SEMISTABLE BUNDLES ON CURVES AND REDUCIBLE REPRESENTATIONS OF THE FUNDAMENTAL GROUP

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0. Introduction

A.A. Bolibruch [3] and V.P. Kostov [8] showed independently that if $\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma) \to GL(n, \mathbb{C})$ is an irreducible representation of the fundamental group, then there is an algebraic bundle E together with an algebraic connection $\nabla : E \to \Omega^1(\log \Sigma) \otimes E$ with underlying local system ρ, with the property that $E \cong \oplus_1^n \mathcal{O}$ is algebraically trivial. Equivalently, E can be taken to be the twist $L \otimes (\oplus_1^n \mathcal{O})$ of a line bundle L by an algebraically trivial bundle. Those twists are the unique semistable bundles on $\mathbb{P}^1$. In [7], it is indeed proven that if $\mathbb{P}^1$ is replaced by a smooth projective complex curve of higher genus, the theorem remains true in this form: there is a (E, ∇) as above with E semistable of degree 0 (and also with E semistable of any degree, even if it is not emphasized in the article). Let us call here for short such an (E, ∇) a *realization* of ρ. Note, it is crucial to require ∇ to have poles only along Σ. If one allows one more pole, then one can for example on $\mathbb{P}^1$ trivialize E even with parameters (see [2], section 4).

On the other hand, on $\mathbb{P}^1_{\mathbb{C}}$, A. Bolibruch [1] constructed representations which cannot be realized on the trivial bundle. The purpose of this note is to show that on a higher genus Riemann surface, there are representations ρ which cannot be realized on a semistable bundle. We show:

Theorem 0.1. *Let X be a smooth projective complex curve of genus g and let $\Sigma \subset X$ be a finite nonempty set.*

If $(g = 0, |\Sigma| \geq 3, n \geq 4)$ or $(g \geq 1, |\Sigma| \geq 1, n \geq 5)$ there exists a representation

$$\rho : \pi_1(X \setminus \Sigma) \to GL(n, \mathbb{C})$$

which cannot be realized by an algebraic connection

$$\nabla : E \to \Omega^1_X(\log \Sigma) \otimes E$$

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with logarithmic poles along Σ *and with* E *semistable.*

The proof is an adaptation of Bolibruch's ideas to the higher genus case, together with the use of Gabber's algebraic view ([7], section 1) on the Bolibruch-Kostov theorem.

Finally, let us remark that Deligne extensions (E, ∇) ([5]) are very natural in geometry. They are not compatible with pull-backs, but appear as direct images of connections, for example, Gauß-Manin connections of semistable families are Deligne extensions with nilpotent residues. On the other hand, the Gauß-Manin bundles tend to be highly instable, as they contain a positive Hodge subbundle. Thus it is not clear what is the rôle of semistability for realization of monodromy (see one computation in section 5).

1. Bolibruch's construction

Throughout the note we use the following notations.

(1.A) X is a smooth projective complex curve, $\Sigma = \{p_1, ..., p_m\} \subset X$ is finite and nonempty.

$$\rho : \pi_1(X \setminus \Sigma) \to GL(n, \mathbb{C})$$

is a representation. E is a vector bundle on X of rank n,

$$\nabla : E \to \Omega^1_X(\log \Sigma) \otimes E$$

is a logarithmic connection on E with underlying local system ρ. We call (E, ∇) a *realization* of ρ.

The eigenvalues of the residue endomorphism

$$\mathrm{res}_{p_i}(\nabla) : E \otimes \mathbb{C}(p_i) \to E \otimes \mathbb{C}(p_i)$$

at p_i are called $\beta_{i1}, ..., \beta_{in}$. They are ordered such that

$$\mathrm{Re}\, \beta_{i1} \leq ... \leq \mathrm{Re}\, \beta_{in} \ .$$

The following theorem is the key to Bolibruch's examples.

Theorem 1.1. *Let* $X, \Sigma, \rho, E, \nabla,$ *and* β_{ij} *be as in (1.A). Suppose that* E *is semistable, that* ρ *is reducible, and that for each* $i \in \{1, ..., m\}$ *the monodromy of* ρ *around* p_i *has only one eigenvalue* λ_i *and only one Jordan block.*

Then $\beta_{i1} = ... = \beta_{in} =: \beta_i$ *for all* i *and the slope* $\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$ *satisfies*

$$e^{2\pi\sqrt{-1}\mu(E)} = \prod_i \lambda_i \ .$$

Said differently, (E, ∇) *is the Deligne extension characterized by the property that* $(\mathrm{res}_{p_i}(\nabla) - \beta_i I)$ *nilpotent.*

We first prove a local statement.

Lemma 1.2. *Let $j : U = X \setminus \{p\} \hookrightarrow X$ be the embedding of the complement of a point on a smooth analytic contractible curve. Let (E, ∇) be a regular connection on U, such that the underlying local monodromy has only one eigenvalue and one Jordan block. Thus E has a filtration $E_i \subset E_{i+1}$ stabilized by ∇. Let $F \subset j_*E$ be a bundle such that $\nabla|_F$ has logarithmic poles in $\{p\}$, and let us denote by β_ℓ its eigenvalues, ordered such that $(\beta_{\ell+1} - \beta_\ell) \in \mathbb{N}$. Let $F_i := j_*E_i \cap F$. Then $F_i \subset F$ is a subbundle, $\nabla|_{F_i}$ has logarithmic poles and its residues are precisely $\{\beta_1, \ldots, \beta_i\}$.*

Proof. If the rank of E is 1, there is of course nothing to prove. Since X is smooth and has dimension 1, $F_i \subset F$ is a subbundle. As is well known, $\nabla|_F$ stabilizes F_i, for it takes values in $\Omega^1(\log\{p\}) \otimes E \cap \Omega^1(\log\{p\}) \otimes j_*E_i$. Furthermore, each (F_i, E_i) satisfies the same assumptions as (F, E). Let us thus first consider F_2. If $(\beta_1 - \beta_2) \in \mathbb{N} \setminus \{0\}$, one performs Gabber's construction ([7], section 1): F_i embedds into F_i', for $i = 1, 2$, with cokernel $\mathbb{C}(p)$, such that ∇_F extends as a logarithmic connection, the residue of which has the new eigenvalues $\beta_1 - 1, \beta_2$. Furthermore if the local generators in $\{p\}$ of F are e_1, e_2 with e_1 generating F_1 and $e_2 \otimes \mathbb{C}(p)$ being an eigenvector to β_2, then the new generators are $(\frac{e_1}{z}, e_2)$. In this basis, one has $\mathrm{res}_p' - \mathrm{res}_p = \mathrm{diag}(-1, 0)$. Repeating the procedure $(\beta_1 - \beta_2)$ times, one reaches a new F_2 with residue $= \mathrm{diag}(\beta_2, \beta_2)$. Now by Deligne [5] again, this implies that the local monodromy underlying E_2 is diagonal (actually even a homothety), a contradiction. Thus $(\beta_1 - \beta_2) \leq 0$. Replacing now E by E/E_1, one proceeds inductively. ☐

Proof of theorem 1.1. Because ρ is reducible, the local system $\ker(\nabla|_{X\setminus\Sigma})$ contains a local subsystem V of some rank ℓ with $0 < \ell < n$. Let $(\mathcal{V}, \nabla|_{\mathcal{V}}) \subset (E|_{X\setminus\Sigma}, \nabla|_{X\setminus\Sigma})$ be the induced algebraic regular connection. Let $j : X \setminus \Sigma \to X$ be the inclusion and

$$F := j_*(\mathcal{V}) \cap E \subset E \ .$$

By lemma 1.2, F is a subbundle of E and ∇ restricts to a logarithmic connection on F with residue eigenvalues $\beta_{i1}, \ldots, \beta_{i\ell}$ at p_i.

On the other hand, one has (see [6], appendix B, for example)

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} = -\deg(E).$$

Thus semistability of E implies

$$\frac{1}{k}\sum_{i=1}^{m}\sum_{j=1}^{\ell}\beta_{ij} = -\mu(F) \geq -\mu(E) = \frac{1}{n}\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij} \ .$$

Together with $\beta_{ij} - \beta_{i\ell} \geq 0$ for $j \geq \ell$ this shows

$$\beta_{i1} = ... = \beta_{in}$$

and

$$e^{2\pi\sqrt{-1}\mu(E)} = \prod_{i=1}^{m} e^{-2\pi\sqrt{-1}\beta_{i1}} = \prod_{i=1}^{m}\lambda_i \ .$$

□

2. Examples of reducible representations

Here we list several representations as in theorem 1.1.

As in 1.A, X is a smooth projective complex curve of genus g, and $\Sigma = \{p_1, ..., p_m\} \subset X$ is a finite nonempty set. One chooses a system of paths $a_1, b_1, ..., a_g, b_g$ and $c_1, ..., c_m$ with a common base point such that $\pi_1(X \setminus \Sigma)$ is generated by them with the single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdot ... \cdot a_g b_g a_g^{-1} b_g^{-1} \cdot c_1 ... c_m = 0 \ .$$

Here c_i is a loop around p_i. Then a representation $\rho : \pi_1(X \setminus \Sigma) \to GL(n, \mathbb{C})$ is given by $n \times n$-matrices $A_1, B_1, ..., A_g, B_g$ and $C_1, ..., C_m$ which satisfy the same relation.

The starting point is to overtake Bolibruch's examples by setting $A_i = B_i = \mathrm{id}$, ((2.1)–(2.3)) and then to modify ((2.4)-(2.5)).

The following representations are all reducible, and the local monodromies C_i around the points p_i have only one eigenvalue λ_i and only one Jordan block.

(2.1) $\rho^{(1)}$: Choose $\nu_i \in \mathbb{C} \setminus \{0\}$, $i = 1, ..., m$, with $\sum_{i=1}^{m} \nu_i = 0$ and a nilpotent $n \times n$-matrix $N^{(1)}$ with $\mathrm{rank}\, N^{(1)} = n - 1$. Define

$$C_i^{(1)} := \exp(\nu_i N^{(1)}) \ , \quad A_i^{(1)} := B_i^{(1)} := \mathrm{id} \ .$$

Then $\lambda_i = 1$.

(2.2) (Bolibruch [1] Example 5.3.1) $\rho^{(2)}$: $m = 3$, $n = 4$, $A_i^{(2)} := B_i^{(2)} :=$ id,

$$C_1^{(2)} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \ C_2^{(2)} := \begin{pmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{pmatrix} \ ,$$

$$C_3^{(2)} := \begin{pmatrix} -1 & 0 & 2 & -1 \\ 4 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix} .$$

Then $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = -1$. A semistable bundle E with a logarithmic connection (with poles only in Σ) which realizes $\rho^{(2)}$ must have slope $\mu(E) \equiv \frac{1}{2} \mod \mathbb{Z}$ by theorem 0.1.

In the case $g = 0$ this is impossible as any semistable bundle has slope in $\mathbb{Z}$. In that case $\rho^{(2)}$ cannot be realized by a logarithmic connection on a semistable bundle (with poles only in Σ) and in particular not by a Fuchsian differential system.

(2.3) $\rho^{(3)}$: $m \geq 3,\ n = 4,\ A_i^{(3)} := B_i^{(3)} := \mathrm{id}$. Define $N^{(3)} := \log C_1^{(2)}$,

$$C_1^{(3)} := ... := C_{m-2}^{(3)} := \exp(2\pi\sqrt{-1}\frac{1}{2m-4}) \exp(\frac{1}{m-2}N^{(3)})$$

$$C_{m-1}^{(3)} := -C_2^2 \ ,\ C_m^{(3)} := C_3^{(2)} \ .$$

Then $\lambda_1 = ... = \lambda_{m-2} = \exp(2\pi\sqrt{-1}\frac{1}{2m-4}),\ \lambda_{m-1} = \lambda_m = -1,\ \prod_i \lambda_i = -1$.

(2.4) $\rho^{(4)}$: $g \geq 1,\ n = 2.\ A_i^{(4)} := B_i^{(4)} := \mathrm{id}$ for $i \geq 2$. Define

$$A_1^{(4)} := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} ,\ B_1^{(4)} := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} ,\ C_i^{(4)} := \begin{pmatrix} 1 & \frac{-1}{m} \\ 0 & 1 \end{pmatrix} .$$

Then $\lambda_i = 1$.

(2.5) $\rho^{(5)}$: $g \geq 1,\ n$ even, $n \geq 4.\ A_i^{(5)} := B_i^{(5)} := \mathrm{id}$ for $i \geq 2$. Define

$$\alpha_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} ,\ \alpha_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,\ \beta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\delta_1 := \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} ,\ \delta_2 := \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix} ,$$

$$A_1^{(5)} := \begin{pmatrix} \alpha_1 & \alpha_2 & & 0 \\ & \alpha_1 & \ddots & \\ & & \ddots & \alpha_2 \\ 0 & & & \alpha_1 \end{pmatrix} ,\ B_1^{(5)} := \begin{pmatrix} \beta & & & 0 \\ & \beta & & \\ & & \ddots & \\ 0 & & & \beta \end{pmatrix} .$$

The matrix

$$A_1^{(5)} B_1^{(5)} A_1^{(5)^{-1}} B_1^{(5)^{-1}} = \begin{pmatrix} \delta_1 & \delta_2 & & * \\ & \delta_1 & \ddots & \\ & & \ddots & \delta_2 \\ 0 & & & \delta_1 \end{pmatrix}$$

has one $n \times n$ Jordan block with eigenvalue -1. Define

$$C_i := \exp(2\pi\sqrt{-1}\frac{1}{2m})\exp\left(\frac{-1}{m}\log(-A_1^{(5)}B_1^{(5)}A_1^{(5)^{-1}}B_1^{(5)^{-1}})\right) .$$

Then $\lambda_i = \exp(2\pi\sqrt{-1}\frac{1}{2m})$, $\prod_i \lambda_i = -1$.

3. The proof of theorem 0.1

Theorem 3.1. *Let X be a smooth projective complex curve,*

$$\Sigma = \{p_1, ..., p_m\} \subset X$$

be a finite nonempty set, and

$$\rho_\ell : \pi_1(X \setminus \Sigma) \to GL(n_\ell, \mathbb{C}) \ , \quad \ell = 1, 2,$$

be two representations with the following properties. Both representations are reducible. Each local monodromy has a single eigenvalue and a single Jordan block. Let λ_i^ℓ be those eigenvalues in p_i. Then $\lambda_i^1 \neq \lambda_i^2$ and $\prod_i \lambda_i^1 \neq \prod_i \lambda_i^2$.

Then $\rho_1 \oplus \rho_2$ cannot be realized by a semistable bundle on X with a logarithmic connection with poles only in Σ.

Proof. Let (E, ∇) be the algebraic regular connection on $X \setminus \Sigma$ with underlying ρ. Then $(E = E_1 \oplus E_2, \nabla = \nabla_1 \oplus \nabla_2)$, where ρ_i underlies (E_i, ∇_i). Let $F \subset j_*E$ be a bundle such that $\nabla|_F$ has logarithmic poles in Σ. Then $F_\ell = j_*E_\ell \cap F \subset F$ is a subbundle, stabilized by $\nabla|_F$. Let us denote by ∇_{F_ℓ} the induced connection. Then its residue is the restriction of the residue of ∇_F to F_ℓ. Since $\lambda_i^1 \neq \lambda_i^2$ in all points p_i, a fortiori none of the eigenvalues of $\mathrm{res}_{p_i}(\nabla_{F_1})$ can be an eigenvalue of $\mathrm{res}_{p_i}(\nabla_{F_2})$. Consequently, one has

$$\mathrm{res}_{p_i}(\nabla_F) = \mathrm{res}_{p_i}(\nabla_{F_1}) \oplus \mathrm{res}_{p_i}(\nabla_{F_2}). \tag{3.1}$$

On the other hand, since $F_1 \cap F_2 \subset F$ is torsion free and supported in Σ, one has $F_1 \cap F_2 = 0$, thus $F_1 \oplus F_2 \subset F$ is a locally free subsheaf, isomorphic to F away of Σ, and thus isomorphic to F by the condition (3.1).

If now moreover F is semistable, then F_ℓ is semistable as well, and one has $\mu(F_1) = \mu(F_2)$. This contradicts theorem 1.1. □

For the proof of theorem 0.1 one applies theorems 1.1 and 3.1 to several combinations of the representations in section 2.
$g = 0, |\Sigma| \geq 3, n \geq 4$: $\rho^{(1)}$ for $n^{(1)} = n - 4$ and $\rho^{(3)}$.
$g \geq 1, |\Sigma| \geq 1, n$ odd, $n \geq 5$: $\rho^{(1)}$ for $n^{(1)} = 1$ and $\rho^{(5)}$ for $n^{(5)} = n - 1$.
$g \geq 1, |\Sigma| \geq 1, n$ even, $n \geq 6$: $\rho^{(4)}$ and $\rho^{(5)}$ for $n^{(5)} = n - 2$.

4. Two-dimensional representations

W. Dekkers [4] showed that, for $X = \mathbb{P}^1_{\mathbb{C}}$ and a finite nonempty subset $\Sigma \subset X$, any two-dimensional representation $\rho : \pi_1(X \setminus \Sigma) \to GL(2, \mathbb{C})$ can be realized on the trivial bundle with poles only in Σ. A.A. Bolibruch gave a simpler proof, using the analogous result for irreducible connections ([3], [8], [7]). We adapt now this to higher genus.

Theorem 4.1. *Let X be a smooth projective complex curve, $\Sigma \subset X$ a finite nonempty set, and*

$$\rho : \pi_1(X \setminus \Sigma) \to GL(2, \mathbb{C})$$

be a two-dimensional representation.

There exists a semistable bundle E of even degree with a logarithmic connection with poles only in Σ which realizes ρ, but not necessarily of odd degree.

Proof. For ρ irreducible see [7]. Suppose that ρ is reducible. Let (E, ∇) be a vector bundle on X with logarithmic connection with poles only in Σ which realizes ρ.

Let $V \subset \ker(\nabla|_{X \setminus \Sigma})$ be a subsystem of rank 1. We denote by $j : X \setminus \Sigma \hookrightarrow X$ the inclusion and define

$$F := j_*(V \otimes \mathcal{O}_{X \setminus \Sigma}) \subset E\ .$$

Then

$$0 \to F \to E \to E/F \to 0$$

is an exact sequence of bundles, and F and E/F are equipped with the induced connection. Then E will be semistable if $\deg F = \deg E/F$.

1st case: For each $p \in \Sigma$ the two eigenvalues of the local monodromy around p coincide. Following Deligne, one can choose (E, ∇) such that at each $p \in \Sigma$ the two residue eigenvalues coincide. Thus in particular, $\deg F = \deg E/F$, and E is semistable.

2nd case: For some $p \in \Sigma$ the two eigenvalues of the local monodromy around p differ. Let (E, ∇) be again a Deligne extension. Then the space $E \otimes \mathbb{C}(p)$ splits into two one-dimensional eigenspaces $F \otimes \mathbb{C}(p)$ and $(E/F) \otimes \mathbb{C}(p)$ of the residue endomorphism $\mathrm{res}_p(\nabla)$. One can apply Gabber's construction [7] (section 1) to either one of these eigenspaces and increase by one either the degree of F or that of E/F. Repeating this one can obtain bundles $E' \supset F'$ with logarithmic connections such that $\deg F' = \deg(E'/F')$. Then E' is semistable. $\square$

Remark 4.2. If $\rho : \pi_1(X \setminus \Sigma) \to GL(2, \mathbb{C})$ is reducible and for any $p \in \Sigma$ the local monodromy around p has only one Jordan block then any semistable bundle (E,∇) with logarithmic connection which realizes ρ is at each point $p\in\Sigma$ a Deligne extension by theorem 1.1. It satisfies $\deg E = 2\deg F\in 2\mathbb{Z}$.

Examples of such representations are given in (2.1) ($n=2$) and in (2.4). Or simply take $0\neq\alpha\in H^0(X,\omega)$ and the connection

$$(\mathcal{O}\oplus\mathcal{O}, d+\begin{pmatrix} 0 & \alpha \\ 0 & 0\end{pmatrix})$$

if the genus is ≥ 1.

5. Some three-dimensional representations

Bolibruch's first class of representations

$$\rho:\pi_1(X\setminus\Sigma)\to GL(n,\mathbb{C})$$

for $X=\mathbb{P}^1_{\mathbb{C}}$, $\Sigma\subset X$ finite, which cannot be realized by a semistable bundle with a logarithmic connection (with poles only in Σ) has the following properties [1] (ch. 2):

(i) ρ is three-dimensional and reducible with a one-dimensional subrepresentation ρ'.
(ii) For each $p_i\in\Sigma$ the local monodromy of ρ around p_i has only one eigenvalue λ_i and one Jordan block.
(iii) If (E'',∇'') realizes $\rho'':=\rho/\rho'$ and if it is a Deligne extension at each point $p\in\Sigma$ then E'' is not semistable.

By theorem 1.1 it is obvious that ρ with (i) – (iii) cannot be realized by a semistable bundle with logarithmic connection (with poles only in Σ).

Remark 5.1. (iii) follows if one knows a single bundle (E'',∇'') with logarithmic connection which realizes ρ/ρ', which is a Deligne extension at each $p\in\Sigma$, and which is not semistable. Then any other bundle which realizes ρ/ρ' and which is a Deligne extension at each $p\in\Sigma$ is obtained from E'' by tensoring with a suitable line bundle.

Remark 5.2. If ρ'' is a two-dimensional representation with (ii) and $\prod_i\lambda_i=1$ and $|\Sigma|\geq 2$ then one can construct easily a three-dimensional representation ρ with (i) and (ii) and $\rho/\rho'=\rho''$.

We use the notations of section 2. Let ρ'' be given by 2×2-matrices $A_1'', B_1'', ..., A_g'', B_g''$ and $C_1'', ..., C_m''$. Define

$$A_i := \begin{pmatrix} 1 & 0 \; 0 \\ 0 & \\ 0 & A_i'' \end{pmatrix}, \; B_i := \begin{pmatrix} 1 & 0 \; 0 \\ 0 & \\ 0 & B_i'' \end{pmatrix},$$

$$C_i := \begin{pmatrix} \lambda_i & \gamma_{i1} \; \gamma_{i2} \\ 0 & \\ 0 & C_i'' \end{pmatrix}$$

for suitable γ_{ij} such that

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdot ... \cdot A_g B_g A_g^{-1} B_g^{-1} \cdot C_1 ... C_m = \mathrm{id}$$

holds. $\gamma_{1j}, ..., \gamma_{m-1j}$ can be chosen freely. γ_{m1} and γ_{m2} are given by two linear functions in $\gamma_{1j}, ..., \gamma_{m-1j}$ such that for each $i = 1, ..., m-1$ the linear parts in γ_{i1}, γ_{i2} of the two functions are together invertible. For generic solutions $\gamma_{1j}, ..., \gamma_{mj}$ the matrices C_i have only one Jordan block.

Bolibruch proved (iii) for his examples by quite involved explicit calculations. Other examples, for higher genus curves X can be obtained as follows.

Let $f : Z \to X$ be a proper semistable, nonisotrivial family of elliptic curves over a curve X. Let $Y \subset Z$ be the union of the bad fibers. The Gauß-Manin bundle

$$R^1 f_* \Omega^\bullet_{Z/X}(\log Y)$$

on X has rank 2, and the Gauss-Manin connection ∇ on it has logarithmic poles with milpotent residues along $\Sigma \subset f(Y)$ (Σ might be smaller, due to bad fibers of f inducing good fibers for the Jacobian family). It contains the positive subbundle $f_*\omega_{Z/X}$, thus is instable. Once such an f is chosen, one obtains other ones by considering the pullback family over any covering of X, étale on Σ. In particular, one can make the genus of X arbitrarily high.

References

[1] D.A. Anosov and A.A. Bolibruch, *The Riemann-Hilbert problem*, Aspects of Mathematics **22**, Vieweg Verlag, 1994.
[2] S. Bloch and H. Esnault, *A Riemann-Roch theorem for flat bundles, with values in the algebraic Chern-Simons theory*, Annals of Maths. **151** (2000), 1025–1070.
[3] A.A. Bolibruch, *On sufficient conditions for the positive solvability of the Riemann-Hilbert problem*, Mathem. Notes of the Ac. of Sci. of USSR **51:2** (1992), 110–117.

[4] W. Dekkers, *The matrix of a connection having regular singularities on a vector bundle of rank 2 on* $\mathbb{P}^1\mathbb{C}$, in Lecture Notes in Math. **712**, Springer Verlag, 1979, pp. 33–43.

[5] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Math. **163** (1970), Springer Verlag.

[6] H. Esnault and E. Viehweg, *Logarithmic de Rham complexes and vanishing theorems*, Invent. math. **86** (1086), 161–194.

[7] H. Esnault and E. Viehweg, *Semistable bundles on curves and irreducible representations of the fundamental group*, in *Algebraic geometry, Hirzebruch 70* (Warsaw, 1998), Contemp. Math. **241**, Amer. Math. Soc., Providence, RI, 1999, pp. 129–138.

[8] V.P. Kostov, *Fuchsian linear systems on* $\mathbb{CP}^1$ *and the Riemann-Hilbert problem*, C. R. Acad. Sci. Paris., Série I, **t. 315** (1992), 143–148.

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