes \mathcal{K}_t^{k+1} \otimes \omega_{C_t}) \to H^0(C_t, \mathcal{F}_x)$$

and hence (6.5).

§ 7. Reformulation of the Main Lemma and another covering construction

In § 5 we formulated the Main Lemma under the conditions introduced in Theorem 0.4. In this section, we consider a slightly more general situation.

(7.1) Using the assumptions and notations introduced in (5.1) and (5.2), we define

$$M_\nu = \operatorname{Max}\{2, |\{\zeta_{\mu,\nu};\ \mu=1, \ldots, M\}|\} - 2.$$

We consider $(n+1)$-tuples $(\underline{a}^{(\mu)}, t_\mu) \in \mathbb{N}^{n+1}$.

If two points have the same ν-th coordinate, i.e. if $\zeta_{\mu,\nu} = \zeta_{\gamma,\nu}$, for $\mu \neq \gamma$, we assume

$$\frac{t_\mu}{a_\nu^{(\mu)}} = \frac{t_\gamma}{a_\nu^{(\gamma)}}.$$

We assume moreover that the $(n+1)$-tuple satisfies (1.6). Under this condition, one knows ((1.9) and (3.5, i)) that $\mathcal{M}'^N$ is full for all $N > 0$.

(7.2) Generalizing the notations introduced in (0.7), we write $\mathcal{F}\left(\frac{a_1}{r}, \ldots, \frac{a_n}{r}\right)$ in the following case:

a) all the a_i are integers and if $a_i \neq 0$, there exists an invertible sheaf $\mathcal{H}_i$ such that $\mathcal{O}_X(0, \ldots, 1, \ldots, 0) = \mathcal{H}_i^r$.

b) Under this condition $\mathcal{F}\left(\frac{a_1}{r}, \ldots, \frac{a_n}{r}\right) = \mathcal{F} \otimes \mathcal{H}_1^{a_1} \otimes \ldots \otimes \mathcal{H}_n^{a}$.

(7.3) **Main Lemma.** *Under the assumptions and notations of* (7.1), *assume that for* $d_1 \geqq d_2 \geqq \ldots d_n \geqq 2$, *the sheaf* $\mathcal{M}(d_1, \ldots, d_n)$ *is weakly positive with respect to* $\mathcal{U}_k$. *Then*

$$\mathcal{M}(d_1 + M_1 \cdot d_{k+1}, \ldots, d_k + M_k \cdot d_{k+1}, d_{k+1}, \ldots, d_n)$$

is weakly positive with respect to $\mathcal{U}_{k+1}$, *where* $\mathcal{U}_k$ *is defined by*

$$\mathcal{U}_k = \{\tau^{-1}(U_1 \times \ldots \times U_{k-1} \times V_k);\ U_i \subseteq \mathbb{P}_i^1 \text{ open and } V_k \subseteq \mathbb{P}_k^1 \times \ldots \times \mathbb{P}_n^1 \text{ open}\}.$$

(7.4) **Corollary.** *If* $\mathcal{M}(d_1, \ldots, d_n)$ *has a non trivial section, then* $\mathcal{M}(d'_1, \ldots, d'_n)$ *is weakly positive over some product open set, where* $d'_i = d_i + M_i \cdot \sum_{j=i+1}^{n} d_j$.

Proof of the corollary. For $k=1$, $\mathcal{U}_1$ is nothing but the set of all open subvarieties $\tau^{-1}(V)$ for any open V in $(\mathbb{P}^1)^n$. Especially, $\mathcal{U}_1$ contains all open sets in $\mathbb{P} - \tau^{-1}(\{\zeta_1, \ldots, \zeta_M\})$. The given section induces an injection $\mathcal{O}_{\mathbb{P}} \to \mathcal{M}(d_1, \ldots, d_n)$. So $\mathcal{M}(d_1, \ldots, d_n)$ is weakly positive with respect to $\mathcal{U}_1$. Now, (7.3) is just the induction step from k to $k+1$.

For $k+1 = n$, one sees that $\mathcal{U}_n$ is by definition the set of all product open sets. Hence, (7.4) is true.

In order to prove (7.3), we need the following proposition, which is a kind of "semi-stable reduction" in our special situation.

(7.5) **Proposition.** *For* $\nu = 1, \ldots, n$, *we choose a point* $z_\nu \in \mathbb{P}_\nu^1$ *in general position and write* $U_\nu = \mathbb{P}_\nu^1 - \{\zeta_{1,\nu}, \ldots, \zeta_{M,\nu}, z_\nu\}$. *Take an integer* $r > 0$. *Then, there exists a* *commutative diagram of non singular varieties*

$$\begin{array}{ccccc} & & C & \xrightarrow{p} & T \\ & \overset{\delta}{\swarrow} & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\ C' = C_1 \times \ldots \times C_n & \overset{\rho}{\searrow} & & & \\ & & \mathbb{P}_1^1 \times \ldots \times \mathbb{P}_n^1 & \xrightarrow{\pi'} & \mathbb{P}_{k+1}^1 \times \ldots \times \mathbb{P}_n^1 \end{array}$$

such that

1) $\rho_\nu: C_\nu \to \mathbb{P}_\nu^1$ *is a finite covering,* $\rho = \rho_1 \times \ldots \times \rho_n$, $\pi' = \pi'_{\{k+1,\ldots,n\}}$ *the projection,* δ *is birational and* σ' *is generically finite.*

2) *There is an invertible sheaf* $\mathcal{N}$ *on* C, *an inclusion* $\sigma^{-1}\mathcal{M}'^l \cdot \mathcal{O}_C \to \mathcal{N}^l$, *for all* $l > 0$, *which is surjective over* $\sigma^{-1}(U_1 \times \ldots \times U_n)$ *and such that the following diagram*

$$\begin{array}{ccccc} \mathcal{M}'^l & \longrightarrow & \sigma_* \mathcal{N}^l & \longrightarrow & \mathcal{M}'^l \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ \mathcal{O}_{(\mathbb{P}^1)^n} & \longrightarrow & \sigma_* \mathcal{O}_C & \longrightarrow & \mathcal{O}_{(\mathbb{P}^1)^n} \end{array}$$

is commutative (*as in* (3.8) *the second row consists of the natural inclusion and the trace map*).

3) *r divides the degree of ρ_ν. In other words, for all* $\varepsilon_\nu \in \{0, 1\}$, *there is a sheaf*

$$\mathcal{O}_C\left(\frac{\varepsilon_1}{r}, \ldots, \frac{\varepsilon_n}{r}\right).$$

4) *p is flat with reduced fibers.*

5) σ *is étale over* $U_1 \times \ldots \times U_n$.

6) *There is a natural inclusion*

$$\omega_{C/T} \to \mathcal{O}_C\left(M_1 + \frac{1}{r}, \ldots, M_k + \frac{1}{r}, 0, \ldots, 0\right)$$

which is surjective over $\sigma^{-1}(U_1 \times \ldots \times U_n)$.

The construction depends on r.

(7.6) *Claim. One may assume that* $r = 1$.

Proof. For a fix $\nu \in \{1, \ldots, n\}$, we set $S_\nu = \{\zeta_{\mu_1,\nu}, z_\nu\}$ if $\zeta_{\mu,\nu} = \zeta_{\mu_1,\nu}$ for $\mu = 1, \ldots, M$ and $S_\nu = \{\zeta_{\mu_2,\nu}, \zeta_{\mu_1,\nu}\}$ for two distinct points $\zeta_{\mu_2,\nu}$ and $\zeta_{\mu_1,\nu}$ otherwise. We can find cyclic coverings $h_\nu: \bar{\mathbb{P}}_\nu^1 \to \mathbb{P}_\nu^1$ of degree r, where $\bar{\mathbb{P}}_\nu^1 = \mathbb{P}^1$, which is totally branched over S_ν and étale outside of S_ν. Set $h = h_1 \times \ldots \times h_n$.

For arbitrary points $p_\nu \in \bar{\mathbb{P}}_\nu^1$, one has

$$h^* \mathcal{O}_{(\mathbb{P}^1)^n}(\varepsilon_1, \ldots, \varepsilon_n) = \mathcal{O}_{(\mathbb{P}^1)^n}(r \cdot \varepsilon_1 \cdot p_1, \ldots, r \cdot \varepsilon_n \cdot p_n),$$

for $\varepsilon_i \in \{0, 1\}$. Hence, 3) is true for every finite covering factorizing over h. The set $h^{-1}(\{\zeta_{\mu,\nu};\ \mu = 1, \ldots, M\} - S_\nu)$ consists of $\bar{M}_\nu = r \cdot M_\nu$ points. So, in 6), we may assume that $r = 1$.

By construction, h is étale over $U_1 \times \ldots \times U_n$. So, in 5), we also may assume that $r=1$.

Now, let $\{\zeta'_\gamma;\ \gamma=1,\ldots,\bar{M}\}$, be the points of $h^{-1}(\{\zeta_1,\ldots,\zeta_M\})$. One has

$$h^*\mathcal{M}' \subseteq \bigcap_{\gamma=1}^{\bar{M}} m_{\zeta'_\gamma}^{(\underline{b}^{(\gamma)},\tau_\gamma)}$$

where, if $h(\zeta'_\gamma)=\zeta_\mu$, the $(n+1)$-tuple $(\underline{b}^{(\gamma)},\tau_\gamma)$ verifies $\tau_\gamma = r\cdot t_\mu$ and

$$b_\nu^{(\gamma)} = \begin{cases} a_\nu^{(\mu)} & \text{if } \zeta_{\mu,\nu}\in S_\nu \\ r\cdot a_\nu^{(\mu)} & \text{otherwise.}\end{cases}$$

By (3.8), one knows that there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}' & \longrightarrow & h_* \bigcap_{\gamma=1}^{\bar{M}} m_{\zeta'_\gamma}^{(\underline{b}^{(\gamma)},\tau_\gamma)} & \longrightarrow & \mathcal{M}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{(\mathbb{P}^1)^n} & \longrightarrow & h_*\mathcal{O}_{(\mathbb{P}^1)^n} & \longrightarrow & \mathcal{O}_{(\mathbb{P}^1)^n}. \end{array}$$

Hence, in 2), we can also assume that $r=1$.

(7.7) We define a non singular cyclic covering $\rho_\nu\colon C_\nu \to \mathbb{P}_\nu^1$ by the following condition:

ρ_ν is ramified in $\zeta_{\mu,\nu}$ with the ramification order $\dfrac{N\cdot a_\nu^{(\mu)}}{t_\mu}$, where N is the smallest common multiple of $\dfrac{t_\mu}{a_\nu^{(\mu)}}$, $\mu=1,\ldots,M$, $\nu=1,\ldots,n$.

This is possible by assumption ((7.1) and (1.6)). Moreover, one may assume that ρ_ν is étale over U_ν. In fact, choose a divisor $D_1=\sum \dfrac{t_\mu}{a_\nu^{(\mu)}}\cdot\zeta_{\mu,\nu}$, where the sum is taken over the different $\zeta_{\mu,\nu}$ on $\mathbb{P}_\nu^1$. Choose an η such that $D=D_1+\eta\cdot z_\nu$ is a divisor of degree $l\cdot N$ for $l\in\mathbb{N}$. Applying the construction (3.13) to $\mathcal{L}=\mathcal{O}_{(\mathbb{P}^1)}(l)$ and $\mathcal{L}^N=\mathcal{O}_{(\mathbb{P}^1)}(D)$, we get ρ_ν.

Set $\rho=\rho_1\times\ldots\times\rho_n$ and $C'=C_1\times\ldots\times C_n$. One has the inclusion

$$\rho^* m_{\zeta_\mu}^{(\underline{a}^{(\mu)},t_\mu)} \to \bigcap n_{\zeta'}^N$$

where the intersection is taken over all ζ' in $\rho^{-1}(\zeta_\mu)$ and $n_{\zeta'}$ is the maximal ideal of ζ'.

Set $\mathcal{N}' = \bigcap_{\zeta'\in\rho^{-1}(\{\zeta_1,\ldots,\zeta_M\})} n_{\zeta'}^N$. Then, one has the inclusion $\rho^*\mathcal{M}'\to\mathcal{N}'$, and from (3.8) we know that for every blowing up of C' making $\mathcal{N}'$ invertible, say $\delta\colon C\to C_1\times\ldots\times C_n$ and $\delta^{-1}\mathcal{N}'\cdot\mathcal{O}_C=\mathcal{N}$, we have the property (3.8) for $\sigma=\rho\cdot\delta$.

(7.8) *Claim.* Let $C'=X\times Y$ be the product of two smooth varieties of dimension k an $(n-k)$ respectively and $p'\colon C'\to Y$ be the projection. Let S be a finite subset of points $c=(x,y)$ of C'. For all c of S, one considers a parameter system near x in X and near y in Y, say $(x_1,\ldots,x_k)$ and $(x_{k+1},\ldots,x_n)$, such that the divisors which are globally defined by $x_i=0$ are smooth.

Define $\Delta_{c,v}$ by $V(x_v)\times Y$ if $v\leqq k$ and by $X\times V(x_v)$ if $v>k$. Similarly define $\Delta'_{c,v}$ by $V(x_v)$ if $v>k$.

Set

$$\Delta=\bigcup_{\substack{c\in S\\ v=1,\dots,n}}\Delta_{c,v}\qquad \Delta'=\bigcup_{\substack{c\in S\\ v=k+1,\dots,n}}\Delta'_{c,v}.$$

Denote by $\omega_{C'}\langle\Delta\rangle$ the sheaf of n-holomorphic forms with logarithmic poles along the normal crossing divisor Δ.

Then there exists a commutative diagram of smooth varieties

$$\begin{array}{ccc} C & \xrightarrow{\;p\;} & T\\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'}\\ C' & \xrightarrow[\;p'\;]{} & Y\end{array}$$

such that

1) *$\delta^{-1}\Delta$, $\delta'^{-1}\Delta'$ are normal crossing divisors $\delta^*\omega_{C'}\langle\Delta\rangle=\omega_C\langle\delta^{-1}\Delta\rangle$, $\delta'^*\omega_Y\langle\Delta'\rangle=\omega_T\langle\delta'^{-1}\Delta'\rangle$.*

2) *δ, δ' are birational, p is flat and has reduced fibers.*

3) *δ is an isomorphism outside of Δ.*

4) *For each point $c\in S$, $\delta^{-1}\mathscr{n}_c\cdot\mathcal{O}_C$ is invertible, where $\mathscr{n}_c$ is the sheaf of maximal ideal at c.*

Proof. Define $\delta'\colon T\to Y$ to be the blowing up of all the reduced points y such that there exists x such that $(x,y)\in S$. Call E_y the corresponding exceptional divisors. Define $\tilde{p}\colon C\to X\times T$ to be the blowing up of all reduced varieties (x,E_y) such that $(x,y)\in S$. Now we may assume the following conditions to be true.

a) $Y=\operatorname{Spec}(B)$, where B is a local ring at y.

b) $T=\operatorname{Spec}\left(B\left[\frac{x_{k+2}}{x_{k+1}},\dots,\frac{x_n}{x_{k+1}}\right]\right)$, E_y is defined by x_{k+1}.

c) $X=\operatorname{Spec}(A)$, where A is a local ring at x such that $(x,y)\in S$.

d) C is either i) $\operatorname{Spec}\left(A\otimes B\left[\frac{x_2}{x_1},\dots,\frac{x_{k+1}}{x_1},\frac{x_{k+2}}{x_{k+1}},\dots,\frac{x_n}{x_{k+1}}\right]\right)$ or

$$\text{ii) } \operatorname{Spec}\left(A\otimes B\left[\frac{x_1}{x_{k+1}},\dots,\frac{x_k}{x_{k+1}},\frac{x_{k+2}}{x_{k+1}},\dots,\frac{x_n}{x_{k+1}}\right]\right).$$

Now, $\delta'^{-1}\Delta'$ is defined by $x_{k+1}\cdot\left(x_{k+1}\cdot\frac{x_{k+2}}{x_{k+1}}\right)\cdot\ldots\cdot\left(x_{k+1}\cdot\frac{x_n}{x_{k+1}}\right)$ and is a normal crossing divisor as well as $\delta^{-1}\Delta$ which is defined by either i)

$$x_1\cdot\left(x_1\cdot\frac{x_2}{x_1}\right)\cdot\ldots\cdot\left(x_1\cdot\frac{x_{k+1}}{x_1}\right)\cdot\left(x_1\cdot\frac{x_{k+1}}{x_1}\cdot\frac{x_{k+2}}{x_{k+1}}\right)\cdot\ldots\cdot\left(x_1\cdot\frac{x_{k+1}}{x_1}\cdot\frac{x_n}{x_{k+1}}\right)$$

or by ii)

$$\left(x_{k+1}\cdot\frac{x_1}{x_{k+1}}\right)\cdot\ldots\cdot\left(x_{k+1}\cdot\frac{x_{k-1}}{x_{k+1}}\right)\cdot x_{k+1}\cdot\left(x_{k+1}\cdot\frac{x_{k+2}}{x_{k+1}}\right)\cdot\ldots\cdot\left(x_{k+1}\cdot\frac{x_n}{x_{k+1}}\right).$$

This computation shows moreover that $\delta^{-1} n_c \cdot \mathcal{O}_C$ is generated either by x_1 or by x_{k+1} and that $\delta'^{-1}\omega_Y\langle\Delta'\rangle$ is generated by

$$\frac{d(x_{k+1})}{x_{k+1}} \wedge \frac{d\left(x_{k+1}\cdot\frac{x_{k+2}}{x_{k+1}}\right)}{x_{k+1}\cdot\frac{x_{k+2}}{x_{k+1}}} \wedge \ldots \wedge \frac{d\left(x_{k+1}\cdot\frac{x_n}{x_{k+1}}\right)}{x_{k+1}\cdot\frac{x_n}{x_{k+1}}}$$

which is a local generator of $\omega_T\langle\delta'^{-1}\Delta'\rangle$.

One sees similarly that the inverse image by δ of a local generator of $\omega_{C'}\langle\Delta\rangle$ is a local generator of $\omega_C\langle\delta^{-1}\Delta\rangle$. Since p has obviously reduced fibers, the only point to show is that p is flat. So, one can assume B to be complete: $B=\mathbb{C}[\![x_{k+1},\ldots,x_n]\!]$. Then, one has $C=C_1\times\operatorname{Spec}(\mathbb{C}[\![x_{k+2},\ldots,x_n]\!])$, $T=T_1\times\operatorname{Spec}(\mathbb{C}[\![x_{k+2},\ldots,x_n]\!])$, $p=p_1\times$ identity.

Since T_1 is 1-dimensional, p_1 is flat, as well as p.

(7.9) We prove that (7.8) implies (7.5). We take

$$\rho'\colon Y=C_{k+1}\times\ldots\times C_n\to\mathbb{P}^1_{k+1}\times\ldots\times\mathbb{P}^1_n,$$

$$X=C_1\times\ldots\times C_k \quad\text{and}\quad S=\rho^{-1}\{\zeta_1,\ldots,\zeta_M,(z_1,\ldots,z_n)\}.$$

For a point $c\in S$, write $c=(c_1,\ldots,c_n)$ and define

$$\Delta_{c,\nu}=C_1\times\ldots\times c_\nu\times\ldots\times C_n \quad\text{and}\quad \Delta'_{c,\nu}=C_{k+1}\times\ldots\times c_\nu\times\ldots\times C_n.$$

We consider the divisors

$$D=\bigcup_{\mu=1}^{M}\bigcup_{\nu=1}^{n} D_{\mu,\nu}\cup D_z$$

where

$$D_{\mu,\nu}=\mathbb{P}^1_1\times\ldots\times\mathbb{P}^1_{\nu-1}\times\zeta_{\mu,\nu}\times\mathbb{P}^1_{\nu+1}\times\ldots\times\mathbb{P}^1_n$$

$$D_z=\bigcup_{\nu=1}^{n}\mathbb{P}^1_1\times\ldots\times\mathbb{P}^1_{\nu-1}\times z_\nu\times\mathbb{P}^1_{\nu+1}\times\ldots\times\mathbb{P}^1_n$$

and

$$D'=\bigcup_{\mu=1}^{M}\bigcup_{\nu=k+1}^{n}\pi' D_{\mu,\nu}\cup D_{z'}$$

where

$$D_{z'}=\bigcup_{\nu=k+1}^{n}\mathbb{P}^1_{k+1}\times\ldots\times z_\nu\times\ldots\times\mathbb{P}^1_n.$$

Since D and D' contain the discriminants of ρ and ρ', one has

$$\omega_{C'}\langle\rho^{-1}D\rangle=\rho^*\omega_{(\mathbb{P}^1)^n}\langle D\rangle=\rho^*\mathcal{O}_{(\mathbb{P}^1)^n}(M_1+1,\ldots,M_n+1)$$

and

$$\omega_{C_{k+1}\times\ldots\times C_n}\langle\rho'^{-1}D'\rangle=\rho'^*\omega_{(\mathbb{P}^1)^{n-k}}\langle D'\rangle=\rho'^*\mathcal{O}_{(\mathbb{P}^1)^{n-k}}(M_{k+1}+1,\ldots,M_n+1).$$

By construction, one has $\rho^{-1}D=\Delta$ and $\rho'^{-1}D'=\Delta'$. One has the inclusions

$$\omega_{C/T}\to\omega_C\langle\delta^{-1}\Delta\rangle\otimes p^*\omega_T\langle\delta'^{-1}\Delta'\rangle^{-1}$$

from (7.8, 2), and

$$\omega_{C/T} \to \delta^*(\omega_{C'}\langle\Delta\rangle \otimes p'^*\omega_{C_{k+1}\times\ldots\times C_n}\langle\Delta'\rangle^{-1}) = \sigma^*\mathcal{O}_{(\mathbb{P}^1)^n}(M_1+1,\ldots,M_k+1,0,\ldots,0)$$

from (7.8, 1). Hence, (7.5, 6) is satisfied and the other points of (7.5) are trivially fulfilled.

§ 8. The proof of the Main Lemma 7.3.

We return now to the situation considered in (7.1), (7.2) and (7.3).

(8.1) *Claim.* In order to prove (7.3) we may assume that $\mathcal{M}(d_1-1, d_2-1, \ldots, d_n-1)$ is generated by its global sections over some $U \in \mathcal{U}_k$.

Proof. For all $\alpha > 0$ we find – by (4.3, 5)) – some $\beta > 0$ such that $\mathcal{M}^{\alpha\cdot\beta}(\alpha\cdot\beta\cdot d_1+\beta, \ldots, \alpha\cdot\beta\cdot d_n+\beta)$ is generated by its global sections over some $U \in \mathcal{U}_k$.

If we consider $\bar{\mathcal{M}} = \mathcal{M}^{\alpha\cdot\beta}$ and $\bar{d}_\nu = \alpha\cdot\beta\cdot d_\nu + \beta + 1$ for $\nu = 1, \ldots, n$, the sheaf $\bar{\mathcal{M}}(\bar{d}_1, \ldots, \bar{d}_n)$ fulfills the condition we ask for in (8.1). If, however,

$$\begin{aligned}&\bar{\mathcal{M}}(\bar{d}_1+M_1\cdot\bar{d}_{k+1}, \ldots, \bar{d}_k+M_k\cdot\bar{d}_{k+1}, \bar{d}_{k+1}, \ldots, \bar{d}_n)\\ &\quad = \mathcal{M}(d_1+M_1\cdot d_{k+1}, \ldots, d_k+M_k\cdot d_{k+1}, d_{k+1}, \ldots, d_n)^{\alpha\cdot\beta}\\ &\quad\quad \otimes\mathcal{O}_{\mathbb{P}}(1,\ldots,1)^{\beta+1}\otimes\mathcal{O}_{\mathbb{P}}(M_1,\ldots,M_k,0,\ldots,0)^{\beta+1}\end{aligned}$$

is weakly positive with respect to $\mathcal{U}_{k+1}$, we obtain (7.3) from (4.3, 2)).

We choose a birational morphism $g: Z \to \mathbb{P}$, Z nonsingular, such that for a normal crossing divisor D the sheaf $g^*\mathcal{M}(d_1-1, \ldots, d_n-1) \otimes \mathcal{O}_Z(-D)$ is generated by its global sections. After (8.1) this is possible and moreover we can choose D in such a way that $g(D) \subseteq \mathbb{P} - U$ for some $U \in \mathcal{U}_k$. Of course, $g^*\mathcal{M}(d_1, \ldots, d_n) \otimes \mathcal{O}_Z(-D)$ is also generated by its global sections and moreover of "$\mathcal{L}$ dimension" n.

(8.2) *Definition.* We write

$$C_N = \operatorname{Supp}\left(\operatorname{Coker}\left(\tau_* g_*\left(\omega_{Z/(\mathbb{P}^1)^n} \otimes \mathcal{O}_Z\left(-\left[\frac{D}{N}\right]\right)\right) \to \mathcal{O}_{(\mathbb{P}^1)^n}\right)\right)$$

and $\delta_{k+1} = \operatorname{Min}\{N \geqq 1;\ (\mathbb{P}^1)^n - C_N$ contains a product open set$\}$.

(8.3) *Claim.* $\delta_{k+1} \leqq d_{k+1}$.

Proof. Of course $C_N \subseteq C_{N'}$ for $N' \leqq N$, and we have to prove that $(\mathbb{P}^1)^n - C_{d_{k+1}}$ does contain some product open set.

Assume this is not the case. Then we can find an irreducible component C of $C_{d_{k+1}}$ such that $(\mathbb{P}^1)^n - C$ does not contain any product open set, and we can choose C in such a way that $\dim(C)$ is maximal. Let us prove first, using the notation of (5.1):

(8.4) *Claim.* There exists a subset $I \subseteq \{1, \ldots, n\}$ such that

1) $\{k+1, \ldots, n\} \nsubseteq I$

2) $\pi'_I|_C$ is generically finite over $(\mathbb{P}^1)^I$ (this means just that $|I|=\dim(C)$ and $\pi'_I(C)=(\mathbb{P}^1)^I$).

Proof. We have the inclusions $C\subseteq\tau\cdot g(D)\subseteq(\mathbb{P}^1)^n-\tau(U)$ for some $U\in\mathscr{U}_k$. Hence C is contained in the union of $\pi'^{-1}_{\{k,\dots,n\}}(B)$ with fibres of π'_ν for $\nu=1,\dots,k-1$ for some divisor B in $\mathbb{P}^1_k\times\dots\times\mathbb{P}^1_n$. C meets every product open set and, being irreducible, it must lie in $\pi'^{-1}_{\{k,\dots,n\}}(B)$. There exists some $\nu_1\in\{k+1,\dots,n\}$ such that $\dim(C)=\dim(\pi'_{\{1,\dots,n\}-\{\nu_1\}}(C))$. Otherwise, for $\nu=k+1,\dots,n$, one could write $C=C_\nu\times\mathbb{P}^1_\nu$ for $C_\nu\subseteq(\mathbb{P}^1)^{\{1,\dots,n\}-\{\nu\}}$ and C would be of the form $C'\times\mathbb{P}^1_{k+1}\times\dots\times\mathbb{P}^1_n$ for $C'\in(\mathbb{P}^1)^{\{1,\dots,k\}}$. For dimension reasons this is only possible if $B=P\times\mathbb{P}^1_{k+1}\times\dots\times\mathbb{P}^1_n$ for a point $P\in\mathbb{P}^1_k$, contradicting the assumption, that C meets every product open set. Using a similar argument, we can find – if necessary – step by step subsets

$$\{1,\dots,n\}\supseteq\{\nu_1,\dots,\nu_s\}\supseteq\{\nu_1,\dots,\nu_{s-1}\}\supseteq\dots\supseteq\{\nu_1\},$$

such that

$$\dim(C)=\dim(\pi'_{\{1,\dots,n\}-\{\nu_1,\dots,\nu_s\}}(C)) \qquad \text{for } s=2,\dots,n-\dim(C).$$

(8.5) We write $I'=\{1,\dots,n\}-I$ and – for a sufficiently general point $P\in(\mathbb{P}^1)^I$ we have $(\mathbb{P}^1)^{I'}\simeq\pi'^{-1}_I(P)\simeq\pi_I^{-1}(P)$ since we assumed τ^{-1} to be an isomorphism outside the points $\zeta_1,\dots,\zeta_M$. Let $Y=g^{-1}\tau^{-1}\pi'^{-1}_I(P)$ and denote $g\cdot\tau|_Y=j$.

We have:

A) $g^*\mathscr{M}(d_1,\dots,d_n)|_Y=j^*\mathcal{O}_{(\mathbb{P}^1)^{I'}}(d_{j_1},\dots,d_{j_s})=\mathcal{O}_Y(d_{j_1},\dots,d_{j_s})$ if

$$I'=\{j_1,\dots,j_s\}.$$

B) $\mathcal{O}_Y(d_{j_1},\dots,d_{j_s})\otimes\mathcal{O}_Y(-D|_Y)$ is generated by its sections and $D_Y=D|_Y$ is a normal crossing divisor.

C) $C_{d_{k+1}}|_{(\mathbb{P}^1)^{I'}}$ contains the isolated points $C|_{(\mathbb{P}^1)^{I'}}$.

We choose some $\eta\gg0$ such that $\eta\cdot d_{k+1}\geqq d_{j_\nu}$ for $\nu=1,\dots,s-1$. By the choice of I in (8.4) we have $j_s\geqq k+1$ and $d_{k+1}\geqq d_{j_s}$. Hence it follows from B) that $\mathcal{O}_Y(\eta\cdot d_{k+1},\eta\cdot d_{k+1},\dots,\eta\cdot d_{k+1},d_{k+1})\otimes\mathcal{O}_Y(-D_Y)$ is also generated by its global sections. Moreover, P being in general position, this sheaf has the "$\mathscr{L}$ dimension" s. (4.6) applied to the exact sequence

$$\begin{aligned}0&\to\mathcal{O}_{(\mathbb{P}^1)^{I'}}(\eta,\dots,\eta,1)\otimes\omega_{(\mathbb{P}^1)^{I'}}\otimes j_*\omega_{Y/(\mathbb{P}^1)^{I'}}\left(-\left[\frac{D_Y}{d_{k+1}}\right]\right)\\&\to\mathcal{O}_{(\mathbb{P}^1)^{I'}}(\eta,\dots,\eta,1)\otimes\omega_{(\mathbb{P}^1)^{I'}}\to\mathscr{F}\to0\end{aligned}$$

gives a surjection

$$H^0((\mathbb{P}^1)^{I'},\mathcal{O}_{(\mathbb{P}^1)^{I'}}(\eta-2,\dots,\eta-2,-1))\to H^0((\mathbb{P}^1)^{I'},\mathscr{F}).$$

By construction $\operatorname{Supp}(\mathscr{F})=C_{d_{k+1}}|_{(\mathbb{P}\ )^{I'}}$ contains some isolated points, hence $\mathscr{F}$ contains a sky-scraper sheaf and $H^0((\mathbb{P}^1)^{I'},\mathscr{F})\neq0$. However,

$$\mathcal{O}_{(\mathbb{P}^1)^{I'}}(\eta-2,\dots,\eta-2,-1)$$

can not have non-trivial sections and this contradiction proves (8.3).

(8.6) For a fixed $r' \geqq k+2$ we set $r = r'^2 \cdot \delta_{k+1}$ and we take

$$\begin{array}{ccccc} & & C & \xrightarrow{\quad p \quad} & T \\ & {\scriptstyle\delta}\swarrow & \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma'} \\ C' = C_1 \times \ldots \times C_n & \xrightarrow{\ \rho\ } & \mathbb{P}^1_1 \times \ldots \times \mathbb{P}^1_n & \xrightarrow{\quad \pi' \quad} & \mathbb{P}^1_{k+1} \times \ldots \times \mathbb{P}^1_n \end{array}$$

to be the diagram (depending on r) of (7.5). Let $U_1 \times \ldots \times U_n$ be the product open set over which σ is étale. The choice of δ_{k+1} allows us to choose a product open subset U of $(U_1 \times \ldots \times U_n) \cap ((\mathbb{P}^1)^n - C_{\delta_{k+1}})$.

We write $X = \sigma^{-1}(U)$. From (7.5, 2)) we have an inclusion

$$\sigma^{-1} \cdot \mathcal{M}'(d_1, \ldots, d_n) \cdot \mathcal{O}_C \to \mathcal{N}(d_1, \ldots, d_n)$$

and this is an isomorphism over X.

(8.7) *Claim.* Let $c_\nu = (M_\nu + k + 3) \cdot \delta_{k+1}$. Then there exists an open subvariety Y' in T such that

$$\mathcal{N}\left(d_1 + M_1 \cdot \delta_{k+1} + \frac{c_1}{r'}, \ldots, d_k + M_k \cdot \delta_{k+1} + \frac{c_k}{r'}, d_{k+1}, \ldots, d_n\right)$$

is weakly positive over $X \cap p^{-1}(Y')$.

Before we start to prove (8.7) we want to show *that* (8.7) *implies* (7.3): We may assume that $Y' = \sigma'^{-1}(V)$ for some open subvariety V of $\mathbb{P}^1_{k+1} \times \ldots \times \mathbb{P}^1_n$. From (4.3, 5)) we know that for every $\alpha > 0$ there exists some $\beta > 0$ such that

$$(\mathcal{N}^r(r \cdot d_1 + M_1 \cdot \delta_{k+1} \cdot r + c_1 \cdot r' \cdot \delta_{k+1}, \ldots, r \cdot d_k + M_k \cdot \delta_{k+1} \cdot r$$
$$+ c_k \cdot r' \cdot \delta_{k+1}, r \cdot d_{k+1}, \ldots, r \cdot d_n))^{2 \cdot \alpha \cdot \beta} \otimes \mathcal{O}_C(1, \ldots, 1)^\beta$$

is generated by its global sections over $X \cap p^{-1}(Y')$. Using (7.5, 2)) we find a map, surjective over $U \cap \pi'^{-1}(V)$,

$$\bigoplus \sigma_* \mathcal{O}_C \to (\mathcal{M}'^r(r \cdot d_1 + M_1 \cdot \delta_{k+1} \cdot r + c_1 \cdot r' \cdot \delta_{k+1}, \ldots, r \cdot d_k$$
$$+ M_k \cdot \delta_{k+1} \cdot r + c_k \cdot r' \cdot \delta_{k+1}, r \cdot d_{k+1}, \ldots, r \cdot d_n))^{2 \cdot \alpha \cdot \beta} \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(\beta, \ldots, \beta).$$

For β big enough the sheaf $\bigoplus \sigma_* \mathcal{O}_C \otimes \mathcal{O}_{(\mathbb{P}^1)^n}(\beta, \ldots, \beta)$ is generated by its global sections and, applying τ^*, we find that

$$(\mathcal{M}^r(r \cdot d_1 + M_1 \cdot \delta_{k+1} \cdot r + c_1 \cdot r' \cdot \delta_{k+1}, \ldots, r \cdot d_k + M_k \cdot \delta_{k+1} \cdot r$$
$$+ c_k \cdot r' \cdot \delta_{k+1}, r \cdot d_{k+1}, \ldots, r \cdot d_n))^{2 \cdot \alpha \cdot \beta} \otimes \mathcal{O}_{\mathbb{P}}(1, \ldots, 1)^{2 \cdot \beta}$$

is generated by its global sections over $\tau^{-1}(U \cap \pi'^{-1}(V))$, which is an element of $\mathcal{U}_{k+1}$.

By definition (4.2) the sheaf

$$\mathcal{M}(d_1 + M_1 \cdot \delta_{k+1}, \ldots, d_k + M_k \cdot \delta_{k+1}, d_{k+1}, \ldots, d_n)^{r' \cdot r' \cdot \delta_{k+1}}$$
$$\otimes \mathcal{O}_{\mathbb{P}}(c_1, \ldots, c_k, 0, \ldots, 0)^{r' \cdot \delta_{k+1}}$$

is weakly positive with respect to $\mathscr{U}_{k+1}$ and using (4.3.2)) we obtain the weak positivity of $\mathscr{M}(d_1+M_1\cdot\delta_{k+1},\ldots,d_k+M_k\cdot\delta_{k+1},d_{k+1},\ldots,d_n)$ with respect to $\mathscr{U}_{k+1}$.

By (8.3) $\delta_{k+1}\leqq d_{k+1}$ and we get (7.3).

We return to the situation described at the beginning of this section. We can find sections $s_1,\ldots,s_\eta$ of $g^*\mathscr{M}(d_1-1,\ldots,d_n-1)$ such that $V(s_\nu)=D+V(s'_\nu)$ for sections s'_ν generating the sheaf $g^*\mathscr{M}(d_1-1,\ldots,d_n-1)\otimes\mathscr{O}_Z(-D)$. These sections induce sections of $\mathscr{N}(d_1-1,\ldots,d_n-1)$ under the natural inclusion

$$\begin{aligned}&H^0(Z,g^*\mathscr{M}(d_1-1,\ldots,d_n-1))\\&\quad=H^0((\mathbb{P}^1)^n,\mathscr{M}'(d_1-1,\ldots,d_n-1))\to H^0((\mathbb{P}^1)^n,\sigma_*\mathscr{N}(d_1-1,\ldots,d_n-1))\end{aligned}$$

and we choose $\mathscr{I}$ to be the ideal-sheaf on C such that $\mathscr{N}(d_1-1,\ldots,d_n-1)\otimes\mathscr{I}$ is generated exactly by the global sections obtained in this way.

(8.8) *Claim.* Let $\tau'\colon V\to C$ be any birational morphism such that $\tau'^{-1}\mathscr{I}\cdot\mathscr{O}_V=\mathscr{O}_C(-B)$ for a normal crossing divisor B. Then $\tau'_*\left(\omega_{V/C}\left(-\left[\frac{B}{\delta_{k+1}}\right]\right)\right)\to\mathscr{O}_C$ is an isomorphism over X.

Proof. From (3.12) we know that (8.8) is independent of the birational morphism τ' choosen. We therefore can assume that we have a commutative diagram

$$\begin{array}{ccc}V & \xrightarrow{\tau'} & C\\ \Big\downarrow{\scriptstyle\sigma''} & & \Big\downarrow{\scriptstyle\sigma}\\ Z & \xrightarrow{\tau\cdot g} & (\mathbb{P}^1)^n.\end{array}$$

Over $\tau'^{-1}(X)$ one has $B=\sigma''^*(D)$ and for simplicity we can assume that both are equal everywhere. Moreover we can choose V such that $\sigma''|_{\tau'^{-1}(X)}$ is étale. Then we have $\omega_{\tau'^{-1}(X)/X}=\sigma''^*\,\omega_{Z/(\mathbb{P}^1)^n}|_{\tau'^{-1}(X)}$ and

$$\left[\frac{B}{\delta_{k+1}}\right]\Bigg|_{\tau'^{-1}(X)}=\sigma''^*\left[\frac{D}{\delta_{k+1}}\right]\Bigg|_{\tau'^{-1}(X)}.$$

We apply "flat base change" ([6], III.9.3) to obtain

$$\begin{aligned}&\sigma^*(\tau\cdot g)_*\,\omega_{Z/(\mathbb{P}^1)^n}\left(-\left[\frac{D}{\delta_{k+1}}\right]\right)\Bigg|_X\\&\quad=\tau'_*\,\sigma''^*\,\omega_{Z/(\mathbb{P}^1)^n}\left(-\left[\frac{D}{\delta_{k+1}}\right]\right)\Bigg|_X=\tau'_*\,\omega_{V/C}\left(-\left[\frac{B}{\delta_{k+1}}\right]\right)\Bigg|_X.\end{aligned}$$

By the choice of X, $\sigma(X)$ does not meet $C_{\delta_{k+1}}$ which implies (8.8).

(8.9) We define (for $r=r'^2\cdot\delta_{k+1}$)

$$\mathscr{F}=\mathscr{O}_C\left(\frac{M_1}{\delta_{k+1}\cdot r'}+\frac{k+2}{r},\ldots,\frac{M_k}{\delta_{k+1}\cdot r'}+\frac{k+2}{r},0,\ldots,0\right)$$

and

$$\gamma = \operatorname{Min}\left\{\begin{matrix}\mu;\ \mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{F}^{\mu \cdot \delta_{k+1}} \text{ is weakly positive with} \\ \text{respect to } \{X \cap p^{-1}(Y);\ Y \subseteq T \text{ open, } Y \neq \emptyset\}\end{matrix}\right\}.$$

This definition makes sense. In fact, if we consider

$$\mathcal{M}'(d_1 + \eta, \ldots, d_k + \eta, d_{k+1}, \ldots, d_n) \qquad \text{for } \eta \gg 0,$$

then this sheaf is generated by its global sections over

$$\pi'^{-1}_{\{1, \ldots, k\}}(U_1 \times \ldots \times U_k).$$

Therefore $\mathcal{N}(d_1 + \eta, \ldots, d_k + \eta, d_{k+1}, \ldots, d_n)$ is generated by its global sections over X for $\eta \gg 0$.

The sheaf $\mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{F}^{\gamma \cdot \delta_{k+1}}$ is weakly positive with respect to the set of open subvarieties of the form $X \cap p^{-1}(Y)$. Hence some power of the sheaf $(\mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{F}^{\gamma \cdot \delta_{k+1}})^{\delta_{k+1}-1} \otimes \mathcal{O}_C(1, \ldots, 1)$ is generated by its global sections over $X \cap p^{-1}(Y)$, for some open Y in T.

In fact, $\mathcal{O}_{C'}(1, \ldots, 1)$ is ample and we can apply (4.3, 5)) to the morphism δ. By our choice of $\mathcal{I}$ some power of

$$\begin{aligned}(\mathcal{N}(d_1, \ldots, d_n) &\otimes \mathcal{F}^{\gamma \cdot \delta_{k+1}})^{\delta_{k+1}-1} \otimes \mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{I} \\ &= (\mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{F}^{\gamma \cdot (\delta_{k+1}-1)})^{\delta_{k+1}} \otimes \mathcal{I}\end{aligned}$$

is generated by its global sections over $X \cap p^{-1}(Y)$.

Let us choose $\mathcal{K} = \mathcal{O}_C\left(\frac{1}{r}, \ldots, \frac{1}{r}, 0, \ldots, 0\right)$, where the first zero occurs on the $(k+1)$-st place.

(8.10). *Claim.* If we take $N = \delta_{k+1}$ and $\mathcal{L} = \mathcal{N}(d_1, \ldots, d_n) \otimes \mathcal{F}^{\gamma \cdot (\delta_{k+1}-1)}$ then the assumptions made in (6.3) are satified.

Proof. We just verified a), and b) is obvious by the choice of $\mathcal{K}$. The condition d) in (6.3) is nothing but (8.8). Hence we are only left with c).

Let x and x' be two points out of $p^{-1}(t) \simeq C_1 \times \ldots \times C_k$ for sufficiently general $t \in T$. We can find some $\nu \in \{1, \ldots, k\}$ and $x_\nu, x'_\nu \in C_\nu$ such that $x_\nu \neq x'_\nu$ and

$$x \in C_1 \times \ldots \times x_\nu \times \ldots \times C_k, \qquad x' \in C_1 \times \ldots \times x'_\nu \times \ldots \times C_k.$$

By the choice of $\mathcal{K}$ we have $\deg_{C_\nu}(\mathcal{K}|_{C_\nu}) \geqq 1$ and (up to numerical equivalence) $\mathcal{K}$ is equal to the sum of $C_1 \times \ldots \times x_\nu \times \ldots \times C_k$ and some divisor in general position.

So we can use (6.4) and we find some open subvariety Y' of T such that $\mathcal{L} \otimes \mathcal{K}^{k+1} \otimes \omega_{C/T}$ is weakly positive over $p^{-1}(Y') \cap X$. From (7.5, 6)) we get an inclusion, isomorphic over X,

$$\omega_{C/T} \otimes \mathcal{K}^{k+1} \to \mathcal{O}_C\left(M_1 + \frac{k+2}{r}, \ldots, M_k + \frac{k+2}{r}, 0, \ldots, 0\right)$$

and $\omega_{C/T}\otimes\mathcal{K}^{k+1}\to\mathcal{F}^{r'\cdot\delta_{k+1}}$, isomorphic over X. In other words,

$$\mathcal{N}(d_1,\dots,d_n)\otimes\mathcal{F}^{\gamma\cdot(\delta_{k+1}-1)+r'\cdot\delta_{k+1}}$$

is weakly positive over $X\cap p^{-1}(Y')$, which by definition of γ is only possible if

$$\gamma\cdot(\delta_{k+1}-1)+r'\cdot\delta_{k+1}>(\gamma-1)\cdot\delta_{k+1} \quad\text{or}\quad \gamma<(r'+1)\cdot\delta_{k+1}.$$

Hence

$$\begin{aligned}&\mathcal{N}(d_1,\dots,d_n)\otimes\mathcal{F}^{(r'+1)\cdot\delta_{k+1}^2}\\ &\quad=\mathcal{N}\Bigg(d_1+\frac{M_1\cdot(r'+1)\cdot\delta_{k+1}}{r'}+\frac{(k+2)\cdot(r'+1)\cdot\delta_{k+1}}{r'^2},\dots,\\ &\qquad d_k+\frac{M_k\cdot(r'+1)\cdot\delta_{k+1}}{r'}+\frac{(k+2)\cdot(r'+1)\cdot\delta_{k+1}}{r'^2},d_{k+1},\dots,d_n\Bigg)\end{aligned}$$

is weakly positive over $X\cap p^{-1}(Y')$. However

$$\frac{M_\nu\cdot(r'+1)}{r'}+\frac{(k+2)\cdot(r'+1)}{r'^2}=M_\nu+\frac{M_\nu+k+2}{r'}+\frac{k+2}{r'^2}$$

is smaller then $M_\nu+\dfrac{M_\nu+k+3}{r'}$ for $r'\geqq k+2$ and we obtain (8.7).

§ 9. Roth's Theorem

It's well known to the specialists how to obtain Roth's theorem assuming (0.4): one constructs by a "pigeon-hole-principle" (or Siegel-type-lemma) an auxiliary polynomial $f\in\mathbb{Q}[X_1,\dots,X_n]$ with an high order vanishing at the point $(\alpha,\dots,\alpha)$ and a relatively small height. Dyson's lemma, applied in all the conjugates of $(\alpha,\dots,\alpha)$ and in an approximation point $\left(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n}\right)$ says that the vanishing order of f at the approximation point has to be small.

Recall that $(d_1\cdot\ldots\cdot d_n\cdot\mathrm{Vol}(I(\underline{d},\underline{a},t)))$ expresses *pointwise* the number of conditions for f to be of multidegree $\underline{d}$ and to have a zero of type $(\underline{a},t)$. One needs the following technical estimation of it.

(9.1) **Lemma.** *Under the assumptions*

$$\alpha_\nu=\frac{1}{d_\nu}\quad\textit{and}\quad t\leqq\frac{n}{2}$$

one has

$$\mathrm{Vol}(I(\underline{d},\underline{a},t))\leqq\exp(-6\cdot n\cdot s^2/c)$$

where $t=(1/2-s)\cdot n$ *and* $c=\frac{1}{n}\sum_{\nu=1}^{n}\left(1+\frac{1}{d_\nu}\right)^2$.

Proof. (copied from a manuscript by M. Mignotte, Publications d'Orsay *n*° 77–74). As in 2.4, ii), one writes

$$\mathrm{Vol}(I(\underline{d}, \underline{a}, t)) = \lim_{d_\nu \to \infty} (d_1 \cdot \ldots \cdot d_n)^{-1} \cdot J(\underline{d})$$

where $J(\underline{d})$ is the number of integral points $\underline{i}$ verifying

$$0 \leqq i_\nu \leqq d_\nu \quad \text{and} \quad \sum_{\nu=1}^{n} \frac{i_\nu}{d_\nu} \leqq (\tfrac{1}{2} - s) \cdot n.$$

If one replaces i_ν by $d_\nu - i_\nu$ one finds $J(\underline{d})$ to be also the number of integral points $\underline{i}$ verifying

$$0 \leqq i_\nu \leqq d_\nu \quad \text{and} \quad \sum_{\nu=1}^{n} \frac{i_\nu}{d_\nu} \geqq (\tfrac{1}{2} + s) \cdot n.$$

For $F_\nu(u) = \sum_{i_\nu=0}^{d_\nu} \exp\left(u \cdot \left(\frac{i_\nu}{d_\nu} - \frac{1}{2}\right)\right)$ and $F(u) = F_1(u) \cdot \ldots \cdot F_n(u)$ one finds for $u > 0$ that $J(\underline{d}) \cdot \exp(s \cdot u \cdot n) \leqq F(u)$.

Adding up and using $1 \leqq \frac{\mathrm{sh}(v)}{v} \leqq \exp\left(\frac{v^2}{6}\right)$, one gets

$$\begin{aligned} F_\nu(u) &= \mathrm{sh}\left(\frac{u \cdot (d_\nu + 1)}{2 \cdot d_\nu}\right) \cdot \left(\mathrm{sh}\left(\frac{u}{2 \cdot d_\nu}\right)\right)^{-1} \\ &\leqq (d_\nu + 1) \cdot \mathrm{sh}\left(\frac{u \cdot (d_\nu + 1)}{2 \cdot d_\nu}\right) \cdot \frac{2 \cdot d_\nu}{u \cdot (d_\nu + 1)} \\ &\leqq (d_\nu + 1) \cdot \exp\left(\left(\frac{u \cdot (d_\nu + 1)}{2 \cdot d_\nu}\right)^2 \cdot \frac{1}{6}\right). \end{aligned}$$

For $u = \frac{12 \cdot s}{c}$ one finds the inequality wanted.

From now on we consider a fixed number field K of degree $d \geqq 2$. In [1], page 279, one finds:

(9.2) **Siegel's Lemma.** *Let N, M and d be positive integers verifying $N > d \cdot M$. Consider M linear forms l_i with coefficients in K. Then there exists a non-zero vector $x \in \mathbb{Q}^N$ such that $l_i(\underline{x}) = 0$ for $i = 1, \ldots, M$ and*

$$h(\underline{x}) \leqq c \cdot (c \cdot N)^{\frac{d \cdot M}{N - d \cdot M}} \cdot \left(\prod_{i=1}^{M} h(l_i)\right)^{\frac{d}{N - d \cdot M}}$$

where h is the height and c depends only on K.

The proof is a "box" or "pigeon-hole" principle applied to the conjugates of the forms l_i.

(9.3) **Corollary** (*see* [1], *Lemma* 6). *Take α_ν, $\nu = 1, \ldots, n$, such that $K = \mathbb{Q}(\alpha_\nu)$ for all ν, and call $\lambda(\alpha_\nu) = \log h(\alpha_\nu)$ the logarithmic height. Take $(\underline{d}, \underline{a}, t)$ such that $d \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t)) < 1$. Then there exists a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ of multi-*

degree $\underline{d}$ *having a zero of type* $(\underline{a}, t)$ *in* $\xi = (\alpha_1, \dots, \alpha_n)$ *verifying*

$$\lambda(f) = \log h(f) \leqq \frac{d \cdot \operatorname{Vol}(I(\underline{d}, \underline{a}, t))}{1 - d \cdot \operatorname{Vol}(I(\underline{d}, \underline{a}, t))} \cdot \left(\sum_{\nu=1}^{n} d_\nu \cdot \lambda(\alpha_\nu) + (\log 2) \cdot \sum_{\nu=1}^{n} d_\nu \right) + o(d_1 + \dots + d_n).$$

To prove it (see [1]), one applies Siegel's lemma to the number of polynomials $N = (d_1 + 1) \cdot \ldots \cdot (d_n + 1)$ of degree $\underline{d}$ and the number of conditions $M = d_1 \cdot \ldots \cdot d_n \cdot \operatorname{Vol}(I(\underline{d}, \underline{a}, t))$.

Keeping the notations (9.3), we take $\alpha = \alpha_1 = \dots = \alpha_n$. Let κ be a positive number such that

(9.5) $$\left| \alpha - \frac{p}{q} \right| \leqq \left(\frac{1}{q} \right)^\kappa$$

has infinitely many solutions. Here $|\ |$ denotes the usual absolute value in $\mathbb{C}$ and we fix an embedding of K in $\mathbb{C}$. Roth's theorem, as stated in the introduction says that $\kappa \leqq 2$.

For any $n \in \mathbb{N}$ and a given constant $Q > 1$ we can choose solutions $\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}$ such that we have

(9.6) *Assumption*

i) $q_n \geqq q_{n-1} \geqq \dots \geqq q_1 \geqq Q$.

ii) For all natural numbers $N \geqq 2 \cdot \log(q_n)$ let $d_\nu = \left[\frac{N}{\log(q_\nu)} \right]$, $a_\nu = d_\nu^{-1}$ and $d'_\nu = d_\nu + (d-1) \cdot \sum_{i=\nu+1}^{n} d_i$. Then we assume that

$$d'_1 \cdot \ldots \cdot d'_n \leqq \left(1 + \frac{1}{2 \cdot n!} \right) \cdot d_1 \cdot \ldots \cdot d_n.$$

ii) can be fulfilled since one has $\frac{d_\nu}{d_{\nu-1}} \leqq 2 \cdot \frac{\log(q_{\nu-1})}{\log(q_\nu)}$, independently of N. In the second half of this paragraph we prove:

(9.7) **Lemma.** *Assume that for* $t > 1$

$$d \cdot \operatorname{Vol}(I(\underline{d}, \underline{a}, t)) = 1 - \frac{1}{2 \cdot n!}.$$

Then one has the inequality

$$n - \kappa \cdot (t-1) \geqq -n \cdot (4 \cdot n! \cdot (\lambda(\alpha) + \log(2)) + \log(|\alpha| + 2)) \cdot \frac{1}{\log(Q)}.$$

The assumption on t is verified for n big enough. On the other hand,

$$\operatorname{Vol}\left(I\left(\underline{d}, \underline{a}, \frac{n}{2} \right) \right) = \frac{1}{2}$$

and $d \geqq 2$, which implies $t \leqq \frac{n}{2}$. (9.1) gives

$$\frac{1}{d} - \frac{1}{2 \cdot d \cdot n!} = \operatorname{Vol}(I(\underline{d}, \underline{a}, t)) \leqq \exp(-6 \cdot n \cdot s^2 \cdot c^{-1})$$

or (if $\underline{d}$ is big enough and $6 \geqq c$)

$$-\log\left(\frac{1}{d}-\frac{1}{2\cdot d\cdot n!}\right)\geqq n\cdot\left(\frac{1}{2}-\frac{t}{n}\right)^2.$$

(9.7) being true for all Q we have

$$\kappa^{-1}\geqq\frac{t-1}{n}\geqq\frac{1}{2}-\frac{1}{n}-\frac{1}{\sqrt{n}}\cdot\sqrt{-\log\left(\frac{1}{d}-\frac{1}{2\cdot d\cdot n!}\right)}.$$

The limit for $n\to\infty$ gives $\kappa\leqq 2$.

Proof of (9.7). Let f be the polynomial of (9.3), applied to the data $\underline{d}$, $\underline{a}$ and t, introduced in (9.6). f has a zero of type $(\underline{a}, t)$ at $(\alpha', \ldots, \alpha')$ for all the conjugates α' of α. Let τ be the biggest real number such that f has a zero of type $(\underline{a}, \tau)$ at $\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)$. Theorem 0.4 gives

$$\operatorname{Vol}(I(\underline{d}, \underline{a}, \tau))\leqq\frac{d'_1\cdot\ldots\cdot d'_n}{d_1\cdot\ldots\cdot d_n}-\operatorname{Vol}(I(\underline{d}, \underline{a}, t))\cdot d\leqq\frac{1}{n!}$$

and hence $\tau\leqq 1$. This means that we can find $\underline{i}\in\mathbb{N}^n$ such that $\sum_{\nu=1}^{n}\frac{i_\nu}{d_\nu}=\tau\leqq 1$ and $C=\Delta^{\underline{i}}f\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right)\neq 0$ where we use the notation

$$\Delta^{\underline{i}}=\frac{1}{i_1!\cdot\ldots\cdot i_n!}\cdot\frac{\partial^{\underline{i}}}{\partial x_1^{i_1}\cdot\ldots\cdot\partial x_n^{i_n}}.$$

By (9.6, ii) we have $i_\nu-d_\nu\geqq -d_\nu\geqq -N\cdot\log(q_\nu)^{-1}$. $\Delta^{\underline{i}}$ does not introduce new denominators and hence

$$|C|\geqq q_1^{i_1-d_1}\cdot\ldots\cdot q_n^{i_n-d_n}\cdot h(f)^{-1}\geqq\exp(-n\cdot N)\cdot h(f)^{-1}. \tag{9.8}$$

On the other hand, the Taylor expansion at $(\alpha, \ldots, \alpha)$ of $\Delta^{\underline{i}}f$ gives

$$C=\sum_{\underline{j}\in\mathbb{N}^n}\Delta^{\underline{j}}\Delta^{\underline{i}}f(\underline{\alpha})\cdot\left(\frac{p_1}{q_1}-\alpha\right)^{j_1}\cdot\ldots\cdot\left(\frac{p_n}{q_n}-\alpha\right)^{j_n}.$$

Let $\underline{j}$ be a n-tuple with $\Delta^{\underline{j}}\Delta^{\underline{i}}f(\underline{\alpha})\neq 0$. Then

$$\begin{aligned}&\frac{1}{N-\log(q_n)}\cdot\sum_{\nu=1}^{n}j_\nu\cdot\log(q_\nu)\\&\qquad\geqq\sum_{\nu=1}^{n}\frac{j_\nu}{\dfrac{N}{\log(q_\nu)}-1}\geqq\sum_{\nu=1}^{n}\frac{j_\nu}{d_\nu}\geqq t-\tau\geqq t-1\end{aligned}$$

and (9.5) implies

$$\left|\alpha-\frac{p_1}{q_1}\right|^{j_1}\cdot\ldots\cdot\left|\alpha-\frac{p_n}{q_n}\right|^{j_n}\leqq(q_1^{j_1}\cdot\ldots\cdot q_n^{j_n})^{-\kappa}\leqq\exp(-\kappa\cdot(t-1)\cdot(N-\log(q_n))).$$

For $R = \sum_{\underline{j} \in \mathbb{N}^n} |\Delta^{\underline{j}} \Delta^{\underline{i}} f(\underline{\alpha})|$ we obtain

$$|C| \leqq \exp(-\kappa \cdot (t-1) \cdot (N - \log(q_n))) \cdot R. \tag{9.9}$$

In order to bound R we consider $\|f\|$, the polynomial obtained from f by replacing the coefficients by their absolute value, the Taylor expansion of $\|f\|$ at $|\underline{\alpha}| + \underline{1} = (|\alpha|+1, \ldots, |\alpha|+1)$ and the Taylor expansion of $\Delta^{\underline{i}} \|f\|$ at $|\underline{\alpha}|$:

$$\|f\|(|\underline{\alpha}| + \underline{2}) = \sum_{\underline{k} \in \mathbb{N}^n} \Delta^{\underline{k}} \|f\|(|\underline{\alpha}| + \underline{1})$$
$$\geqq \Delta^{\underline{i}} \|f\|(|\underline{\alpha}| + \underline{1})) = \sum_{\underline{j} \in \mathbb{N}^n} \Delta^{\underline{j}} \Delta^{\underline{i}} \|f\|(|\underline{\alpha}|) \geqq R.$$

On the other hand $\|f\|(|\underline{\alpha}| + \underline{2})$ is bounded by

$$\|f\|(\underline{1}) \cdot (|\alpha|+2)^{d_1 + \ldots + d_n}$$
$$\leqq (d_1+1) \cdot \ldots \cdot (d_n+1) \cdot h(f) \cdot (|\alpha|+2)^{d_1 + \ldots + d_n}.$$

Hence (9.8) and (9.9) imply

$$-\kappa \cdot (t-1) \cdot (N - \log(q_n)) + \lambda(f)$$
$$+ \log(|\alpha|+2) \cdot (d_1 + \ldots + d_n) + o(d_1 + \ldots + d_n) \geqq -n \cdot N - \lambda(f).$$

Using (9.3) and replacing $d \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t)) \cdot (1 - \mathrm{Vol}(I(\underline{d}, \underline{a}, t)))^{-1}$ by the upper bound $2 \cdot n!$ we obtain

$$n \cdot N - \kappa \cdot (t-1) \cdot (N - \log(q_n)) + o(d_1 + \ldots + d_n)$$
$$\geqq -(4 \cdot n! \cdot (\lambda(\alpha) + \log(2)) + \log(|\alpha|+2)) \cdot \sum_{\nu=1}^{n} d_\nu.$$

Our choice of $\underline{d}$ gives $N \cdot \frac{1}{\log(Q)} \geqq d_\nu$ for all ν and the right hand side of the inequality is bigger than

$$-N \cdot (4 \cdot n! \cdot (\lambda(\alpha) + \log(2)) + \log(|\alpha|+2)) \cdot n \cdot \frac{1}{\log(Q)}.$$

These inequalities are true for all $N \geqq 2 \cdot \log(q_n)$ and hence we obtain (9.7).

§ 10. Remarks about possible ameliorations of (0.4)

In the proof of (0.4) we did not try to obtain the "optimal" inequality. In this section we want to give just some hints how one might obtain a better bound for the volumes if the points are in special positions. We work out further improvements in the two variable case. In the higher dimensional situation one would have to do quite a lot of calculations to get similar improvements and – since we do not see any applications where this would be of any advantage – we did not try to do them. At the end of this section we discuss the necessity of assumption a) and b) in (0.4).

(10.1) In the proof of (7.3) in §8 we introduced some number δ_{k+1} and it was only at the end (after (8.7)) that we replaced δ_{k+1} by the bigger number d_{k+1} (see (8.3)). Moreover in the definition of δ_{k+1} in (8.2) it is enough to take:

$$\delta_{k+1} = \mathrm{Min}\,\{N \geqq 1;\, (\mathbb{P}^1)^n - C_N \text{ contains an element of } \mathscr{U}_{k+1}\}.$$

Nevertheless it seems to be quite difficult to improve the inequality for δ_{k+1} in special cases. Only for $k=1$ we have:

(10.2) *Claim.* Assume that the polynomial f in (0.4) has the decomposition $f = h(x_1) \cdot g_1(x_1, \ldots, x_n)^{m_1} \cdot \ldots \cdot g_s(x_1, \ldots, x_n)^{m_s}$ where the g_i are irreducible and two by two distinct, then

$$\delta_2 - 1 \leqq m = \mathrm{Max}\,\{m_1, \ldots, m_s\}.$$

Proof. We just have to show that $(\mathbb{P}^1)^n - C_{m+1}$ contains an element of $\mathscr{U}_2$, i.e. an open set of the form $U \times V$ where $U \subseteq \mathbb{P}^1_1$ and $V \subseteq \mathbb{P}^1_2 \times \ldots \times \mathbb{P}^1_n$.

We can take, for example, $U = \mathbb{P}^1_1 - \overline{V(h(x_1))}$ and

$$V = \pi'_{\{2, \ldots, n\}}(\mathrm{Sing}\,(D_{\mathrm{red}}))$$

where D is the zero-set of f in $(\mathbb{P}^1)^n$.

In (7.3) (for $k=2$) we can replace d_2 by $m+1$ or copying the limit process of (8.1) even by m (Just replace f by f^N, and $m+1$ by $N \cdot m + 1$). As in §5 one gets

(10.3) **Corollary.** *Under the assumptions of* (0.4) *one has*

$$\begin{aligned}
&\sum_{\mu=1}^{M} \mathrm{Vol}\,(I(\underline{d}, \underline{a}, t_\mu)) \\
&\quad \leqq \left(1 + (M'-2) \cdot \frac{m}{d_1} + \sum_{i=3}^{n} (M'-2) \cdot \frac{d_i}{d_1}\right) \cdot \prod_{j=2}^{n} \left(1 + (M'-2) \cdot \sum_{i=j+1}^{n} \frac{d_i}{d_j}\right) \\
&\qquad - (d_1 \cdot \ldots \cdot d_n)^{-1} \cdot \lim_{N \to \infty} N^{-n} \cdot h^0((\mathbb{P}^1)^n, \mathscr{L}'^{(\underline{d}')}(d'_1, \ldots, d'_n)^N).
\end{aligned}$$

Of course, one can replace the obvious bound $m \leqq d_2$ by a better bound only if one knows something about the position of the points $\zeta_1, \ldots, \zeta_M$.

In the proof of (0.4) we replaced the term

$$\lim_{N \to \infty} N^{-n} \cdot h^0((\mathbb{P}^1)^n, \mathscr{L}'^{(\underline{d}')}(d'_1, \ldots, d'_n)^N)$$

by zero. One can do better, as we want to explain in the proof of

(10.4) **Theorem** (*see* [1]). *Assume that for* $n=2$ *the assumptions of* (0.4) *are satisfied. Then*

$$\sum_{\mu=1}^{M} \mathrm{Vol}\,(I(\underline{d}, \underline{a}, t_\mu)) \leqq 1 + \frac{M'-2}{2} \cdot \frac{d_2}{d_1}.$$

Proof. Let $P_1 \in \mathbb{P}^1_1$ be a point in general position.

If, using the notation introduced in (2.5), $\pi_1'^{-1}(P_1)$ meets the support of $C_{\zeta_\gamma}^{(\underline{d},\underline{a},t_\gamma)}$, then $a_1 \cdot d_1 < t_\gamma$. We may (replacing f by some power of f) assume that a_2 divides t_γ and d_1 and we can find some natural number s such that $a_1 \cdot d_1 = t_\gamma - a_2 \cdot s$ and $(x_2 - \zeta_{\gamma,2})^s$ divides f. The polynomial $\bar{f} = f \cdot (x_2 - \zeta_{\gamma,2})^{-s}$ is of multidegree $(\delta_1 = d_1,\ \delta_2 = d_2 - s)$ and has a zero of type $(\underline{a}, \tau_\mu)$ at ζ_μ, where $\tau_\gamma = t_\gamma - s \cdot a_2$ and $\tau_\mu = t_\mu$ for $\mu \neq \gamma$. We have $a_1 \cdot \delta_1 = \tau_\gamma$ and (using (2.7)) $a_2 \cdot \delta_2 \geqq \tau_\mu$ for $\mu \neq \gamma$. After the description of the volumes in (2.4, ii)) this implies that for $\mu \neq \gamma$

$$\delta_1 \cdot \delta_2 \cdot \mathrm{Vol}(I(\underline{\delta}, \underline{a}, \tau_\mu)) = d_1 \cdot d_2 \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t_\mu))$$

and moreover

$$\delta_1 \cdot \delta_2 \cdot \mathrm{Vol}(I(\underline{\delta}, \underline{a}, \tau_\gamma)) = d_1 \cdot d_2 \cdot \mathrm{Vol}(I(\underline{d}, \underline{a}, t_\gamma)) - s \cdot d_1.$$

Together we find that the inequality (10.4) for the tuples $(\underline{\delta}, \underline{a}, \tau_\mu)$ implies the inequality for $(\underline{d}, \underline{a}, t_\mu)$.

Hence we may assume that the support of $C_{\zeta_\mu}^{(\underline{d},\underline{a},t_\mu)}$ does not meet $\pi_1'^{-1}(P_1)$ for $\mu = 1, \ldots, M$. Using the notations of (5.4) the sheaf $\mathscr{L}(d_1', d_2')$ on $\mathbb{P}'$ is arithmetically positive. By (4.7) (or [10], 3.1), for $q > 0$, $h^q(\mathbb{P}', \mathscr{L}(d_1', d_2')^N)$ is bounded from above by a linear polynomial in N and the Riemann-Roch-Theorem for surfaces implies that (see [6], p. 362)

$$\begin{aligned}\lim_{N\to\infty} & N^{-2} \cdot h^0((\mathbb{P}^1)^2, \mathscr{L}'^{(\underline{d}')}(d_1', d_2')^N) \\ &= \lim_{N\to\infty} N^{-2} \cdot h^0(\mathbb{P}', \mathscr{L}(d_1', d_2')^N) = \tfrac{1}{2} \cdot c_1(\mathscr{L}(d_1', d_2'))^2\end{aligned}$$

where $c_1(\ )^2$ denotes the selfintersection-number.

The arithmetical positivity implies that $c_1(\mathscr{L}(d_1', d_2')) \cdot B \geqq 0$ for every effective divisor B on $\mathbb{P}'$.

Let $H = \tau'^{-1} \pi_1'^{-1}(P_1)$. The condition on the support of $C_{\zeta_\mu}^{(\underline{d},\underline{a},t_\mu)}$ implies that $c_1(\mathscr{L}(d_1', d_2')) \cdot H = d_2' = d_2$. f induces a section of $\mathscr{L}(d_1', d_2')$ whose zero-set is of the form $B + (M' - 2) \cdot d_2 \cdot H$ for an effective divisor B. Hence

$$c_1(\mathscr{L}(d_1', d_2'))^2 \geqq c_1(\mathscr{L}(d_1', d_2')) \cdot ((M' - 2) \cdot d_2 \cdot H) = (M' - 2) \cdot d_2^2.$$

The inequality given in (10.3) or (5.13) implies therefore (10.4).

(10.5) In Theorem (0.4) we made two quite restrictive assumptions:

a) *the coordinates $\zeta_{1,\nu}, \ldots, \zeta_{M,\nu}$ must be two by two distinct,*

b) *the hyperplanes are given by $\underline{a}^{(\mu)} = \underline{a}$ for all μ.*

The "weak positivity statement" (5.3) or (7.4) was obtained however under the hypothesis

a') *if $\zeta_{\mu,\nu} = \zeta_{\gamma,\nu}$, for $\mu \neq \gamma$ and some ν, one has*

$$a_\nu^{(\mu)} \cdot t_\gamma = a_\nu^{(\gamma)} \cdot t_\mu,$$

without using b).

Simple examples (for $n = 2$ one can take polynomials of the form $f_1(x_1) \cdot f_2(x_2)$) show that neither (5.4) nor the inequalities (0.4) or (5.13) remain true if we replace a) by a′) or if we leave out the assumption b). The reason is that in the proof of (5.4) (case (5.9) and case (5.10)) we had to use the combinatorial statements obtained in § 2. To be more precise, the arguments given in § 5 show

(10.6) **Proposition.** *Using the notations introduced in* (2.6) *we assume that the hypothesis* a′) *given above is satisfied and moreover that*

i) *For* $\mu \neq \gamma$ $\operatorname{Supp}(C_{\zeta_\mu}^{(\underline{d}, \underline{a}^{(\mu)}, t_\mu)}) \cap \operatorname{Supp}(C_{\zeta_\gamma}^{(\underline{d}, \underline{a}^{(\gamma)}, t_\gamma)}) = \emptyset$.

ii) *For any subset* $I = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$ *and for any point* $Q \in (\mathbb{P}^1)^{\{1, \ldots, n\} - I}$ *let* $\beta = \beta_{I, Q}: (\mathbb{P}^1)^I \to (\mathbb{P}^1)^n$ *be the natural embedding. Then we assume that*

$$H^0((\mathbb{P}^1)^I, \operatorname{Im}(\beta^* \mathscr{L}'^{(\underline{d})} \to \mathcal{O}_{(\mathbb{P}^1)^I}) \otimes \mathcal{O}_{(\mathbb{P}^1)^I}(d_{i_1}, \ldots, d_{i_s})) \neq 0.$$

Then for

$$M_\nu = \operatorname{Max}\{2, |\{\zeta_{\mu,\nu};\ \mu = 1, \ldots, M\}|\} - 2$$

and $d'_i = d_i + \sum_{j=i+1}^{n} M_i \cdot d_j$ *one has the inequality*

$$\sum_{\mu=1}^{M} d_1 \cdot \ldots \cdot d_n \cdot \operatorname{Vol}(I(\underline{d}, \underline{a}^{(\mu)}, t_\mu)) \leqq d'_1 \cdot \ldots \cdot d'_n.$$

However, even if the hypothesis a) given in (10.5) is satisfied, the only cases where one can verify (10.6) i) and ii) without using (2.8) and (2.9) are:

- If $a_\nu^{(\mu)} \cdot d_\nu \geqq t_\mu$ for all ν and μ, i.e. if $\mathscr{M}' = \mathscr{L}'^{(\underline{d})}$.
- If $\underline{a}^{(\mu)} = \underline{a}$ for $\mu = 1, \ldots, M-1$ and if t_M is very small.

The assumption (10.6, i)) is not only used in (5.9) but also in (5.11) where it enables us to count the dimension of a certain cokernel pointwise. Hence without this assumption one can not expect to find an inequality similar to (0.4) or (10.6). The reason is, that without (10.6, i)) the conditions which force a polynomial to have a zero of type $(\underline{a}, t_\mu)$ at the point ζ_μ depend too much on those for the other points.

The argument given in (9.7) carries over to the case where one considers good approximations of different algebraic numbers $\alpha_1, \ldots, \alpha_n \in K$, as long as each of them is a generator of K (this just implies the condition a) in (0.4) or (10.5)). In order to use good approximations of $\alpha_1, \ldots, \alpha_k$ to bound approximations of $\alpha_{k+1}, \ldots, \alpha_n$ one would like to get rid of this condition and to be able to consider numbers out of smaller numberfields too. In this special case, the dependence of the conditions, mentioned above, also appears in Siegel's Lemma (see (9.2) and [1], page 279) where the rank of the map Φ defined by the linear forms l_i and all there conjugates can be bounded by a constant smaller than $d \cdot M$ (using the notation of (9.2)). Hence in order to generalise (9.7), it might be easier to try to use (7.4) and arguments similar to the ones given in § 5 to bound the sum of the rank of Φ and the volume corresponding to the zero of f at the approximation point directly.

If one wants to obtain the theorems of W.M. Schmidt about simultaneous approximations in a way similar to our proof of the theorem of Roth, one seems to have to consider $(\mathbb{P}^m)^n$ instead of $(\mathbb{P}^1)^n$, and to generalize Dyson's Lemma to this situation. The problem turns out to be (6.2) where one would need a description of the open subvariety Y.

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