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ALGEBRAIC DIFFERENTIAL CHARACTERS

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0. Introduction

In [5], Cheeger and Simons defined on a $\mathcal{C}^\infty$ manifold X a group of differential characters $\hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$, which is an extension of the global $\mathbb{R}$ valued closed forms of degree $2n$ having $\mathbb{Z}$ periods, by the group $H^{2n-1}(X, \mathbb{R}/\mathbb{Z})$. (In fact, they write $\hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$ but the notation $2n$ rather than $(2n-1)$ fits better with weights in algebraic geometry). Similarly, there is a group of (complex) differential characters $\hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$, presented as an extension of the global $\mathbb{C}$ valued closed forms of degree $2n$ having $\mathbb{Z}$ periods, by the group $H^{2n-1}(X, \mathbb{C}/\mathbb{Z})$. The group $\hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$) is also presented as an extension of the Betti cohomology group $H^{2n}(X, \mathbb{Z})$ by global $\mathbb{R}$ valued (resp. $\mathbb{C}$ valued) differential forms of degree $2n-1$, modulo the closed ones with $\mathbb{Z}$ periods.

They define a ring structure on $\hat{H}^{2\bullet}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{H}^{2\bullet}(X, \mathbb{C}/\mathbb{Z})$) and show the existence of functorial and additive classes $\hat{c}_n(E, \nabla)_{\mathbb{R}} \in \hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$ (resp. $c_n(E, \nabla) \in \hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$) for a $\mathcal{C}^\infty$ bundle with a connection ∇, lifting the closed form $P_n(\nabla^2, \ldots, \nabla^2)$ with $\mathbb{Z}$ periods, where P_n is the homogeneous symmetric invariant polynomial of degree n which is the n-th symmetric function in the entries on diagonal matrices. When the connection ∇ is flat, that is when $\nabla^2 = 0$, then $\hat{c}_n(E, \nabla)_{\mathbb{R}} \in H^{2n-1}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{c}_n(E, \nabla) \in H^{2n-1}(X, \mathbb{C}/\mathbb{Z})$).

The aim of this article is to develop a similar construction when X is an algebraic smooth variety over a field k and (E, ∇) is an algebraic bundle with an algebraic connection. We define a group of *algebraic differential characters* $AD^n(X)$, which is an extension of global closed algebraic forms of degree $2n$, whose cohomology class in $\mathbb{H}^{2n}(X, \Omega_X^{\geq 2n})$ is algebraic, by the group $\mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \to \ldots)$, introduced and studied in [7], [3]. When $k = \mathbb{C}$, $AD^n(X)$ maps to the group of *analytic differential characters* $D^n(X)$, defined as an extension of global analytic forms of degree $2n$, with $\mathbb{Z}(n)$ periods, by

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$H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$. The group $D^n(X)$ maps to the Deligne cohomology group $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$. (The above description is valid when X is proper, otherwise one has to modify it a little bit). The group $AD^n(X)$ (resp. $D^n(X)$) is also presented as an extension of the subgroup Ker $CH^n(X) \to \mathbb{H}^n(X, \Omega^n_X \to \ldots \to \Omega^{2n-1}_X)$ of the Chow group $CH^n(X)$ (resp. of the subgroup Ker $H^{2n}(X_{\rm an}, \mathbb{Z}(n)) \to H^n(X_{\rm an}, \Omega^n_X \to \ldots \to \Omega^{2n-1}_X))$ of the Betti cohomology group $H^{2n}(X_{\rm an}, \mathbb{Z}(n)))$ by $\mathbb{H}^{n-1}(X, \Omega^n_X \to \ldots \to \Omega^{2n-1}_X)/H^{n-1}(X, \mathcal{K}_n)$ (resp. $\mathbb{H}^{n-1}(X_{\rm an}, \Omega^n_X \to \ldots \to \Omega^{2n-1}_X)/H^{2n-1}(X_{\rm an}, \mathbb{Z}(n)))$. There is a ring structure on $AD^\bullet(X), D^\bullet(X)$. If (E, ∇) is an algebraic bundle with an algebraic connection, we define functorial and additive classes $c_n(E, \nabla) \in AD^n(X)$, lifting $P_n(\nabla^2, \ldots, \nabla^2)$ and the algebraic Chern classes $c_n(E) \in CH^n(X)$. The group $AD^n(X)$ maps to $H^0(X, \Omega^{2n-1}_X/d\Omega^{2n-2}_X)$. The class $c_n(E, \nabla)$ lifts the algebraic Chern-Simons class $w_n(E, \nabla)$, related to the algebraic equivalence relation on cycles, defined in [3]. When ∇ is flat, then $c_n(E, \nabla) \in \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \to \ldots)$ and is the class defined in [7].

We give two constructions of the classes. The first one is a *generalized splitting principle*. In [6] and [7], we had defined a modified splitting principle when ∇ is flat. However, our construction had the disavantage to use the flag bundle of E, rather than simply its projective bundle $\mathbb{P}(E)$, and to introduce a "τ cohomology" on the flag bundle which is not a free module over the corresponding "τ cohomology" of the base X. We correct those two points here by introducing a slightly more complicated "τ cohomology" on $\mathbb{P}(E)$, also defined when ∇ is not flat, which is free over the corresponding "τ cohomology" on X. The product structure is then naturally defined following the recipe explained in [1] and [8]. The whole construction is now very closed to the construction of Hirzebruch-Grothendieck for bundles without supplementary structure.

The second construction relies on the generalized *Weil algebra* defined by Beilinson and Kazhdan in [2]. To a bundle E, they associate functorially a Weil algebra complex $\Omega^\bullet_{X,E}$ together with a group $H^{2n}(X, U_E(n))$, which is an extension of the homogeneous symmetric invariant polynomials P of degree n mapping to

$$\mathrm{Im}(H^{2n}(X_{\rm an}, \mathbb{Z}(n)) \to H^{2n}(X_{\rm an}, \mathbb{C}))$$

via the Weil homomorphism, by $H^{2n-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(n))$. By evaluating their construction on the universal simplicial bundle, they defined functorial classes

$$c_n^{BK}(E) \in H^{2n}(X, U_E(n)).$$

A connection ∇ on E defines a map $\nabla : H^n(X, U_E(n)) \to D^n(X)$, and thereby classes in $D^n(X)$.

We modify their construction to make it more algebraic. We obtain in this way $c_n^{ABK}(E,\nabla) \in AD^n(X)$. We show that $c_n(E,\nabla) = c_n^{ABK}(E,\nabla)$, reflecting the fact that the splitting principle and the universal bundle construction define the same Chern classes for bundles without supplementary structure.

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1. Notations

1) X is a smooth algebraic variety over a field k, D is a normal crossing divisor, $j : U = X - D \to X$ is the embedding.
2) $\lambda : X \to \bar{X}$ is a good compactification, such that $\Delta = \bar{X} - X$ and $\bar{X} - U = \bar{D}$ are normal crossing divisors, $\bar{j} = \lambda \circ j : U \to \bar{X}$ is the embedding.
3) $(\Omega^\bullet_X(\log D), \Omega^{\geq n}_X(\log D))$ is the de Rham complex with logarithmic poles along D, filtered by its stupid filtration, and $\Omega^{<n}_X(\log D)$ is its quotient.
4) $\tau_0 : \Omega^\bullet_X(\log D) \to N^\bullet$ is a map of complexes, where $N^\bullet$ is a differential graded algebra, such that the sheaves N^n are locally free, as well as $B^b = \mathrm{Ker}\ \Omega^b_X(\log D) \to N^b$, such that $N^0 = \mathcal{O}_X$, and that if a is the smallest degree for which $B^a \neq 0$, then $B^b = B^a \wedge \Omega^{b-a}_X(\log D)$ for $b \geq a$ (see [3] (3.11) and [6] (2.1)).
5) $\mathcal{K}_n$ is the image of the Zariski sheaf $\mathcal{K}_n^M$ of Milnor K theory in $K_n^M(k(X))$, with its $d\log$ map $\mathcal{K}_n \xrightarrow{d\log} \Omega^{\geq n}_X[n]$, and its induced map $\tau_0 \circ d\log : \mathcal{K}_n \to N^{\geq n}[n]$. $N^\infty\mathcal{K}_n$ is the complex $(\mathcal{K}_n \to N^n \to N^{n+1} \to ...)$.
6) (E,∇) is an algebraic bundle of rank r with an algebraic τ_0 connection, also called a N^1 valued connection

$$\nabla : E \to N^1 \otimes_{\mathcal{O}_X} E,$$

that is a k linear map verifying the Leibniz rule

$$\nabla(\lambda e) = \tau_0 \circ d(\lambda) \otimes e + \lambda \nabla e.$$

∇^2 is the $\mathcal{O}_X$ linear map $\nabla \circ \nabla : E \to N^2 \otimes E$, call the curvature of ∇, with

$$\nabla : N^n \otimes E \to N^{n+1} \otimes E$$

defined by the sign convention

$$\nabla(\alpha \otimes e) = \tau_0 \circ d(\alpha) \otimes e + (-1)^n \alpha \wedge \nabla e.$$

If $\nabla^2 = 0$, one defines the τ_0 de Rham complex

$$(N^\bullet \otimes E, \nabla).$$

7) If $k = \mathbb{C}$, $a : X_{\mathrm{an}} \to X_{\mathrm{zar}}$ is the identity from X endowed with the analytic topology to X endowed with the Zariski topology. For a sheaf $\mathcal{F}$ on X_{zar}, we denote by $\mathcal{F}_{\mathrm{an}}$ the sheaf $a^*\mathcal{F}$. When the context is clear, we still write $\mathcal{F}$ for $\mathcal{F}_{\mathrm{an}}$.
8) $H^p_{\mathcal{D}}(X, \mathbb{Z}(q))$ is the Deligne-Beilinson cohomology of X.
9) $\mathcal{F}^a\mathcal{H}^b$ is the Zariski sheaf associated to $F^aH^b_{DR}(U)$ (the Hodge filtration on the de Rham cohomology), $\mathcal{H}^n(\mathbb{C}/\mathbb{Z}(m))$ to $H^n(U, \mathbb{C}/\mathbb{Z}(m))$, $\mathcal{F}^a_{\mathbb{Z}(c)}\mathcal{H}^b = \mathrm{Ker}\ (\mathcal{F}^a\mathcal{H}^b \to \mathcal{H}^b(\mathbb{C}/\mathbb{Z}(c)))$, $\mathcal{H}^a_{\mathcal{D}}(b)$ to $H^a_{\mathcal{D}}(U, \mathbb{Z}(b))$.
10) For $G = GL(r)$, of $\mathcal{G} = M(r)$, we denote by $P_n \in S^n(\mathcal{G}^*)^G$ the symmetric invariant polynomial of degree n which is the n-th symmetric function of the diagonal entries on diagonal matrices. We denote by $G_e \subset G$ the matrices fixing the subspace spanned by the r'' first canonical basis vectors, and by $\mathcal{G}_e \subset \mathcal{G}$ the corresponding algebra.
11) If E is a vector bundle, we denote by $\mathcal{G}_E$ the endomorphisms of E, and by $\mathcal{G}^*_E$ its dual (of course isomorph to $\mathcal{G}_E$). If

$$(e) : 0 \to E'' \to E \to E' \to 0$$

is an exact sequence of bundles with $r'' = \mathrm{rank} E'', r' = \mathrm{rank} E'$, we denote by $\mathcal{G}_{(E,e)} \subset \mathcal{G}_E$ the endomorphisms respecting (e) and by $\mathcal{G}^*_{(E,e)}$ its dual.

2. Algebraic Differential Characters

Definition 2.0.1. *Let τ_0 be as in 1, 4). We denote by ι the natural embedding $N^{\geq 2n}[n] \to N^{\geq n}[n]$. We define on X the complex*

$$C(n)_{\tau_0} = \mathrm{cone}\ (\mathcal{K}_n \oplus N^{\geq 2n}[n] \xrightarrow{d\log \oplus -\iota} N^{\geq n}[n])[-1].$$

If $k = \mathbb{C}$, we define on $\bar{X}_{\text{an}}$

$$\begin{aligned} DR(n)_{\tau_0} &= \text{cone } (\Omega_{\bar{X}}^{\leq n}(\log \bar{D}) \to R\lambda_*(N^{\geq n}[1])[-1] \\ &= \mathcal{O}_X \to \cdots \to \Omega_{\bar{X}}^{n-1}(\log \bar{D}) \to \lambda_* N^n \to \lambda_* N^{n+1} \cdots \end{aligned}$$

$$C(n)_{\tau_0}^{\text{an}} = \text{cone } (R\lambda_*\mathbb{Z}(n) \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{\epsilon \oplus -\iota} DR(n)_{\tau_0})[-1]$$

where ϵ is induced by the map

$$\epsilon : R\lambda_*\mathbb{Z}(n) \to \Omega_{\bar{X}}^{\bullet}(\log \bar{D}).$$

We define

$$\begin{aligned} AD_{\tau_0}^n(X) &= \mathbb{H}^n(X, C(n)_{\tau_0}) \\ D_{\tau_0}^n(X) &= \mathbb{H}^{2n}(X, C(n)_{\tau_0}^{\text{an}}) \quad \textit{if} \;\; k = \mathbb{C}. \end{aligned}$$

When $\tau_0 =$ identity and $D = \phi$, we simply write $AD^n(X)$ and $D^n(X)$.

Proposition 2.0.2.

1) *The group $D_{\tau_0}^n(X)$ does not depend on λ (this justifies the notation).*
2) *There are exact sequences*
   i)

$$0 \to \mathbb{H}^n(X, N^\infty \mathcal{K}_n) \to AD_{\tau_0}^n(X) \to \text{Ker } (H^0(X, N_{cl}^{2n}) \to \frac{\mathbb{H}^{2n}(X, N^{\geq n})}{CH^n(X)}) \to 0$$

   ii)

$$0 \to \mathbb{H}^{2n}(X_{\text{an}}, \text{cone } (R\lambda_*\mathbb{Z}(n) \to DR(n)_{\tau_0})[-1]) \to D_{\tau_0}^n(X) \to \text{Ker } (H^0(X_{\text{an}}, N_{cl}^{2n}) \to \frac{\mathbb{H}^{2n}(\bar{X}_{\text{an}}, DR(n)_{\tau_0})}{H^{2n}(X_{\text{an}}, \mathbb{Z}(n))}) \to 0$$

For $\tau_0 =$ identity, $D = \phi$ and X proper, this reads

$$0 \to H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \to D^n(X) \to \text{Ker } (H^0(X_{\text{an}}, \Omega_{cl}^{2n}) \to H^{2n}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))) \to 0$$

For $\tau_0 =$ identity, $D = \phi$ and X non-proper, then one has a splitting $\mathbb{H}^m(X_{\text{an}}, DR(n)) = H_{DR}^m(X) \oplus R(m, n)$ and

$$\text{Ker } \mathrm{D^n(X)} \xrightarrow{\text{restriction}} \mathbb{H}^{2n}(\mathrm{X_{an}}, \mathrm{C(n)^{an}}|_{\mathrm{X_{an}}}) = \mathrm{R(2n-1, n)}.$$

3) *If $k = \mathbb{C}$, there is a commutative diagram*

$$\begin{array}{ccc} AD^n_{\tau_0}(X) & \xrightarrow{\alpha} & CH^n(X) \\ {\scriptstyle\psi_{\tau_0}}\downarrow & & \downarrow{\scriptstyle\psi} \\ D^n_{\tau_0}(X) & \xrightarrow{\beta} & H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \end{array}$$

Proof.

1) As usual, if λ and λ' are two good compactifications, one constructs a third one λ_1 dominating λ and λ', with $\sigma : \bar{X}_1 \to \bar{X}$. One just has to compare the λ and the λ_1 constructions. Then σ induces a map

$$\sigma^* : \mathbb{H}^m(\bar{X}, C(n)^{\mathrm{an}}_{\tau_0,\lambda}) \to \mathbb{H}^m(\bar{X}_1, C(n)^{\mathrm{an}}_{\tau_0,\lambda_1})$$

(where we put a λ index to underline the dependance), and from the cone definition of $C(n)^{\mathrm{an}}_{\tau_0,\lambda}$, one just has to see that

$$\sigma^* : \mathbb{H}^m(\bar{X}, DR(n)_{\tau_0,\lambda}) \to \mathbb{H}^m(\bar{X}_1, DR(n)_{\tau_0,\lambda_1})$$

is an isomorphism. But one has a long exact sequence

$$\to \mathbb{H}^m(X_{\mathrm{an}}, N^{\geq n}) \to \mathbb{H}^m(\bar{X}, DR(n)_{\tau_0\lambda}) \to H^m_{DR}(X)/F^n H^m_{DR}(X) \to ...$$

and σ^* induces an isomorphism on the two terms $\mathbb{H}^m(X_{\mathrm{an}}, N^{\geq n})$ and $H^m_{DR}(X)/F^n H^m_{DR}(X)$.

2) i) We just regard the exact triangle

$$\to N^\infty \mathcal{K}_n \to C(n)_{\tau_0} \to N^{\geq 2n}[n] \xrightarrow{[1]}$$

ii) Similarly we regard the exact triangle

$$\to \text{cone } (R\lambda_* \mathbb{Z}(n) \to DR(n)_{\tau_0})[-1] \to C(n)^{\mathrm{an}}_{\tau_0} \to R\lambda_* N^{\geq 2n}_{\mathrm{an}}[n] \xrightarrow{[1]} .$$

When $\tau_0 =$ identity and $D = \phi$, the maps

$$\to \Omega^\bullet_{\bar{X}_{\mathrm{an}}}(\log \bar{D}) \to DR(n) \to R\lambda_*(\Omega^\bullet_{X_{\mathrm{an}}})$$

define a splitting

$$\mathbb{H}^m(\bar{X}_{\mathrm{an}}, DR(n)) = H^m_{DR}(X) \oplus R(m, n),$$

with $H^m(X_{\mathrm{an}}, \mathbb{Z}(n))$ mapping to $H^m_{DR}(X)$. This shows that the restriction map is injective on the right hand side of the exact sequence ii) and that the Kernel of the left hand side is $R(2n-1, n)$.

3) α is induced by

$$C(n)_{\tau_0} \to \mathcal{K}_n$$

and β by

$$C(n)^{\mathrm{an}}_{\tau_0} \to \text{cone } (R\lambda_*\mathbb{Z}(n) \to \Omega^{<n}_{\bar{X}}(\log \bar{D}))[-1].$$

The cycle map is compatible with restriction to open sets. From the commutative diagrams of exact sequences

$$\begin{array}{ccccccccc}
 & & & & \mathbb{H}^{2n-1}(X, N^{\geq n}/N^{\geq 2n}) & = & \mathbb{H}^{2n-1}(X, N^{\geq n}/N^{\geq 2n}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & H & \longrightarrow & AD^n_{\tau_0}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & CH^n(\bar{X}) & \longrightarrow & CH^n(X) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{H}^n(X, N^{\geq n}/N^{\geq 2n}) & = & \mathbb{H}^n(X, N^{\geq n}/N^{\geq 2n}) & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & & \mathbb{H}^{2n-1}(X_{\mathrm{an}}, N^{\geq n}/N^{\geq 2n}) & = & \mathbb{H}^{2n-1}(X_{\mathrm{an}}, N^{\geq n}/N^{\geq 2n}) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ker}^{\mathrm{an}} & \longrightarrow & H^{\mathrm{an}} & \longrightarrow & D^n_{\tau_0}(X) \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ker}^{\mathrm{an}} & \longrightarrow & H^{2n}_{\mathcal{D}}(\bar{X}, \mathbb{Z}(n)) & \longrightarrow & H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathbb{H}^{2n}(X_{\mathrm{an}}, N^{\geq n}/N^{\geq 2n}) & = & \mathbb{H}^{2n}(X_{\mathrm{an}}, N^{\geq n}/N^{\geq 2n})
\end{array}$$

with

$$H = \mathbb{H}^n(\bar{X}, \bar{C}(n)_{\tau_0})$$
$$H^{\mathrm{an}} = \mathbb{H}^{2n}(\bar{X}, \bar{C}(n)^{\mathrm{an}}_{\tau_0})$$
$$\bar{C}(n)_{\tau_0} = \text{cone } (\mathcal{K}_n \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{d\log \oplus -\iota} R\lambda_*(N^{\geq n}))[-1]$$
$$\bar{C}(n)^{\mathrm{an}}_{\tau_0} = \text{cone } (\mathbb{Z}(n) \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{\epsilon \oplus -\iota} \bar{D}R(n)_{\tau_0})[-1]$$
$$\bar{D}R(n)_{\tau_0} = \text{cone } (\Omega^{<n}_{\bar{X}} \to R\lambda_* N^{\geq n}[1])[-1].$$

One sees that it is enough to lift $\psi_{\bar{X}} : CH^n(\bar{X}) \to H^{2n}_{\mathcal{D}}(\bar{X}, \mathbb{Z}(n))$ to $\psi_{\bar{X},\tau_0} : H \to H^{\mathrm{an}}$. One has

$$R^i a_* \bar{C}(n)^{\mathrm{an}}_{\tau_0} = \mathcal{H}^{i-1}(\mathbb{C}/\mathbb{Z}(n)) \text{ for } i \le n-1.$$

By the Bloch-Ogus vanishing theorem ([4], (6.2)), this implies

$$\mathbb{H}^{2n}(\bar{X}, \bar{C}(n)^{\mathrm{an}}_{\tau_0}) = \mathbb{H}^{2n}(\bar{X}, \mathrm{cone}(\ \tau_{\le n-1} Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0} \to Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0}))$$

One has an exact sequence

$$0 \to \mathcal{H}^{n-1}(\mathbb{C}/\mathbb{Z}(n)) \to \mathcal{H}^n_{\mathcal{D}}(\mathbb{Z}(n)) \to \mathcal{F}^n_{\mathbb{Z}(n)}\mathcal{H}^n \to 0$$

and the map of complexes

$$(\mathcal{H}^n_{\mathcal{D}}(\mathbb{Z}(n)) \to (a_*\lambda_* N)^{\ge n}/(a_*\lambda_* N)^{\ge 2n}[n]) \to \mathrm{cone}(\ \tau_{\le n-1} Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0} \to Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0})$$

factors through

$$(\mathcal{F}^n_{\mathbb{Z}(n)}\mathcal{H}^n \to (a_*\lambda_* N)^{\ge n}/(a_*\lambda_* N)^{\ge 2n}[n]) \xrightarrow{\sigma} \mathrm{cone}(\ \tau_{\le n-1} Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0} \to Ra_* \bar{C}(n)^{\mathrm{an}}_{\tau_0}).$$

To obtain $\psi_{\bar{X},\tau_0} : H \to H^{\mathrm{an}}$, we first map $\mathcal{K}_n \to (\lambda_* N)^{\ge n}/(\lambda_* N)^{\ge 2n}[n]$ to $\mathcal{F}^n_{\mathbb{Z}(n)}\mathcal{H}^n \to (a_*\lambda_* N)^{\ge n}/(a_*\lambda_* N)^{\ge 2n}[n]$ via the $d\log$ map on $\mathcal{K}_n$ and the change of topology map on $\lambda_* N^i$, and then we apply σ.

□

2.1. **Products.** In this section, we want to define products on $AD^n_{\tau_0}(X)$ and $D^n_{\tau_0}(X)$, compatibly with the products in the Chow groups and in the Deligne cohomology. To this aim we follow the pattern explained in [1] and [8].

Definition 2.1.1. *Let $\alpha \in \mathbb{R}$. We define*

$$C(m)_{\tau_0} \times C(n)_{\tau_0} \xrightarrow{\cup_\alpha} C(m+n)_{\tau_0}$$

by

$$
\begin{aligned}
x \cup_\alpha y &= \{x, y\} && x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
&= 0 && x \in \mathcal{K}_m, y \in N^{\geq 2n}[n] \\
&= (1-\alpha) d\log x \wedge y && x \in \mathcal{K}_m, y \in N^{\geq n}[n] \\
&= 0 && x \in N^{\geq 2m}[m], y \in \mathcal{K}_n \\
&= x \wedge y && x \in N^{\geq 2m}[m], y \in N^{\geq 2n}[n] \\
&= (-1)^{\deg x} \alpha x \wedge y && x \in N^{\geq 2m}[m], y \in N^{\geq n}[n] \\
&= \alpha x \wedge d\log y && x \in N^{\geq m}[m], y \in \mathcal{K}_n \\
&= (1-\alpha) x \wedge y && x \in N^{\geq m}[m], y \in N^{\geq 2n}[n] \\
&= 0 && x \in N^{\geq m}[m], y \in N^{\geq n}[n]
\end{aligned}
$$

Proposition 2.1.2. 1) *These formulae define for each* $\alpha \in \mathbb{R}$ *a product, compatibly with the product on* $\mathcal{K}_n, N^{\geq 2n}$*, and with the product*

$$
\mathcal{K}_m \times N^{\geq 2n} \xrightarrow{d\log \times 1} \Omega_X^{\geq m}[m] \times N^{\geq 2n} \xrightarrow{\tau_0} N^{\geq m+2n}[m]
$$
$$
N^{\geq 2m} \times \mathcal{K}_n \xrightarrow{1 \times d\log} N^{\geq 2m} \times \Omega^{\geq n}[n] \xrightarrow{\tau_0} N^{2m+n}[n]
$$

2) *One has* $x \cup_\alpha y = (-1)^{\deg x \deg y} y \cup_{(1-\alpha)} x$.
3) *For* α *and* β*, the products* $\cup_\alpha$ *and* $\cup_\beta$ *are homotopic. We denote by* $\cup$ *the induced product in cohomology. In particular*
   a) $\cup$ *is commutative on* $AD^n_{\tau_0}(X)$.
   b) *The restriction of* $\cup$ *to* $\mathbb{H}^n(X, N^\infty \mathcal{K}_n)$ *(2.0.2, 2 ii)) is given by*

$$
\begin{aligned}
&\{x, y\} && x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
&d\log x \wedge y && x \in \mathcal{K}_m, y \in N^{\geq n}[n] \\
&0 && \textit{otherwise}
\end{aligned}
$$

Proof. The verification is exactly as in [8], (3.2), (3.3), (3.5), where one replaces

$$\begin{array}{lll} \mathbb{Z}(n) & \text{by} & \mathcal{K}_n \\ F^n & \text{by} & N^{\geq 2n}[n] \\ \Omega_X^\bullet & \text{by} & N^{\geq n}[n]. \end{array}$$

In particular, the homotopy between $\cup_\alpha$ and $\cup_\beta$ is given by

$$\begin{array}{lll} h(x \otimes y) & = (-1)^\mu(\alpha - \beta) x \wedge y & \text{if } \ x \in (N^{\geq m}[m])^{\mu-1} y \in (N^{\geq n}[n])^{\mu'-1} \\ & = 0 & \text{otherwise} \end{array}$$

for

$$h : (C(m)_{\tau_0} \otimes_{\mathbb{Z}} C(n)_{\tau_0})^\ell \to C(m+n)^\ell_{\tau_0}.$$

□

3. The splitting principle for bundles with connection

3.1. **The τ complex.** Let $\tau_0 : \Omega^\bullet_X(\log D) \to N^\bullet$ and (E, ∇) be as in 1 4), 6), with $B^b = B^a \wedge \Omega_X^{b-a}(\log D)$. Let $\pi : P := \mathbb{P}(E) \to X$ be the projective bundle of E, and $D' := \pi^{-1}(D)$.

Definition 3.1.1.

$$\Omega^n_{\mathbb{P},\tau_0}(\log D') = \frac{\pi^* N^n \oplus \Omega^n_{\mathbb{P}}(\log D')/\pi^* B^a \wedge \Omega^{n-a}_{\mathbb{P}}(\log D')}{\pi^*(\Omega^n_X(\log D)/B^a \wedge \Omega^{n-a}_X(\log D))}.$$

For example, if $\Omega^\bullet_X(\log D)$ surjects onto $N^\bullet$, then this is

$$\mathcal{F}^n = \Omega^n_{\mathbb{P}}(\log D')/\pi^* B^a \wedge \Omega^{n-a}_{\mathbb{P}}(\log D'),$$

and in general it is the push-down of this sheaf via

$$\begin{array}{ccc} \pi^*(\Omega^n_X(\log D)/B^a \wedge \Omega^{n-a}_X(\log D)) & \xrightarrow{\subset} & \mathcal{F}^n \\ \subset \big\downarrow & & \\ N^n. & & \end{array}$$

The Leibniz rule implies that $\pi^* B^a \wedge \Omega^{n-a}_{\mathbb{P}}(\log D')$ is a subcomplex of $\Omega^\bullet_{\mathbb{P}}(\log D')$, and therefore τ_0 induces a map of differential graded algebras, still denoted by τ_0:

Definition 3.1.2.

$$\tau_0 : (\Omega^\bullet_{\mathbb{P}}(\log D'), d) \to (\Omega^\bullet_{\mathbb{P},\tau_0}(\log D'), \tau_0 \circ d).$$

Then ∇ induces a τ_0 connection, still denoted by ∇:

$$\nabla : \pi^*E \to \Omega^1_{\mathbb{P},\tau_0}(\log D') \otimes \pi^*E.$$

Recall from [6], § 2, that the connection

$$\nabla : E \to N^1 \otimes E$$

induces a splitting

$$\tau : \Omega^1_{\mathbb{P},\tau_0}(\log D') \to \pi^*N^1$$

of the exact sequence

$$0 \to \pi^*N^1 \to \Omega^1_{\mathbb{P},\tau_0}(\log D') \to \Omega^1_{\mathbb{P}/X} \to 0,$$

such that the π^*N^1 valued connection $\tau \circ \nabla$ on π^*E respects the canonical filtration

$$0 \to \Omega^1_{\mathbb{P}/X}(1) \to \pi^*E \to \mathcal{O}(1) \to 0,$$

i.e.:

$$\tau \circ \nabla(\Omega^1_{\mathbb{P}/X}(1)) \subset \pi^*N^1 \otimes \Omega^1_{\mathbb{P}/X}(1).$$

The induced connection on $\Omega^1_{\mathbb{P}/X} = \Omega^1_{\mathbb{P}/X}(1) \otimes \mathcal{O}(1)^*$ is then given by applying first the splitting into $\Omega^1_{\mathbb{P},\tau_0}(\log D')$, then applying the differential d of $\Omega^\bullet_{\mathbb{P},\tau_0}(\log D')$, then projecting onto the factor $\pi^*N^1 \otimes \Omega^1_{\mathbb{P}/X}$ of $\Omega^1_{\mathbb{P},\tau_0}(\log D')$. We write for short

Definition 3.1.3.

$$\tau d : \Omega^1_{\mathbb{P}/X} \to \pi^*N^1 \otimes \Omega^1_{\mathbb{P}/X}.$$

Lemma 3.1.4. *In the splitting*

$$\Omega^n_{\mathbb{P},\tau_0}(\log D') = \oplus_{a=0}^n \Omega^a_{\mathbb{P}/X} \otimes \pi^*N^{n-a}$$

one has

$$\begin{aligned}\tau \circ d(\Omega^a_{\mathbb{P}/X} \otimes \pi^*N^{n-a}) \\ \subset \Omega^{a-1}_{\mathbb{P}/X} \otimes \pi^*N^{n-a+2} \oplus \Omega^a_{\mathbb{P}/X} \otimes \pi^*N^{n-a+1} \\ \oplus\, \Omega^{a+1}_{\mathbb{P}/X} \otimes \pi^*N^{n-a}\end{aligned}$$

Proof. The map $\tau \circ d$ has 3 components:

$$d_{\rm rel} : \Omega^1_{\mathbb{P}/X} \to \Omega^2_{\mathbb{P}/X}$$
$$\tau d$$
$$\mu : \Omega^1_{\mathbb{P}/X} \to \pi^*N^2,$$

where μ is $\mathcal{O}_{\mathbb{P}}$ linear (see [3], (4.3.2) for the study of μ). One just applies the Leibniz rule. $\square$

Definition 3.1.5. 1) *We define a decreasing filtration* Φ *on* $\Omega^n_{\mathbb{P},\tau_0}(\log D')$ *by*

$$\Phi^\ell\Omega^n_{\mathbb{P},\tau_0}(\log D') = \oplus_{a=\ell}^n \Omega^a_{\mathbb{P}/X}\otimes \pi^*N^{n-a}$$

for $\ell \le n$

$$\Phi^\ell\Omega^n_{\mathbb{P},\tau_0}(\log D') = 0 \quad \text{for} \quad \ell > n,$$
$$\Phi^0\Omega^n_{\mathbb{P},\tau_0}(\log D') = \Omega^n_{\mathbb{P},\tau_0}(\log D')$$

fulfilling

$$d\Phi^\ell\Omega^n_{\mathbb{P},\tau_0}(\log D') \subset \Phi^{\ell-1}\Omega^{n+1}_{\mathbb{P},\tau_0}(\log D')$$

2) *We define the complex*

$$\begin{aligned} (M^{\ge n})^\ell &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \ell < n \\ &= \Omega^\ell_{\mathbb{P},\tau_0}(\log D')/\Phi^{2n-\ell}\Omega^\ell_{\mathbb{P},\tau_0}(\log D') \\ &(= \oplus_{a<2n-\ell}\Omega^a_{\mathbb{P}/X}\otimes \pi^*N^{\ell-a}) \qquad\qquad . \end{aligned}$$

In particular, $(M^{\ge n})^\ell = 0$ *for* $\ell \ge 2n$.

3) *We define the complex*

$$\begin{aligned} \Gamma(n)_\tau &= \text{cone}\ (\mathcal{K}_n \xrightarrow{\tau\circ d\log} M^{\ge n}[n])[-1] \\ &= \text{cone}\ (\mathcal{K}_n \oplus (M^{\ge n})^{\ge 2n}[n] \xrightarrow{\tau\circ d\log\oplus 0} M^{\ge n}[n])[-1] \end{aligned}$$

endowed with the map

$$\pi^{-1} : C(n)_{\tau_0} \to R\pi_*\Gamma(n)_\tau$$

induced by

$$\pi^{-1}\mathcal{K}_n \to \mathcal{K}_n$$
$$\pi^{-1}N^{\ge 2n} \to 0 = (M^{\ge n})^{\ge 2n},$$
$$\pi^{-1}N^{\ge n} \to M^{\ge n}.$$

4) *We define the multiplication*

$$\Gamma(m)_\tau \times \Gamma(m)_\tau \to \Gamma(m+n)_\tau$$

as in 2.1.1, replacing $N^{\ge n}$ *by* $M^{\ge n}$*, and* $N^{\ge 2n}$ *by* 0*, and observing that for* $x \in \Omega^m_{\mathbb{P},\tau_0}(\log D')$*,* $y \in \oplus_{a<2n-\ell}\Omega^a_{\mathbb{P}/X}\otimes\pi^*N^{\ell-a}$*, then*

$x \wedge y \in$

$$\begin{aligned} &\oplus_{a<2n+m-\ell}\, \Omega^a_{\mathbb{P}/X}\otimes \pi^*N^{m+\ell-a} \\ &\qquad = \oplus_{a<2(n+m)-(\ell+m)}\Omega^a_{\mathbb{P}/X}\otimes\pi^*N^{m+\ell-a}. \end{aligned}$$

Thus concretely

$$\begin{aligned} x \cup_\alpha y &= \{x, y\} \quad \textit{for} \quad x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= (1-\alpha) d\log x \wedge y \in (M^{\geq m+n})^{m+\ell} \\ &\qquad \textit{for} \quad x \in \mathcal{K}_m, y \in (M^{\geq n})^\ell \\ &= \alpha x \wedge d\log y \in (M^{\geq m+n})^{n+\ell} \\ &\qquad \textit{for} \quad x \in (M^{\geq m})^\ell, y \in \mathcal{K}_n \\ &= 0 \ \textit{for} \quad x \in M^{\geq m}[m], y \in M^{\geq n}[n]. \end{aligned}$$

Again $\cup_\alpha$ does not depend on α on cohomology. We denote it by $\cup$. In particular, the product induces an action

$$\mathbb{H}^a(\mathbb{P}, M^{\geq b}[b]) \times \mathbb{H}^{a'}(\mathbb{P}, \Gamma(b')_\tau) \xrightarrow{\cup} \mathbb{H}^{a+a'}(\mathbb{P}, M^{\geq (b+b')}[b+b'])$$

making Image $\mathbb{H}^{\bullet-1}(\mathbb{P}, N^{\geq \bullet\bullet}[\bullet\bullet])$ in $\mathbb{H}^\bullet(\mathbb{P}, \Gamma(\bullet\bullet)_\tau)$ into an ideal of square 0.

5) *The group $\mathbb{H}^1(\mathbb{P}, \Gamma(1)_\tau) = \mathbb{H}^1(\mathbb{P}, \mathcal{K}_1 \to \pi^* N^1)$ is the group of isomorphism classes of rank 1 bundles with a $\pi^* N^1$ valued connection. We denote by ξ the class of $(\mathcal{O}(1), \tau \circ \nabla)$ in it (3.1)*

Theorem 3.1.6. *The maps*

$$\pi^{-1} : AD^a_{\tau_0}(X) \to \mathbb{H}^a(\mathbb{P}, \Gamma(a)_\tau)$$

are injective for all $a \geq 0$. One has a splitting

$$\mathbb{H}^m(\mathbb{P}, \Gamma(n)_\tau) = \oplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, C(n-i)_{\tau_0}) \cup \xi^i$$

(where r is the rank of E (1, 6)).

Proof. We denote by $[\xi]$ the class of $\mathcal{O}(1)$ in $H^1(\mathbb{P}, \mathcal{K}_1)$. One has the projective bundle formula for the cohomology of the sheaves $\mathcal{K}$

$$\mathbb{H}^m(\mathbb{P}, \mathcal{K}_n) = \oplus_{i=0}^{r-1} H^{m-i}(X, \mathcal{K}_{n-i}) \cup [\xi]^i.$$

One also has obviously

$$\mathbb{H}^m(\mathbb{P}, M^{\geq n}[n]) = \oplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, N^{\geq n-i}[n-i]) \cup \xi^i.$$

One regards the exact sequences

$$\begin{aligned} &\to H^{m-1}(\mathbb{P}, \mathcal{K}_n) \to \mathbb{H}^{m-1}(\mathbb{P}, M^{\geq n}[n]) \to \\ &\qquad \mathbb{H}^m(\mathbb{P}, \Gamma(n)_\tau) \to H^m(\mathbb{P}, \mathcal{K}_n) \to \mathbb{H}^m(\mathbb{P}, M^{\geq n}[n]) \to \end{aligned}$$

and

$$\begin{aligned}\to \oplus_{i=0}^{r-1} H^{m-1-i}(X, \mathcal{K}_{n-i}) \to \oplus_{i=0}^{r-1}\mathbb{H}^{m-1-i}(X, N^{\geq r-1-i}[r-1-i]) \\ \to \oplus_{i=0}^{r-1}\mathbb{H}^{m-i}(X, C(n-i)_{\tau_0}) \to \oplus_{i=0}^{r-1} H^{m-i}(X, \mathcal{K}_{n-i}) \\ \to \oplus_{i=0}^{r-1}\mathbb{H}^{m-i}(X, M^{\geq(n-i)}[n-i]) \to .\end{aligned}$$

The map π^{-1} induces an isomorphism on the $\mathcal{K}$ and M cohomology, thus on the Γ cohomology as well. □

Definition 3.1.7. *Let $(E, \nabla), \tau_0, \xi$ be as in 3.1.5. We define*

$$c_0(E, \nabla) = 1 \in AD^0_{\tau_0}(X) = \mathbb{Z}$$

and

$$c_n(E, \nabla) \in AD^n_{\tau_0}(X), n = 1, \ldots, r$$

by the formula

$$\sum_{n=0}^{r}(-1)^n c_n(E, \nabla) \cup \xi^{r-n} = 0$$

via 3.1.6. We define $c_n(E, \nabla) = 0$ for $n > m$.

Theorem 3.1.8.

1) **Functoriality:** *Let $\pi : Y \to X$ be a morphism of smooth varieties, such that $D' = \pi^{-1}D$ is a normal crossing divisor. Let $\tau_0 : \Omega^\bullet_Y(\log D') \to \Omega^\bullet_{Y,\tau_0}(\log D')$ be the map of differential graded algebras with $\Omega^n_{Y,\tau_0}(\log D')$ defined as in 3.1.1 by*

$$\Omega^n_{Y,\tau_0}(\log D') = \frac{\pi^* N^n \oplus \Omega^n_Y(\log D')/\pi^* B^a \wedge \Omega^{n-a}_Y(\log D')}{\pi^*(\Omega^n_X(\log D)/B^a \wedge \Omega^{n-a}_X(\log D))}.$$

Then ∇ induces on π^*E a $\Omega^1_{Y,\tau_0}(\log D')$ valued connection $\pi^*\nabla$, and π^{-1} induces a map

$$\pi^{-1} : AD^n_{\tau_0}(X) \to AD^r_{\tau_0}(Y).$$

Then $c_n(\pi^*E, \pi^*\nabla) = \pi^{-1}c_n(E, \nabla)$. For all further maps

$$\tau : \Omega^\bullet_{Y,\tau_0}(\log D') \to M^\bullet$$

of differential graded algebras, one has

$$c_n(\pi^*E, \pi^*\nabla) = \text{Im } c_n(\pi^*E, \pi^*\nabla)$$

in $AD^n_{\tau\circ\tau_0}(X)$.

2) **First class:** *If E is of rank 1, and τ_0, ∇ are as in 1, 4), 6), then $c_1(E, \nabla)$ is the isomorphism class of (E, ∇) in the group of isomorphism classes of rank 1 bundles with a τ_0 connection $AD^1_{\tau_0}(X) = \mathbb{H}^1(X, \mathcal{K}_1 \to \pi^* N^1)$.*
3) **Whitney product formula:** *Let*
$$0 \to (E'', \nabla'') \to (E, \nabla) \to (E', \nabla') \to 0$$
be an exact sequence of bundles with τ_0 connection, that is the bundle sequence is exact and
$$\nabla E'' \subset N^1 \otimes E'' \ , \quad \nabla'' = \nabla|_{E''}, \nabla'$$
is the quotient connection. Then one has
$$c_n(E, \nabla) = \oplus_{a=0}^n c_a(E', \nabla') \cup c_{n-a}(E'', \nabla'')$$
in $AD^n_{\tau_0}(X)$ for all $n \geq 0$.
4) **Characterization of the classes:** *The classes $c_n(E, \nabla) \in AD^n_{\tau_0}(X)$ are uniquely determined by the functoriality property, the definition of the first class and the Whitney product formula.*

Proof. 1) and 2) are clear, and 4) is clear once one knows 3). For 3) , we mimic the classical proof [9], 6.10. One has

$$\begin{array}{ccccc} \mathbb{P}' := \mathbb{P}(E') & \xrightarrow{i} & P & \xleftarrow{j} & U : \mathbb{P} - \mathbb{P}' \\ & & & & \downarrow p \\ & & & & \mathbb{P}'' := \mathbb{P}(E'') \end{array}$$

where p is an affine filtration with the obvious notations $\Gamma(m)_{\tau'}, r'', \ldots$ one defines

$$\alpha = \sum_{n=0}^{r'} (-1)^n c_n(E', \nabla') \cup \xi^{r'-n} \in \mathbb{H}^{r'}(\mathbb{P}, \Gamma(r')_\tau)$$

$$\beta = \sum_{n=0}^{r''} (-1)^n c_n(E'', \nabla'') \cup \xi^{r''-n} \in \mathbb{H}^{r''}(\mathbb{P}, \Gamma(r'')_\tau)$$

Then $\gamma = \alpha \cup \beta \in \mathbb{H}^r(\mathbb{P}, \Gamma(r)_\tau)$ fulfills

$$j^* \beta = p^{-1}(\sum_{n=0}^{r''} (-1)^n c_n(E'', \nabla'') \cup \xi''^{r''-n}) = 0$$

and $\alpha|_{\mathbb{P}'} = 0$. Let $\beta' \in \mathbb{H}^r_{\mathbb{P}'}(\mathbb{P}, \Gamma(r)_\tau)$, with $\rho(\beta') = \beta$ in the localization sequence

$$\mathbb{H}^r_{\mathbb{P}'}(\mathbb{P}, \Gamma(r)_\tau) \xrightarrow{\rho} \mathbb{H}^r(\mathbb{P}, \Gamma(r)_\tau) \xrightarrow{j^*} \mathbb{H}^r(U, \Gamma(r)_\tau).$$

Since the product

$$\mathbb{H}^a(\mathbb{P},\Gamma(b)_\tau)\times\mathbb{H}^{a'}(\mathbb{P},\Gamma(b')_\tau)\to\mathbb{H}^{a+a'}(\mathbb{P},\Gamma(b+b')_\tau)$$

induces a compatible product

$$\mathbb{H}^a(\mathbb{P},\Gamma(b)_\tau)\times\mathbb{H}^{a'}_{\mathbb{P}'}(\mathbb{P},\Gamma(b')_\tau)\to\mathbb{H}^{a+a'}_{\mathbb{P}'}(\mathbb{P},\Gamma(b+b')_\tau)$$

one has $\gamma=\rho(\alpha\cup\beta')$. On the other hand, one has a Gysin isomorphism

$$\mathbb{H}^{a-r''}(\mathbb{P}',\Gamma(b-r'')_{\tau''})\xrightarrow{i_*}\mathbb{H}^a_{\mathbb{P}'}(\mathbb{P},\Gamma(b)_\tau).$$

To see this, one observes that i_* induces an isomorphism on the $\mathcal{K}$ cohomology. For the $M^{\geq b}$ cohomology, one makes a dévissage with the stupid filtration. This then reduces to the obvious isomorphism

$$\mathbb{H}^a(\mathbb{P}',\Omega^b_{\mathbb{P}'/X}\otimes\pi'^*N^c)\xrightarrow{i_*}\mathbb{H}^{a+r''}_{\mathbb{P}'}(\mathbb{P},\Omega^{b+r''}_{\mathbb{P}/X}\otimes\pi^*N^c).$$

We write $\beta'=i_*(\beta'')$ with $\beta''\in\mathbb{H}^{r'}(\mathbb{P},\Gamma(r')_\tau)$, and $\gamma=\rho(\alpha\cup i_*\beta'')$. It remains to see that

$$x\cup i_*y=i_*(x|_{\mathbb{P}'}\cup y)$$

for $x\in\mathbb{H}^a(\mathbb{P},\Gamma(b)_\tau)$, $y\in\mathbb{H}^{a'-r''}(\mathbb{P}',\Gamma(b'-r'')_{\tau'})$, as $\alpha|_{\mathbb{P}'}=0$. One checks that the diagram

$$\begin{array}{ccc}\Gamma(b)_\tau\times\text{cone }(\Gamma(b')_\tau\to Rj_*\Gamma(b')_\tau)[-1] & \xrightarrow{\cup} & \text{cone }(\Gamma(b+b')\to Rj_*\Gamma(b+b')_\tau)[-1]\\ \text{restriction}\times\text{residue}\big\downarrow & & \big\downarrow\text{residue}\\ \Gamma(b)_{\tau'}|\mathbb{P}'\times\Gamma(b'-r'')_{\tau'}|\mathbb{P}' & \xrightarrow{\cup} & \Gamma(b+b'-r'')_{\tau'}|\mathbb{P}'\end{array}$$

is commutative. □

3.2. **Comparison with the algebraic Chern-Simons invariants of** [3].

Theorem 3.2.1. 1) *Let (τ_0,E,∇) be as in 3.1.6, with τ_0 surjective. Then the image of $c_n(E,\nabla)$ in $H^0(X,N^{2n}_{cl})$ (2.0.2 2,i)) is $P_n(\nabla^2,\ldots,\nabla^2)$ (1, 10)).*

2) *Assume $\tau_0=$ identity. Then the image of $c_n(E,\nabla)$ under*

$$AD^n_{\tau_0}(X)\to H^0(X,\Omega^{2n-1}_X(\log D)/d\Omega^{2n-2}_X(\log D))$$

coincides with the algebraic Chern-Simons class $w_n(E,\nabla)$ defined in [3], *§ 2.*

Proof. 1) Let $p:G\to X$ be the flag bundle of $E,\Delta=p^{-1}(D)$. Define as in [3], (4.3.3) the sheaf

$$\bar{M}^{2n-\ell}=p^*N^{2n+\ell}/\mu(\Omega^1_{G/X}\otimes p^*N^{2n+\ell-2})$$

where μ is as in the proof of 3.1.4, for $\ell \geq 0$. One defines $\Omega^\bullet_{G,\tau_0}(\log D)$ as in 3.1.8, 1), and $\Gamma(n)_\tau$ similarly. Let

$$\bar{\Gamma}(n)_\tau = \text{cone } (\mathcal{K}_n \to \bar{M}^{\geq n}[n])[-1]$$

with

$$\begin{aligned} (\bar{M}^{\geq n})^\ell &= (M^{\geq n})^\ell & \ell \leq 2n-1 \\ &= \bar{M}^\ell & \ell \geq 2n. \end{aligned}$$

Then $\bar{\Gamma}(n)_\tau$ maps to $\Gamma(n)_\tau$. We lift the product in $\Gamma(n)_\tau$ to $\bar{\Gamma}(n)_\tau$ defined in 3.1.5 by setting

$$\begin{aligned} x \cup_\alpha y &= \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= (1-\alpha) d\log x \wedge y & x \in \mathcal{K}_m, y \in (\bar{M}^{\geq n}[n])^\ell \\ &= \alpha x \wedge d\log y & x \in (\bar{M}^{\geq m}[m])^\ell, y \in \mathcal{K}_n \\ &= 0 & x \in \bar{M}^{\geq m}[m], y \in \bar{M}^{\geq n}[n]. \end{aligned}$$

Here the product $d\log x \wedge y$ $(x \wedge d\log y)$ means $\tau(d\log x \wedge y)$ $(\tau(x \wedge d\log y))$ for $m + \ell \geq 2(m+n)$ $(\ell + n \geq 2(m+n))$. The product being compatible with the natural product of $(\bar{M}^{\geq n})^{\geq 2n}$, the image of $p^{-1}c_n(E, \nabla)$ in $H^0(G, \bar{M}^{2n} \to \bar{M}^{2n+1} \to ...)$ is the n-th symmetric product of the image $F(\ell_i)$ of

$$\ell_i = c_1(L_i, \tau \circ \nabla) \in \mathbb{H}^1(G, \mathcal{K}_1 \to p^* N^1)$$

in

$$\mathbb{H}^2(G, (\bar{M}^{\geq 1})^{\geq 2}) = \mathbb{H}^0(G \frac{p^* N^2}{\mu(\Omega^1_{G/X})} \to ...).$$

In other words, considering as in [3], § 4, the quotient differential graded algebra

$$\Omega^\bullet_{G,\tau_0}(\log \Delta) \to (\mathcal{O}_G \to \bar{M}^{\geq 1}),$$

$F(\ell_i)$ is the curvature of the $(\bar{M}^{\geq 1})^1 = p^* N^1$ valued connection $\tau \circ \nabla$ on L_i. As in [3], (4.7.8), this is exactly $p^* P_n(\nabla^2, \dots, \nabla^2)$. Now, as N^{2n} si locally free, $H^0(X, N^{2n})$ injects into $H^0(U, N^{2n})$ for any open $\phi \neq U \subset X$. Thus we may assume that $\nabla : E \to N^1 \otimes E$ lifts to $\bar{\nabla} : E \to \Omega^1_X \otimes E$ as $\Omega^1_X(\log D) \to N^1$ is surjective, and it is enough to prove 1) for $\bar{\nabla}$. As in [3], § 4, we may further assume that there is a morphism $\varphi : X \to T$, where T is an affine space, such that $(E, \bar{\nabla}) = \varphi^*(\mathcal{E}, \psi)$, where $\mathcal{E}$ is the trivial bundle, and ψ is such that if $q : P \to T$ is the flag bundle of $\mathcal{E}$, and $(\bar{M}^{\geq 1})$

is defined on p, then the map $\Omega_T^{2n} \to p_*(\bar{M}^{\geq 1})^{2n}$ is injective ([3], Proposition 4.9.1). Since

$$AD^n(T) = \frac{H^0(T, \Omega_T^{2n-1})}{dH^0(T, \Omega_T^{2n-2})} = H^0(X, \frac{\Omega_T^{2n-1}}{d\Omega_T^{2n-2}}) \subset H^0(T, \Omega_T^{2n}) \subset H^0(P, (\bar{M}^{\geq 1})^{2n})$$

we see that

$$\text{Im } c_n(\mathcal{E}, \psi) \text{ in } H^0(P, (\bar{M}^{\geq 1})^{2n}) = p^* P_n(F(\psi), \dots, F(\psi))$$

implies that

$$\text{Im } c_n(\mathcal{E}, \psi) \text{ in } H^0(T, \Omega_T^{2n}) = P_n(F(\psi), \dots, F(\psi)).$$

2) From the exact sequence

$$0 \to H^0(X - D, \mathcal{H}^{2n-1}) \to H^0(X, \frac{\Omega_X^{2n-1}(\log D)}{d\Omega_X^{2n-1}(\log D)}) \to H^0(X, \Omega_X^{2n}(\log D))$$

and the injectivity

$$H^0(X - D, \mathcal{H}^{2n-1}) \to H^0(U - D, \mathcal{H}^{2n-1})$$

for any open $0 \neq U \subset X$ ([4], (4.2.2)) one sees that the question is local. We may check it on $(T, \mathcal{E}, \psi)$ as in 1). But on T, the algebraic Chern-Simons invariant is the only class mapping to $P_n(F(\psi), \dots, F(\psi))$.

□

3.3. Comparison with classes of bundles with flat connections.

Theorem 3.3.1. *Let (E, ∇, τ_0) be as in 3.1.6, with τ_0 surjective and with $\nabla^2 = 0$. Then the class $c_n(E, \nabla) \in \mathbb{H}^n(X, N^\infty \mathcal{K}_n)$ (2.0.2) are the same as the classes defined in* [7] *(see also* [3]*, (3.11)).*

Proof. We consider the flag bundle $p : G \to X$, with $\Delta = p^{-1}(D)$. We have the quotient maps

$$\Omega_G^\bullet(\log \Delta) \to \Omega_{G,\tau_0}^\bullet(\log \Delta) \to p^* N^\bullet$$

when $\Omega_{G,\tau_0}^\bullet(\log \Delta)$ is as in 3.1.8, 1). We also have the filtration Φ on $\Omega_{G,\tau_0}^\bullet(\log \Delta)$ defined as in 3.1.5, for G replacing $\mathbb{P}$, and the corresponding complex $\Gamma(n)_\tau$. One has a quotient map

$$\Gamma(n)_\tau \to \text{cone } (\mathcal{K}_n \oplus (p^* N^{\geq 2n})[n] \to (p^* N^{\geq n})[n])[-1]$$

and the multiplication $\cup_0$ on $\Gamma(n)_\tau$ defined in 3.1.5 maps to

$$\begin{aligned} x \cup_0 y &= \{x, y\} && x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= \tau d \log x \wedge y && x \in \mathcal{K}_m, y \in p^* N^\ell \\ &= 0 && \text{otherwise.} \end{aligned}$$

By the Whitney product formula 3.1.8, 3), the class $c_n(E, \nabla) \in AD^n_{\tau_0}(X)$ is the n-th symmetric function of the rank one subquotients

$$(L_i, \tau \circ \nabla) \in \mathbb{H}^1(G, \mathcal{K}_1 \to (p^* N^1)_{cl}) \subset \mathbb{H}^1(G, \mathcal{K}_1 \to p^* N^1).$$

Since the multiplication on $\Gamma(n)_\tau$, restricted to the elements of $\mathbb{H}^n(G, \Gamma(n)_\tau)$ mapping to $\mathbb{H}^n(G, \mathcal{K}_n \to (p^* N^n)_{cl})$, maps to the multiplication on $(\mathcal{K}_n \to (p^* N^n)_{cl})$ defined in [7], p. 51 in order to construct the classes of flat bundles, one has that the image of $c_n(E, \nabla)$ in $\mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n)$ is the class defined in [7]. One concludes by the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^n(X, N^\infty \mathcal{K}_n) & \xrightarrow{\subset} & AD^n_{\tau_0}(X) \\ \subset \downarrow & & \subset \downarrow \\ \mathbb{H}^n(G, \mathcal{K}_n \to p^* N^n \to \cdots \to p^* N^{2n-1}) & \longleftarrow & \mathbb{H}^n(G, \Gamma(n)_\tau), \end{array}$$

where the left vertical injective arrow is the composition of the injection $\mathbb{H}^n(X, N^\infty \mathcal{K}_n) \to \mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n)$ as in [7], p.51, p.52, with the injection $\mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n) \to \mathbb{H}^n(G, \mathcal{K}_n \to p^* N^n \to \cdots \to p^* N^{2n-1})$, knowing that

$$\mathrm{Im}\ \mathrm{c_n(E, \nabla)\ in\ H^0(X, N^{2n}_{cl}) = P_n}(\nabla^2, \ldots, \nabla^2) = 0$$

3.2.1, 1). □

4. The Weil Algebra

4.1. **Construction of Beilinson-Kazhdan.**

1. In the unpublished note [2], Beilinson and Kazhdan construct a complex generalizing the Weil homomorphism

$$\begin{aligned} (S^n \mathcal{G}^*)^G &\to H^{2n}_{DR}(X) \\ P &\mapsto P(\nabla^2, \ldots, \nabla^2) \end{aligned}$$

associated to (E, ∇). We give a short account of their construction. Instead of considering reductive groups in general, we consider the subgroup $G_e \subset G$ of matrices fixing the flag

$$e : 0 \to k^{r''} \to k^r \to k^{r'} \to 0.$$

We denote by (E, e) and exact sequence

$$0 \to E'' \to E \to E' \to 0$$

with rank $E'' = r''$, with the convention $(E, 0) = E$, $G_0 = G$ (see 1, 10), 11)). Let

$$0 \to \Omega^1_X \xrightarrow{\iota} \Omega^1_{X,(E,e)} \xrightarrow{\pi} \mathcal{G}^*_{(E,e)} \to 0$$

be the (functorial) Atiyah sequence of (E, e), whose splitting is equivalent to a Ω^1_X valued connection on E compatible with e. Then $\Omega^1_{X,(E,e)} = (p_* \Omega^1_{\mathcal{E}_e})^G_e$, where $p : \mathcal{E}_e \to X$ is the total space of the G_e torseur (1, 10)) associated to (E, e). Let $\Omega^\bullet_{X,(E,e)}$ be the sheaf of commutative differential graded algebras generated by $\mathcal{O}_X$ in degree 0, Ω^1_X in degree 1, such that for $f \in \mathcal{O}_X, df$ is the Kähler differential in $\Omega^1_X \subset \Omega^1_{X,(E,e)}$. Then

$$\Omega^n_{X,(E,e)} = \oplus_{a+b=n} \Omega^{a,b}_{X,(E,e)}$$

with

$$\Omega^{a,b}_{X,(E,e)} = \Lambda^{a-b} \Omega^1_{X,(E,e)} \otimes S^b \mathcal{G}^*_{(E,e)}$$

with differential $d = d' + d''$,

$$d' : \Omega^{a,b}_{X,(E,e)} \to \Omega^{a+1,b}_{X,(E,e)}$$

being induced from $\mathcal{E}_e$ via $\Lambda^i \Omega^1_{X,(E,e)} = (p_* \Omega^i_{\mathcal{E}_e})^G$, and

$$d'' : \Omega^{a,b}_{X,(E,e)} \to \Omega^{a,b+1}_{X,(E,e)}$$

being defined by $d''|\Omega^{10}_{X,(E,e)} = d''|\Omega^1_{X,(E,e)} = \pi$,

$$d''(x_1 \wedge \ldots \wedge x_{a-b} \otimes \sigma) = \sum_{i=1}^{a-b} (-1)^i x_1 \wedge \ldots \wedge \hat{x}_i \ldots \wedge x_{a-b} \pi(x_i) \cdot \sigma$$

for

$$x_1 \wedge \ldots \wedge x_{a-b} \in \Lambda^{a-b} \Omega^1_{X,(E,e)}, \sigma \in S^b \mathcal{G}^*_{(E,e)}.$$

2. One defines the decreasing filtration

$$F^n \Omega^\bullet_{X,(E,e)} = \oplus_{a \geq n} \Omega^{a,b}_{X,(E,e)}.$$

Then

$$(\Omega^\bullet_X, \Omega^{\geq n}_X) \to (\Omega^\bullet_{X,(E,e)}, F^n)$$

is a filtered quasi-isomorphism.

3. There is a canonical map
$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \to S^n(\mathcal{G}^*_{(E,e)}) = \Omega^{n,n}_{X,(E,e)}$$
with Im $w^n \subset (\Omega^{n,n}_{X,(E,e)})_{cl} \subset F^n\Omega^\bullet_{X,(E,e)}[2n]$ which defines a map of commutative differential graded algebras
$$w : \oplus_n S^n(\mathcal{G}_e^*)^{G_e}[-2n] \to \Omega^\bullet_{X,(E,e)}.$$
This is called the *Weil homomorphism* of (E,e) and $\Omega^\bullet_{X,(E,e)}$ is called its *Weil algebra*. (Locally on X, $T_{\mathcal{E}_e/X} \simeq \mathcal{G}_e \times G_e \times X$. The gluing functions act via the adjoint representation on $\mathcal{G}_e$ and the multiplication on G_e. Therefore $(S^n\mathcal{G}_e^*)^{G_e}$ defines functions on $T_{\mathcal{E}_e/X}$. This defines w^n. To see that $d'(\text{Im } w^n) = d''(\text{Im } w^n) = 0$, one may assume that $\mathcal{E}_e \simeq G_e \times X$, and since Im w^n does not depend on X, one also may assume that $\mathcal{E}_e \simeq G_e$. Then one knows that $d' : S^n\mathcal{G}_e^* \to \mathcal{G}_e^* \otimes S^n\mathcal{G}_e^*$ vanishes on $(S^n\mathcal{G}_e^*)^{G_e}$, and $d''|S^n\mathcal{G}_e^* = 0$ by definition). In particular, w^n defines a map
$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \to \mathbb{H}^{2n}(X, F^n\Omega^\bullet_{X,(E,e)}) = \mathbb{H}^{2n}(X, \Omega^{\geq n}_X).$$
4. When $k = \mathbb{C}$, one sets
$$S^n(\mathcal{G}_e^*)^{G_e}_{\mathbb{Z}(n)} = \text{Ker } S^n(\mathcal{G}_e^*)^{G_e} \to H^{2n}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)).$$
For $E_e = E_{un,e}$, the universal bundle $E_{un,e} = G_e^{\Delta_\ell} \times k^r/G_e$ on the simplicial $BG_e = G_e^{\Delta_\ell}/G_e$, w^n is an isomorphism, and one has
$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \xrightarrow{\sim} \mathbb{H}^{2n}(BG_e, \Omega^{\geq n}) = \mathbb{H}^{2n}(BG_e, \Omega^\bullet),$$
the last isomorphism coming from
$$H^i(BG_e, \Omega^j) = 0 \; i \neq j.$$
This remains true on $BG_{e,\text{an}}$. This implies that $S^n(\mathcal{G}_e^*)^{G_e}_{\mathbb{Z}(n)} \xrightarrow{\sim} H^{2n}(BG_{e,\text{an}}, \mathbb{Z}(n))$, the group $H^{2n}(BG_{e,\text{an}}, \mathbb{Z}(n))$ being torsion free.
5. When $k = \mathbb{C}$, they define the *Weil cohomology* by
$$U_{(E,e)}(n) = \text{cone } (\mathbb{Z}(n) \oplus S^n(\mathcal{G}_e^*)^{G_e}[-2n] \to \Omega^\bullet_{X,(E,e)})[-1],$$
with the exact sequence
$$0 \to H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \to H^{2n}(X_{\text{an}}, U_{(E,e)}(n)) \to S^n(\mathcal{G}_e^*)^{G_e}_{\mathbb{Z}(n)} \to 0.$$
The complexes $U_{(E,e)}(n)$ have a multiplication defined as in [1], [8] for the Deligne cohomology, compatible with the map of complexes
$$U_{(E,e)}(n) \to \text{cone } (Z(n) \oplus F^n\Omega^\bullet_{X,(E,e)} \to \Omega^\bullet_{X,(E,e)})[-1]$$

induced by w^n. This induces a map, still denoted by w^n, from the Weil cohomology to the analytic Deligne cohomology.

6. A Ω^1_X valued connection $\nabla : (E,e) \to \Omega^1_X \otimes (E,e)$ on (E,e) is a splitting of the $\Omega^1_{X,(E,e)}$ sequence, or, equivalently, a quotient map of commutative differential graded algebras
$$\nabla : \Omega^\bullet_{X,(E,e)} \to \Omega^\bullet_X$$
which is a quasi-isomorphism inversing $\Omega^\bullet_X \to \Omega^\bullet_{X,(E,e)}$. (Then $\nabla^2 = 0$ if and only if ∇ induces a quasi-isomorphism of filtered complexes). Thus ∇ maps $U_{(E,e)}(n)$ to
$$C(n)^{\mathrm{an}}|X = \text{cone } (\mathbb{Z}(n) \oplus \Omega^{\geq 2n}_X \to \Omega^\bullet_X)[-1]$$
via
$$S^n(\mathcal{G}^*_e)^{G_e}[-2n] \xrightarrow{w^n} F^{2n}\Omega^\bullet_{X,(E,e)} \xrightarrow{\nabla} \Omega^{\geq 2n}_X.$$
This defines classes of (E,∇,e) in
$$\mathbb{H}^{2n}(X_{\mathrm{an}}, C(n)^{\mathrm{an}}|X)$$
(2.0.2, 2)), mapping to the classes in the analytic Deligne cohomology (but not in $D^n(X)$ if X is not proper). The image of this class is $P_n(\nabla^2,\ldots,\nabla^2)$ (see 1, 10)), since ∇ maps $\mathcal{G}^*_{(E,e)}$ to Ω^2_X via the curvature. In particular, if $\nabla^2 = 0$, then those classes are in $H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$. The authors claim that this is exactly $\hat{c}_n(E,\nabla)$ (without saying why) for $e = 0$.

4.2. **Algebraic construction.** We want to make the construction of Beilinson-Kazhdan algebraic.

Remark 4.2.1. 1) *The construction of the filtered quasi-isomorphism*
$$(\Omega^\bullet_X, \Omega^{\geq n}_X) \to (\Omega^\bullet_{X,E}, F^n\Omega^\bullet_{X,E})$$
is algebraic, as well as the Weil morphism w^n. If (E,∇,τ_0) is as in 1, 4), 6), one first defines $\Omega^\bullet_{X,E}(\log D)$ by replacing the Atiyah extension by the logarithmic Atiyah extension
$$0 \to \Omega^1_X(\log D) \to \Omega^1_{X,E}(\log D) \to \mathcal{G}^*_E \to 0.$$
Then one defines $\Omega^\bullet_{X,E,\tau_0}(\log D)$ by replacing the logarithmic Atiyah extension by the N^1 valued Atiyah extension
$$0 \to N^1 \to \Omega^1_{X,E,\tau_0}(\log D) \to \mathcal{G}^*_E \to 0$$
obtained by push forward through $\Omega^1_X(\log D) \to N^1$ of the logarithmic extension. Thus
$$\Omega^n_{X,E,\tau_0}(\log D) = \frac{N^n \oplus \Omega^n_{X,E}(\log D)}{\Omega^n_X(\log D)}$$

This defines a filtered quasi-isomorphism

$$(N^\bullet, N^{\geq n}) \to (\Omega^\bullet_{X,E,\tau_0}(\log D), F^n\Omega^\bullet_{X,E,\tau_0}(\log D))$$

where F^n is defined as in 4.1 2):

$$F^n\Omega^\bullet_{X,E,\tau_0}(\log D) = \oplus_{a\geq n}\Omega^{a,b}_{X,E,\tau_0}(\log D)$$
$$\Omega^{a,b}_{X,E,\tau_0}(\log D) = \Lambda^{a-b}\Omega^1_{X,E,\tau_0}(\log D) \otimes S^b(\mathcal{G}^*_E).$$

We still denote by w^n the induced Weil homomorphism

$$(S^m\mathcal{G}^*)^G \to S^n(\mathcal{G}^*_E) \to (\Omega^{n,n}_{X,E,\tau_0}(\log D))_{cl} \to F^n\Omega^1_{X,E,\tau_0}(\log D)[2n].$$

When $\pi : Y \to X$ is a morphism of (simplicial) smooth varieties, such that $D' = \pi^{-1}(D)$ is a normal crossing divisor, then one has a map

$$\pi^{-1} : \Omega^\bullet_{X,E,\tau_0}(\log D) \to R\pi_*\Omega^\bullet_{Y,\pi^*E,\tau_0}(\log D')$$

induced by

$$\pi^{-1} : \Omega^\bullet_{X,\tau_0}(\log D) \to R\pi_*\Omega^\bullet_{Y,\tau_0}(\log D')$$

(3.1.1).

2) *Then a N^1 valued connection on E defines a quotient of commutative graded algebras*

$$\nabla : \Omega^\bullet_{X,E,\tau_0}(\log D) \to N^\bullet$$

which is an inverse to the quasi-isomorphism

$$N^\bullet \to \Omega^\bullet_{X,E,\tau_0}.$$

Then $\nabla^2 = 0$ is equivalent to ∇ being a filtered quasi-isomorphism.

Definition 4.2.2. *Let E be a rank r vector bundle and τ_0 be as in 1 4). We define*

$$ABK^n_{E,\tau_0} = \text{cone } (\mathcal{K}_n \otimes S^n(\mathcal{G}^*)^G[-n] \xrightarrow{d\log \oplus -w^n} F^n\Omega^\bullet_{X,E,\tau_0}(\log D)[n])[-1].$$

For τ_0 = identity, $D \neq \phi$ we write simply ABK^n_E.

Lemma 4.2.3. *For $X = BG, E = E_{un}$ (4.1 4)) one has*

$$\mathbb{H}^n(BG, ABK^n_{E_{un}}) = H^n(BG, \mathcal{K}_n).$$

For $X = BG_e$, and $E_{un,e} = E_{un}|_{BG_e}$ with its canonical filtration coming from this flag, one has

$$\mathbb{H}^n(BG_e, ABK^n_{E_{un,e}}) = H^n(BG_e, \mathcal{K}_n).$$

Proof. We just write the exact sequence

$$\to \mathbb{H}^{2n-1}(X, N^{\geq n}) \to \mathbb{H}^n(X, ABK^n_{E,\tau_0,e}) \\ \to H^n(X, \mathcal{K}_n) \oplus S^n(\mathcal{G}_e^*)^G \to \mathbb{H}^{2n}(X, N^{\geq n})$$

and apply 4.1, 4). □

Definition 4.2.4. *Let (E, τ_0) be as in 4.2.2. We define*

$$c_n^{ABK}(E) \in \mathbb{H}^n(X, ABK^n_{E,\tau_0}(\log D))$$

as the inverse image under a map $[E] : X_\bullet \to BG$ defined on a simplicial model of X by the transition functions of E, of

$$c_n(E_{un}) \in H^n(BG, \mathcal{K}_n) = \mathbb{H}^n(BG, ABK^n_{E_{un}}).$$

As usual, all possible classes of $[E]$ are homotop and this definition does not depend on the choice of $[E]$. However, one has to apply here the functoriality of the complexes $ABK^n_{E,\tau_0}(\log D)$ as explained in 4.2.1 1).

Let (E, ∇, τ_0) be now as in 1, 4), 6). Then ∇ induces a map, still denoted by ∇, from $ABK^n_{E,\tau_0}(\log D)$ to

$$C(n)_{\tau_0} = \text{cone } (\mathcal{K}_n \oplus N^{\geq 2n}[n] \to N^{\geq n}[n])[-1]$$

(4.2.1 2)).

Definition 4.2.5.

$$c_n^{ABK}(E, \nabla) = \nabla c_n^{ABK}(E) \in AD^n_{\tau_0}(X).$$

Theorem 4.2.6. *One has*

$$c_n^{ABK}(E, \nabla) = c_n(E, \nabla) \in AD^n_{\tau_0}(X).$$

Proof. We want to apply the characterization of the classes 3.1.8.

1) The functoriality of the class $c_n^{ABK}(E)$ is clear. This implies the functoriality of the classes $c_n^{ABK}(E, \nabla)$.
2) Let E be of rank 1. Then $\mathcal{G}_E = \mathcal{O}_X$, and $(\mathcal{G}^*)^G = k$. One has

$$\mathbb{H}^1(X, ABK^1_E) = \mathbb{H}^1(X, \mathcal{K}_1 \xrightarrow{d\log} \Omega^1_{X,E} \longrightarrow \Lambda^2\Omega^1_{X,E} \oplus \frac{\mathcal{O}_X}{k}).$$

One considers the principal $\mathbb{G}_m$ bundle $p : \mathcal{E} = E - \{0\} \to X$ with local trivialization $\mathcal{E}|_{U_i} = \mathbb{G}_m \times U_i$, and gluing $t_i = \xi_{ij} t_j$, t_i parameter of $\mathbb{G}_m$ and $\xi_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. This defines a $\Omega^1_{X,E} = (p_*\Omega^1_{\mathcal{E}})^{\mathbb{G}_m}$ connection on E via $\frac{dt_i}{t_i} - \frac{dt_j}{t_j} = \frac{d\xi_{ij}}{\xi_{ij}}$, with local form $\frac{dt_i}{t_i}$. Then $d'(\frac{dt_i}{t_i}) = 0$, while $d''(\frac{dt_i}{t_i})$, which is the image of $\frac{dt_i}{t_i}$ in $\mathcal{O}_X$, is the residue of $\frac{dt_i}{t_i}$ along the zero section of E. Therefore, this is

$1 \in k \subset \mathcal{O}$. Thus $(d' + d'')(\frac{dt_i}{t_i}) = 0$ in $\frac{\mathcal{O}_X}{k}$ and $(\xi_{ij}, \frac{dt_i}{t_i})$ defines the class of E in $\mathbb{H}^1(X, ABK_E^1)$. Now if ∇ is a τ_0 connection on E, it maps $\frac{dt_i}{t_i}$ to the local form α_i of the connection, Thus

$$\nabla(\xi_{ij}, \frac{dt_i}{t_i}) = (\xi_{ij}, \alpha_i) = c_1(E, \nabla) \in AD^1_{\tau_0}(X).$$

3) In order to understand the Whitney product formula, one has first to introduce a product on $ABK^n_{E,\tau_0}(\log D)$. Again, one takes the same definition as in 2.1.1:

$$ABK^m_{E,\tau_0}(\log D) \times ABK^n_{E,\tau_0}(\log D) \xrightarrow{\cup_\alpha} ABK^{m+n}_{E,\tau_0}(\log D)$$

is given by

$$\begin{aligned}
x \cup_\alpha y \quad &= \{x, y\} && x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
&= 0 && x \in \mathcal{K}_m, y \in S^n \\
&= (1-\alpha) d\log x \wedge y && x \in \mathcal{K}_m, y \in F^n \\
&= 0 && x \in S^m, y \in \mathcal{K}_n \\
&= x \wedge y && x \in S^m, y \in S^n \\
&= (-1)^{\deg\, x} \alpha x \wedge y && x \in S^m, y \in F^n \\
&= \alpha x \wedge d\log y && x \in S^m, y \in \mathcal{K}_n \\
&= (1-\alpha) x \wedge y && x \in F^m, y \in S^n \\
&= 0 && x \in F^m, y \in F^n,
\end{aligned}$$

where we shortened the notations by

$$\begin{aligned}
S^m = &\quad S^m(\mathcal{G}^*)^G[-m] \\
F^m = &\quad F^m \Omega^\bullet_{X,E,\tau_0}(\log D)[m].
\end{aligned}$$

Then a N^1 valued connection ∇ maps $ABK^n_{E,\tau_0}(\log D)$ to $C(n)_{\tau_0}$ (4.2.1 2)) compatibly to the product.

4) Let $e : 0 \to E'' \to E \to E' \to 0$ be an exact sequence of bundles, with rank $E'' = r''$, rank $E' = r'$, $r = r' + r''$. Let $\mathcal{G}_{E,e} \xrightarrow{\rho} \mathcal{G}_E$ be the endomorphisms of E respecting the extension e, and $\mathcal{G}_{E,e} \xrightarrow{p' \oplus p''} \mathcal{G}_{E'} \oplus \mathcal{G}_{E''}$ be the restrictions of the compatible endomorphisms to those of E' and of E''. This defines maps

$$\mathcal{G}^*_E \xrightarrow{\rho^*} \mathcal{G}^*_{E,e} \xleftarrow{(p' \oplus p^n)^*} \mathcal{G}^*_{E'} \oplus \mathcal{G}^*_{E''}$$

and maps

$$ABK^n_E \xrightarrow{\rho^*} ABK^n_{E,e} \xleftarrow{p'^* \oplus p''^*} ABK^n_{E'} \oplus ABK^n_{E''}$$

where one defines $ABK^n_{E,e}$ in the following way. Let $G_e \subset G$ be the matrices of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, A \in GL(r''), B \in GKL(r'),$$

and $\mathcal{G}_e \subset \mathcal{G}$ be the corresponding algebra. Then by restriction of the structure group of E from G to G_e, one considers $\mathcal{E}_e$ the corresponding G_e torseur and one does the same construction as in 4.1 with $\mathcal{E}_e$ replacing $\mathcal{E}$. Now $E_{un}|_{BG_e}$ is the extension

$$e_{un} : 0 \to E''_{un} \to E_{un}|_{BG_e} \to E'_{un} \to 0$$

and one has

$$H^n(BG, \mathcal{K}_n) = \mathbb{H}^n(BG, ABK^n_{E_{un}}) \\ \xrightarrow{\rho^*} H^n(BG_e, ABK^n_{(E_{un}, e_{un})}) = H^n(BG_e, \mathcal{K}_n)$$

(Lemma 4.2.3) and

$$H^n(BG_e, \mathcal{K}_n) = \oplus_{a=0}^n p'^* H^a(BG', \mathcal{K}_a) \cup p''^* H^{n-a}(BG'', \mathcal{K}_{n-a}).$$

Since the product on ABK is compatible with the product on $\mathcal{K}$, this implies

$$H^n(BG_e, ABK^n_{(E_{un}, e_{un})}) = \\ \oplus_{a=0}^n \, p'\mathbb{H}^a(BG', ABK^a_{E'_{un}}) \cup p''\mathbb{H}^{n-a}(BG'', ABK^{n-a}_{E''_{un}})$$

and one has the decomposition

$$\rho^* c_n^{ABK}(E_{un}) = \oplus_{a=0}^n p'^* c_a^{ABK}(E'_{un}) \cup p''^* c_{n-a}^{ABK}(E''_{un}).$$

By functoriality, one obtains the similar relation for the classes of E:

$$\rho^* c_n^{ABK}(E) = \oplus_{a=0}^n p'^* c_a^{ABK}(E') \cup p''^* c_{n-a}^{ABK}(E'').$$

If E now has a τ_0 connection ∇, the map $\nabla : \Omega^\bullet_{X,E,\tau_0}(\log D) \to N^\bullet$ (4.2.1 2)) factors through

$$\Omega^\bullet_{X,E,\tau_0}(\log D) \xrightarrow{\rho^*} \Omega^\bullet_{X,E,e,\tau_0}(\log D) \xrightarrow{\nabla_\rho} N^\bullet$$

when ∇ is compatible with the exact sequence. This shows the Whitney product formula.

□

Remark 4.2.7. *The construction of the tautological $\Omega^1_{X,E}$ valued connection on E of rank r goes as in the second section of the proof of 4.2.6 for $r = 1$: One considers a local trivialization $\mathcal{E}|_{U_i} \simeq G \times U_i$ of the principal G bundle $\mathcal{E}$. Then the gluing is given by $g_i = g_{ij} g_j$, $g_{ij} \in G(U_i \cap U_j)$, where g_i is the tautological G valued function on $G : g_i(x) = x$. Then*

$$dg_i g_i^{-1} - g_{ij} dg_j g_j^{-1} g_{ij}^{-1} = dg_{ij} g_{ij}^{-1}$$

is the equation of the connection.

References

[1] Beilinson, A.: Higher regulators and values of L functions, J. Soviet Math. 30 (1985), 2036-2070.

[2] Beilinson, A.; Kazhdan, D.: Appendix B. Chern classes, unpublished.

[3] Bloch, S.; Esnault, H.: Algebraic Chern-Simons theory, to appear in American J. of Mathematics.

[4] Bloch, S.; Ogus, A.: Gersten's conjecture and the homology of schemes, Ann. Sci. Éc. N. S. (4) 7 (1974), 181-202.

[5] Cheeger, J.; Simons, J.: Differential characters and geometric invariants, Lecture Notes in Mathematics 1167, Springer, Berlin (1980), 50-80.

[6] Esnault, H.: Characteristic class of flat bundles, I. Topology 27 (3) (1988), 323-352.

[7] Esnault, H.: Characteristic class of flat bundles, II. K-Theory 6 (1992), 45-56.

[8] Esnault, H.; Viehweg, E.: Deligne-Beilinson cohomology, in "Beilinson conjectures on special values of L functions", Perspectives in Mathematics 4 (1988), 43-91.

[9] Iversen, B.: Cohomology of sheaves, Springer Universitext (1986).

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