

# ALGEBRAIC THEORY OF CHARACTERISTIC CLASSES OF BUNDLES WITH CONNECTION

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## 0. INTRODUCTION

The Weil algebra homomorphism

$$w = \oplus w_n : \oplus_n S^n(\mathfrak{g}(\mathbb{C})^*)^{G(\mathbb{C})} \rightarrow \oplus_n H_{DR}^{2n}(BG(\mathbb{C}))$$

assigns a de Rham cohomology class of the classifying space  $BG(\mathbb{C})$  to a  $G(\mathbb{C})$  invariant polynomial on the dual  $\mathfrak{g}(\mathbb{C})^*$  of the Lie algebra associated to the  $\mathbb{C}$ -valued points of an algebraic group  $G$  (see [20], Chapter XII).

In the unpublished note [3], Beilinson and Kazhdan give an algebraic description of  $w$  as an iterated Atiyah extension (see [1]).

Let  $p : \mathcal{E} \rightarrow X$  be a (simplicial) principal  $G$  bundle on the (simplicial) smooth algebraic variety  $X$  over a ring  $k$  of characteristic zero. The exact sequence

$$0 \rightarrow p^*\Omega_X^1 \rightarrow \Omega_{\mathcal{E}}^1 \rightarrow \Omega_{\mathcal{E}/X}^1 \rightarrow 0$$

of regular one forms induces the Atiyah extension

$$(0.0.1) \quad 0 \rightarrow \Omega_X^1 \rightarrow \Omega_{X,\mathcal{E}}^1 \rightarrow \mathfrak{g}_{\mathcal{E}}^* \rightarrow 0$$

where

$$\Omega_{X,\mathcal{E}}^1 = (p_*\Omega_{\mathcal{E}}^1)^G, \mathfrak{g}_{\mathcal{E}}^* = (p_*\Omega_{\mathcal{E}/X}^1)^G = \mathcal{E} \times_G \mathfrak{g}^*.$$

Here  $G$  acts via the adjoint representation on  $\mathfrak{g}^*$ .

For example, if  $G = GL(r)$ , and  $E = \mathcal{E} \times_G k^r$ , the corresponding bundle of  $k$ -modules  $k^r$ , then  $\mathfrak{g}_{\mathcal{E}}^* = \text{End } E$ , the endomorphisms of  $E$ .

Then 0.0.1 induces a  $n$ -extension

$$\begin{aligned} 0 \rightarrow \Omega_X^n \rightarrow \Lambda^n \Omega_{X,\mathcal{E}}^1 \rightarrow \Lambda^{n-1} \Omega_{X,\mathcal{E}}^1 \otimes \mathfrak{g}_{\mathcal{E}}^* \rightarrow \dots \\ \rightarrow \Lambda^{n-i} \Omega_{X,\mathcal{E}}^1 \otimes S^i \mathfrak{g}_{\mathcal{E}}^* \rightarrow \dots \rightarrow S^n \mathfrak{g}_{\mathcal{E}}^* \rightarrow 0 \end{aligned}$$

which defines a connecting homomorphism

$$(0.0.2) \quad H^0(X, S^n \mathfrak{g}_{\mathcal{E}}^*) \rightarrow H^n(X, \Omega_X^n).$$

Evaluated on  $X = BG = (G^{\ell+1}/G)_{\ell}$ , the  $\mathbb{Z}$ -simplicial scheme classifying  $G$ -principal bundles, and  $\mathcal{E} = (G^{\ell+1})_{\ell}$ , the universal  $G$ -principal bundle, one has

$$H^0(BG, S^n \mathfrak{g}_{\mathcal{E}}^*) = S^n(\mathfrak{g}^*)^G,$$

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and, if  $G$  is reductive, the natural maps

$$\begin{aligned} \mathbb{H}^{2n}(BG(L), \Omega_{BG}^{\geq n}) &\rightarrow H^n(BG(L), \Omega_{BG}^n) \\ \mathbb{H}^{2n}(BG(L), \Omega_{BG}^{\geq n}) &\rightarrow \mathbb{H}^{2n}(BG(L), \Omega_{BG}^\bullet) = H_{DR}^{2n}(BG(L)) \end{aligned}$$

are isomorphisms for any algebraically closed field  $L$  of characteristic 0. The Weil homomorphism  $w_n$  is just the connecting homomorphism 0.0.2, where one identifies the right hand side with the de Rham cohomology via those two isomorphisms.

Chern-Weil theory assigns to a  $C^\infty$  manifold  $X$  and a bundle  $E$  of rank  $r$  with a connection  $\nabla$ , a morphism

$$[\nabla]^* : \oplus_n S^n(\mathfrak{g}(\mathbb{C})^*) \rightarrow \oplus_n H^0(X, \Omega_{\infty, \text{cl}}^{2n}),$$

where  $\Omega_\infty^i$  is the sheaf of  $C^\infty$  forms of degree  $i$  containing the sheaf  $\Omega_{\infty, \text{cl}}^i$  of closed forms, and  $G = GL(r)$ , such that

$$[\nabla]^*(P) = P(\nabla^2, \dots, \nabla^2)$$

is a closed form, the de Rham class of which is  $[E]^*(w(P))$ . Here

$$[E]^* : \oplus_n H_{DR}^{2n}(BG) \rightarrow \oplus_n H_{DR}^{2n}(X)$$

is the map induced by  $E$ .

The theory of secondary classes of Chern-Simons and Cheeger-Simons is a factorization of  $[\nabla]^*$ . In a spirit closed to the algebraic definition of  $w$ , they both have an algebraic incarnation. The purpose of this survey is to describe it. For the analytic side of the theory, we refer to [16] and [28].

## 1. CHERN-SIMONS THEORY

### 1.1. Classical theory. [10]

Given a  $C^\infty$  manifold  $X$ , a bundle  $E$  of rank  $r$ , a connection  $\nabla$ ,  $P \in S^n(\mathfrak{g}(\mathbb{C})^*)^{G(\mathbb{C})}$ ,  $G = GL(r)$ , Chern and Simons consider the principal  $G$  bundle

$$p : \mathcal{E} \rightarrow X$$

together with the canonical trivialization  $p^*E = \oplus_1^r \mathcal{O}_\infty$ . In this canonical basis,  $p^*\nabla$  becomes a  $r \times r$  matrix  $A$  of one forms, with curvature  $F(A)$ . They define

$$TP(A) = n \int_0^1 P(A, \underbrace{F(tA), \dots, F(tA)}_{(n-1) \text{ times}}) dt \in H^0(\mathcal{E}, \Omega_\infty^{2n-1}),$$

a functorial solution to the equation

$$d? = P(F(A), \dots, F(A)).$$

In order to define classes of  $(E, \nabla)$  (or equivalently of the local system  $E^\nabla$ ) on  $X$ , and not only on  $\mathcal{E}$ , which of course depends on  $E$ , they assume  $\nabla^2 = 0$ . In this case,  $TP(A)$  defines a class  $[TP(A)]$  in  $H_{DR}^{2n-1}(\mathcal{E})$ . Assuming further that  $P$  has  $\mathbb{Z}$ -periods, that is

$$P \in \text{Ker} (S^n(\mathfrak{g}(\mathbb{C})^*)^{G(\mathbb{C})} \rightarrow H_{DR}^{2n}(BG(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n))),$$

where  $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}$ , then the restriction of  $[TP(A)]$  to any  $C^\infty$  section of  $p$  is a well defined class

$$TP(\nabla) \in H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)).$$

We denote it simply by  $Tc_n(\nabla)$  if  $P$  is the polynomial defining the  $n$ -th Chern class.

**1.2. Algebraic Theory.** Let  $X$  be a smooth algebraic variety over a field  $k$  of characteristic 0,  $(E, \nabla)$  be a bundle of rank  $r$  with a connection,  $P \in S^n(\mathfrak{g}(k)^*)^{G(k)}$ ,  $G = GL(r)$ . Given on a Zariski open set  $U \subset X$  a local trivialization of  $E$ ,  $\nabla$  becomes a  $r \times r$  matrix of one forms

$$A \in H^0(U, M(r \times r, \Omega^1)),$$

and one can define

$$TP(A) \in H^0(U, \Omega^{2n-1})$$

as before with

$$dTP(A) = P(\nabla^2, \dots, \nabla^2) \in H^0(X, \Omega_{\text{cl}}^{2n}).$$

The point is that if  $g \in H^0(U, GL(r, \mathcal{O}))$  is a gauge transformation, then

$$TP(A) - TP(g^{-1}dg + g^{-1}Ag)$$

is locally exact for  $n \geq 2$ , and thus viewed as a class in  $H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$ , or in the subgroup  $H^0(X, \mathcal{H}_{DR}^{2n-1}) \subset H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$  if  $\nabla^2 = 0$ , with  $\mathcal{H}_{DR}^{2n-1} = \Omega_{\text{cl}}^{2n-1}/d\Omega^{2n-2}$ , it depends only on  $\nabla$ . We denote it by

$$TP(\nabla) \in H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$$

for  $n \geq 2$ . (In [7], it is denoted by  $w_n(E, \nabla, P)$ ). If  $P$  is the polynomial defining the  $n$ -th Chern class, we simply denote it by  $Tc_n(\nabla)$ .

Due to the Bloch-Ogus theory [9], the restriction maps to the generic point are injective:

$$\begin{aligned} H^0(X, \Omega^{2n-1}/d\Omega^{2n-2}) &\subset \Omega_{k(X)}^{2n-1}/d\Omega_{k(X)}^{2n-2} \\ H^0(X, \mathcal{H}_{DR}^{2n-1}) &\subset H_{DR}^{2n-1}(k(X)), \end{aligned}$$

for  $n \geq 2$ .

Thus  $TP(\nabla)$  is recognized at the generic point of  $X$ . One asks what kind of algebraic class of  $E$  it controls.

To this aim, one first shows that  $TP(\nabla)$  in fact lies in a subgroup

$$E_n^{0,2n-1} \subset H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$$

arising from the coniveau spectral sequence. When  $k$  is algebraically closed, the group  $E_n^{0,2n-1}$  maps to the group

$$CH_{1,DR}^n(X) = \left( \frac{\{\oplus \mathbb{Z}Z, Z \text{ prime cycle of codimension } n\}}{\sim_{DR}} \right) \otimes_{\mathbb{Z}} k$$

where  $Z \sim_{DR} 0$  if there is a divisor  $W$  containing the support of  $Z$  such that the class of  $Z$  in  $H_{DR,W}^{2n}(X)$ , the de Rham cohomology of  $X$  with supports along  $W$ , vanishes. Clearly,  $CH_{1,DR}^n(X)$  is a quotient of the Chow group with

$k$ -coefficients  $CH^n(X) \otimes_{\mathbb{Z}} k$ , and for  $n = 2$ , it coincides with the Griffiths group  $\otimes k$ .

We denote by  $c_P(E)$  the class of  $E$  associated to  $P$  in  $CH^n(X) \otimes_{\mathbb{Z}} k$ .

**Proposition 1.2.1.** (see [7], Proposition 5.4.1) *Let  $k$  be an algebraically closed field of characteristic 0. The image of  $TP(\nabla)$  in  $CH_{1,DR}^n(X)$  equals the image of  $c_P(E)$ .*

[Strictly speaking, the proof in loc. cit. assumes  $k = \mathbb{C}$ ,  $\nabla^2 = 0$ , and deals with

$$CH_{1,\text{Betti}}^n(X) = \frac{\{\oplus \mathbb{Z}Z, Z \text{ prime cycle of codimension } n\}}{\sim_{\text{Betti}}}$$

where  $Z \sim_{\text{Betti}} 0$  if there is a divisor  $W$  containing the support of  $Z$  such that the class of  $Z$  in  $H_{\text{Betti},W}^{2n}(X, \mathbb{Z}(n))$ , the Betti cohomology of  $X$  with supports along  $W$ , vanishes. Then  $CH_{1,\text{Betti}}^n(X)$  is a quotient of  $CH^n(X)$ .

It is straightforward to generalize. First, the sequence (5.4.3) of loc. cit. has an obvious de Rham version. Then one has that

$$Tc_n(\nabla) \in \text{Im } \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d \log} \Omega^n \rightarrow \dots \rightarrow \Omega^{2n-1}) \xrightarrow{d} H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$$

([17], section 2.2). The map  $d$  factors through

$$\mathbb{H}^n(X, 0 \rightarrow 0 \rightarrow \Omega^{n+1}/d\Omega^n \rightarrow \Omega^{n+2} \rightarrow \dots \rightarrow \Omega^{2n-1})$$

and

$$E_n^{0,2n-1} = \text{Im } \mathbb{H}^n(X, 0 \rightarrow 0 \rightarrow \Omega^{n+1}/d\Omega^n \rightarrow \Omega^{n+2} \rightarrow \dots \rightarrow \Omega^{2n-1}) \rightarrow H^0(X, \Omega^{2n-1}/d\Omega^{2n-2}).$$

Finally, to prove that the image of  $TP(\nabla)$  in  $CH_{1,DR}^n(X)$  is the correct one, one replaces loc. cit. (5.4.5) by the exact sequence

$$\begin{aligned} 0 \rightarrow (0 \rightarrow \Omega^n/d\Omega^{n-1} \rightarrow \Omega^{n+1} \rightarrow \dots \rightarrow \Omega_{\text{cl}}^{2n-2}) \\ \rightarrow (\mathcal{K}_n \rightarrow \Omega^n/d\Omega^{n-1} \rightarrow \Omega^{n+1} \rightarrow \dots \rightarrow \Omega^{2n-1}) \\ \rightarrow \mathcal{K}_n \oplus \Omega^{2n-1}/d\Omega^{2n-2}[-n] \rightarrow 0. \end{aligned}$$

Here and in section 2,  $\mathcal{K}_n$  is the Zariski sheaf

$$\text{Im } (\mathcal{K}_n^M \rightarrow i_{k(X)^*} K_n^M(k(X)))$$

where  $i_{k(X)} : \text{Spec } k(X) \rightarrow X$  is the inclusion of the generic point and  $K_n^M$  is the Milnor  $K$ -theory.]

The main theorem is now

**Theorem 1.2.2.** (see [7], Theorem 5.6.2) *Let  $X$  be projective smooth over  $\mathbb{C}$ ,*

$$P \in \text{Ker } (S^n(\mathfrak{g}(\mathbb{C})^*)^{G(\mathbb{C})} \rightarrow H^{2n}(BG(\mathbb{C}), \mathbb{C}/\mathbb{R}(n))),$$

$n \geq 2$ , and let  $(E, \nabla)$  be a flat bundle on  $X$ . Then  $TP(\nabla) = 0$  if and only if the image of  $c_P(E)$  in  $CH_{1,\text{Betti}}^n(X) \otimes_{\mathbb{Z}} \mathbb{R}$  vanishes.

**1.3. Question.** When  $\nabla$  is not flat, there are many examples of non-vanishing  $TP(\nabla)$  classes. However, when  $\nabla$  is flat, we don't know any ( $n \geq 2$ ).

This raises the question of whether  $TP(\nabla)$  always vanishes under the assumption of Theorem 1.2.2. It is related to a question arising from Nori's work ([24], Introduction), often called Nori's conjecture: if

$$Z \in \text{Ker} (CH^2(X) \rightarrow H_{\mathcal{D}}^4(X, \mathbb{Q}(2))),$$

where  $X$  is projective smooth over  $\mathbb{C}$ , and  $H_{\mathcal{D}}^a(b)$  is the Deligne cohomology, is  $Z \otimes \mathbb{Q}$  algebraically equivalent to 0?

Since for  $n = 2$ ,  $\sim_{\text{Betti}}$  is the algebraic equivalence, a generalization of Nori's question to any codimension  $n$  is: if

$$Z \in \text{Ker} (CH^n(X) \rightarrow H_{\mathcal{D}}^{2n}(X, \mathbb{Q}(n))),$$

where  $X$  is projective smooth over  $\mathbb{C}$ , does the image of  $Z$  in  $CH_{1, \text{Betti}}^n(X) \otimes \mathbb{Q}$  vanish?

Reznikov's theorem [25] asserts that if  $X$  is projective smooth over  $\mathbb{C}$  and  $E$  is flat on  $X$ , then

$$c_n(E) \in \text{Ker} (CH^n(X) \rightarrow H_{\mathcal{D}}^{2n}(X, \mathbb{Q}(n)))$$

for  $n \geq 2$ . In view of this and of 1.2.2, the question for  $Z = c_n(E)$ ,  $E$  flat, is equivalent to the question of the vanishing of  $Tc_n(\nabla)$  for any flat structure  $\nabla$  on  $E$ .

## 2. CHEEGER-SIMONS THEORY

### 2.1. Classical theory: differential characters. [11]

Chern-Weil theory provides an invariant  $P(\nabla^2, \dots, \nabla^2) \in H^0(X, \Omega_{\infty, \text{cl}}^{2n})$  when  $\nabla^2 \neq 0$  and Chern-Simons theory an invariant  $TP(\nabla) \in H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n))$  when  $\nabla^2 = 0$  and  $P$  has  $\mathbb{Z}$ -periods. Chern-Simons theory defines invariants for  $(E, \nabla)$  combining the two. To this aim, Chern and Simons define the ancestor of Deligne cohomology, the group of differential characters

$$\hat{H}^{2n}(X) = \mathbb{H}^{2n}(X, \mathbb{Z}(n)) \rightarrow \mathcal{O}_{\infty} \rightarrow \dots \rightarrow \Omega_{\infty}^{2n-1},$$

which is an extension of

$$\text{Ker} (H^0(X, \Omega_{\infty, \text{cl}}^{2n}) \rightarrow H^{2n}(X, \mathbb{C}/\mathbb{Z}(n)))$$

with  $H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n))$ . (The notation differs from theirs, as well as the hypercohomology presentation).

This group is functorial, and there is a natural ring structure on  $\oplus_n \hat{H}^{2n}(X)$ . For  $P$  with  $\mathbb{Z}$ -periods, they define  $\hat{c}_P \in \hat{H}^{2n}(X)$ , by saying that in bounded rank and dimension, there is a classifying space for bundles with connection. Since a  $C^{\infty}$  bundle always carries a  $C^{\infty}$  connection, this space also classifies bundles and thus has no odd dimensional cohomology. This implies that on this space, the differential characters inject into closed differential forms. Chern-Weil theory provides then the classes of the universal connection.

## 2.2. Algebraic theory: algebraic differential characters. [17]

Let  $X$  be a smooth algebraic variety defined over a field  $k$  of characteristic 0. One defines the group of algebraic differential characters by

$$AD^n(X) := \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d \log} \Omega^n \rightarrow \dots \rightarrow \Omega^{2n-1}),$$

an extension of

$$\text{Ker} (H^0(X, \Omega_{\text{cl}}^{2n}) \rightarrow \frac{\mathbb{H}^{2n}(X, \Omega^{\geq n})}{\text{Im } CH^n(X)})$$

by

$$\mathbb{H}^n(X, \Omega^\infty \mathcal{K}_n) := \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d \log} \Omega^n \rightarrow \dots \rightarrow \Omega^{\dim X}).$$

This group is functorial, and  $AD(X) = \bigoplus_n AD^n(X)$  has a natural ring structure. Over  $k = \mathbb{C}$ , it maps to  $\hat{H}^{2n}(X)$ , compatibly with the extension. In general, it also maps to  $CH^n(X)$  and  $H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$  for  $n \geq 2$ . For  $n = 1$ ,  $AD^1(X)$  is the group of isomorphism classes of  $(E, \nabla)$ , where  $E$  is a rank 1 bundle and  $\nabla$  is a connection, and  $AD^1(X) \rightarrow H^0(X, \Omega_{\text{cl}}^2)$  is the curvature map.

**Theorem 2.2.1.** [17] *There are functorial classes  $c_n(E, \nabla) \in AD^n(X)$ , such that  $c_1(E, \nabla)$  is the class of  $(\det E, \det \nabla)$  in  $AD^1(X)$ , such that  $c_n(E, \nabla)$  lifts  $c_n(E) \in CH^n(X)$ ,  $Tc_n(\nabla) \in H^0(X, \Omega^{2n-1}/d\Omega^{2n-2})$  for  $n \geq 2$ , and  $\hat{c}_n(E, \nabla) \in \hat{H}^{2n}(X)$  if  $k = \mathbb{C}$ . Those classes verify the Whitney product formula.*

Classically, there are two ways of constructing classes of vector bundles in a cohomology theory: via the splitting principle, knowing  $c_1(\mathcal{O}(1))$ , where  $\mathcal{O}(1)$  is the tautological bundle on the projective bundle  $\pi : \mathbb{P} = \mathbb{P}(E) \rightarrow X$  to  $E$ , and knowing the freeness of the cohomology of  $\mathbb{P}$  as a module over the cohomology of  $X$ . Or via the (simplicial) classifying space  $BG$ , and defining classes of the universal (simplicial) bundle  $EG$ . For connections,  $AD(\mathbb{P})$  is not a free module over  $AD(X)$ ,  $\mathcal{O}(1)$  does not have a connection, nor does  $EG$  have an algebraic connection.

Nonetheless, one can define the classes  $c_n(E, \nabla)$  via a modified splitting principle and via a universal  $BG$  construction as well.

For the splitting principle, one has to consider connections with values in a differential graded algebra. This leads to a definition of general groups of algebraic differential characters fulfilling the splitting principle, and thereby the unicity of the classes of Theorem 1.2.2. As an illustration, let us explain the objects for  $n = 2$ .  $\nabla$  defines a splitting

$$\tau : \Omega_{\mathbb{P}}^1 \rightarrow \pi^* \Omega_X^1$$

of the exact sequence of forms such that  $\tau \circ \nabla$  is compatible with  $\pi^* E \rightarrow \mathcal{O}(1)$ . Thus it defines a class

$$\xi = (\mathcal{O}(1), \tau \circ \nabla) \in \mathbb{H}^1(\mathbb{P}, \mathcal{K}_1 \xrightarrow{\tau \circ d \log} \pi^* \Omega_X^1).$$

One observes that with respect to the splitting  $\tau$ , one has

$$d(\Omega_{\mathbb{P}/X}^2) \subset \Omega_{\mathbb{P}/X}^3 \oplus \Omega_{\mathbb{P}/X}^2 \otimes \pi^* \Omega_X^1 \oplus \Omega_{\mathbb{P}/X}^1 \otimes \pi^* \Omega_X^2.$$

In particular, the complex

$$\mathcal{K}_2 \xrightarrow{d \log} \frac{\Omega_{\mathbb{P}}^2}{\Omega_{\mathbb{P}/X}^2} \xrightarrow{d} \pi^* \Omega_X^3$$

is well defined. With respect to a natural product  $\cup$ , its cohomology equals  $AD^2(X) \oplus AD^1(X) \cup \xi$  (see [17], section 2).

For the universal construction, one first constructs a cohomology associated to a smooth (simplicial) algebraic variety  $X$  and a (simplicial) vector bundle  $E$ , which depends on  $E$ , such that its value on  $(BG, EG)$  is “tautological”, and such that a connection  $\nabla$  in  $E$  maps this cohomology to  $AD^n(X)$ .

In the analytic context, Beilinson und Kazhdan ([3]) developed this procedure to recover the Cheeger-Simons classes in  $\hat{H}^{2n}(X)$ . They construct an analytic cohomology associated to  $X$  and  $E$ , which maps to  $\hat{H}^{2n}(X)$  via a connection in  $\nabla$  on  $E$ . The main ingredient, which they construct and which we use in our algebraization of their construction, is the following filtered differential graded algebra.

Let  $p : \mathcal{E} \rightarrow X$  be the principal  $G$  bundle to  $E$ , and  $\Omega_{X,\mathcal{E}}^1$  be as in 0.0.2. Then

$$\begin{aligned} \Omega_{X,\mathcal{E}}^n &= \bigoplus_{a+b=n} \Omega_{X,\mathcal{E}}^{a,b} \\ \Omega_{X,\mathcal{E}}^{a,b} &= \Lambda^{a-b} \Omega_{X,\mathcal{E}}^1 \otimes S^b \mathfrak{g}_{\mathcal{E}}^* \\ F^n \Omega_{X,\mathcal{E}}^\bullet &= \bigoplus_{a \geq n} \Omega_{X,\mathcal{E}}^{a,b} [a+b]. \end{aligned}$$

There is a natural differential  $\Omega_{X,\mathcal{E}}^n \rightarrow \Omega_{X,\mathcal{E}}^{n+1}$  extending the Kähler differential, and the natural injection

$$(\Omega_{X,\mathcal{E}}^\bullet, \Omega_{X,\mathcal{E}}^{\geq n}) \rightarrow (\Omega_{X,\mathcal{E}}^\bullet, F^n \Omega_{X,\mathcal{E}}^\bullet)$$

is a filtered quasi-isomorphism. A connection  $\nabla$  is equivalent to an inverse quasi-isomorphism

$$[\nabla] : \Omega_{X,\mathcal{E}}^\bullet \rightarrow \Omega_X^\bullet$$

and the integrability condition is equivalent to

$$[\nabla](F^n \Omega_{X,\mathcal{E}}^\bullet) \subset \Omega_X^{\geq n}.$$

The Weil homomorphism can be understood as a map

$$S^n(\mathfrak{g}^*)^G \xrightarrow{w_n} F^n \Omega_{X,\mathcal{E}}^\bullet[2n]$$

and the tautological cohomology needed is

$$\mathbb{H}^n(X, \text{cone}(\mathcal{K}_n \otimes S^n(\mathfrak{g}^*)^G[-n]) \xrightarrow{d \log \oplus -w_n} F^n \Omega_{X,\mathcal{E}}^\bullet[n])[-1]$$

([17], section 3).

### 2.3. Questions.

1. Since there are flat bundles  $(E, \nabla)$  such that  $c_n(E) \neq 0$  in  $CH^n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ , for example  $E = \bigoplus^r L_i$ ,  $L_i \in \text{Pic}^0 X$ ,  $X$  abelian variety ([6]), there are classes  $c_n(E, \nabla)$  which are not vanishing for flat bundles. The question is what is the coniveau of those classes, that is the largest codimension  $a$  such that a codimension  $a$  subvariety  $Z \subset X$  exists with  $c_n(E, \nabla)|_{X-Z} = 0$ . This question is related to the vanishing of  $Tc_n(\nabla)$  in section 1.
2. The group  $AD^n(X)$  mixes  $\mathcal{K}$ -cohomology with cohomology of differential forms. It would be more powerful to mix  $\mathcal{K}$ -cohomology with something related to Betti or étale cohomology. This probably would solve the question on the vanishing of  $Tc_n(\nabla)$ .

### 3. RIEMANN-ROCH THEOREMS

Let  $f : X \rightarrow S$  be a projective smooth morphism of relative dimension  $d$  over an algebraically closed field  $k$  of characteristic 0, with  $S$  and  $X$  smooth. Mumford [23] observed that if  $d = 1$ , then

$$(3.0.1) \quad c_n(R^1 f_*(\Omega_{X/S}^\bullet)) = 0 \text{ in } CH^n(S) \otimes_{\mathbb{Z}} \mathbb{Q},$$

applying the Grothendieck-Riemann-Roch theorem to the single sheaves of the relative de Rham complex  $\Omega_{X/S}^\bullet$ . The same argument shows

$$(3.0.2) \quad c_n\left(\sum_i (-1)^i R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S})\right) = (-1)^d f_*(c_d(\Omega_{X/S}^1) \cdot c_n(E)) \text{ in } CH^n(S) \otimes_{\mathbb{Z}} \mathbb{Q},$$

if  $E$  is a bundle on  $X$ , endowed with a flat relative connection  $\nabla_{X/S}$ . If  $\nabla_{X/S}$  comes from a flat global connection  $\nabla$ , the Gauß-Manin bundles

$$R^i f_*(\Omega_{X/S}^\bullet \otimes E)$$

carry the Gauß-Manin connection  $GM(\nabla)$ .

If  $k = \mathbb{C}$ , the work of Bismut-Lott [4] and Bismut [5] shows that 3.0.2 is true in  $\hat{H}^{2n}(X) \otimes \mathbb{Q}$ :

$$(3.0.3) \quad \hat{c}_n\left(\sum_i (-1)^i [R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S}), GM(\nabla)]\right) = (-1)^d f_*(c_d(\Omega_{X/S}^1) \cdot \hat{c}_n(E, \nabla)) \text{ in } \hat{H}^{2n}(X) \otimes \mathbb{Q}.$$

For  $n = 1$ , there is an analogy with the situation where  $S$  is a finite field  $\mathbb{F}_q$ ,  $(E, \nabla)$  is a local system  $V$ . Then the work of Deligne [12], [13], and subsequent work by Laumon [21], S. Saito [26], T. Saito [27], show that 3.0.3 for  $n = 1$  remains true:

$$(3.0.4) \quad \det \sum_i (-1)^i H_{\text{ét}}^i(X, V) = (-1)^d \det V|_{c_d(\Omega_X^1)}$$

as Frobenius-modules over  $\mathbb{F}_q$ .



Both 3.0.3 and 3.0.4 involve classes of the local system. The classes  $Tc_n(\nabla)$  and  $c_n(E, \nabla)$  reflect also the choice of the algebraic structure  $E$ . One shows

**Theorem 3.0.1.** [8]

a)

$$c_1\left(\sum_i (-1)^i [R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S}), GM(\nabla)]\right) = (-1)^d f_*(c_d(\Omega_{X/S}^1) \cdot c_1(E, \nabla)) \text{ in } AD^1(S) \otimes \mathbb{Q}$$

b)

$$Tc_n\left(\sum_i (-1)^i [R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S}), GM(\nabla)]\right) = (-1)^d f_*(c_d(\Omega_{X/S}^1) \cdot Tc_n(E, \nabla)) \text{ in } H^0(S, \mathcal{H}_{DR}^{2n-1}) \text{ if } n \geq 2.$$

More generally, Theorem 3.0.1 remains true if one replaces  $\nabla$  by a flat connection with logarithmic poles along a relative normal crossing divisor  $Y = \bigcup_i Y_i$ ,  $c_d(\Omega_{X/S}^1)$  by the relative top Chern class

$$c_d(\Omega_{X/S}^1(\log Y), \text{res}_{Y_i}) \in \mathbb{H}^d(X, \mathcal{K}_d \rightarrow \oplus_i \mathcal{K}_d|_{Y_i} \rightarrow \oplus_{i < j} \mathcal{K}_d|_{Y_i \cap Y_j} \rightarrow \dots)$$

as defined by T. Saito in [27], using the existence of the residue maps  $\text{res}_{Y_i} : \Omega_{X/S}^1(\log Y) \rightarrow \mathcal{O}_{Y_i}$ , and the classes  $c_n(E, \nabla), Tc_n(\nabla)$  by the corresponding classes involving logarithmic poles, which we haven't discussed here at all.

The introduction of logarithmic poles is necessary in order to have sufficiently many bundles with connections with the help of which one can reduce the problem to curves as in [21], [26], [27].

Another generalization of Theorem 3.0.1 is

**Theorem 3.0.2.** [8] *Under the assumptions*

- (i)  $S = \text{Spec } K$ ,  $K$  function field over  $k$  and
- (ii)  $\nabla^2 \in \mathbb{H}^0(X, f^*\Omega_S^2 \otimes \text{End } E)$ ,

*one has the same conclusion as in Theorem 3.0.1.*

Note that the assumption (ii) of 3.0.2 allows to define Gauß-Manin connections on  $S$ . Under the weaker assumption  $\nabla_{X/S}^2 = 0$ , no natural connection is defined on  $R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S})$ . However one has

**Theorem 3.0.3.** [8] *Under the assumptions 3.0.2 (i) and  $d = 1$  (thus (ii) is automatically fulfilled), there is a naturally defined connection on*

$$\det\left(\sum_{i=0}^{i=2} (-1)^i R^i f_*(\Omega_{X/S}^\bullet \otimes E, \nabla_{X/S})\right).$$

*With respect to this connection, the conclusion of Theorem 3.0.1, a) holds true.*

In  $AD^n(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ , one obtains the Riemann-Roch formula in the very trivial case  $X = Y \times S$ ,  $f = \text{projection}$ , by deforming the relative de Rham cohomology to some relative Higgs cohomology, and keeping the Gauß-Manin connection [18].

### 3.1. Remarks and Questions.

- 1) Mumford's observation 3.0.1 generalizes to

$$c_n(R^1 f_* \Omega_{X/S}^\bullet) = 0 \text{ in } CH^n(S) \otimes_{\mathbb{Z}} \mathbb{Q},$$

as computed by van der Geer [19], applying the Grothendieck-Riemann-Roch theorem to powers of a principal polarization of the corresponding family of abelian varieties. Then 3.0.2 implies

$$c_n(R^2 f_* \Omega_{X/S}^\bullet) = 0 \text{ in } CH^n(S) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for a family of smooth projective surfaces as well. Does one have

$$c_n(R^i f_* \Omega_{X/S}^\bullet) = 0 \text{ in } CH^n(S) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for a smooth projective family over a smooth base  $S$ ?

The contribution of the singularities of  $f$  at infinity is difficult to understand. Mumford [23] shows actually that the Gauß-Manin bundle with logarithmic singularities of a semi-stable curve has torsion Chern classes in the Chow group. In general, even the analytic statement in Deligne cohomology is not understood (see [16], 3.6 Questions).

- 2) Deligne [13] gave a proof of 3.0.4 for  $d = 1$ , rank  $V = 1$  using the geometry of the abelian variety  $\text{Pic}^0 X$ . In view of the shape of the Riemann-Roch formula, containing the expression  $c_d(\Omega_{X/S}^1)$ , it would be natural to try to understand it using the geometry of the moduli of Higgs bundles, Hitchin's map and Higgs cohomology.
- 3) Gauß-Manin bundles, when  $\nabla^2 = 0$ , are direct images of special holonomic  $\mathcal{D}$ -modules with regular singularities under special projective morphisms. A general Riemann-Roch formula would require a Chern-Simons and Cheeger-Simons theory for  $\mathcal{D}$ -modules, the algebraic cycle part of which should be given by [22].

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