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Recent developments on characteristic classes of flat bundles on complex algebraic manifolds

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Refined characteristic classes of flat bundles have been recently considered by several authors. The most striking example is given by Reznikov's solution to Bloch's conjecture on smooth projective varieties (see section 3). A purely algebraic theory has been developed as well (see section 4). In this note, we present an account of this. We recall at the beginning older work on topological classes (see section 2), necessary to understand the background of more recent results and questions.

1 Generalities

1.1

If $X_{\rm an}$ is a complex analytic manifold, a rank n flat bundle $(E_{\rm an}, \nabla_{\rm an})$ is a rank n analytic bundle $E_{\rm an}$ endowed with a $\mathbb{C}$ linear map

$$\nabla_{\rm an} : E_{\rm an} \to \Omega^1_{X_{\rm an}} \otimes_{\mathcal{O}_{X_{\rm an}}} E_{\rm an}$$

with values in the analytic one forms $\Omega^1_{X_{\rm an}}$, fulfilling the Leibniz condition

$$\nabla_{\rm an}(\lambda e) = d\lambda \otimes e + \lambda \nabla_{\rm an}(e)$$

for λ and e local sections of the analytic functions $\mathcal{O}_{X_{\rm an}}$ and $E_{\rm an}$. $\nabla_{\rm an}$ extends to

$$\nabla_{\rm an} : \Omega^i_{X_{\rm an}} \otimes_{\mathcal{O}_{X_{\rm an}}} E_{\rm an} \to \Omega^{i+1}_{X_{\rm an}} \otimes_{\mathcal{O}_{X_{\rm an}}} E_{\rm an}$$

via the sign convention

$$\nabla_{\rm an}(\omega e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla_{\rm an}(e)$$

for $\omega \in \Omega^i_{X_{\rm an}}$. $\nabla_{\rm an}$ is flat or integrable if the $\mathcal{O}_{X_{\rm an}}$ linear map

$$\nabla^2_{\rm an} : E_{\rm an} \to \Omega^2_{X_{\rm an}} \otimes_{\mathcal{O}_{X_{\rm an}}} E,$$

called curvature, vanishes. The local existence of n linearly independent solutions to the system $\nabla_{\rm an}$ of linear differential equations guarantees that $L = {\rm Ker}\nabla_{\rm an}$ is a local system of complex vector spaces of dimension n over $X_{\rm an}$. The corresponding monodromy representation $\rho : \pi_1(X, x) \to GL(n, \mathbb{C})$ is defined by following solutions in L along loops $\gamma \in \pi_1(X, x)$. The correspondence

$$L \to (E_{\rm an} = L \otimes_{\mathbb{C}} \mathcal{O}_{X_{\rm an}}, \nabla_{\rm an} = 1 \otimes d)$$
$$(E_{\rm an}, \nabla_{\rm an}) \to L = {\rm Ker}\nabla_{\rm an}$$

is known under the name of Riemann-Hilbert correspondence. (Here d is the Kähler differential).

In this article, we study flat bundles $(E_{\rm an}, \nabla_{\rm an})$ on analytic manifolds which arise from an algebraic structure. Crucial to our purpose is Deligne's theorem:

Theorem 1.1 [10] *Let X be a complex algebraic manifold, and $(E_{\rm an}, \nabla_{\rm an})$ be a flat bundle on the associated analytic manifold $X_{\rm an}$. Then there is an algebraic bundle E and an algebraic connection*

$$\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$$

such that $(E, \nabla)_{\rm an} = (E_{\rm an}, \nabla_{\rm an})$.

A connection $\nabla_{\rm an} : E_{\rm an} \to \Omega^1_{X_{\rm an}} \otimes_{\mathcal{O}_{X_{\rm an}}} E_{\rm an}$ can be interpreted as a $\mathcal{O}_{X_{\rm an}}$ splitting of the Atiyah extension

$$0 \to \Omega^1_{X_{\rm an}} \otimes E_{\rm an} \to \mathcal{P}^1(E_{\rm an}) \to E_{\rm an} \to 0$$

of principal parts of $E_{\rm an}$ (see [1], [10]). One says that ∇ is an algebraic connection if it is induced by an algebraic splitting of the Atiyah extension $\mathcal{P}^1(E)$ of principal parts of E. It implies automatically that the curvature is algebraic.

Thus Deligne's theorem is an application of Serre's GAGA theorem [22], when X is proper. But when X is not proper, it contains a large part of the theory of differential equations with regular singularities.

A more precise formulation is

Theorem 1.2 *In the above situation, for each good compactification $j : X \hookrightarrow \bar{X}$ such that $\bar{X}$ is smooth and $\bar{X} - X = D$ is a normal crossing divisor, there are algebraic extensions $(\bar{E}, \bar{\nabla})$, $\bar{\nabla} : \bar{E} \to \Omega^1_{\bar{X}}(\log D) \otimes \bar{E}$ of $(E_{\rm an}, \nabla_{\rm an})$.*

(One can even determine all possible such extensions [16], Appendix C).

1.2

Let $CH^i(X)$ be the Chow group of codimension i cycles on X modulo rational equivalence. The algebraicity of flat bundles 1.1 allows to define Chern classes

$$c_i^{CH}(E) \in CH^i(X).$$

This gives at least two guide lines in the interest for algebraic bundles in algebraic geometry.

- They grasp the part of the topology of $X_{\rm an}$ encoded in finite representations of the fundamental group.
- They possibly give interesting algebraic cycles on X.

The purpose of this article is to give the state of the art on the second question. The first one had a spectacular development in the last decade, due to work by Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau, Hitchin, Corlette, Simpson and others (see [23]).

1.3 Examples

Flat bundles in algebraic geometry arise naturally as Gauß-Manin bundles or as bundles coming from finite coverings.

• Let $\varphi : Y \to X$ be a smooth proper family over X smooth. Then for each $i \geq 0$,

$$R^i\varphi_*\Omega^\bullet_{Y/X}$$

is endowed with the Gauß-Manin connection. When one has a flat (in the sense of algebraic geometry) compactification $\bar{\varphi} : \bar{Y} \to \bar{X}$ of φ such that $X \hookrightarrow \bar{X}$ is a good compactification, then

$$R^i\bar{\varphi}_*\Omega^\bullet_{\bar{Y}/\bar{X}}(\log(\bar{Y} - Y))$$

is a natural logarithmic extension of $R^i\varphi_*\Omega^\bullet_{Y/X}$.

• Let $\varphi : Y \to X$ be an étale covering over X smooth. Then $\varphi_*(\mathcal{O}_Y, d)$ splits into a sum of flat bundles. When one has a flat compactification $\bar{\varphi} : \bar{Y} \to \bar{X}$ of φ such that $X \hookrightarrow \bar{X}$ is a good compactification, then $\varphi_*(\mathcal{O}_Y, d_{\log})$ splits into a sum of flat bundles with a logarithmic connection along $\bar{X} - X$ extending the factors over X, where

$$d_{\log} : \mathcal{O}_{\bar{Y}} \to \Omega^1_{\bar{Y}}(\log(\bar{Y} - Y)).$$

2 Topology

The topological Chern classes $c_i^{\mathbb{Z}}(E) \in H^{2i}(X_{\text{an}}, \mathbb{Z})$ of flat bundles are torsion. In fact, Chern-Weil theory expressing the class in

$$H^{2i}(X_{\text{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{2i}(X_{\text{an}}, \mathbb{C}) = \text{ de Rham cohomology group } H^{2i}_{DR}(X_{\text{an}})$$

as the cohomology class of a smooth differential form associated to any connection implies that $c_i^{\mathbb{Z}}(E) \otimes \mathbb{Q} = 0$. Grothendieck [19], Théorème 4.8, gives bounds for the torsion of $c_i^{\mathbb{Z}}(E)$ in terms of invariants of the Galois group of the field of definition of (X, E) by reduction modulo a prime p.

A deep explanation of the torsion of $c_i^{\mathbb{Z}}(E)$ when X is proper is given by

Theorem 2.1 (Deligne-Sullivan) ([11]) *Let X be proper smooth over $\mathbb{C}$ (in fact a compact polyedron would be sufficient). Let (E, ∇) be a flat bundle. Then there is an étale cover $Y \to X$ (explicitely described in terms of a subring A of $\mathbb{C}$ of finite type such that the monodromy representation*

$$\rho : \pi_1(X, x) \to GL(n, A)$$

is A valued) such that $E|_Y$ is trivialized as a C^∞ bundle.

This is in contrast to the algebraic situation. Deligne gives the following example. Let X be a smooth proper curve, and E be a non torsion element in $\mathrm{Pic}^0 X$, the group of rank 1 vector bundles of degree $(= c_1)$ 0. Then if $Y \to X$ is finite and $E|_Y \simeq \mathcal{O}_Y$, E is a direct factor of $\mathcal{O}_Y$ as a $\mathcal{O}_X$ module. If $Y \to X$ is étale, this contradicts that E is not torsion. If $Y \to X$ ramifies, then a power of E has to be an ideal sheaf on X, so is not in $\mathrm{Pic}^0 X$.

3 Analytic Structure

3.1 Deligne-Beilinson cohomology

The Deligne-Beilinson cohomology $H^a_{\mathcal{D}}(X, \mathbb{Z}(b))$ ([2], [17]) encodes the topological cohomology $H^j(X_{\mathrm{an}}, \mathbb{Z})$ and its Hodge filtration F^i coming from the analytic structure of the algebraic manifold. It is naturally presented as an extension

$$(1) \qquad 0 \to \frac{H^{a-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})}{F^b} \to H^a_{\mathcal{D}}(X, \mathbb{Z}(b)) \to \mathrm{Ker}(F^b H^a(X_{\mathrm{an}}, \mathbb{C}) \to H^a(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})) \to 0$$

The right hand side group is discrete, whereas the left hand side group is endowed with the classical topology coming from $\mathbb{C}$. For example, when X is proper, then the group $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is an extension of the image in the de Rham cohomology of the group of codimension i Hodge cycles by a group, which is itself an extension of the torsion in $H^{2i}(X_{\mathrm{an}}, \mathbb{Z})$ by Griffiths' intermediate jacobian

$$H^{2i-1}(X_{\mathrm{an}}, \mathbb{C})/H^{2i-1}(X_{\mathrm{an}}, \mathbb{Z}) + F^i.$$

The latter is a complex torus, sometimes (e.g. for $i = 1$ or $i = d, d = \dim X$) an abelian variety. (It would be easier to say that $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is an extension of the Hodge cycles by the intermediate jacobian, but in the above presentation, we distributed the torsion differently).

There is a cycle map

$$CH^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)).$$

So flat bundles have classes $c_i^{\mathcal{D}}(E) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})/F^i$ (see equation 1). For example, for $i = 1$, $c_i^{\mathcal{D}}(E) \in \frac{H^1(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z})}{F^1}$. When $i = 1$ and X is proper, then the intermediate jacobian is just the classical jacobian $\mathrm{Pic}^0 X$.

3.2

A key conjecture by Bloch and Beilinson ([4], [2]) asserts that if X is projective smooth, there should exist on $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ a filtration

$$0 = F^{i+1} \subseteq F^i \subset \ldots \subset F^1 \subset F^0 = CH^i(X) \otimes \mathbb{Q}$$

compatible with products and correspondences. Then F^1 should be $\mathrm{Ker} CH^i(X) \otimes \mathbb{Q} \to H^{2i}(X, \mathbb{C})$, in particular $F^1 CH^1(X) \otimes \mathbb{Q} = \mathrm{Pic}^0(X) \otimes \mathbb{Q}$, F^2 should be $\mathrm{Ker} CH^i(X) \otimes \mathbb{Q} \to H^{2i}_{\mathcal{D}}(X, \mathbb{Q}(i))$. So if one had a splitting principle for flat bundles, in the sense that one could reduce the computation of $c_i^{CH}(E) \in CH^i(X)$ to the one of a direct sum of rank 1 flat bundles then one would have

"Conjecture" (Bloch) [3] $c_i^{CH}(E) \in F^i CH^i(X) \otimes \mathbb{Q}$ when (E, ∇) is flat on X projective smooth.

As one does not know the existence of the filtration, and as flat bundles are far from "splitting" as sums of rank 1 flat bundles, the "conjecture" does not mean anything precise, except of

Conjecture (Bloch) [3] $c_i^{\mathcal{D}}(E) \otimes \mathbb{Q} = 0$ for $i \geq 2$ when X is projective smooth.

3.3

In his seminal paper [3] introducing the dilogarithm function in algebraic geometry, not only Bloch states the above conjectures, but he proves at the same time

Theorem 3.1 (Bloch) [3] *If $i = 2$ and X is proper smooth, there is a countable subgroup $\Delta^* \subset \mathbb{C}/\mathbb{Q}$ such that*

$$c_2^{\mathcal{D}} \text{ (flatbundles) } \in \text{Im } H^3(X_{\text{an}}, \Delta^*) \text{ in } \frac{H^3(X_{\text{an}}, \mathbb{C}/\mathbb{Z})}{F^2}.$$

In fact, even if X is not proper, Bloch shows that

$$\text{Im } c_2^{\mathcal{D}} \text{ (flat bundles) in } \frac{H^3(X_{\text{an}}, \mathbb{C}/\mathbb{Q})}{\mathbb{H}^3(X_{\text{an}}, \Omega^{\geq 2}_{X_{\text{an}}})} \in \text{Im } H^3(X, \Delta^*).$$

Also, if X is proper, one can refer only to the Hodge theory to show

Theorem 3.2 [15] *On X proper smooth, the classes $c_i^{\mathcal{D}}$ (bundles with algebraic connection) form a countable subset of $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$.*

(This theorem is of no use as one does not know any example of an algebraic bundle on a proper smooth variety which is not flat but carries a non flat algebraic connection).

Reznikov made a major progress answering positively Bloch's conjecture. His result is stronger. Not only $c_i^{\mathcal{D}}(E)$ is torsion for $i \geq 2$ but some good analytic classes in $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$ mapping to $c_i^{\mathcal{D}}(E)$ are torsion as well (see 3.5).

3.4 Secondary Analytic Classes

A rank 1 flat bundle is equivalent to a representation

$$\rho \in \text{Hom } (\pi_1(X, x), \mathbb{C}^*) = H^1(X_{\text{an}}, \mathbb{C}/\mathbb{Z}).$$

This class ρ maps to the class $c_1^{\mathcal{D}}(E) \in \frac{H^1(X_{\text{an}}, \mathbb{C}/\mathbb{Z})}{F^1}$ when the algebraic bundle E underlies the analytic one (see theorem 1.1)). General functorial and additive classes in $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$ can be constructed for $i \geq 1$. I know of four ways.

1. The original one of Cheeger-Simons [7] consists in defining a group of differential characters $\hat{H}^j(X, \mathbb{R}/\mathbb{Z})$ (or $\hat{H}^j(X, \mathbb{C}/\mathbb{Z})$) on a differentable manifold X, which is an extension of global C^∞ $\mathbb{R}$ (or $\mathbb{C}$) valued forms of degree j with $\mathbb{Z}$ periods by $H^{j-1}(X, \mathbb{R}/\mathbb{Z})$ (or $H^{j-1}(X, \mathbb{C}/\mathbb{Z})$). They show that flat bundles (E, ∇) on the C^∞ manifold X have classes $\hat{c}_j(E, \nabla) \in H^{2j-1}(X, \mathbb{R}/\mathbb{Z}) \subset \hat{H}^{2j}(X, \mathbb{R}/\mathbb{Z})$. (One defines exactly similarly classes of flat bundles in $H^{2j-1}(X, \mathbb{C}/\mathbb{Z}) \subset \hat{H}^{2j}(X, \mathbb{C}/\mathbb{Z})$). Those classes are smooth and do not refer to an analytic structure. If X is a proper smooth algebraic manifold, then Bloch for unitary bundles ([3]) and Gillet-Soulé for non unitary ones ([18]) show that those classes map to $c_i^{\mathcal{D}}(E)$.
2. Karoubi ([20]) constructed classes with K theory.
3. Beilinson ([2]) observes that the Deligne-Beilinson cohomology $H^{2i}_{\mathcal{D}}(BG_\bullet, \mathbb{Z}(i))$ of the discrete simplicial scheme $BG_\bullet$, $G = GL(n, \mathbb{C})$, is just $H^{2i-1}(BG_\bullet, \mathbb{C}/\mathbb{Z})$. This defines classes by universality.
4. One can also develop a modified splitting principle ([12], [13]) to define functorial and additive classes $c_i^{\rm an}(E, \nabla) \in H^{2i-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z})$.

One shows ([14], [12], [20]) that the classes defined by the splitting principle, by universality and by K theory are the same.

3.5

Now we can express Reznikov's theorem.

Theorem 3.3 (Reznikov) [21] *On X projective smooth, $\hat{c}_i(E, \nabla)$ and $c_i^{\rm an}(E, \nabla)$ lie in*

$$H^{2i-1}(X_{\rm an}, \mathbb{Q}/\mathbb{Z}) \subset H^{2i-1}_{\mathcal{D}}(X, \mathbb{Z}(i))$$

for $i \geq 2$.

One may apply his method to show

Theorem 3.4 [9] *On X smooth $\hat{c}_i(E, \nabla)$ and $c_i^{\rm an}(E, \nabla)$ lie in $H^{2i-1}(X_{\rm an}, \mathbb{Q}/\mathbb{Z})$ (which no longer injects into $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ for X not proper) for $i \geq 1$ when (E, ∇) is a $\mathbb{Q}$ variation of Hodge structure, for example when (E, ∇) is a Gauß-Manin bundle associated to a smooth proper family (see (1.3)).*

In fact, in [9], there is a version of this on a topological manifold X.

3.6 Questions

There are many questions raised by Reznikov's answer.

- When X is not proper, what about $c_i^{\mathcal{D}}(E)$, $c_i^{\rm an}(E, \nabla)$?
- For (E, ∇) a $\mathbb{Q}$ variation of Hodge structure, one has the vanishing theorem 3.4. Let $j : X \to \bar{X}$, $(\bar{E}, \bar{\nabla})$ be as in theorem 1.1. The de Rham classes $c_i^{DR}(\bar{E})$ of $\bar{E}$ are expressable in terms of cycles supported in D with coefficients depending on the residues of $\bar{\nabla}$ along D ([16], Appendix B). Let $\mathbb{Q} \subset A \subset \mathbb{C}$ be the smallest field containing those coefficients. For example, when the residues are nilpotent, $c_i^{DR}(\bar{E}) = 0$, and $A = \mathbb{Q}$. Does one have $c_i^{\mathcal{D}}(\bar{E}) = 0$ in $H^{2i}_{\mathcal{D}}(\bar{X}, A(i))$? (See also [9]).

• When one knows that $c_i^{\mathrm{an}}(E,\nabla)\in H^{2i-1}(X_{\mathrm{an}},\mathbb{Q}/\mathbb{Z})$, what does the torsion reflect exactly? (Of course this torsion is an upper bound for the torsion of the classes $c_i^{\mathbb{Z}}(E)$ (2)).
• Grothendieck's coniveau filtration of a cohomology theory H fulfilling localization is defined by

$$N^aH(X)=\{x\in H,\ \exists\ \text{subscheme } Z\subset X,\ \operatorname{codim} Z\geq a, x|_{X-Z}=0 \text{ in } H(X-Z)\}.$$

Bloch-Ogus theory [6] implies that $N^jH^{2i-1}(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z})=0$ for $j\geq i$. So the lowest piece possibly non vanishing is $N^{i-1}H^{2i-1}(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z})$. (For $i=1$, $N^0H^1(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z})=H^1(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z})$ but for $i\geq 2$, the lowest piece is smaller than the whole group). When (E,∇) comes from a finite representation of the fundamental group, then

$$c_i^{\mathrm{an}}(E,\nabla)\in N^{i-1}H^{2i-1}(X_{\mathrm{an}},\mathbb{Q}/\mathbb{Z})$$

for all $i\geq 1$. ([14]). The proof requires the existence of algebraic classes ([13], section 4). Does one have in general

$$c_i^{\mathrm{an}}(E,\nabla)\in N^{i-1}H^{2i-1}(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z})?$$

4 Algebraic Structures

4.1 Secondary Algebraic Classes

A natural question is to find classes lifting the secondary analytic classes $c_i^{\mathrm{an}}(E,\nabla)$ as well as the most possible algebraic classes $c_i^{CH}(E)\in CH^i(X)$. This is the purpose of [13].

We denote by $\mathcal{K}_i^m$ the image of the Zariski sheaf of Milnor K theory in the constant sheaf of Milnor K theory of the field $k(X)$ of rational functions on X. (Here X is defined over any field k). The $d\log:\mathcal{K}_1\to\Omega_X^1$ map defined by $d\log f=\frac{df}{f}$ extends to a map $d\log:\mathcal{K}_i^m\to\Omega_X^i$.

Theorem 4.1 [13] *Let (E,∇) be a flat connection with logarithmic poles along D. Then there are functorial and additive classes*

$$c_i(E,\nabla)\in H^i:=\mathbb{H}^i(X,\mathcal{K}_i^m\xrightarrow{d\log}\Omega_X^i(\log D)\to\Omega_X^{i+1}(\log D)\to\ldots)$$

lifting $c_i^{CH}(E)\in CH^i(X)=H^i(X,\mathcal{K}_i^m)$. When the field of definition of X is $\mathbb{C}$, there are maps

$$H^i\to\mathbb{H}^{2i}(X_{\mathrm{an}},\mathbb{Z}\to\mathcal{O}_{X_{\mathrm{an}}}\to\ldots$$
$$\ldots\to\Omega^i_{X_{\mathrm{an}}}\to\Omega^{i+1}_{X_{\mathrm{an}}}(\log D)\to\ldots)\to H^{2i-1}(X_{\mathrm{an}}-D_{\mathrm{an}},\mathbb{C}/\mathbb{Z})$$

taking $c_i(E,\nabla)$ to $c_i^{\mathrm{an}}(E,\nabla)$, compatibly with the image $c_i^{\mathcal{D}}(E)$ of $c_i^{CH}(E)$.

4.2

The algebraic classes $c_n(E,\nabla)$ define torsion free classes

$$\gamma_n(E,\nabla)\in\mathbb{H}^{n-1}(X,\frac{\Omega_X^n(\log D)}{\Omega^n_{\mathrm{clsd}}}\to\Omega_X^{n+1}(\log D)\to\ldots)$$

which are difficult to understand, and classes

$$\theta_n(E,\nabla) \in H^0(X, \frac{\Omega_X^{2n-1}(\log D)_{\text{clsd}}}{d\Omega_X^{2n-2}(\log D)}) = H^0(X-D, \mathcal{H}^{2n-1}).$$

Here $\mathcal{H}^j$ is the Zariski sheaf associated to the presheaf $H^j_{DR}(U)$ defined by Bloch and Ogus [6].

The rest of this note is devoted to the classes $\theta_n(E,\nabla)$ studied in [5]. It turns out that they define an algebraic Chern-Simons theory.

4.3

Locally, the bundle E is trivialized: $E \simeq \oplus^r \mathcal{O}_X$, so a connection ∇ (not necessarily integrable) is determined by a $r \times r$ matrix A of one forms. The curvature ∇^2 is then $\nabla^2 = F(A) = dA - A^2$. Let P be an invariant homogeneous polynomial of degree n on matrices $M \in M(r \times r, k)$ (X might be defined on any field k of characteristic zero), that is an element

$$P \in \text{Hom}_k(\text{End } (k^r)^{\otimes n}, k),$$

invariant under the diagonal adjoint action of $GL(r,k)$. Then the forms

$$TP(A) = n\int_0^1 P(A \wedge F(tA))dt,$$

with $d\ TP(A) = P(F(A))$, defined by Chern-Simons ([8]), glue together to a well defined class

$$\omega_n(E,\nabla,P) \in H^0(X, \frac{\Omega_X^{2n-1}}{d\Omega_X^{2n-2}}).$$

When ∇ is flat, then

$$\omega_n(E,\nabla,P) \in H^0(X,\ \mathcal{H}^{2n-1}) \subset H^0(X, \frac{\Omega_X^{2n-1}}{d\Omega_X^{2n-1}}).$$

Theorem 4.2 *If $D = \phi$, then*

$$\theta_n(E,\nabla) = \omega_n(E,\nabla,P_n)(=: \omega_n(E,\nabla)),$$

where P_n is the invariant polynomial describing the n-th Chern class.

4.4

The comparison 4.2 allows to make a purely algebraic Chern-Simons theory on X over $k = \mathbb{C}$.

We introduce the generalized Griffiths group $\text{Griff}^n(X)$ as the group of codimension n cycles homologous to zero on X modulo those homologous to zero on some divisor in X. For example $\text{Griff}^2(X)$ is the classical Griffiths group of codimension 2 cycles homologous to zero modulo those algebraically equivalent to zero.

There is an extension

$$0 \to \frac{H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z})}{N^1 H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z})} \to \mathcal{E} \xrightarrow{d_n} \mathrm{Griff}^n(X) \to 0,$$

where $\mathcal{E} \subset H^0(X, \mathcal{H}^{2n-1}(\mathbb{Z}))$, and $\mathcal{H}^j(\mathbb{Z})$ is the Zariski sheaf associated to the presheaf $H^i(U_{\mathrm{an}}, \mathbb{Z})$. Then

Theorem 4.3 *Let (E, ∇) be a flat connection on X proper smooth over $\mathbb{C}$. Then $\omega_n(E, \nabla) \in \mathcal{E} \otimes \mathbb{Q}$, $d_n(\omega_n(E, \nabla))$ is the Chern class $c_n(E) \in \mathrm{Griff}^n(X) \otimes \mathbb{Q}$, and $\omega_n(E, \nabla) \otimes \mathbb{Q} = 0$ if and only if $c_n(E) \otimes \mathbb{Q} = 0$.*

The proof relies on the comparison 4.2, on Reznikov's theorem 3.3, and on the definition of the mixed Hodge structure on the (infinite dimensional) $\mathbb{Z}$ module $\mathcal{E}$.

4.5

To make the theory more flexible, one has to introduce logarithmic poles. In fact,

$$\omega_n(E, \nabla) \in H^0(X, \mathcal{H}^{2n-1}) \subset H^0(X - D, \mathcal{H}^{2n-1})$$

and lies in $\mathcal{E} \otimes \mathbb{C}$. Its image $d_n(c_n(E, \nabla))$ in the Griffiths group differs from the algebraic class of E by a class supported in D, whose precise shape is clear for $n = 2$ (and is vanishing if ∇ has nilpotent residues along D).

4.6 Rigidity

The link of $\omega_n(E, \nabla)$ with the algebraic class in the Griffiths group (if $D = \phi$) forces $\omega_n(E, \nabla)$ to be invariant in a deformation of (E, ∇) over X. But in fact, a stronger rigidity holds true: one can allow X to vary in a one dimensional family.

4.7 Vanishing – Non Vanishing

The invariants

$$\omega_n(E, \nabla) \in H^0(X, \frac{\Omega_X^{2n-1}}{d\Omega^{2n-2}})$$

are certainly non vanishing when ∇ is not flat, as

$$d\omega_n(E, \nabla) = c_n(E) \in H^0(X, \Omega^{2n}_{\mathrm{clsd}}).$$

However, the only examples we have for an integrable connection are vanishing classes:

Gauß-Manin systems of curves (even with logarithmic poles), general weight one Gauß-Manin systems, weight two Gauß-Manin systems of surfaces, finite monodromy.

In characteristic p large enough, one can define $\omega_n(E, \nabla)$ and all Gauß-Manin systems of proper smooth varieties vanish.

One may raise the question of whether on X smooth proper, $\omega_n(E, \nabla)$ always vanishes for $n \geq 2$.

References

[1] Atiyah, M.: Complex analytic connections in fibre bundles, Trans. Am. Math. Soc. **85** (1957), 181–207

[2] Beilinson, A.: Higher regulators and values of *L*-functions, J. Soviet Math. **30** (1985), 2036–2070

[3] Bloch, S.: Applications of the dilogarithm function in algebraic *K*-theory and algebraic geometry, Int. Symp. on Alg. Geometry, Kyoto (1977), 103–114

[4] Bloch, S.: Lectures on Algebraic Cycles, Duke University Publications (1980)

[5] Bloch, S., Esnault, H.: Algebraic Chern-Simons theory, preprint 1996, 47 pages

[6] Bloch, S., Ogus, A.: Gersten's conjecture and the homology of schemes, Ann. Sc. ENS, 4^0 série **7** (1974), 181–202

[7] Cheeger, J., Simons, J.: Differential characters and geometric invariants, in "Geometry and Topology, Proceedings of the special year, University of Maryland 1983-84", Lecture Notes in Math. **1167** (1985), Springer, 50–80

[8] Chern, S.-S., Simons, J.: Characteristic forms and geometric invariants, Ann. of Math. **99** (1974), 181-202

[9] Corlette, K., Esnault, H.: Classes of local systems of $\mathbb{Q}$ hermitian vector spaces, preprint (1995)

[10] Deligne, P.: Équations Différentielles à Points Singuliers Réguliers, Springer Lecture Notes **163** (1970)

[11] Deligne, P., Sullivan, D.: Fibrés vectoriels complexes à groupe structural discret, CRAS **281** (1975), 1081–1083

[12] Esnault, H.: Characteristic classes of flat bundles, Topology **27** (1988), 323–352

[13] Esnault, H.: Characteristic classes of flat bundles, II, *K*-Theory **6** (1992), 45–56

[14] Esnault, H.: Coniveau of Classes of Flat Bundles trivialized on a Finite Smooth Covering of a Complex Manifold, *K*-Theory **8** (1994), 483–497

[15] Esnault, H., Srinivas, V.: Chern classes of vector bundles with holomorphic connections on a complete smooth variety, J. Diff. Geom. **36** (1992), 257–267

[16] Esnault, H., Viehweg, E.: Logarithmic De Rham complexes and vanishing theorems, Inv. math. **86** (1986), 161–194

[17] Esnault, H., Viehweg, E.: Deligne-Beilinson cohomology, in "Beilinson's Conjectures on Special Values of *L*-Functions", Perspectives in Mathematics **4** (1988), 43–91

[18] Gillet, H., Soulé, Ch.: Arithmetic Chow groups and differential characters, in "Algebraic *K*-Theory: connections with geometry and topology", ed. Jardine and Snaith (1989), Kluwer Academic Publishers, 30-68

[19] Grothendieck, A.: Classes de Chern et représentations linéaires des groupes discrets, in "Dix exposés sur la cohomologie des schémas", North-Holland (1968), 215–305

[20] Karoubi, M.: Classes caractéristiques de fibrés feuilletés, holomorphes et algébriques, *K*-Theory **8** (1994), 153–211

[21] Reznikov, A.: All regulators of flat bundles are torsion, Annals of Math. **139** (1994), 1–14

[22] Serre, J.-P.: Géométrie Algébrique et Géométrie Analytique, Ann. Inst. Fourier **6** (1956), 1–42

[23] Simpson, C.: Higgs bundles and local systems, Publ. Math. IHES **75** (1992), 5–95

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