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THE STEINBERG CURVE

By Hélène Esnault and Marc Levine

Abstract. Let E and E' be elliptic curves over $\mathbb{C}$. We construct non-torsion 0-cycles in the kernel of the Albanese mapping $\mathrm{CH}_0(E \times E')_{\deg 0} \to E \times E'$, which are not detectable by a certain class of cohomology theories, including the cohomology of the analytic motivic complex involving the dilogarithm function defined by S. Bloch. This is in contrast to the étale version of Bloch's complex defined by S. Lichtenbaum, which contains the Chow group.

0. Introduction. Let X be a smooth projective algebraic variety over $\mathbb{C}$, and let X_{an} be the associated compact complex manifold. The equivalence of the category of algebraic coherent sheaves on X with the category of analytic sheaves on X_{an}, proved in [16], yields the isomorphism

$$\mathrm{CH}^1(X) \cong H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}),$$

where $\mathrm{CH}^1(X)$ is the group of divisors modulo linear equivalence, and $\mathcal{O}_{X_{\mathrm{an}}}$ is the sheaf of analytic functions. This isomorphism, together with the exponential sequence

$$0 \to (2\pi i)\mathbb{Z} \to \mathcal{O}_{X_{\mathrm{an}}} \to \mathcal{O}^*_{X_{\mathrm{an}}} \to 1,$$

yields a direct connection of $\mathrm{CH}^1(X)$ with the Hodge theory of X_{an}.

It is natural to ask if the group of codimension p cycles on X modulo rational equivalence, $\mathrm{CH}^p(X)$, admits a similar description for $p > 1$. Presumably with this in mind, S. Bloch has introduced a complex for the analytic topology, denoted $\mathcal{B}(2)$. There is a natural map

$$\mathbb{H}^*(X_{\mathrm{an}}, \mathcal{B}(2)) \to H^*_{\mathcal{D}}(X, \mathbb{Z}(2)),$$

where $H^*_{\mathcal{D}}(X, \mathbb{Z}(2))$ is the weight two Deligne cohomology, as well as a cycle class

$$\mathrm{CH}^2(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \mathcal{B}(2)),$$

factorizing the cycle class to Deligne cohomology $\mathbb{H}^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ [5]. Our purpose in this article is to construct non-torsion cycles in $\mathrm{CH}^2(X)$ (X a product of two

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elliptic curves) which vanish in $\mathbb{H}^4(X_{\rm an}, \mathcal{B}(2))$. In fact, we construct non-torsion cycles which vanish in $\mathbb{H}^4(X_{\rm an}, \Gamma(2)_{\rm an})$, where $\Gamma(2)_{\rm an}$ is any complex of sheaves on $X_{\rm an}$ satisfying a modest list of axioms.

The cycles we construct come from the *Steinberg curve* on the product $E \times E'$ of elliptic curves. In the degenerate case of the product of nodal cubic curves $E_0 \times E_0$, the Steinberg curve is just the line $x + y = 1$ on $\mathbb{C}^* \times \mathbb{C}^*$, where $\mathbb{C}^*$ is the nonsingular locus of E_0. In general, we have the Tate parametrizations $\mathbb{C}^* \times \mathbb{C}^* \to E \times E'$, and the Steinberg curve is the image of $\{x + y = 1\} \subset \mathbb{C}^* \times \mathbb{C}^*$ in $E \times E'$. It turns out that, using the definition of $\mathrm{CH}_0(E_0 \times E_0)$ given by [8], $K_2(\mathbb{C})$ is a summand of $\mathrm{CH}_0(E_0 \times E_0)_0$, with the Steinberg curve parametrizing the classical Steinberg relation. However, the fact that the Steinberg curve is *nonalgebraic* unless $E = E' = E_0$ implies that the analog of the Steinberg relation is *not* satisfied in $\mathrm{CH}_0(E \times E')$, unless $E = E' = E_0$. Since one can use the cover $\mathbb{C}^* \times \mathbb{C}^* \to E \times E'$ to compute cohomology in the analytic topology, the analog of the Steinberg relation *is* satisfied in $\mathbb{H}^4(E_{\rm an} \times E'_{\rm an}, \Gamma(2)_{\rm an})$, where $\Gamma(2)_{\rm an}$ is as above.

Our results are in contrast with various results on cohomology theories defined via the étale topology, or using coefficients mod n. For instance, Lichtenbaum [9] has defined an étale version of weight two motivic cohomology, which receives the codimension two Chow groups injectively. Also, Raskind and Spieß [14] have shown that, for smooth elliptic curves E, E' defined over a p-adic field k, there is a surjective map of $K_2(k)/n$ onto the mod n Albanese kernel in $\mathrm{CH}_0(E \times E')/n$ for n prime to p.

We recall some well-known facts on the Tate parametrization of elliptic curves in §1. In §2 we introduce the Steinberg curve, and show that it parametrizes the Steinberg relation in $\mathrm{CH}_0(E_0 \times E_0)$. We then consider $\mathrm{CH}_0(E \times E')$, with at least one of E, E' smooth, and show that the 0-cycle in the Albanese kernel corresponding to a point u of the Steinberg curve is non-torsion in $\mathrm{CH}_0(E \times E')$ (outside of countably many points u). In §3, we list our axioms for the complex $\Gamma(2)_{\rm an}$, and show that the "Steinberg relation" holds in $\mathbb{H}^4(E_{\rm an} \times E'_{\rm an}, \Gamma(2)_{\rm an})$. In §4, we consider the problem of constructing a non-torsion cycle which vanishes in both $\mathbb{H}^4(X_{\rm an}, \mathcal{B}(2))$ and in the absolute Hodge cohomology $H^2(X, \Omega^2_{X/\mathbb{Q}})$. We give such an example for $X = E \times E_0$, $E \neq E_0$, but we are not able to handle the smooth case.

1. Tate curves and line bundles. For a scheme X over $\mathbb{C}$, we let $X_{\rm an}$ denote the set of $\mathbb{C}$-points with the classical topology. We let $\mathcal{O}_{X_{\rm an}}$ denote the sheaf of holomorphic functions on $X_{\rm an}$.

We begin by describing a construction of the universal analytic Tate curve over $\mathbb{C}$. We first form the analytic manifold $\hat{\mathcal{C}}^*$ as the quotient of the disjoint union $\sqcup_{i=-\infty}^{\infty} U_i$, with each $U_i = \mathbb{C}^2$, by the equivalence relation

$$(x, y) \in U_i \setminus \{Y = 0\} \sim \left(\frac{1}{y}, xy^2\right) \in U_{i+1} \setminus \{X = 0\}.$$

The function $\tilde{\pi}(x, y) = xy$ is globally defined on $\hat{\mathcal{C}}^*$. Letting $D \subset \mathbb{C}$ be the disk $\{|z| < 1\}$, we define $\mathcal{C}^* = \tilde{\pi}^{-1}(D)$, so $\tilde{\pi}$ restricts to the analytic map $\pi\colon \mathcal{C}^* \to D$. We let $\tilde{0}\colon D \to \mathcal{C}^*$ be the section $z \mapsto (z, 1) \in U_0$.

Let $D^* \subset D$ be the punctured disk $z \neq 0$. Since the map $(x, y) \mapsto (\frac{1}{y}, xy^2)$ is an automorphism of $(\mathbb{C}^*)^2$, the open submanifold $\pi^{-1}(D^*)$ of $\mathcal{C}^*$ is isomorphic to $(\mathbb{C}^*)^2$, and the restriction of the map π is just the map $(x, y) \mapsto xy$. Thus, the projection $p_2\colon (\mathbb{C}^*)^2 \to \mathbb{C}^*$ gives an isomorphism of the fiber $\mathcal{C}^*_t := \pi^{-1}(t)$ with $\mathbb{C}^*$, for $t \in D^*$.

The fiber $\pi^{-1}(0)$, on the other hand, is an infinite union of projective lines. Indeed, define the map $f_i\colon \mathbb{CP}^1 \to \mathcal{C}^*_0$ by sending $(a : 1) \in \mathbb{CP}^1 \setminus \infty$ to $(0, a) \in U_i$, and $\infty = (1 : 0)$ to $(0, 0) \in U_{i+1}$, and let $C_i = f_i(\mathbb{CP}^1)$. Then $\pi^{-1}(0) = \cup_{i=-\infty}^{\infty} C_i$, with $\infty \in C_i$ joined with $0 \in C_{i+1}$. Note in particular that the value $\tilde{0}(0)$ of the zero section avoids the singularities of $\pi^{-1}(0)$.

Define the automorphism ϕ of $\mathcal{C}^*$ over D by sending $(x, y) \in U_i$ to $(x, y) \in U_{i-1}$. This gives the action of $\mathbb{Z}$ on $\mathcal{C}^*$, with n acting by ϕ^n. It is easy to see that this action is free and proper, so the quotient space $\mathcal{E} := \mathcal{C}^*/\mathbb{Z}$ exists as a bundle $\pi\colon \mathcal{E} \to D$. The section $\tilde{0}\colon D \to \mathcal{C}^*$ induces the section $0\colon D \to \mathcal{E}$.

Take $t \in D^*$. Identifying $\mathcal{C}^*_t$ with $\mathbb{C}^*$ as above, we see that ϕ restricts to the automorphism $z \mapsto tz$. Thus, the fiber $\mathcal{E}_t := \pi^{-1}(t)$ for $t \in D^*$ is the *Tate elliptic curve* $\mathbb{C}^*/t^{\mathbb{Z}}$, with identity $0(t)$. On $\mathcal{C}^*_0$, however, ϕ is the union of the "identity" isomorphisms $C_i \to C_{i-1}$. Thus $\phi(\infty \in C_i) = 0 \in C_i$, so the restriction of $\mathcal{C}^*_0 \to \mathcal{E}_0$ to C_0 identifies $\mathcal{E}_0$ with the nodal curve $\mathbb{CP}^1/0 \sim \infty$. We let $* \in \mathcal{E}_0$ denote the singular point. Then $\tilde{0}(0) \in \mathcal{E}_0 \setminus *$.

The map $(t, w) \in D \times \mathbb{C}^* \mapsto (\frac{t}{w}, w) \in U_0$ gives an isomorphism $\psi\colon D \times \mathbb{C}^* \to U_0 \setminus \{Y = 0\}$ over D. The composition

$$D \times \mathbb{C}^* \to U_0 \setminus \{Y = 0\} \subset \mathcal{C}^* \xrightarrow{q} \mathcal{E}$$

defines the map $p\colon D \times \mathbb{C}^* \to \mathcal{E}$ over D, with image $\mathcal{E} \setminus \{*\}$.

Take $u \in \mathbb{C}^*$. We have the local system on $\mathcal{E}$

$$\mathcal{L}_u := \mathcal{C}^* \times \mathbb{C}/(z, \lambda) \sim (\phi(z), u\lambda) \to \mathcal{E},$$

and the associated holomorphic line bundle $\mathcal{L}^{\mathrm{an}}_u$ on $\mathcal{E}$.

Let E_t be the algebraic elliptic curve associated to the analytic variety $\mathcal{E}_t$, let $L_u(t)$ and $L^{\mathrm{an}}_u(t)$ denote the restriction of $\mathcal{L}_u$ and $\mathcal{L}^{\mathrm{an}}_u$ to $\mathcal{E}_t$, and let $L^{\mathrm{alg}}_u(t)$ be the algebraic line bundle on E_t corresponding to $L^{\mathrm{an}}_u(t)$ via [16]. The restriction of p to $t \times \mathbb{C}^*$ defines the map $p_t\colon \mathbb{C}^* \to E_{t\,\mathrm{an}}$. For $t \neq 0$, p_t is a covering space of $E_{t\,\mathrm{an}}$. The map $p_0\colon \mathbb{C}^* \to E_{0\,\mathrm{an}}$ is the analytic map associated to the algebraic open immersion

$$\mathbb{P}^1 \setminus \{0, \infty\} \xrightarrow{j} \mathbb{P}^1 \to \mathbb{P}^1/0 \sim \infty = E_0.$$

If E is an elliptic curve over $\mathbb{C}$, then $E_{\mathrm{an}} \cong \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice spanned by 1 and some τ in the upper half plane. Taking $t = e^{2\pi i\tau}$ gives the isomorphism $E_{\mathrm{an}} \cong \mathcal{E}_t$, so each elliptic curve over $\mathbb{C}$ occurs as an E_t for some (in fact for infinitely many) $t \in D^*$.

Sending $u \in \mathbb{C}^*$ to the isomorphism class of $L_u^{\mathrm{alg}}(t)$ defines a homomorphism $\tilde{p}_t\colon \mathbb{C}^* \to \mathrm{Pic}\,(E_t)$. We denote the identity $0(t) \in E_t$ simply by 0 if t is given.

LEMMA 1.1. *For all $t \in D$, $c_1(L_u^{\mathrm{alg}}(t)) = (p_t(u)) - (0)$.*

Proof. We first handle the case $t \neq 0$. Let $q\colon \mathbb{C} \to E := E_t$ be the map $q(z) = p_t(e^{2\pi iz})$, let $\tau \in \mathbb{C}$ be an element with $e^{2\pi i\tau} = t$, and let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and τ. The map q identifies E with $\mathbb{C}/\Lambda$, and $L_u(t)$ with the local system defined by the homomorphism $\rho\colon \Lambda \to \mathbb{C}^*$, $\rho(a + b\tau) = u^b$.

There is a unique cocycle θ in $Z^1(\Lambda, H^0(\mathbb{C}, \mathcal{O}^*_{\mathbb{C}_{\mathrm{an}}}))$ with $\theta(1) = 1$, $\theta(\tau) = e^{-2\pi iz}$; let L be the corresponding holomorphic line bundle on E. Computing $c_1^{\mathrm{top}}(L) \in H^2(E, \mathbb{Z})$ by using the exponential sequence, we find that $\deg(L) = 1$. By Riemann-Roch, we have $H^0(E, L) = \mathbb{C}$; let $\Theta(z)$ be the corresponding global holomorphic function on $\mathbb{C}$, i.e.,

$$\Theta(z + 1) = \Theta(z),\ \ \Theta(z + \tau) = e^{-2\pi iz}\Theta(z),$$

and the divisor of Θ on E is (x), with $L \cong \mathcal{O}_E(x)$.

Take $v, w \in \mathbb{C}$ with $u = e^{2\pi iv}$ and $q(w) = x$. Let $f(z) = \frac{\Theta(z+w-v)}{\Theta(z+w)}$. Then

$$f(z + 1) = f(z),\ \ f(z + \tau) = uf(z),$$

and $\mathrm{Div}(f) = (p(u)) - (0)$. Thus, multiplication by f defines an isomorphism

$$\times f\colon\ \mathcal{O}_{E_{\mathrm{an}}}((p(u)) - (0)) \to L_u^{\mathrm{an}}.$$

The proof for $E_0 = \mathbb{P}^1/0 \sim \infty$ is essentially the same, where we replace $\frac{\Theta(z+w-v)}{\Theta(z+w)}$ with the rational function $\frac{X-u}{X-1}$. □

Thus, the image of $\tilde{p}_t$ in $\mathrm{Pic}\,(E_t)$ is $\mathrm{Pic}^0\,(E)$. After identifying the smooth locus of E_t^0 of E_t with $\mathrm{Pic}^0\,(E_t)$ by sending $x \in E_t^0$ to the class of the invertible sheaf $\mathcal{O}_{E_t}((x) - (0))$, we have $\tilde{p}_t = p_t$.

2. The Albanese kernel and the Steinberg relation. Let X be a smooth projective variety. We let $\mathrm{CH}_0\,(X)$ denote the group of zero cycles on X, modulo rational equivalence, $F^1\,\mathrm{CH}_0\,(X)$ the subgroup of cycles of degree zero, and $F^2\,\mathrm{CH}_0\,(X)$ the kernel of the Albanese map $\alpha_X\colon F^1\,\mathrm{CH}_0\,(X) \to \mathrm{Alb}\,(X)$. The choice of a point $0 \in X$ gives a splitting to the inclusion $F^1\,\mathrm{CH}_0\,(X) \to \mathrm{CH}_0\,(X)$.

Let E, E' be smooth elliptic curves. As $\mathrm{Alb}\,(E \times E') = E \times E'$, the inclusion $F^2\,\mathrm{CH}_0\,(E \times E') \to F^1\,\mathrm{CH}_0\,(E \times E')$ is split by sending $(x, y) - (0, 0)$ to $(x, y) -$

$(x,0) - (0,y) + (0,0)$. Thus $F^2\,\mathrm{CH}_0\,(E \times E')$ is generated by zero-cycles of the form $(x,y) - (x,0) - (0,y) + (0,0)$. Choosing an isomorphism $E \cong E_t$, $E' \cong E_{t'}$, we have the covering spaces $p\colon\ \mathbb{C}^* \to E_{\mathrm{an}}$, $p'\colon\ \mathbb{C}^* \to E'_{\mathrm{an}}$, and the map

$$\begin{aligned} p * p'\colon\ \mathbb{C}^* \otimes \mathbb{C}^* &\to F^2\,\mathrm{CH}_0\,(E \times E') \\ u \otimes v &\mapsto p(u) * p'(v) := \\ (p(u), p'(v)) - (p(u),0) &- (0, p'(v)) + (0,0). \end{aligned} \tag{2.1}$$

By the theorem of the cube [11], the map $p * p'$ is a well-defined group homomorphism, and thus is surjective.

In case one or both of E, E' is the singular curve E_0, we will need to use the theory of zero-cycles mod rational equivalence defined in [8]. If X is a reduced, quasi-projective variety over a field k with singular locus X_{sing}, the group $\mathrm{CH}_0\,(X)$ (denoted $\mathrm{CH}_0\,(X, X_{\mathrm{sing}})$ in [8]) is defined as the quotient of the free abelian group on the regular closed points of X, modulo the subgroup generated by zero-cycles of the form $\mathrm{Div} f$, where f is a rational function on a dimension one closed subscheme D of X such that:

(1) No irreducible component of D is contained in X_{sing}.

(2) In a neighborhood of each point of $D \cap X_{\mathrm{sing}}$, the subscheme D is a complete intersection.

(3) f is in the subgroup $\mathcal{O}^*_{D, D\cap X_{\mathrm{sing}}}$ of $k(D)^*$.

It follows in particular from these conditions that $\mathrm{Div} f$ is a sum of regular points of X.

For X a reduced curve, sending a regular closed point $x \in X$ to the invertible sheaf $\mathcal{O}_X(x)$ extends to give an isomorphism $\mathrm{CH}_0\,(X) \cong \mathrm{Pic}\,(X)$.

We extend the definition of $F^i\,\mathrm{CH}_0$ to $E \times E'$ with either $E = E_0$, $E' = E_0$ or $E = E' = E_0$, by defining $F^1\,\mathrm{CH}_0\,(E \times E')$ as the subgroup of $\mathrm{CH}_0\,(E \times E')$ generated by the differences $[x] - [y]$, and $F^2\,\mathrm{CH}_0\,(E \times E')$ the subgroup generated by expressions $[(x,y)] - [(x,0)] - [(0,y)] + [(0,0)]$, where x is a smooth point of E and y a smooth point of E'. The surjection $p * p'\colon\ \mathbb{C}^* \otimes \mathbb{C}^* \to F^2\,\mathrm{CH}_0\,(E \times E')$ is then defined by the same formula as (2.1).

PROPOSITION 2.1. (The Steinberg relation) *Take $E = E' = E_0$. Then*

(1) *There is an isomorphism $\mathrm{CH}_0\,(E_0 \times E_0) \cong \mathbb{Z} \oplus (\mathbb{C}^* \times \mathbb{C}^*) \oplus K_2(\mathbb{C})$, sending $F^2\,\mathrm{CH}_0\,(E_0 \times E_0)$ onto the summand $K_2(\mathbb{C})$.*

(2) *$p(u) * p(1-u) = 0$ in $\mathrm{CH}_0\,(E_0 \times E_0)$ for all $u \in \mathbb{C} \setminus \{0,1\}$.*

Proof. Let X be a quasi-projective surface over a field k. By [7], there is an isomorphism $\phi\colon\ H^2(X, \mathcal{K}_2) \to \mathrm{CH}_0\,(X)$. The product $\mathcal{O}^*_X \otimes \mathcal{O}^*_X \to \mathcal{K}_2$ gives the cup product

$$H^1(X, \mathcal{O}^*_X) \otimes H^1(X, \mathcal{O}^*_X) \xrightarrow{\cup} H^2(X, \mathcal{K}_2).$$

In addition, let D, D' be Cartier divisors which intersect properly on X, and suppose that $\operatorname{supp} D \cap \operatorname{supp} D' \cap X_{\mathrm{sing}} = \emptyset$. Then

$$\phi(\mathcal{O}_X(D) \cup \mathcal{O}_X(D')) = [D \cdot D'], \tag{2.2}$$

where $\cdot$ is the intersection product and $[-]$ denotes the class in CH_0.

Since $L_u^{\mathrm{alg}} = \mathcal{O}_{E_0}(p(u) - 0)$, (2.2) implies

$$p(u) * p(1-u) = \rho(p_1^* L_u^{\mathrm{alg}} \cup p_2^* L_{1-u}^{\mathrm{alg}}),$$

so to prove (2), it suffices to show that $p_1^* L_u^{\mathrm{alg}} \cup p_2^* L_{1-u}^{\mathrm{alg}} = 0$ in $H^2(E_0 \times E_0, \mathcal{K}_2)$.

Write X for $E_0 \times E_0$. Let $\bar{\mathcal{K}}_2$ be the image of $\mathcal{K}_2$ in the constant sheaf $K_2(\mathbb{C}(X))$. By Gersten's conjecture [13, §7, Theorem 5.11], the surjection π: $\mathcal{K}_2 \to \bar{\mathcal{K}}_2$ is an isomorphism at each regular point of X, hence π induces an isomorphism on H^2.

Let q: $\mathbb{P}^1 \to E_0$ be the normalization, giving the normalization $q \times q$: $\mathbb{P}^1 \times \mathbb{P}^1 \to X$. Let i: $* \to E_0$ be the inclusion of the singular point. We have the exact sequence of sheaves on E_0

$$q_* \mathcal{K}_1 \xrightarrow{\beta} i_* K_1(\mathbb{C}) \to 0 \tag{2.3}$$

and the exact sequence of sheaves on X:

$$(q \times q)_* \mathcal{K}_2 \xrightarrow{\alpha} (i \times q)_* \mathcal{K}_2 \oplus (q \times i)_* \mathcal{K}_2 \to (i \times i)_* K_2(\mathbb{C}) \to 0, \tag{2.4}$$

with augmentations ϵ_1: $\mathcal{K}_1 \to (2.3)$, ϵ_2: $\bar{\mathcal{K}}_2 \to (2.4)$. The various cup products in K-theory give the map of complexes

$$p_1^*(2.3) \otimes p_2^*(2.3) \to (2.4), \tag{2.5}$$

compatible with the cup product

$$p_1^* \mathcal{K}_1 \otimes p_2^* \mathcal{K}_1 \to \bar{\mathcal{K}}_2. \tag{2.6}$$

The augmentation ϵ_1: $\mathcal{K}_1 \to \ker \beta$ is an isomorphism. The augmentation ϵ_2: $\bar{\mathcal{K}}_2 \to \ker \alpha$ is an injection, and the cokernel is supported on $* \times *$. Indeed, by [6, Lemma 1.15 and Corollary 1.16], there is an isomorphism of sheaves on $X \setminus \{* \times *\}$,

$$\mathcal{I}/\mathcal{I}^2 \otimes (q \times q)_* \Omega^1_{(q \times q)^{-1}(X_{\mathrm{sing}})/X_{\mathrm{sing}}} \cong \ker \alpha / \bar{\mathcal{K}}_2,$$

where $\mathcal{I}$ is the ideal sheaf of X_{sing}. Since $(q \times q)^{-1}(X_{\mathrm{sing}}) \to X_{\mathrm{sing}}$ is étale away from $* \times *$, the relative differentials vanish, verifying our claim. Thus,

$\epsilon_2\colon \bar{\mathcal{K}}_2 \to \ker\alpha$ induces an isomorphism on H^2, and the complexes (2.3) and (2.4) give rise to maps

$$\delta_2\colon K_2(\mathbb{C}) \to H^2(X, \ker\alpha) = H^2(X, \bar{\mathcal{K}}_2) = H^2(X, \mathcal{K}_2)$$
$$\delta_1\colon \mathbb{C}^* = K_1(\mathbb{C}) \to H^1(E_0, \mathcal{K}_1).$$

The compatibility of (2.5) with (2.6) yields the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}^* \otimes \mathbb{C}^* & \xrightarrow{\ \cup\ } & K_2(\mathbb{C}) \\ \Big\downarrow{\scriptstyle \delta_1 \otimes \delta_1} & & \Big\downarrow{\scriptstyle \delta_2} \\ H^1(E_0, \mathcal{K}_1) \otimes H^1(E_0, \mathcal{K}_1) & \xrightarrow[p_1^* \cup p_2^*]{} & H^2(X, \mathcal{K}_2) \end{array}$$

Since $L_v^{\mathrm{alg}} = \delta_1(v)$ for each $v \in \mathbb{C}^*$, we have

$$p_1^* L_u^{\mathrm{alg}} \cup p_2^* L_{1-u}^{\mathrm{alg}} = \delta_2(\{u, 1-u\}) = 0.$$

completing the proof of (2). Similarly, since

$$p_1^* L_u^{\mathrm{alg}} \cup p_2^* L_v^{\mathrm{alg}} = \delta_2(\{u, v\}),$$

we see that $F^2 \mathrm{CH}_0(X)$ is the image of $K_2(\mathbb{C})$ in $H^2(X, \mathcal{K}_2)$.

For (1), we have the isomorphisms

$$H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{K}_2) \cong \mathbb{Z}, \ \ H^1(\mathbb{P}^1, \mathcal{K}_2) \cong \mathbb{C}^*,$$

the generator of $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{K}_2)$ being the class of a point in $\mathrm{CH}_0(\mathbb{P}^1 \times \mathbb{P}^1)$, and the map $\mathbb{C}^* \to H^1(\mathbb{P}^1, \mathcal{K}_2)$ being induced by the Gysin map $\mathbb{C}^* = H^0(\operatorname{Spec}\mathbb{C}, \mathcal{K}_1) \to H^1(\mathbb{P}^1, \mathcal{K}_2)$ for the inclusion of a point (this follows from the projective bundle formula for $\mathcal{K}$-cohomology). Using Gersten's conjecture *loc. cit.* and the Gersten resolution of $\mathcal{K}_2$ [13, §7, Proposition 5.8], we have

$$R^j(q \times q)_* \mathcal{K}_2 = 0 = R^j(i \times q)_* \mathcal{K}_2; \quad j > 0$$

and

$$H^j(\mathbb{P}^1, \mathcal{K}_2) = 0; \quad j > 1.$$

Fix a smooth point y of E_0. The inclusion $y \times y \to X$ induces the map $\mathbb{Z} \to \mathrm{CH}_0(X)$. The inclusions $E_0 \times y \to X$, $y \times E_0 \to X$ induce the map

$$\mathbb{C}^* \times \mathbb{C}^* = F^1 \mathrm{CH}_0(E_0) \times F^1 \mathrm{CH}_0(E_0) \to \mathrm{CH}_0(X);$$

one checks that these maps correspond to the terms $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ and $H^1(\mathbb{P}^1,\mathcal{K}_2)\times H^1(\mathbb{P}^1,\mathcal{K}_2)$ in the spectral sequence arising from the resolution (2.4) of $\ker\alpha$. Thus, this spectral sequence gives the exact sequence

$$\begin{aligned} H^1(X,\ker\alpha) &\xrightarrow{\gamma} H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)\to\mathbb{Z}\oplus\mathbb{C}^*\times\mathbb{C}^*\oplus K_2(\mathbb{C})\\ &\to H^2(X,\mathcal{K}_2)\to 0. \end{aligned}$$

To complete the proof of (1), we need only show that γ is surjective.

Let $x = q^{-1}(y)$, giving the inclusions i_1: $\mathbb{P}^1\times x\to\mathbb{P}^1\times\mathbb{P}^1$, i_2: $x\times\mathbb{P}^1\to\mathbb{P}^1\times\mathbb{P}^1$. By the projective bundle formula, $H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ is isomorphic to $\mathbb{C}^*\oplus\mathbb{C}^*$, with each $\mathbb{C}^*$ given as the image under Gysin of the maps

$$i_{j*}\colon\ \mathbb{C}^* = H^0(\mathbb{P}^1,\mathcal{K}_1)\to H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2);\quad j=1,2.$$

We can factor say i_{1*} as the composition

$$\mathbb{C}^* = H^0(\operatorname{Spec}\mathbb{C},\mathcal{K}_1)\xrightarrow{i_{x*}} H^1(\mathbb{P}^1,\mathcal{K}_2)\xrightarrow{p_2^*} H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2).$$

Since y is a smooth point of E_0, we have

$$H^1_y(E_0,\mathcal{K}_2)\cong H^1_x(\mathbb{P}^1,\mathcal{K}_2)\cong H^0(x,\mathcal{K}_1),$$

so we have the Gysin map $H^0(\operatorname{Spec}\mathbb{C},\mathcal{K}_1)\xrightarrow{i_{y*}} H^1(E_0,\mathcal{K}_2)$ with $q^*\circ i_{y*} = i_{x*}$. Also, the map

$$(q\times q)^*\circ p_2^*\colon\ \mathcal{K}_{2E_0}\to\mathcal{K}_{2\mathbb{P}^1\times\mathbb{P}^1}$$

factors through $\ker\alpha$. Thus, we see that the factor $i_{1*}(\mathbb{C}^*)$ of $H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ is in the image of γ. The factor $i_{2*}(\mathbb{C}^*)$ is handled similarly. □

In contrast to Proposition 2.1, the Steinberg relation is *not* satisfied in $\mathrm{CH}_0\,(E\times E')$ if at least one of E, E' is smooth. To show this, we first require the following lemma:

LEMMA 2.2. *Let* s: $\mathbb{C}\setminus\{0,1\}\to E\times E'$ *be the analytic map* $s(u) = (p(u), p'(1-u))$. *Then* $s(\mathbb{C}\setminus\{0,1\})$ *is not contained in any algebraic curve on* $E\times E'$, *except in case* $E = E' = E_0$.

Proof. We first consider the case in which both E and E' are smooth elliptic curves, $E = E_t$, $E' = E_{t'}$, where t and t' are in $\mathbb{C}^*$ and $|t|<1$, $|t'|<1$. We have the maps

$$p\colon\ \mathbb{C}^*\to E,\ p'\colon\ \mathbb{C}^*\to E',$$

which are group homomorphisms with $\ker p = t^{\mathbb{Z}}$, $\ker p' = t'^{\mathbb{Z}}$.

Suppose that $s(\mathbb{C}^*)$ is contained in an algebraic curve $D \subset E \times E'$. For each $x \in E$, $(x \times E') \cap D$ is a finite set, hence, for each $u \in \mathbb{C} \setminus \{0, 1\}$, the set of points of $\mathbb{C}^* \times \mathbb{C}^*$ of the form $(t^n u, 1 - t^n u)$ has finite image in $E \times E'$. Thus, for each u, there are integers n, m and p, depending on u, such that $n \neq m$ and

$$1 - t^m u = t'^p (1 - t^n u). \tag{2.7}$$

Since there are uncountably many u, there is a single choice of n, m and p for which (2.7) holds for uncountably many u. But then

$$(t'^p t^n - t^m) u = t'^p - 1. \tag{2.8}$$

If $t'^p t^n - t^m = 0$, then $|t'| = 1$, contradicting the condition $|t'| < 1$. If $t'^p t^n - t^m \neq 0$, then we can solve (2.8) for u, so (2.7) only holds for this single u, a contradiction.

If say $E' = E_0$, then $p'\colon \mathbb{C}^* \to E'$ is injective, and we have the infinite set of points $p'(1 - t^n u)$ in the image of s, all lying over the single point $p(u)$. □

THEOREM 2.3. *Let $E = E_t$, $E' = E_{t'}$, with at least one of E, E' nonsingular. Then, for all u outside a countable subset of $\mathbb{C} \setminus \{0, 1\}$, $p(u) * p'(1 - u)$ is not a torsion element in $F^2 \operatorname{CH}_0(E \times E')$.*

Proof. We first give the proof in case E and E' are both nonsingular. For a quasi-projective $\mathbb{C}$-scheme X, we let $S^n X$ denote the nth symmetric power of X. For X smooth, we have the map

$$\begin{aligned} \rho_n\colon S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) &\to \operatorname{CH}_0(X) \\ \left(\sum_{i=1}^n x_i, \sum_{j=1}^n y_j\right) &\mapsto \left[\sum_{i=1}^n x_i - \sum_{j=1}^n y_j\right]. \end{aligned}$$

For each integer $n \geq 1$, we have the morphism

$$\begin{aligned} \phi_n\colon E \times E' &\to S^{2n}(E \times E') \times S^{2n}(E \times E') \\ (x, y) &\mapsto (n(x, y) + n(0, 0), n(x, 0) + n(0, y)), \end{aligned}$$

By [15, Theorem 1], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of Zariski closed subsets of $E \times E'$.

On the other hand, since $p_g(E \times E') = 1$, the Albanese kernel $F^2 \operatorname{CH}_0(E \times E')$ is "infinite dimensional" [10]; in particular, $F^2 \operatorname{CH}_0(E \times E')_{\mathbb{Q}} \neq 0$. Since $F^2 \operatorname{CH}_0(E \times E')$ is generated by cycles of the form $p(u) * p(v)$, it follows that $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of *proper* closed subsets of $E \times E'$. If D is a proper Zariski closed subset of $E \times E'$, then, by Lemma 2.2, $s^{-1}(D)$ is a proper closed analytic subset of $\mathbb{C} \setminus \{0, 1\}$, hence $s^{-1}(D)$ is countable. Thus, the set of $u \in \mathbb{C} \setminus \{0, 1\}$ such that $p(u) * p'(1 - u)$ is torsion is countable, which completes the proof in case both E and E' are nonsingular.

If say $E' = E_0$, we use essentially the same proof. We let X be the open subscheme $E \times (E_0 \setminus \{*\})$ of $E \times E_0$. We have the map ρ_n: $S^nX(\mathbb{C}) \times S^nX(\mathbb{C}) \to \mathrm{CH}_0\,(E \times E_0)$ defined as above. By [8, Theorem 4.3], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of closed subsets D_i of X. By [17], we have the similar infinite dimensionality result for $\mathrm{CH}_0\,(E \times E_0)$ as in the smooth case, from which it follows that each D_i is a proper closed subset of X. Thus, the closure of each D_i in $E \times E_0$ is a proper algebraic subset of $E \times E_0$. The same argument as in the smooth case finishes the proof. $\square$

3. Indetectability. The zero-cycle $p(u) * p(1 - u)$ is indetectable by cohomology theories based on the sheaf $\mathcal{O}^*_{E_{\mathrm{an}} \times E'_{\mathrm{an}}}$. We first consider the following abstract situation.

The exponential sequence

$$0 \to \mathbb{Z}(1) \xrightarrow{\iota} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$$

defines a projective resolution of the group $\mathbb{C}^*$, so the complex

$$\mathbb{C}^*(1) \xrightarrow{\iota \otimes \mathrm{id}} \mathbb{C} \otimes \mathbb{C}^*$$

represents the derived tensor product $\mathbb{C}^* \otimes^L \mathbb{C}^*$. Let $\Gamma_0(2)$ be the complex:

$$\begin{aligned} \mathbb{Z}[\mathbb{C} \setminus \{0,1\}] \oplus \mathbb{C}^*(1) &\to \mathbb{C} \otimes \mathbb{C}^* \\ (u, 2\pi i n \otimes z) &\mapsto \log(1-u) \otimes u + 2\pi i n \otimes z, \end{aligned}$$

with $\mathbb{C} \otimes \mathbb{C}^*$ in degree two (we make some choice of $\log(1-u)$ for each $u \in \mathbb{C} \setminus \{0,1\}$).

Let $X = E \times E'$, and let $\Gamma(2)_{\mathrm{an}}$ be a complex of sheaves on X_{an} with the following properties:

(3.1)

(1) There is a group homomorphism cl: $\mathrm{CH}_0\,(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}})$.

(2) There is a map ρ: $\mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}[-2] \to \Gamma(2)_{\mathrm{an}}$ in the derived category of sheaves $D(\mathrm{Sh}_{X_{\mathrm{an}}})$.

(3) The composition

$$\mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}[-2] \to \Gamma(2)_{\mathrm{an}}$$

extends to a map $\Gamma_0(2) \to \Gamma(2)_{\mathrm{an}}$ in $D(\mathrm{Sh}_{X_{\mathrm{an}}})$.

(4) The composition

$$\begin{aligned} \mathrm{Pic}\,(X) \otimes \mathrm{Pic}\,(X) &\cong H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}) \otimes H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}) \\ &\xrightarrow{\cup} \mathbb{H}^2(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}) \xrightarrow{\rho} \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}}) \end{aligned}$$

agrees with the composition

$$\mathrm{Pic}\,(X) \otimes \mathrm{Pic}\,(X) \xrightarrow{\cup} \mathrm{CH}_0\,(X) \xrightarrow{\mathrm{cl}} \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}}).$$

Remark 3.1. To justify the axioms above, at least in case X is smooth, we note the following: Let Y be a scheme smooth and of finite type over a field k. There is a complex of sheaves $\Gamma_Y(q)$ on Y_{Zar} whose hypercohomolgy computes the *motivic cohomology* of Y,

$$H^p(Y, \mathbb{Z}(q)) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \Gamma_X(q)).$$

The complexes $\Gamma_Y(q)$ have products $\Gamma_Y(q) \otimes^L \Gamma_Y(q') \to \Gamma_Y(q+q')$ in the derived category, and the assignment $Y \mapsto \Gamma_Y(q)$ extends to a functor from smooth $\mathbb{C}$-schemes of finite type to the derived category of sheaves on the big Zariski site of smooth schemes of finite type over $\mathbb{C}$. $\Gamma_Y(0) = \mathbb{Z}_{Y_{\mathrm{Zar}}}$ and $\Gamma_Y(1) = \mathcal{O}_Y^*[-1]$ in the derived category. For details, see e.g. [19].

Suppose we have complexes of sheaves on Y_{an}, $\Gamma_Y(q)_{\mathrm{an}}$, for $q = 1, 2$, functorial in Y, with natural products $\Gamma_Y(1)_{\mathrm{an}} \otimes^L \Gamma_Y(1)_{\mathrm{an}} \to \Gamma_Y(2)_{\mathrm{an}}$, and with $\Gamma_Y(1) = \mathcal{O}^*_{Y_{\mathrm{an}}}$. Suppose in addition we have maps in the derived category of sheaves on the big Zariski site of smooth quasi-projective $\mathbb{C}$-schemes

$$\theta_q\colon\ \Gamma_{(-)}(q) \to R\epsilon_*\Gamma(-)(q)_{\mathrm{an}};\ \ q = 1, 2,$$

where ϵ is the change of topology morphism, such that the θ are compatible with the products. Finally, suppose that θ_1 is the canonical map adjoint to the inclusions $\epsilon^*\mathcal{O}_Y^* \to \mathcal{O}^*_{Y_{\mathrm{an}}}$. Then $\Gamma(2)_{\mathrm{an}} := \Gamma_X(2)_{\mathrm{an}}$ satisfies the axioms above. Indeed, since $K_2(\mathbb{C}) = H^2(\mathrm{Spec}\,\mathbb{C}, \mathbb{Z}(2)) = H^2(\Gamma_{\mathrm{Spec}\,\mathbb{C}}(2))$ (see [12], [18]) the product map

$$\mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \Gamma_{\mathrm{Spec}\,\mathbb{C}}(2)$$

extends to a map $\Gamma_0(2) \to \Gamma_{\mathrm{Spec}\,\mathbb{C}}(2)$. Composing this with the map $p_X^*\colon\ \Gamma_{\mathrm{Spec}\,\mathbb{C}}(2) \to \Gamma_X(2)$ given by the structure morphism, and then with $\theta_2(X)$, verifies axiom (3). The remaining axioms follow from the isomorphisms

$$H^2(X, \mathbb{Z}(1)) \cong \mathrm{CH}^1\,(X) \cong \mathrm{Pic}\,(X), H^4(X, \mathbb{Z}(2)) \cong \mathrm{CH}^2\,(X),$$

compatible with the various products.

THEOREM 3.2. *Let $E = E_t$ and $E' = E_{t'}$, and let $\Gamma(2)_{\mathrm{an}}$ be a complex of sheaves on $E_{\mathrm{an}} \times E'_{\mathrm{an}}$ satisfying the conditions (3.1). Then* $\mathrm{cl}\,(p(u) * p(1-u)) = 0$ *for all* $u \in \mathbb{C} \setminus \{0, 1\}$.

Proof. We give the proof in case both E and E' are nonsingular; the singular case is similar, but easier, and is left to the reader.

Since

$$p(u) * p(1-u) = [p_1^* c_1(L_u^{\mathrm{alg}})] \cap [p_2^* c_1(L_{1-u}^{\mathrm{alg}})],$$

it follows from (3.1)(4) that we need to show that $\rho([L_u^{\mathrm{an}}] \cup [L_{1-u}^{\mathrm{an}}]) = 0$. The class $[L_u^{\mathrm{an}}] \in H^1(E_{\mathrm{an}}, \mathcal{O}^*_{E_{\mathrm{an}}})$ is the image of $[L_u] \in H^1(E_{\mathrm{an}}, \mathbb{C}^*)$ under the map of sheaves $\mathbb{C}^* \to \mathcal{O}^*_{E_{\mathrm{an}}}$, and similarly for L_{1-u} and L_{1-u}^{an}. Thus, by (3.1)(3), it suffices to see that $p_1^*[L_u] \cup p_2^*[L_{1-u}] \in \mathbb{H}^2(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \mathbb{C}^* \otimes^L \mathbb{C}^*)$ vanishes in $\mathbb{H}^4(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma_0(2))$.

The $\mathbb{Z}$-covers $p\colon \mathbb{C}^* \to E = E_t$, $p'\colon \mathbb{C}^* \to E' = E_{t'}$ give natural maps

$$\begin{aligned} \alpha&\colon H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E_{\mathrm{an}}, \mathbb{C}^*), \\ \beta&\colon H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E'_{\mathrm{an}}, \mathbb{C}^*). \end{aligned}$$

Similarly, the $\mathbb{Z}^2$-cover $p \times p'\colon \mathbb{C}^* \times \mathbb{C}^* \to E \times E'$ gives the natural map

$$\gamma\colon \mathbb{H}^*(\mathbb{Z}^2, H^0(\mathbb{C}^* \times \mathbb{C}^*, \Gamma_0(2))) \to \mathbb{H}^*(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma_0(2)).$$

Letting $\iota\colon \mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \Gamma_0(2)$ denote the natural map, the maps above are compatible with the respective cup products:

$$\iota \circ (\alpha(a) \cup \beta(b)) = \gamma \circ \iota(a \cup b).$$

Each $v \in \mathbb{C}^*$ gives the corresponding homomorphism $v\colon \mathbb{Z} \to \mathbb{C}^*$, $v(n) = v^n$. Since $[L_u] \in H^1(E_{\mathrm{an}}, \mathbb{C}^*)$ is $\alpha(u\colon \mathbb{Z} \to \mathbb{C}^*)$ and $[L_{1-u}] \in H^1(E'_{\mathrm{an}}, \mathbb{C}^*)$ is $\beta(1-u\colon \mathbb{Z} \to \mathbb{C}^*)$, it suffices to show that $\iota(p_1^* u \cup p_2^*(1-u)) = 0$ in $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2))$, where $p_1^* u, p_2^*(1-u)\colon \mathbb{Z}^2 \to \mathbb{C}^*$ are the respective homomorphisms $(a,b) \mapsto u^a$, and $(a,b) \mapsto (1-u)^b$.

We have the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}^2, H^q(\Gamma_0(2))) \Longrightarrow \mathbb{H}^{p+q}(\mathbb{Z}^2, \Gamma_0(2)).$$

Since $\mathbb{Z}^2$ has cohomological dimension two, and since $H^q(\Gamma_0(2)) = 0$ for $q \neq 1, 2$, it follows that the natural map $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2)) \to H^2(\mathbb{Z}^2, H^2(\Gamma_0(2)))$ is an isomorphism. Since $H^2(\Gamma_0(2)) = K_2(\mathbb{C})$, we need to show that the image of $p_1^* u \cup p_2^*(1-u)$ in $H^2(\mathbb{Z}^2, K_2(\mathbb{C}))$ is zero.

By definition of the cup product in group cohomology, we have

$$\begin{aligned} [p_1^* u \cup p_2^*(1-u)]((a,b),(c,d)) &= p_1^* u(a,b) \otimes p_2^*(1-u)(c-a, d-b) \\ &= u^a \otimes (1-u)^{d-b}, \end{aligned}$$

which clearly vanishes in $K_2(\mathbb{C})$. □

As an immediate consequence of Theorem 3.2, we have:

COROLLARY 3.3. *Let E, E' and $\Gamma(2)_{\mathrm{an}}$ be as in Theorem 3.2. Then the composition*

$$\mathbb{C}^* \otimes \mathbb{C}^* \xrightarrow{p*p'} \mathrm{CH}_0\,(E \times E') \xrightarrow{\mathrm{cl}} \mathbb{H}^4(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}})$$

factors through the surjection $\mathbb{C}^* \otimes \mathbb{C}^* \to K_2(\mathbb{C})$.

Example 3.4. In [3], S. Bloch defines a quotient complex $\mathcal{B}(2)_X$ of the analytic complex $\mathcal{O}^*_{X_{\mathrm{an}}}(1) \xrightarrow{\iota\otimes\mathrm{id}} \mathcal{O}_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}}$ fulfilling $\mathcal{H}^i(\mathcal{B}(2)) = 0$ for $i \neq 1, 2$,

$$\mathcal{H}^1(\mathcal{B}(2)) = \mathrm{Im}\left(r\colon\ K_{3,\mathrm{ind}}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}(2)\right) =: \Delta^*(1),$$

where r is the regulator map, and $\mathcal{H}^2(\mathcal{B}(2)) = \mathcal{K}_{2,\mathrm{an}}$. He shows in the same article that $r(K_{3,\mathrm{ind}}(\mathbb{C})) = r(K_{3,\mathrm{ind}}(\mathbb{Q}))$, thus $\Delta^*(1)$ is a countable subgroup of $\mathbb{C}/\mathbb{Z}(2)$, and also that $\mathcal{B}(2)$ maps to the complex $\mathbb{Z}(2) \to \mathcal{O}_{X_{\mathrm{an}}} \to \Omega^1_{X_{\mathrm{an}}}$ which computes the Deligne cohomology $H^*_{\mathcal{D}}(X, 2)$ when X is projective smooth over $\mathbb{C}$. In fact, the cycle map $\mathrm{CH}^2\,(X) \to H^4_{\mathcal{D}}(X, 2)$ is shown to factor through $\mathbb{H}^4(X_{\mathrm{an}}, \mathcal{B}(2))$ [5]. Bloch asked in [4] whether the cycle map $\mathrm{CH}^2\,(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \mathcal{B}(2))$ could possibly be injective. The computations of this article show that it is not. Indeed, the complex $\mathcal{B}(2)_X$ is defined as

$$\mathcal{B}(2)_X := \mathcal{O}^*_{X_{\mathrm{an}}}(1) \xrightarrow{\iota\otimes\mathrm{id}} \mathcal{O}_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}}/\epsilon(\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]),$$

where $\epsilon\colon\ \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \to \mathbb{C} \otimes \mathbb{C}^*$ is the map defined on generators $a \in \mathbb{C} \setminus \{0, 1\}$ by

$$\epsilon(a) = \log(1 - a) \otimes a - \left[2\pi i \otimes \exp\left(\frac{-1}{2\pi i}\int_0^a \log(1 - t)\frac{dt}{t}\right)\right].$$

Let us take $\Gamma(2)_{\mathrm{an}} = \mathcal{B}(2)$. We now verify the conditions 3.1. The complex $\mathcal{O}^*_{X_{\mathrm{an}}}(1) \xrightarrow{\iota\otimes\mathrm{id}} \mathcal{O}_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}}$ represents $\mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}$, so the evident surjection of complexes gives us a map $\mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}} \to \mathcal{B}(2)_X$. Also, the complexes $\mathcal{B}(2)_X$ are clearly contravariantly functorial in X, so to verify (3), it suffices to extend the map $\mathbb{C}^* \otimes^L \mathbb{C}^* \to \mathcal{B}(2)_{\mathrm{Spec}\,\mathbb{C}}$ to a map $\Gamma_0(2) \to \mathcal{B}(2)_{\mathrm{Spec}\,\mathbb{C}}$. We have the evident surjection

$$(\mathbb{C}^*(1) \to \mathbb{C} \otimes \mathbb{C}^*) \to \mathcal{B}(2)_{\mathrm{Spec}\,\mathbb{C}},$$

which we extend to the map $\Gamma_0(2) \to \mathcal{B}(2)_{\mathrm{Spec}\,\mathbb{C}}$ by using the map $\tilde{\epsilon}\colon\ \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \to \mathbb{C}^*(1)$ defined on generators by

$$\tilde{\epsilon}(a) = 2\pi i \otimes \exp\left(\frac{-1}{2\pi i}\int_0^a \log(1 - t)\frac{dt}{t}\right).$$

The condition (1) is given by [5]. Indeed, one computes the Leray spectral sequence associated to $\alpha\colon X_{\rm an} \to X_{\rm Zar}$ and the first term entering $\mathbb{H}^4(\mathcal{B}(2))$ is

$$E^{2,2} = H^2_{\rm Zar}(\mathbb{R}^2\alpha_*\mathcal{B}(2)) = H^2(\mathcal{K}_{2,\mathbb{Z}}),$$

where $\mathcal{K}_{2,\mathbb{Z}} := \operatorname{Ker}\left(\alpha_*\mathcal{K}_{2,\rm an} \xrightarrow{d\log\wedge d\log} H^2(\mathbb{C}/\mathbb{Z}(2))\right)$. Then the cycle map cl is induced by $\mathcal{K}_2 \to \mathcal{K}_{2,\mathbb{Z}}$ on $X_{\rm Zar}$, which is obviously compatible with the product in Pic. Thus we have (4).

Hence we can apply Theorem 2.3 to yield a 0-cycle $p(u) * p(1-u)$ on $E \times E'$, where both E and E' are smooth elliptic curves, which is non-torsion in the Chow group $\mathrm{CH}_0(E \times E')$, but which dies in $\mathbb{H}^4(\mathcal{B}(2))$ by Theorem 3.2.

In [9], S. Lichtenbaum constructs an étale version $\Gamma(2)$ of S. Bloch's analytic complex $\mathcal{B}(2)$, the cohomology of which contains $\mathrm{CH}^2(X)$. This contrasts with the examples discussed above.

Over a p-adic field, W. Raskind and M. Spieß [14] show that the Albanese kernel modulo n of a product of two Tate elliptic curves is dominated by $K_2(k)/n$. This result is not immediately comparable to ours, but is obviously related.

Remark 3.5. Since, $K_2(\overline{\mathbb{Q}}) = 0$, it follows from Corollary 3.3 that $\mathrm{cl}(p(u) * p(v)) = 0$ in $\mathbb{H}^4(E_{\rm an} \times E'_{\rm an}, \Gamma(2)_{\rm an})$ for all $u, v \in \overline{\mathbb{Q}}^*$, and all $\Gamma(2)_{\rm an}$ satisfying the conditions (3.1), in particular for $\Gamma(2)_{\rm an} = \mathcal{B}(2)$. It would be interesting to know if $p(u) * p(v) \in F^2\,\mathrm{CH}_0(E \times E')$ is non-torsion for some $u, v \in \overline{\mathbb{Q}}^*$.

4. The relative situation. In this section, we study the cycles constructed in Section 2 on $X = E \times E_0$, where as there, E is smooth, and E_0 is a nodal curve. We extend the definition of Bloch's complex $\mathcal{B}(2)$ to this case by using a *relative* complex $\overline{\mathcal{B}}(2)$, and use Theorem 3.2 to show that the cycles $p(u) * p(1-u)$ die in $\mathbb{H}^4(X_{\rm an}, \overline{\mathcal{B}}(2))$. Using some results from transcendence theory, we are able to construct examples of non-torsion cycles on X which not only die in $\mathbb{H}^4(X_{\rm an}, \overline{\mathcal{B}}(2))$, but vanish as well in the absolute Hodge cohomology $H^2(X, \Omega^2_{X/\mathbb{Q}})$.

Let $\nu = 1 \times q\colon E \times \mathbb{P}^1 \to X$ be the normalization. We define

$$\overline{\mathcal{K}_2} = \operatorname{Ker}\left(\nu_*\mathcal{K}_2 \overset{|_{E\times 0} - |_{E\times\infty}}{\longrightarrow} \mathcal{K}_2|_E\right) \tag{4.1}$$

LEMMA 4.1. *One has*

$$\mathrm{CH}_0(X) = H^2(X, \overline{\mathcal{K}_2}),$$

and the Chow group $\mathrm{CH}_0(X)$ *fits into an exact sequence*

$$0 \to H^1(E, \mathcal{K}_2) \xrightarrow{\gamma} \mathrm{CH}_0(X) \xrightarrow{\nu^*} \mathrm{CH}_0(E \times \mathbb{P}^1) = \mathrm{Pic}(E) \otimes \mathrm{Pic}(\mathbb{P}^1) \to 0.$$

Moreover, the map γ *is defined by*

$$\gamma\left(\sum_{x\in E^{(1)}} x\otimes\lambda_x\right)=\sum_{x\in E^{(1)}}(x,p_0(\lambda_x))-(x,0).$$

Proof. As in the proof of Proposition 2.1, the map $\nu^*\colon\ \mathcal{K}_2\to\bar{\mathcal{K}_2}$ is surjective, and the kernel is supported in codimension 1. Thus ν^* induces an isomorphism on H^2.

On the other hand,

$$H^1(E\times\mathbb{P}^1,\mathcal{K}_2)=H^1(E,\mathcal{K}_2)\oplus H^0(E,\mathcal{K}_1)\cup c_1(\mathcal{O}(1)).$$

The term $H^1(E,\mathcal{K}_2)$ maps to $0\in H^1(E,\mathcal{K}_2)$ via the difference of the restrictions to $E\times 0$ and $E\times\infty$, while $c_1(\mathcal{O}(1))$ restricts to 0 to either $E\times 0$ or $E\times\infty$. This shows the long exact sequence associated to the short one defining $\bar{\mathcal{K}_2}$ yields the exact sequence in the statement of the lemma.

Finally, the value $\gamma(x\otimes\lambda_x)$ of the map is given by the boundary morphism $\mathbb{C}^*\to H^1(X,\mathcal{O}_X^*)$ induced by the normalization sequence

$$0\to\mathcal{O}_X^*\to q_*\mathcal{O}_{\mathbb{P}^1}^*\xrightarrow{|_0-|_\infty}\mathbb{C}^*\to 0$$

on the right argument λ_x. The formula for γ thus follows from Lemma 1.1. ◻

Let $\mathrm{Nm}\colon\ H^1(E,\mathcal{K}_2)\to\mathbb{C}^*$ be the norm map, defined by

$$\mathrm{Nm}\left(\sum_{x\in E^{(1)}} x\otimes\lambda_x\right):=\prod_{x\in E^{(1)}}\lambda_x. \tag{4.2}$$

We set

$$V(E)=\mathrm{Ker}\mathrm{Nm}. \tag{4.3}$$

One has:

LEMMA 4.2. $F^2\,\mathrm{CH}_0\,(X)=\gamma\,(V(E))$.

Proof. By the definition given in §2, $F^2\,\mathrm{CH}_0\,(X)$ is generated by the expressions $[(x,y)]-[(x,0)]-[(0,y)]+[(0,0)]$, with $x\in E(\mathbb{C})$ and $y\in E_0(\mathbb{C})\setminus\{*\}$. By the formula for γ given in Lemma 4.1, this expression is $\gamma(x\otimes y-0\otimes y)$, after identifying $y\in\mathbb{C}^*$ with $p_0(y)\in E_0(\mathbb{C})$. Clearly $V(E)$ is generated by the elements of $H^1(E,\mathcal{K}_2)$ of the form $x\otimes y-0\otimes y$, whence the lemma. ◻

Next, we want to map $\mathrm{CH}_0(X)$ to a relative version of S. Bloch's analytic motivic cohomology. So we define

$$\bar{\mathcal{B}}(2) := \mathrm{Ker}\left(\nu_*\mathcal{B}(2) \xrightarrow{|_{E\times 0}-|_{E\times\infty}} \mathcal{B}(2)|_E\right). \tag{4.4}$$

In particular, $\bar{\mathcal{B}}(2)$ is an extension of

$$\bar{\mathcal{K}}_{2,\mathrm{an}} = \mathrm{Ker}\left(\nu_*\mathcal{K}_{2,\mathrm{an}} \xrightarrow{|_{E\times 0}-|_{E\times\infty}} \mathcal{K}_{2,\mathrm{an}}|_E\right),$$

placed in degree 2, by $\Delta^*(1)$, placed in degree 1. In other words, $\underline{\mathcal{B}}(2)$ is the pull-back of $\bar{\mathcal{B}}(2)$ via the map $\nu^*\colon \mathcal{K}_{2,\mathrm{an}} \to \bar{\mathcal{K}}_{2,\mathrm{an}}$, and in particular, $\bar{\mathcal{B}}(2)$ receives the complex $\Gamma_0(2)$ as explained in Example 3.

Considering again the Leray spectral sequence attached to the identity $\alpha\colon X_{\mathrm{an}} \to X_{\mathrm{zar}}$, we see that

$$\bar{\mathcal{K}}_{2,\mathbb{Z}} := \mathrm{Ker}\left(\alpha_*\bar{\mathcal{K}}_{2,\mathrm{an}} \to \mathcal{H}^2(\mathbb{C}/\mathbb{Z}(2))\right) \tag{4.5}$$

receives $\bar{\mathcal{K}}_2$ and that the first map of the spectral sequence is then

$$H^2(X, \bar{\mathcal{K}}_{2,\mathbb{Z}}) \to \mathbb{H}^4(X_{\mathrm{an}}, \bar{\mathcal{B}}(2)). \tag{4.6}$$

In conclusion, we have shown:

LEMMA 4.3. *One has a cycle map*

$$\psi_X\colon \mathrm{CH}_0(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \bar{\mathcal{B}}(2))$$

compatible with the cycle map

$$\psi_{E\times\mathbb{P}^1}\colon \mathrm{CH}_0(E\times\mathbb{P}^1) \to \mathbb{H}^4((E\times\mathbb{P}^1)_{\mathrm{an}}, \mathcal{B}(2))$$

on the normalization. Moreover, ψ_X fulfills the conditions described in (3.1).

Proof. We just have to verify the condition (4) of (3.1). From the normalization sequence

$$0 \to \mathcal{O}_X^* \to \nu_*\mathcal{O}_{E\times\mathbb{P}^1}^* \xrightarrow{|_{E\times 0}-|_{E\times\infty}} \mathcal{O}_E^* \to 0,$$

one has a natural map

$$\mathcal{O}_{X_{\mathrm{an}}}^* \otimes \mathcal{O}_{X_{\mathrm{an}}}^* \to \bar{\mathcal{K}}_{2,\mathrm{an}}$$

which obviously fulfills (3.1)(4). □

Now we can apply Theorem 3.2 to conclude:

THEOREM 4.4. *The 0-cycles defined by the Steinberg curve on $E \times E_0$ die in the analytic motivic cohomology $\mathbb{H}^4(X_{\mathrm{an}}, \underline{\overline{\mathcal{B}}}(2))$.*

Let K be a subfield of $\mathbb{C}$. We next consider for any algebraic variety Z defined over K, the cycle map with values in the absolute Hodge cohomology

$$H^m(Z, \mathcal{K}_2) \xrightarrow{d\log \wedge d\log} H^m(Z, \Omega^2_{Z/\mathbb{Q}}) \tag{4.7}$$

induced by the absolute $d\log$ map

$$\mathcal{O}_Z^* \xrightarrow{d\log} \Omega^1_{Z/\mathbb{Q}}. \tag{4.8}$$

This cycle map is obviously compatible with the map γ, and with extension of scalars.

Let $E \to \operatorname{Spec} K$ be an elliptic curve over a subfield K of $\mathbb{C}$. We have the exact sheaf sequence

$$0 \to \mathcal{O}_E \otimes \Omega^1_{K/\mathbb{Q}} \to \Omega^1_{E/\mathbb{Q}} \to \Omega^1_{E/K} \to 0,$$

which induces a two-term filtration $F^*\Omega^2_{E/\mathbb{Q}}$ of $\Omega^2_{E/\mathbb{Q}}$ with $F^2\Omega^2_{E/\mathbb{Q}} = \mathcal{O}_E \otimes \Omega^2_{K/\mathbb{Q}}$. This gives us the natural maps

$$\gamma_1\colon \quad H^*(E, \mathcal{O}_E) \otimes \Omega^1_{K/\mathbb{Q}} \to H^*(E, \Omega^1_{E/\mathbb{Q}})$$
$$\gamma_2\colon \quad H^*(E, \mathcal{O}_E) \otimes \Omega^2_{K/\mathbb{Q}} \to H^*(E, \Omega^2_{E/\mathbb{Q}}).$$

We have the norm map Nm: $H^1(E, \mathcal{K}_2) \to H^0(K, \mathcal{K}_1) = K^*$ as in (4.2), but over K; we let $V(E) \subset H^1(E, \mathcal{K}_2)$ be the kernel of Nm (see (4.3)).

LEMMA 4.5. *Let K be an algebraically closed subfield of $\mathbb{C}$, $E \to \operatorname{Spec} K$ an elliptic curve over K. Then the cycle map with values in absolute Hodge cohomology maps $V(X)$ to the subgroup $\gamma_2[H^1(E, \mathcal{O}_E) \otimes \Omega^2_{E/\mathbb{Q}}]$ of $H^1(E, \Omega^2_{E/\mathbb{Q}})$.*

Proof. The kernel of the composition

$$\operatorname{Pic}(E) = H^1(E, \mathcal{K}_1) \xrightarrow{d\log} H^1(E, \Omega^1_{E/\mathbb{Q}}) \to H^1(E, \Omega^1_{E/K}) \cong K$$

is the composition

$$\operatorname{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \subset K,$$

hence the $d\log$ map sends $\operatorname{Pic}^0(E)$ to the subgroup $\gamma_1[H^1(E, \mathcal{O}_E) \otimes \Omega^1_{K/\mathbb{Q}}]$ of $H^1(E, \Omega^1_{E/\mathbb{Q}})$.

Take $\tau \in \mathrm{Pic}^0(E)$, $u \in H^0(E, \mathcal{K}_1) = K^*$, and let $\xi = \tau \cup u \in H^1(E, \mathcal{K}_2)$. Then

$$d\log(\xi) = d\log(\tau) \cup d\log(u).$$

Since $d\log\colon K^* \to \Omega^1_{K/\mathbb{Q}}$ is just the absolute $d\log$ map, we see that $d\log(\xi)$ lands in the image of the cup product map

$$\left[H^1(E, \mathcal{O}_E) \otimes \Omega^1_{K/\mathbb{Q}}\right] \otimes \Omega^1_{K/\mathbb{Q}} \to H^1(E, \Omega^2_{E/\mathbb{Q}}),$$

which is $\gamma_2(H^1(E, \mathcal{O}_E) \otimes \Omega^2_{K/\mathbb{Q}})$.

Since K is algebraically closed, the cup product $\mathrm{Pic}(E) \otimes K^* \to H^1(E, \mathcal{K}_2)$ is surjective, from which one sees that the cup product maps $\mathrm{Pic}^0(E) \otimes K^*$ onto $V(E)$. Combining this with the computation above completes the proof. □

From the surjectivity of the cup product $\mathrm{Pic}^0(E) \otimes K^* \to V(E)$ for K algebraically closed, we see that the injection $H^1(E, \mathcal{K}_2) \to \mathrm{CH}_0(X)$ sends $V(E)$ isomorphically onto $F^2\,\mathrm{CH}_0(X)$.

Let K be a subfield of $\mathbb{C}$. We say that an element ξ of $\mathrm{CH}_0(X)$ is *defined over* K if there is an K-scheme X^0, an element ξ^0 of $\mathrm{CH}_0(X^0)$ and an isomorphism $\alpha\colon X^0_{\mathbb{C}} \to X$ such that $\xi = \alpha_*(\xi^0_{\mathbb{C}})$. From Lemma 4.5 and the compatibility of $d\log$ with extension of scalars, we have:

LEMMA 4.6. *Take $K = \mathbb{C}$, and let ξ be an element of $F^2\,\mathrm{CH}_0(X) = V(E)$. If ξ is defined over a field of transcendence degree one over $\mathbb{Q}$, then ξ vanishes under the cycle map to absolute Hodge cohomology.*

COROLLARY 4.7. *If E is an elliptic curve with complex multiplication, then there are non-torsion cycles $\xi \in F^2\,\mathrm{CH}_0(X)$ dying in the analytic motivic cohomology as well as in absolute Hodge cohomology.*

Proof. By the remark above, we may replace $F^2\,\mathrm{CH}_0(X)$ with $V(E)$. Let $\bar{E}$ be a model for E, with equation $y^2 = 4x^3 - ax - b$ defined over a number field $K \subset \mathbb{C}$. Let $\omega = \frac{dx}{y}$ be the standard global one-form on $\bar{E}$.

Choosing an isomorphism $\bar{E}_{\mathbb{C}} \cong E_{\mathbb{C}}$ defines the period lattice $L_\omega \subset \mathbb{C}$ for ω. Choose a basis for L_ω of the form $\{\Omega, \tau\Omega\}$, and let $t = e^{2\pi i\tau}$. Let

$$\mathcal{P}\colon\ \mathbb{C} \to \mathbb{CP}^1$$

be the Weierstraß P-function for the lattice L_ω.

The map $\times\Omega^{-1}\colon \mathbb{C} \to \mathbb{C}$ gives rise to the isomorphism of Riemann surfaces

$\alpha_{\mathrm{an}}\colon\ \overline{E}^{\mathrm{an}}_{\mathbb{C}} \to E^{\mathrm{an}}_t$ making the diagram

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\times\Omega^{-1}} & \mathbb{C} \\
\Big\downarrow{\scriptstyle(\mathcal{P},\mathcal{P}')} & & \Big\downarrow{\scriptstyle \exp} \\
 & & \mathbb{C}^* \\
 & & \Big\downarrow{\scriptstyle p} \\
\overline{E}^{\mathrm{an}}_{\mathbb{C}} & \xrightarrow[\alpha_{\mathrm{an}}]{} & E^{\mathrm{an}}_t
\end{array}$$

commute, i.e.,

$$p(u) = \alpha_{\mathrm{an}}\left(\mathcal{P}\left(\frac{\Omega}{2\pi i}\log u\right), \mathcal{P}'\left(\frac{\Omega}{2\pi i}\log u\right)\right).$$

We let

$$\alpha\colon\ \overline{E}_{\mathbb{C}} \to E_t$$

be the corresponding isomorphism of algebraic elliptic curves over $\mathbb{C}$.

By [1, théorème 1], $\mathcal{P}(\frac{\Omega}{2\pi i}\log u)$ has transcendence degree 1 over $\overline{\mathbb{Q}}$ for all $u \in \mathbb{N}$, $u \geq 2$. (We thank Y. André for giving us this reference). Fix a $u \geq 2$, let K be the algebraic closure of the field $\mathbb{Q}(\mathcal{P}(\frac{\Omega}{2\pi i}\log u))$, and let $x \in \overline{E}(K)$ be the point $(\mathcal{P}(\frac{\Omega}{2\pi i}\log u), \mathcal{P}'(\frac{\Omega}{2\pi i}\log u))$. Then x is a generic point of $\overline{E}$ over $\overline{\mathbb{Q}}$.

We take

$$\xi := p(u) * p(1-u).$$

By construction, $\xi = \alpha(\xi_K \times_K \mathbb{C})$, where $\xi_K \in H^1(\overline{E}, \mathcal{K}_2)$ is the element $[(x)-(0)] \cup [1-u]$. Here $[(x)-(0)]$ denotes the class in $\mathrm{Pic}\,(E) = H^1(\overline{E}, \mathcal{K}_1)$, and $[1-u]$ denotes the class in $H^0(\overline{E}, \mathcal{K}_1) = K^*$. Since K has transcendence degree one over $\overline{\mathbb{Q}}$, the class of ξ in the absolute Hodge cohomology of E vanishes, by Lemma 4.6. By Theorem 4.4, ξ dies in the analytic motivic cohomology of E as well. It remains to show that ξ is a non-torsion element of $H^1(E_K, \mathcal{K}_2)$.

We give an analytic proof of this using the regulator map with values in Deligne-Beilinson cohomology.

Let Y be a smooth projective surface over $\mathbb{C}$, and let $\mathrm{NS}\,(Y)$ denote the Néron-Severi group of divisors modulo homological equivalence. Then Hodge theory implies that

$$\mathrm{NS}\,(Y) = \{(z,\varphi) \in (H^2(Y_{\mathrm{an}}, \mathbb{Z}(1)) \times F^1H^2(Y_{\mathrm{an}}, \mathbb{C})), z \otimes \mathbb{C} = \varphi\},$$

and that

$$\mathrm{NS}\,(Y) \cap F^2 H^2_{DR}(Y) = \emptyset.$$

We note that the map $\mathrm{Pic}\,(Y) \otimes \mathbb{C}^* \to H^3_{\mathcal{D}}(Y, \mathbb{Z}(2))$ induced by the cup product in Deligne cohomology factors through $\mathrm{NS}\,(Y) \otimes \mathbb{C}^*$, and that the induced map $\iota\colon\ \mathrm{NS}\,(Y) \otimes \mathbb{C}^* \to H^3_{\mathcal{D}}(Y, \mathbb{Z}(2))$ is injective. Indeed,

$$H^3_{\mathcal{D}}(Y, \mathbb{Z}(2)) = H^2(Y_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(2))/F^2.$$

Now take $Y = E \times E$, and let $U \subset E$ be the complement of a nonempty finite set Σ of points of E. Let $[E \times 0]$ be the class of $E \times 0$ in $\mathrm{NS}\,(Y)$, and let $\gamma\colon\ \mathbb{C}^* \to \mathrm{NS}\,(Y) \otimes \mathbb{C}^*$ be the map $\gamma(v) = [E \times 0] \otimes v$. Let

$$\iota_U\colon\ \mathrm{NS}\,(Y) \otimes \mathbb{C}^* \to H^3_{\mathcal{D}}(E \times U, \mathbb{Z}(2))$$

be the composition of ι with the restriction map $H^3_{\mathcal{D}}(Y, \mathbb{Z}(2)) \to H^3_{\mathcal{D}}(E \times U, \mathbb{Z}(2))$. We claim that the sequence

$$\mathbb{C}^* \xrightarrow{\gamma} \mathrm{NS}\,(Y) \otimes \mathbb{C}^* \xrightarrow{\iota_U} H^3_{\mathcal{D}}(E \times U, \mathbb{Z}(2))$$

is exact. Indeed, we have the localization sequence

$$\oplus_{s \in \Sigma} H^1_{\mathcal{D}}(E \times s, \mathbb{Z}(1)) \xrightarrow{\oplus_s \iota_s} H^3_{\mathcal{D}}(Y, \mathbb{Z}(2)) \to H^3_{\mathcal{D}}(E \times U, \mathbb{Z}(2)) \to,$$

the isomorphism $H^1_{\mathcal{D}}(E \times s, \mathbb{Z}(1)) \cong \mathbb{C}^*$ and the identity

$$\iota_s(v) = \gamma(v), \ \ v \in \mathbb{C}^*,$$

which proves our claim.

In particular, let $[\Xi] = [\Delta - \{0\} \times E] \otimes v$, where Δ is the diagonal, v is an element of $\mathbb{C}^*$ which is not a root of unity, and $[\Delta - \{0\} \times E]$ is the class in $\mathrm{NS}\,(Y)$. Since $[\Delta - \{0\} \times E]$ is not torsion in $\mathrm{NS}\,(Y)/[E \times \{0\}]$, we see that $[\Xi]$ has non-torsion image $[\Xi_{\mathbb{C}(E)}]$ in

$$H^3_{\mathcal{D}}(E \times_{\mathbb{C}} \mathbb{C}(E), \mathbb{Z}(2)) := \varinjlim_{\emptyset \neq U \subset E} H^3_{\mathcal{D}}(E \times U, \mathbb{Z}(2)),$$

where the limit is over nonempty Zariski open subsets U of E.

Let Ξ be the image of $(\Delta - 0 \times E) \otimes v$ in $H^1(Y, \mathcal{K}_2)$. Then $[\Xi]$ is the image of Ξ under the regulator map $H^1(Y, \mathcal{K}_2) \to H^3_{\mathcal{D}}(Y, \mathbb{Z}(2))$. Similarly, letting $\Xi_{\mathbb{C}(E)}$ be the pull-back of Ξ to $E \times_{\mathbb{C}} \mathbb{C}(E)$, $[\Xi_{\mathbb{C}(E)}]$ is the image of $\Xi_{\mathbb{C}(E)}$ under the regulator map $H^1(E \times_{\mathbb{C}} \mathbb{C}(E), \mathcal{K}_2) \to H^3_{\mathcal{D}}(E \times_{\mathbb{C}} \mathbb{C}(E), \mathbb{Z}(2))$. Thus, $\Xi_{\mathbb{C}(E)}$ is a non-torsion element of $H^1(E \times_{\mathbb{C}} \mathbb{C}(E), \mathcal{K}_2)$ for each non-torsion element $v \in \mathbb{C}^*$.

Let $\bar{\Delta}$ be the diagonal in $\bar{E} \times \bar{E}$, let $\bar{\xi}$ be the image of $(\bar{\Delta} - 0 \times \bar{E}) \otimes (1-u)$ in $H^1(E, \mathcal{K}_2)$, and let $\xi_{\bar{\mathbb{Q}}(E)}$ be the image of $\bar{\xi}$ in $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2)$. Clearly, after choosing a complex embedding $\bar{\mathbb{Q}} \subseteq \mathbb{C}$, $\Xi_{\mathbb{C}(E)}$ (for $v = 1-u$) is the image of $\xi_{\bar{\mathbb{Q}}(E)}$ under the extension of scalars $\bar{\mathbb{Q}}(\bar{E}) \to \mathbb{C}(\bar{E}) \cong \mathbb{C}(E)$, hence $\xi_{\bar{\mathbb{Q}}(E)}$ is a non-torsion element of $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2)$.

Since x is a geometric generic point of $\bar{E}$ over $\bar{\mathbb{Q}}$, there is an embedding σ: $\bar{\mathbb{Q}}(E) \to \mathbb{C}$ such that x: $\operatorname{Spec}\mathbb{C} \to \bar{E}$ is the composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\bar{\mathbb{Q}}(E) \to \bar{E}$. Thus, ξ is the image of $\bar{\xi}$ under $(\mathrm{id} \times x)^*$: $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{E}, \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$, and hence ξ is the image of $\xi_{\bar{\mathbb{Q}}(E)}$ under the map $\mathrm{id} \times \sigma_*$: $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$ induced by the extension of scalars σ.

Since the kernel of $\mathrm{id} \times \sigma_*$ is torsion, it follows that ξ is a non-torsion element of $H^1(E, \mathcal{K}_2)$, as desired. □

Remark 4.8. Going back to $X = E \times E'$, where both elliptic curves are smooth, we are lacking the transcendence theorem which would force the existence of a cycle $0 \neq \xi = p(u) * p(1-u) \in F^2 \mathrm{CH}_0(X)$ dying both in $\mathbb{H}^4(X, \mathcal{B}(2))$ and in absolute Hodge cohomology.

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