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Germs of de Rham cohomology classes which vanish at the generic point

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Abstract. We show that hypergeometric differential equations, unitary and Gauß–Manin connections give rise to de Rham cohomology sheaves which do not admit a Bloch–Ogus resolution [1]. The latter is in contrast to Panin's theorem [8] asserting that corresponding étale cohomology sheaves do fulfill Bloch–Ogus theory. © Académie des Sciences/Elsevier, Paris

Germes de classes de cohomologie de de Rham qui s'annulent au point générique

Résumé. *Nous montrons que les systèmes d'équations hypergéométriques, les connexions unitaires et de Gauß–Manin donnent lieu à des faisceaux de cohomologie de de Rham qui n'ont pas de résolution de Bloch–Ogus* [1]. *Ce dernier exemple contraste avec le théorème de Panin* [8] *affirmant que des faisceaux semblables en cohomologie étale vérifient la théorie de Bloch–Ogus.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Soit (E, ∇) une connexion plate sur une variété lisse S sur un corps k algébriquement clos en caractéristique 0. La restriction des faisceaux de cohomologie $\mathcal{H}^i_{\mathrm{DR}}((E, \nabla))$ à leur valeur au point générique de S est trivialement injective pour $i = 0, 1$. Afin de montrer que pour les exemples de (E, ∇) évoqués plus haut, cela n'est plus nécessairement le cas pour $i = 2$, nous forçons l'existence de germes de sections de la façon suivante (*voir* [5]). On remplace S par l'éclatement d'un point. Cela introduit un diviseur exceptionnel sur lequel la connexion est triviale, et donc acquiert des sections. Par le morphisme de Gysin, ces sections fournissent des sections non nulles dans $\mathrm{H}^2_{\mathrm{DR}}(S, (E, \nabla))$ qui en particulier s'annulent au point générique de S. Que ces sections ne s'annulent pas dans le germe du faisceau en un point du diviseur exceptionnel provient d'une hypothèse convenable de résidus dans le cas hypergéométrique, de généricité dans le cas unitaire, et de la théorie de Hodge dans le cas de Gauß–Manin pour une famille à forte variation.

Note présentée par Christophe Soulé.

Let S be a smooth algebraic variety defined over an algebraically closed field k. Bloch–Ogus theory [1] provides an acyclic resolution of the Zariski sheaves of étale cohomology $\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(j))$ if $\operatorname{char} k = 0$ or if $(\operatorname{char} k, n) = 1$, of de Rham cohomology $\mathcal{H}^i_{\text{DR}}$ if $\operatorname{char} k = 0$, and of Betti cohomology $\mathcal{H}^i_{\text{B}}$ if $k = \mathbb{C}$. Here $\mathcal{H}^i$ denotes the Zariski sheaf associated to the presheaf $U \mapsto \mathrm{H}^i(U)$. The first level of the resolution says that the restriction map to the generic point $i_\eta : \eta = \operatorname{Spec} k(S) \to S$,

$$\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n(j)) \longrightarrow i_{\eta *}\mathrm{H}^i_{\text{ét}}(\eta, \mathbb{Z}/n(j)),$$

is injective (and similarly for de Rham and Betti cohomologies).

Bloch–Ogus theory extends in an obvious way to the sheaves of cohomology $\mathcal{H}^i(L)$ with values in a local system of complex vector spaces L of finite monodromy, or equivalently (*see* [6]) to the de Rham cohomology sheaves $\mathcal{H}^i_{\text{DR}}((E, \nabla))$ of a locally free sheaf E with a flat connection ∇, the monodromy of which is finite with respect to one embedding $k \subset \mathbb{C}$ (and hence to all).

A remarkable generalization of the Bloch–Ogus theory for étale cohomology has been given by I. Panin [8]. Let $f : X \to S$ be a projective smooth morphism and let L be a local system of free $\mathbb{Z}/n$-modules of finite rank (where n is prime to $\operatorname{char} k$ if $\operatorname{char} k > 0$). Then the Zariski sheaves $\mathcal{H}^i_{\text{ét}}(f, L(j))$ associated to the presheaves $U \longmapsto \mathrm{H}^i_{\text{ét}}(f^{-1}(U), L(j))$ have a Bloch–Ogus acyclic resolution on S. In particular, the restriction to the generic point

$$\mathcal{H}^i_{\text{ét}}(f, L(j)) \longrightarrow i_{\eta *}\mathrm{H}^i_{\text{ét}}(f^{-1}(\eta), L(j))$$

is injective, as in the classical case "f = identity and $L = \mathbb{Z}/n$".

This raises the question of a similar theorem for the de Rham cohomology in characteristic zero. In this Note we give negative examples:

0.1. Bundles E with a flat connection ∇ for which

$$\mathcal{H}^2_{\text{DR}}((E, \nabla)) \longrightarrow i_{\eta *}\mathrm{H}^2_{\text{DR}}(\eta, (E, \nabla)) \tag{0.1}$$

is not injective, or equivalently (over $\mathbb{C}$), local systems L of complex vector spaces for which $\mathcal{H}^2(L) \to i_{\eta *}\mathrm{H}^2(\eta, L)$ is not injective (*see* 2.1 and 3.1).

0.2. Smooth projective morphisms $f : X \to S$ for which

$$\mathcal{H}^4_{\text{DR}}(f) \longrightarrow i_{\eta *}\mathrm{H}^4_{\text{DR}}(f^{-1}(\eta)) \tag{0.2}$$

is not injective (*see* 1.3 and 1.4). Here $\mathcal{H}^i_{\text{DR}}(f)$ denotes the Zariski sheaf associated to $U \mapsto \mathrm{H}^i_{\text{DR}}(f^{-1}(U))$. Or equivalently, over $\mathbb{C}$,

$$\mathcal{H}^4_{\text{B}}(f) \longrightarrow i_{\eta *}\mathrm{H}^4_{\text{B}}(f^{-1}(\eta)) = i_{\eta *} \varinjlim_{U \subset S} \mathrm{H}^4_{\text{B}}(f^{-1}(U))$$

is not injective, with a similar notation for Betti cohomology.

As $\mathrm{R}^\bullet f_*\mathbb{C} = \oplus_j \mathrm{R}^j f_*\mathbb{C}[-j]$ over $k = \mathbb{C}$ [3], [6] implies that (0.2) is non-injective if

$$\mathcal{H}^{4-j}_{\text{DR}}((\mathrm{R}^j f_*\Omega^\bullet_{X/S}, \nabla)) \longrightarrow i_{\eta *}\mathrm{H}^{4-j}_{\text{DR}}(\eta, (\mathrm{R}^j f_*\Omega^\bullet_{X/S}, \nabla))$$

is non-injective for some j, where ∇ is the Gauß–Manin connection. In the second example we verify this for $j = 2$.

0.3. CONSTRUCTION OF A SECTION. – Let (E', ∇') be defined over a smooth variety S' of dimension at least two, and let $\delta : S \to S'$ be the blow-up of a point $p \in S'$, with exceptional divisor F. Let $(E, \nabla) = \delta^*(E', \nabla')$ be the pullback connection. Then the restriction map

$$\mathrm{H}^1_{\mathrm{DR}}(S, (E, \nabla)) = \mathrm{H}^1_{\mathrm{DR}}(S', (E', \nabla')) \longrightarrow \mathrm{H}^1_{\mathrm{DR}}(S' - p, (E', \nabla')) = \mathrm{H}^1_{\mathrm{DR}}(S - F, (E, \nabla))$$

is an isomorphism. Hence the Gysin map

$$i_F : \mathrm{H}^2_{\mathrm{DR},F}(S, (E, \nabla)) = \mathrm{H}^0_{\mathrm{DR}}(F, (E, \nabla)|_F) = k^{\mathrm{rank}\, E} \longrightarrow \mathrm{H}^2_{\mathrm{DR}}(S, (E, \nabla))$$

is injective, and any section $i_F(\sigma)$, $\sigma \in k^{\mathrm{rank}\, E}$, vanishes at the generic point η of S.

To show that the maps (0.1) and (0.2) need not be injective, we will show that for certain σ, the image $i_F(\sigma)$ is non-zero in the stalk $\mathcal{H}^2_{\mathrm{DR}}((E, \nabla))_q$ for all $q \in F$. The latter is equivalent to saying that for any divisor $C \subset S$, with $F \not\subset C$,

$$i_F(\sigma) \notin i_C \mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)). \tag{0.3}$$

For any smooth dense open subscheme C_0 of C and for $\lambda : S_0 = S - (C - C_0) \to S$ one has

$$\mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)) = \mathrm{H}^2_{\mathrm{DR},C_0}(S_0, (E, \nabla)) = \mathrm{H}^0_{\mathrm{DR}}(C_0, (E, \nabla)|_{C_0}) = \mathrm{H}^0_{\mathrm{DR}}(\tilde{C}, \nu^*(E, \nabla)), \tag{0.4}$$

where $\nu : \tilde{C} \to S$ is the normalization of $C \subset S$. One way to think of this is analytically. Let L be the kernel of ∇, let $\lambda : S_0 \to S$, $j_0 : S_0 - C_0 \to S_0$, and $j = \lambda \circ j_0$. Then $\mathrm{H}^2_C(S, L) = \mathrm{H}^0(S, \mathrm{R}^1 j_* L)$ and

$$\mathrm{H}^2_{C_0}(S_0, L) = \mathrm{H}^0(S, \lambda_* \mathrm{R}^1 j_{0*} L) = \mathrm{H}^0(S_0, \mathrm{R}^1 j_{0*} L) = \mathrm{H}^0(C_0, L|_{C_0}).$$

As $S - S_0$ has codimension ≥ 2 in S, $\mathrm{R}^i \lambda_* L = 0$ for $i = 1, 2$. Thus by the Leray spectral sequence for $j = \lambda \circ j_0$, the restriction map $\mathrm{R}^1 j_* L \to \lambda_* \mathrm{R}^1 j_{0*} L$ is an isomorphism. Since $\mathrm{H}^0(C_0, L|_{C_0}) = \mathrm{H}^0(\tilde{C}, \nu^* L)$ this concludes the proof of (0.4). In other words, we can do as if C was smooth.

This way to force geometrically the existence of sections which have nothing to do with the connection was used by the first author in [5] for example 2.1. We thank I. Panin for explaining to us the proof of his manuscript [8].

1. Gauß–Manin systems

ASSUMPTION 1.1. – *Let $\varphi : Y \to B$ be a semi-stable family of curves of genus $g \geq 1$ over a smooth projective curve B with $\varphi_* \omega_{Y/B}$ ample, Y smooth, and defined over an algebraically closed field k of characteristic zero.*

Let B_0 be the open subscheme of B with $Y_0 = \varphi^{-1}(B_0)$ smooth over B_0, and as in 0.3 let $\delta : S \to S' = B_0 \times B_0$ be the blow-up of a point $p = (b_1, b_2) \in B_0 \times B_0$ with exceptional divisor F. Let $f : X = (Y_0 \times Y_0) \times_{(B_0 \times B_0)} S \longrightarrow S$ be the pullback family. We consider the Gauß–Manin connection $(\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla)$. On the de Rham cohomology

$$\mathrm{H}^0_{\mathrm{DR}}(F, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla)|_F) = \mathrm{H}^2_{\mathrm{DR}}(Y_{b_1} \times Y_{b_2}),$$

one has the F-filtration which defines a pure Hodge-structure after base extension from k to $\mathbb{C}$.

CLAIM 1.2. – *For $\sigma \in \{F^0 - F^1\}\mathrm{H}^2_{\mathrm{DR}}(Y_{b_1} \times Y_{b_2})$ and for all $q \in F$ the image $i_F(\sigma)$ is non-zero in the stalk $\mathcal{H}^2_{\mathrm{DR}}((\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla))_q$.*

Proof. – Let $C \subset S$ be a reduced curve with $F \not\subset C$, let $\nu : \tilde{C} \to S$ be the normalization of C, let $\overline{\delta} : \overline{S} \to B \times B$ be the blow-up of p, and let $n : \Gamma \to \overline{S}$ be the normalization of the closure of C in $\overline{S}$. Let us denote by:

$$\overline{h} : X_\Gamma = (Y \times Y) \times_{(B \times B)} \Gamma \longrightarrow \Gamma,$$
$$h : X_{\tilde{C}} = (Y_0 \times Y_0) \times_{(B_0 \times B_0)} \tilde{C} \longrightarrow \tilde{C}.$$
$$X_F = X \times_S F = \mathbb{P}^1 \times Y_{b_1} \times Y_{b_2} \longrightarrow \mathbb{P}^1$$

the induced families of surfaces. The Gysin map $i_{\tilde{C}} : \mathrm{H}^2_{\mathrm{DR}}(X_{\tilde{C}}) \to \mathrm{H}^4_{\mathrm{DR}}(X)$, followed by the restriction map $\rho_F : \mathrm{H}^4_{\mathrm{DR}}(X) \longrightarrow \mathrm{H}^4_{\mathrm{DR}}(X_F) = \mathrm{H}^4_{\mathrm{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2})$, equals the restriction map

$$\rho_{\nu^{-1}(F)} : \mathrm{H}^2_{\mathrm{DR}}(X_{\tilde{C}}) \longrightarrow \bigoplus_{c \in \nu^{-1}(F)} \mathrm{H}^2_{\mathrm{DR}}(X_{\tilde{C}} \times_{\tilde{C}} \{c\}) = \bigoplus_{c \in \nu^{-1}(F)} \mathrm{H}^2_{\mathrm{DR}}(\{\nu(c)\} \times Y_{b_1} \times Y_{b_2}),$$

followed by the sum of the Gysin maps

$$\bigoplus_{c \in \nu^{-1}(F)} \mathrm{H}^2_{\mathrm{DR}}(\{\nu(c)\} \times Y_{b_1} \times Y_{b_2}) \longrightarrow \mathrm{H}^4_{\mathrm{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}).$$

On the other hand, $\rho_{\nu^{-1}(F)}$ factors through the surjective map

$$\rho : \mathrm{H}^2_{\mathrm{DR}}(X_{\tilde{C}}) \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(\tilde{C}, (\mathrm{R}^2 h_* \Omega^\bullet_{X_{\tilde{C}}/\tilde{C}}, \nabla)) = \mathrm{H}^2_{\mathrm{DR},C}(S, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla)),$$

and $\rho_F \circ i_{\tilde{C}} = \rho'_F \circ i_C \circ \rho$ where ρ'_F is the restriction map

$$\rho'_F : \mathrm{H}^2_{\mathrm{DR}}(S, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla)) \longrightarrow \mathrm{H}^2_{\mathrm{DR}}(F, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}), \nabla)|_F) = \mathrm{H}^4_{\mathrm{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}),$$

and $i_C : \mathrm{H}^2_{\mathrm{DR},C}(S, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla)) \longrightarrow \mathrm{H}^2_{\mathrm{DR}}(S, (\mathrm{R}^2 f_* \Omega^\bullet_{X/S}, \nabla))$ as in 0.3. By definition, $\overline{h}$ is a semi-stable family of surfaces, with singular fibres $Z = \overline{h}^{-1}(\infty)$ for $\infty = n^{-1}(\overline{S} - S) = \Gamma - \tilde{C}$. Hence, all Gauß–Manin bundles, in particular $\mathrm{R}^2 \overline{h}_* \Omega^\bullet_{X_\Gamma/\Gamma}(\log Z)$, have nilpotent residues. This says that the eigenvalues are 0, and thus they are the Deligne extension [2] of their restriction to $\tilde{C}$. Therefore one has

$$\mathrm{H}^0_{\mathrm{DR}}(\tilde{C}, (\mathrm{R}^2 h_* \Omega^\bullet_{X_{\tilde{C}}/\tilde{C}}, \nabla)) = \mathrm{H}^0(\Gamma, \Omega^\bullet_\Gamma(\log \infty) \otimes \mathrm{R}^2 \overline{h}_* \Omega^\bullet_{X_\Gamma/\Gamma}(\log Z)),$$

and $\rho(\{F^0/F^1\} \mathrm{H}^2_{\mathrm{DR}}(X_{\tilde{C}})) = \rho(\mathrm{H}^2(X_{\tilde{C}}, \mathcal{O}_{X_{\tilde{C}}})) \subset \mathrm{H}^0(\Gamma, \mathrm{R}^2 \overline{h}_* \mathcal{O}_{X_\Gamma})$. The sheaf

$$\mathrm{R}^2 \overline{h}_* \mathcal{O}_{X_\Gamma} = (\overline{\delta} \circ n)^* (\mathrm{pr}_1^* \mathrm{R}^1 \varphi_* \mathcal{O}_Y \otimes \mathrm{pr}_2^* \mathrm{R}^1 \varphi_* \mathcal{O}_Y)$$

is dual to $(\overline{\delta} \circ n)^* (\mathrm{pr}_1^* \varphi_* \omega_{Y/B} \otimes \mathrm{pr}_2^* \varphi_* \omega_{Y/B})$. Since $\overline{\delta} \circ n$ is finite, the latter is ample and $\mathrm{H}^0(\Gamma, \mathrm{R}^2 \overline{h}_* \mathcal{O}_{X_\Gamma}) = 0$. Thus, since the restriction and Gysin maps are morphisms of mixed Hodge structures (*see* [4], Theorem 3.2.5 for the covariant restriction ρ_F, and note that the Gysin map $i_{\tilde{C}}$ is dual to the covariant pullback map $\mathrm{H}^4_{\mathrm{DR}}(X) \to \mathrm{H}^4(X_{\tilde{C}})$),

$$\mathrm{im}(\rho_F \circ i_{\tilde{C}}) = \mathrm{im}(\rho'_F \circ i_C) \subset F^3 \mathrm{H}^4_{\mathrm{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}). \qquad (1.1)$$

On the other hand, $\rho'_F \circ i_F$ is the multiplication by $(-1) = \deg(\mathcal{O}_F(F))$. Hence for $\sigma \in \{F^0 - F^1\}\mathrm{H}^2_{\mathrm{DR}}(Y_{b_1} \times Y_{b_2})$, one has

$$\rho'_F \circ i_F(\sigma) \in \{F^2 - F^3\}\mathrm{H}^4_{\mathrm{DR}}(\mathbb{P}^1 \times Y_{b_1} \times Y_{b_2}). \tag{1.2}$$

(1.1) and (1.2) imply $i_F(\sigma) \notin i_C \mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla))$, and, as explained in 0.3, this proves the claim 1.2.

Example 1.3. – There are families which satisfy the assumption 1.1, for example, non-isotrivial semi-stable families of elliptic curves.

For $g \geq 3$ one can even assume that $\varphi : Y \to B$ is smooth. Hence there exist smooth families of surfaces $f : X \to S$ with S projective, for which the map (0.2) is not injective.

Proof. – Let $M_{g,3}$ and A_g be the moduli spaces of curves of genus g with level 3 structure and of g-dimensional principally polarized Abelian varieties, respectively. For $g \geq 3$ the image of $M_{g,3}$ in the Baily–Borel compactification of A_g is a projective manifold whose boundary has codimension larger than or equal to two. Hence $M_{g,3}$ has a projective compactification with the same property. Taking hyperplane intersections one obtains a smooth projective curve B in $M_{g,3}$, and thereby a smooth family of curves $\varphi : Y \to B$.

In order to show that for B in general position, $\varphi_* \omega_{Y/B}$ is ample, we may assume that $k = \mathbb{C}$.

The monodromy representation of the fundamental group of B is irreducible. Indeed, the fundamental group of M_g maps surjectively onto the fundamental group $\mathrm{Sp}(2g, \mathbb{Z})$ of A_g. The latter acts via the standard representation on $\mathbb{C}^{2g}$, in particular irreducibly. On the other hand, the fundamental group of B maps surjectively to the one of $M_{g,3}$, hence to a subgroup H of finite index of $\mathrm{Sp}(2g, \mathbb{Z})$. Thus H is Zariski dense in $\mathrm{Sp}(2g, \mathbb{C})$.

On the other hand, by [7], 4.10, the sheaf $\varphi_* \omega_{Y/B}$ is the direct sum of an ample vector bundle and a vector bundle, flat with respect to the Gauß–Manin connection. The irreducibility of the monodromy representation implies that the latter is trivial.

2. Hypergeometric equations

Example 2.1. – As in 0.3 let $S' = \mathbb{P}^2 - D$, where D is the union of three lines H_1, H_2, and H_3 in general position. Choose $a_1, a_2 \in k$, $a_3 = -a_1 - a_2$ such that the elements $1, a_i, a_j \in k$ are $\mathbb{Q}$-linearly independent, for $1 \leq i < j \leq 3$.

Let $\omega \in \mathrm{H}^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log D))$ be the unique form with $\mathrm{res}_{\mathrm{H}_i}\omega = a_i$ for $i = 1, 2, 3$, and $(E', \nabla') = (\mathcal{O}_{S'}, d + \omega)$. As in 0.3, consider the blow-up $\delta : S \to S'$ with exceptional divisor F and the pullback (E, ∇). We take a section $0 \neq \sigma \in k = \mathrm{H}^0(F, (E, \nabla)|_F)$ and regard its image $i_F(\sigma)$ under the Gysin map.

Claim 2.2. – For all reduced curves $C \subset S$ not containing F, one has $\mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)) = 0$. In particular, $0 \neq i_F(\sigma) \in \mathcal{H}^2_{\mathrm{DR}}((E, \nabla))_q$ for all $q \in F$.

Proof. – Let C_0 be the smooth locus of C. As in (0.4), $\mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)) = \mathrm{H}^0_{\mathrm{DR}}(C_0, (E, \nabla)|_{C_0})$, and since (E, ∇) has rank one, the claim 2.2 is equivalent to $(E, \nabla)|_{C_0} \neq (\mathcal{O}_{C_0}, d)$.

Let $\overline{\delta} : \overline{S} \to \mathbb{P}^2$ be the blow-up of p, and let $n : \Gamma \to \overline{S}$ be the normalization of the closure of C in $\overline{S}$. For $\infty = n^{-1}\overline{\delta}^{-1}(D)$, and for some $m_i \in \mathbb{N}$ sufficiently large, one has

$$\mathrm{H}^0_{\mathrm{DR}}(C_0, (E, \nabla)|_{C_0}) = \mathrm{H}^0(\Gamma, \Omega^\bullet_\Gamma(\log \infty) \otimes (\overline{\delta} \circ n)^*(\mathcal{O}_{\mathbb{P}^2}(\textstyle\sum m_i \mathrm{H}_i), d + \omega)).$$

The residues of $(\overline{\delta} \circ n)^*(\mathcal{O}_{\mathbb{P}^2}(\sum m_i \mathrm{H}_i), d + \omega)$ along $x \in \Gamma$ are in

$$(\mathbb{N} - \{0\}) \cdot (a_i - m_i) \qquad \text{for } \overline{\delta}(n(x)) \in \mathrm{H}_i - \bigcup_{j \neq i} \mathrm{H}_j,$$
$$(\mathbb{N} - \{0\}) \cdot (a_i - m_i) + (\mathbb{N} - \{0\}) \cdot (a_j - m_j) \qquad \text{for } \overline{\delta}(n(x)) \in \mathrm{H}_i \cap \mathrm{H}_j.$$

Indeed, the residue of $(\mathcal{O}_{\mathbb{P}^2}(\sum m_i \mathrm{H}_i), d+\omega)$ along H_i is $a_i - m_i$, and Γ lies on a surface X obtained by a sequence of blow-ups $X \to S$. As well known, if z is a smooth point on a variety Z with local coordinates $x_1, \dots, x_n$, and (E, ∇) a differential equation defined around z by $\omega = \sum_{i=1}^n b_i \frac{\mathrm{d}x_i}{x_i} + \eta$ for a regular form η on Z, then the pullback differential equation on the blow-up of z has residue $\sum_{i=1}^n b_i$ along the exceptional locus.

By the assumption (i) in 2.1, the residues of $(\bar{\delta} \circ n)^*(\mathcal{O}_{\mathbb{P}^2}(\sum m_i \mathrm{H}_i), d+\omega)$ can not be in $\mathbb{Q}$, and a fortiori $(E, \nabla)|_{C_0}$ can not be trivial.

3. Unitary rank one sheaves

Example 3.1. – Let B_1 and B_2 be two non-isogeneous elliptic curves, defined over $k = \mathbb{C}$, let $L_i \in \mathrm{Pic}^0(B_i)$ be non-torsion, and let ∇_i be the unique unitary connection on L_i. Using the notations introduced in 0.3, we choose $S' = B_1 \times B_2$ and

$$(E', \nabla') = (\mathrm{pr}_1^* L_1 \otimes \mathrm{pr}_2^* L_2, \mathrm{pr}_1^* \nabla_1 \otimes \mathrm{pr}_2^* \nabla_2).$$

Then for the pullback (E, ∇) of (E', ∇') on the blow-up S of a point p, the map (0.1) will not be injective. In fact, for the exceptional divisor F on S one has $\mathrm{H}^0_{\mathrm{DR}}(F, (E, \nabla)|_F) = \mathbb{C}$ whereas for all reduced curves $C \subset S$ with $F \not\subset C$ one finds $\mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)) = 0$.

Proof. – Let again C_0 denote the smooth locus of C and let $n : \Gamma \to S$ be the normalization. By (0.4)

$$\mathrm{H}^2_{\mathrm{DR},C}(S, (E, \nabla)) = \mathrm{H}^0_{\mathrm{DR}}(C_0, (E, \nabla)|_{C_0}) = \mathrm{H}^0_{\mathrm{DR}}(\Gamma, n^*(E, \nabla)) \subset \bigoplus_j \mathrm{H}^0(\Gamma_j, \mathrm{p}_{j,1}^* L_1 \otimes \mathrm{p}_{j,2}^* L_2),$$

where Γ_j are the irreducible components of Γ, and where $\mathrm{p}_{j,i}$ denotes the restriction of $\mathrm{pr}_i \circ \delta \circ n : \Gamma \to B_i$ to Γ_j. If $\mathrm{p}_{j,i}$ is dominant, the image B_i' of $\mathrm{p}_{j,i}^* : \mathrm{Pic}^0(B_i) \longrightarrow \mathrm{Pic}^0(\Gamma_j)$ is isogeneous to B_i and it is the Zariski closure of the subgroup generated by $\mathrm{p}_{j,i}^* L_i$. Hence if one of the projections, say $\mathrm{p}_{j,1}$, maps Γ_j to a point, $\mathrm{p}_{j,1}^* L_1 \otimes \mathrm{p}_{j,2}^* L_2 = \mathrm{p}_{j,2}^* L_2$ has no global section.

If both, $\mathrm{p}_{j,1}$ and $\mathrm{p}_{j,2}$ are dominant, the two elliptic curves B_1' and B_2' are not isogeneous, hence $B_1' \cap B_2'$ is finite, and $\mathrm{H}^0(\Gamma_j, \mathrm{p}_{j,1}^* L_1 \otimes \mathrm{p}_{j,2}^* L_2) = 0$.

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