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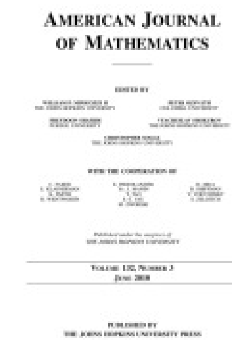
American Journal of Mathematics, Volume 119, Number 4, August 1997, pp. 903-952 (Article)

Published by The Johns Hopkins University Press
DOI: 10.1353/ajm.1997.0021



ALGEBRAIC CHERN-SIMONS THEORY

By Spencer Bloch and Hélène Esnault

Abstract. A theory of secondary characteristic classes analogous to the classical Chern-Simons theory is developed for algebraic vector bundles. Applications are made to questions involving finer charcterisitic classes for bundles with connection and to the Griffiths group of algebraic cycles.

0. Introduction.

0.1. Secondary (Chern-Simons) characteristic classes associated to bundles with connection play an important role in differential geometry. We propose to investigate a related construction for algebraic bundles. Nonflat algebraic connections for bundles not admitting flat structures on complex projective manifolds are virtually nonexistent (we know of none), and a deep theorem of Reznikov ([18]) implies that Chern-Simons classes are torsion for flat bundles on such spaces. On the other hand, it is possible (in several different ways, cf. 1.1 below) given a vector bundle E on X to construct an affine fibration $f\colon\ Y \to X$ (i.e. locally over X, $Y \cong X \times \mathbb{A}^n$) such that f^*E admits an algebraic connection. Moreover, one can arrange that Y itself be an affine variety. Since pullback f^* induces an isomorphism from the *Chow motive* of X to that of Y, one can in some sense say that every algebraic variety is equivalent to an affine variety, and every vector bundle is equivalent to a vector bundle with an algebraic connection. Thus, an algebraic Chern-Simons theory has some interest. Speaking loosely, the content of such a theory is that a closed differential form τ representing a characteristic class like the Chern class of a vector bundle on a variety X will be Zariski-locally exact, $\tau \mid U_i = d\eta_i$. The choice of a connection on the bundle enables one to choose the primitives η_i canonically up to an exact form. In particular, $(\eta_i - \eta_j) \mid U_i \cap U_j$ is exact. When X is affine, a different choice of connection will change the η_i by a global form η.

0.2. Unless otherwise noted, all our spaces X will be smooth, quasi-projective varieties over a field k of characteristic 0. Given a bundle of rank N with connection (E, ∇) on X and an invariant polynomial P of degree n on the

Manuscript received March 1, 1996; revised September 23, 1996.

Research of the first author supported in part by the National Science Foundation; research of the second author supported in part by the DFG.

American Journal of Mathematics 119 (1997), 903–952.

Lie algebra of GL_N (cf. [3]), we construct classes

$$w_n(E,\nabla,P)\in\Gamma(X,\Omega_X^{2n-1}/d\Omega_X^{2n-2});\ n\geq 2. \tag{0.2.1}$$

Here Ω_X^i is the Zariski sheaf of Kähler i-forms on X, and d: $\Omega_X^i\to\Omega_X^{i+1}$ is exterior differentiation. Zariski locally, these classes are given explicitly in terms of universal polynomials in the connection and its curvature. They satisfy the basic compatibility:

> $dw_n(E,\nabla,P)$ is a closed $2n$-form representing the characteristic class in de Rham cohomology associated to P by Chern-Weil theory. Note that dw_n is not necessarily exact, because w_n is not a globally defined form.

The simplest example is to take E trivial of rank 2 and to assume the connection on the determinant bundle is trivial. The connection is then given by a matrix of 1-forms $A=\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}$. Taking $P(M)=\mathrm{Tr}\,(M^2)$ one finds

$$w_2(E,\nabla,P)=2\alpha\wedge d\alpha-4\alpha\beta\gamma+\beta d\gamma+\gamma d\beta \tag{0.2.2}$$

or, if A is integrable,

$$w_2(E,\nabla,P)=-2\alpha\wedge d\alpha=-2\alpha\beta\gamma.$$

One particularly important invariant polynomial P_n maps a diagonal matrix to the nth elementary symmetric function in its entries. We write

$$w_n(E,\nabla):=w_n(E,\nabla,P_n). \tag{0.2.3}$$

For example, $P_2(M):=\frac{1}{2}(\mathrm{Tr}M)^2-\mathrm{Tr}(M^2)$. In fact, when ∇ is integrable, $w_n(E,\nabla,P)=\lambda w_n(E,\nabla)$ for some coefficient $\lambda\in\mathbb{Q}$ (see 2.3.3).

When $k=\mathbb{C}$, $w_n(E,\nabla)$ is linked to the Chern class in $A^n(X)$, where $A^n(X)$ denotes the group of algebraic cycles modulo a certain adequate equivalence relation, homological equivalence on a divisor. For example, $A^2(X)$ is the group of codimension 2 cycles modulo algebraic equivalence. When $n=2$ and X is affine, there is an isomorphism

$$\varphi\colon\ \Gamma(X,\Omega_X^3/d\Omega_X^2)/\Gamma(X,\Omega_X^3)\cong A^2(X)\otimes_{\mathbb{Z}}\mathbb{C}. \tag{0.2.4}$$

(This result, which we will not use, follows easily from results in [2].)

Writing $c_{2,\mathrm{cycle}}(E)$ for the second Chern class of E in $A^2(X)$, we have

$$\varphi(w_2(E,\nabla))=c_{2,\mathrm{cycle}}(E)\otimes 1 \tag{0.2.5}$$

0.3. Suppose now the connection ∇ on E is integrable, i.e. E is flat. Let $\mathcal{K}_i^m$ denote the Zariski sheaf, image of the Zariski-Milnor K sheaf in the constant sheaf $K_i^M(k(X))$. One has a map dlog: $\mathcal{K}_i^m \to \Omega^i_{X,\mathrm{clsd}}$. Functorial and additive classes

$$c_i(E,\nabla) \in \mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega^i \to \Omega^{i+1} \to \cdots) \tag{0.3.1}$$

were constructed in [8]. One has a natural map of complexes

$$\sigma\colon \{\mathcal{K}_i^m \to \Omega^i \to \Omega^{i+1} \to \cdots\} \to \Omega_X^{2i-1}/d\Omega_X^{2i-2}[-i]. \tag{0.3.2}$$

We prove in Section 4

$$w_i(E,\nabla) = \sigma(c_i(E,\nabla)) \in \Gamma(X, \Omega^{2i-1}/d\Omega^{2i-2}). \tag{0.3.3}$$

In the case of an integrable connection, the classes $w_n(E,\nabla)$ are closed. We are unable to answer the following

Basic question 0.3.1. Are the classes $w_i(E,\nabla,P)$ all zero for an integrable connection ∇?

0.4. We continue to assume ∇ integrable. We take $k = \mathbb{C}$, and X smooth and projective. We define the (generalized) Griffiths group $\mathrm{Griff}^n(X)$ to be the group of algebraic cycles of codimension n homologous to zero, modulo those homologous to zero on a divisor. (For $n = 2$, this is the usual Griffiths group of codimension 2 algebraic cycles homologous to zero modulo algebraic equivalence.) Our main result is

THEOREM 0.4.1. *We have* $w_n(E,\nabla) = 0$ *if and only if* $c_n(E) = 0$ *in* $\mathrm{Griff}^n(X)\otimes\mathbb{Q}$.

The proof of this theorem is given in Section 5.

The idea is that one can associate to any codimension n cycle Z homologous to zero an extension of mixed Hodge structures of $\mathbb{Q}(0)$ by $H^{2n-1}(X,\mathbb{Q}(n))$. One gets a quotient extension

$$0 \to H^{2n-1}(X,\mathbb{Q}(n))/N^1 \to E \to \mathrm{Griff}^n(X) \otimes \mathbb{Q}(0) \to 0$$

where N^1 is the subspace of "coniveau" 1, the group on the right has the trivial Hodge structure and where

$$E \subset H^0(X, \mathcal{H}^{2n-1}(\mathbb{Q}(n))).$$

Using the classes (0.3.1) and the comparison (0.3.3) we show

$$w_n(E,\nabla) \in F^0E \cap E(\mathbb{R}).$$

Furthermore, $w_n(E, \nabla) \in E(\mathbb{C})$ maps to the class of $c_n(E)$. Since the kernel of this extension is pure of weight -1 it follows easily that $w_n = 0 - c_n = 0$. In fact, Reznikov's theorem ([18]) implies

$$w_n(E, \nabla) \in E(\mathbb{Q}).$$

0.5. Through its link to the Griffiths group, it is clear that the classes $w_n(E, \nabla)$, when ∇ is integrable, are rigid in a variation of the flat bundle (E, ∇) over X. But in fact, a stronger rigidity (see 2.4.1) holds true: one can allow a 1 dimensional variation of X as well.

0.6. Examples (including Gauss-Manin systems of semi-stable families of curves, weight 1 Gauss-Manin systems, weight 2 Gauss-Manin systems of surfaces, and local systems with finite monodromy) for which the classes $w_n(E, \nabla)$ vanish are discussed in Section 7.

It is possible (cf. Section 7) to define $w_n(E, \nabla, P)$ in characteristic p for p large relative to n. In arithmetic situations, the resulting classes are compatible with reduction mod p. When the bundle (E, ∇) in characteristic p comes via Gauss-Manin from a smooth, proper family of schemes over X, we show using work of Katz ([15]) that $w_n(E, \nabla, P) = 0$. A longstanding conjecture of Ogus ([17]) would imply that a class in $\Gamma(X, \mathcal{H}^n)$ in characteristic 0 (where $\mathcal{H}$ is the Zariski sheaf of de Rham cohomology), which vanished when reduced mod p for almost all p was 0. Thus, Ogus's conjecture would imply an affirmative answer to 0.3 for Gauss-Manin systems.

0.7. In concrete applications, one frequently deals with connections ∇ with logarithmic poles. Insofar as possible, we develop our constructions in this context (see Section 6). The most striking remark is that even if ∇ has logarithmic poles, $w_n(E, \nabla)$ does not have any poles (see Theorem 6.1.1).

Acknowledgments. We thank E. Looijenga for making us aware of the similarity between our invariant w_2 with Witten's invariant for 3-manifolds (see [20]), and G. van der Geer for communicating [12] to us. The second author thanks E. Viehweg for support and encouragement.

1. Affine fibrations. An affine bundle Y over a scheme X is, by definition, a $\mathcal{V}$-torseur for some vector bundle $\mathcal{V}$. Such things are classified by $H^1(X, \mathcal{V})$. In particular, Zariski-locally, $Y \cong X \times \mathbb{A}^n$. Pullback from X to Y is an isomorphism on Chow motives, and hence on any Weil cohomology; e.g. $H_{\mathrm{DR}}(X) \cong H_{\mathrm{DR}}(Y)$, $H_{\text{ét}}(X) \cong H_{\text{ét}}(Y)$, etc. The following is known as "Jouanolou's trick." We recall the argument from [14].

Proposition 1.0.1. *Let X be a quasi-projective variety. Then there exists an affine bundle $Y \to X$ such that Y is an affine variety.*

Proof. Let $X \subset \bar{X}$ be an open immersion with $\bar{X}$ projective. Let $\tilde{X}$ be the blowup of $\bar{X} - X$ on $\bar{X}$. $\tilde{X}$ is projective, and $X \subset \tilde{X}$ with complement D a Cartier divisor. Suppose we have constructed $\pi\colon\ \tilde{Y} \to \tilde{X}$ an affine bundle with $\tilde{Y}$ affine. Since the complement of a Cartier divisor in an affine variety is affine (the inclusion of the open is acyclic for coherent cohomology, so one can use Serre's criterion) it follows that $\pi^{-1}(X) \to X$ is an affine bundle with $Y := \pi^{-1}(X)$ affine. We are thus reduced to the case X projective. Let $P(N) \to \mathbb{P}^N$ be an affine bundle with $P(N)$ affine. Given a closed immersion $X \hookrightarrow \mathbb{P}^N$, we may pull back $P(N)$ over X, so we are reduced to the case $X = \mathbb{P}^N$. In this case, one can take $Y = \mathrm{GL}_{N+1}/\mathrm{GL}_N \times \mathrm{GL}_{N+1}$. □

An exact sequence of vector bundles $0 \to G \to F \to E \to 0$ on X gives rise to an exact sequence of Hom bundles

$$0 \to \mathrm{Hom}\,(E, G) \to \mathrm{Hom}\,(E, F) \to \mathrm{Hom}\,(E, E) \to 0$$

and so an isomorphism class of affine bundles

$$\partial(\mathrm{Id}_E) \in H^1(X, \mathrm{Hom}\,(E, G)).$$

Of particular interest is the Atiyah sequence. Let X be a smooth variety, and let $\mathcal{I} \subset \mathcal{O}_X \otimes \mathcal{O}_X$ be the ideal of the diagonal. Let $\mathcal{P}_X := \mathcal{O}_X \otimes \mathcal{O}_X/\mathcal{I}^2$, and consider the exact sequence

$$0 \to \Omega^1_X \to \mathcal{P}_X \to \mathcal{O}_X \to 0$$

obtained by identifying $\mathcal{I}/\mathcal{I}^2 \cong \Omega^1$ in the usual way. Note that $\mathcal{P}_X$ has two distinct $\mathcal{O}_X$-module structures, given by multiplication on the left and right. These two structures agree on Ω^1 and on $\mathcal{O}_X$. Given E a vector bundle on X, we consider the sequence (Atiyah sequence)

$$0 \to E \otimes_{\mathcal{O}_X} \Omega^1_X \to E \otimes_{\mathcal{O}_X} \mathcal{P}_X \to E \to 0. \tag{1.0.1}$$

The tensor in the middle is taken using the left $\mathcal{O}_X$-structure, and then the sequence is viewed as a sequence of $\mathcal{O}_X$-modules using the *right* $\mathcal{O}_X$-structure.

PROPOSITION 1.0.2. *Connections on E are in* $1-1$ *correspondence with splittings of the Atiyah sequence* (1.0.1).

Proof. (See [1] and [5].) As a sequence of sheaves of abelian groups, the Atiyah sequence is split by $e \mapsto e \otimes 1$. Let $\theta\colon\ E \to E \otimes \mathcal{P}_X$ be an $\mathcal{O}$-linear splitting. Define

$$\nabla(e) := \theta(e) - e \otimes 1 \in E \otimes \Omega^1_X.$$

We have

$$\begin{aligned}\nabla(f\cdot e) &:= \theta(e)\cdot(1\otimes f)-(e\otimes 1)(f\otimes 1)\\ &= (1\otimes f)\cdot\nabla(e)+(e\otimes 1)\cdot(1\otimes f-f\otimes 1)\\ &= f\nabla(e)+df\wedge e,\end{aligned}$$

which is the connection condition. Conversely, given a connection ∇, the same argument shows that $\theta(e)=\nabla(e)+e\otimes 1$ is an $\mathcal{O}$-linear splitting. □

COROLLARY 1.0.3. *Let E be a vector bundle on a smooth affine variety X. Then E admits an algebraic connection.*

Proof. An exact sequence of vector bundles on an affine variety admits a splitting. □

1.1. In conclusion, given a vector bundle E on a smooth variety X, there exist two sorts of affine bundles $\pi\colon Y\to X$ such that π^*E admits a connection. We can take Y to be the Atiyah torseur associated to E, in which case the connection is canonical, or we can take Y to be affine, in which case all vector bundles admit (noncanonical) connections.

2. Chern-Simons. We begin by recalling in an algebraic context the basic ideas involving connections and the Chern-Weil and Chern-Simons constructions.

2.1. Connections and curvature. Let R be a k-algebra of finite type (R and k commutative with 1). A connection ∇ on a module E is a map $\nabla\colon E\to E\otimes_R\Omega^1_{R/k}$ satisfying $\nabla(f\cdot e)=e\otimes df+f\cdot\nabla e$. More generally, if $D\subset \operatorname{Spec}R$ is a Cartier divisor, of equation f, one defines the module $\Omega^1_{R/k}(\log D)$ of Kähler 1-forms with logarithmic poles along D, as the submodule of forms w with poles along D such that $w\cdot f$ and $w\wedge df$ are regular [6]. A connection with log poles along D is a k linear map $\nabla\colon E\to E\otimes\Omega^1_{R/k}(\log D)$ fulfilling the Leibniz relations. When E has a global basis $E=R^N$, ∇ can be written in the form $d+A$, where A is an $N\times N$-matrix of 1-forms. Writing $e_i=(0,\dots,1,\dots,0)$ we have

$$\nabla(e_i)=\sum_j e_j\otimes a_{ij}.$$

The map ∇ extends to a map $\nabla\colon E\otimes\Omega^i\to E\otimes\Omega^{i+1}$ defined by $\nabla(e\otimes\omega)=\nabla(e)\wedge\omega+e\otimes d\omega$. The curvature of the connection is the map $\nabla^2\colon E\to E\otimes\Omega^2$. The curvature is R-linear and is given in the case $E=R^N$ by

$$\begin{aligned}\nabla^2(e_i) &= \sum_j e_j\otimes da_{ij}+\sum_{j,\ell} e_\ell\otimes a_{j\ell}\wedge a_{ij}\\ &= (0,\dots,1,\dots,0)\cdot(dA-A^2).\end{aligned}$$

The curvature matrix $F(A)$ is defined by $F(A) = dA - A^2$. (Note that the definition $F(A) = dA + A^2$ is also found in the literature, e.g. in [3].)

Given $g \in \mathrm{GL}_N(R)$, let $\gamma = g^{-1}$. We can rewrite the connection $\nabla = d + A$ in terms of the basis $\epsilon_i := e_i \cdot g = (g_{i1}, \dots, g_{iN})$, replacing A and $F(A)$ by

$$dg \cdot g^{-1} + gAg^{-1} = -\gamma^{-1} d\gamma + \gamma^{-1} A \gamma \tag{2.1.1}$$

$$F(dg \cdot g^{-1} + gAg^{-1}) = gF(A)g^{-1}. \tag{2.1.2}$$

A connection is said to be integrable or flat if $\nabla^2 = 0$. For a connection on R^N this is equivalent to $F(A) = 0$.

2.2. We recall some basic ideas from [3]. Let $\mathcal{G}$ be a Lie algebra over a field k of characteristic 0, and let G be the corresponding algebraic group. (The only case we will use is $G = \mathrm{GL}_N$.) Write $\mathcal{G}^\ell := \underbrace{\mathcal{G} \otimes \cdots \otimes \mathcal{G}}_{\ell \text{ factors}}$. G acts diagonally on $\mathcal{G}^\ell$ by the adjoint action on each factor, and an element P in the linear dual $(\mathcal{G}^\ell)^*$ is said to be invariant if it is symmetric and invariant under this diagonal action. For a k-algebra R we consider the module $\Lambda^{r,\ell} := \mathcal{G}^\ell \otimes_k \Omega^r_{R/k}$ of r-forms on R with values in $\mathcal{G}^\ell$. Let x_i denote tangent vector fields, i.e. elements in the R-dual of Ω^1. We describe two products $\wedge$: $\Lambda^{r,\ell} \otimes_R \Lambda^{r',\ell'} \to \Lambda^{r+r',\ell+\ell'}$ and $[\ \ ]$: $\Lambda^{r,1} \otimes_R \Lambda^{r',1} \to \Lambda^{r+r',1}$. In terms of values on tangents, these are given by

$$\varphi \wedge \psi(x_1, \dots, x_{r+r'}) = \sum_{\pi,\text{shuffle}} \sigma(\pi)\varphi(x_{\pi_1}, \dots, x_{\pi_r}) \otimes \psi(x_{\pi_{r+1}}, \dots, x_{\pi_{r+r'}}) \tag{2.2.1}$$

$$[\varphi, \psi](x_1, \dots, x_{r+r'}) = \sum_{\pi,\text{shuffle}} \sigma(\pi)[\varphi(x_{\pi_1}, \dots, x_{\pi_r}), \psi(x_{\pi_{r+1}}, \dots, x_{\pi_{r+r'}})]. \tag{2.2.2}$$

Here $\sigma(\pi)$ is the sign of the shuffle. These operations satisfy the identities (for $P \in (\mathcal{G}^\ell)^*$ symmetric, i.e. invariant under the action of the symmetric group in ℓ letters but not necessarily invariant under G)

$$[\varphi, \psi] = (-1)^{rr'+1}[\psi, \varphi] \tag{2.2.3}$$

$$[[\varphi, \varphi], \varphi] = 0 \tag{2.2.4}$$

$$d[\varphi, \psi] = [d\varphi, \psi] + (-1)^r[\varphi, d\psi] \tag{2.2.5}$$

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^r \varphi \wedge d\psi \tag{2.2.6}$$

$$d(P(\varphi)) = P(d\varphi) \tag{2.2.7}$$

$$P(\varphi \wedge \psi \wedge \rho) = (-1)^{rr'} P(\psi \wedge \varphi \wedge \rho). \tag{2.2.8}$$

If P is invariant, we have in addition for $\varphi_i \in \Lambda^{r_i,1}$ and $\psi \in \Lambda^{1,1}$

$$\sum_{i=1}^{\ell} (-1)^{r_1+\cdots+r_i} P(\varphi_1 \wedge \cdots \wedge [\varphi_i, \psi] \wedge \cdots \wedge \varphi_\ell) = 0. \tag{2.2.9}$$

By way of example, we note that if $A = (a_{ij}), B = (b_{ij})$ are matrices of 1-forms, then writing AB (or A^2 when $A = B$) for the matrix of 2-forms with entries

$$\sum_\ell a_{i\ell} \wedge b_{\ell j}$$

we have

$$\begin{aligned}
[A,A](x_1,x_2)_{ij} &= ([A(x_1),A(x_2)] - [A(x_2),A(x_1)])_{ij} \\
&= 2\,(A(x_1)A(x_2) - A(x_2)A(x_1))_{ij} \\
&= 2\sum_\ell \big(a_{i\ell}(x_1)a_{\ell j}(x_2) - a_{i\ell}(x_2)a_{\ell j}(x_1)\big) \\
&= 2\sum_\ell a_{i\ell} \wedge a_{\ell j}(x_1,x_2) = 2A^2(x_1,x_2),
\end{aligned}$$

whence

$$A^2 = \frac{1}{2}[A,A].$$

In the following, for $\varphi \in \Lambda^{r,\ell}$ we frequently write φ^n in place of $\varphi \wedge \cdots \wedge \varphi$ (n-times). The signs differ somewhat from [3] because of our different convention for the curvature as explained above.

THEOREM 2.2.1. ([3]) *Let $P \in (\mathcal{G}^\ell)^*$ be invariant. To a matrix A of* 1*-forms over a ring R, we associate a matrix of* 2*-forms depending on a parameter t*

$$\varphi_t := tF(A) - \frac{1}{2}(t^2 - t)[A,A].$$

Define

$$TP(A) = \ell \int_0^1 P(A \wedge \varphi_t^{\ell-1})dt \in \Omega^{2\ell-1}_{R/k}. \tag{2.2.10}$$

For example, for

$$P(M) = \mathrm{Tr} M^2,\ \ell = 2,\ TP(A) = \mathrm{Tr}\left(AdA - \frac{2}{3}A^3\right) \tag{2.2.11}$$

Then $dTP(A) = P(F(A)^\ell)$. *The association* $A \mapsto TP(A)$ *is functorial for maps of rings* $R \to S$. *If* $A \mapsto T'P(A)$ *is another such functorial mapping satisfying*

$$dT'P(A) = dTP(A) = P(F(A)^\ell),$$

then

$$T'P(A) - TP(A) = d\rho$$

is exact.

Proof. The first assertion follows from Proposition 3.2 of [3], noting that $\Omega(A)$ in their notation is $-F(-A)$ in ours. For the second assertion, we may assume by functoriality that R is a polynomial ring, so $H^{2\ell-1}_{DR}(R/k) = (0)$. The form $T'P(A) - TP(A)$ is closed, and hence exact. □

PROPOSITION 2.2.2. *With notation as above, let* $g \in \mathrm{GL}_N(R)$, *and assume* $\ell \geq 2$. *Then* $TP(dg \cdot g^{-1} + gAg^{-1}) - TP(A)$ *is Zariski-locally exact, i.e. there exists an open cover* $\mathrm{Spec}\,(R) = \bigcup U_i$ *such that the above expression is exact on each* U_i.

Proof. The property of being Zariski-locally exact is compatible under pull-back, so we may argue universally. The matrix A of 1-forms (resp. the element g) is pulled back from the coordinate ring of some affine space $\mathbb{A}^m$ (resp. from the universal element in GL_N with coefficients in the coordinate ring of GL_N), so we may assume R is the coordinate ring of $\mathbb{A}^m \times \mathrm{GL}_N$.

Let η be a closed form on a smooth variety T. Let $f\colon S \to T$ be surjective, with S quasi-projective. Then η is locally exact on T if and only if $f^*\eta$ is locally exact on S. Indeed, given $t \in T$ we can find a section $S'' \subset S$ such that the composition $f'\colon S' \to T$, where $S' \to S''$ is the normalization, is finite over some neighborhood $t \in U$. Assuming $f^*\eta$ is locally exact, it follows that $f^*\eta \mid f'^{-1}(U)$ is locally exact, and so by a trace argument (we are in characteristic zero) that $\eta \mid U$ is locally exact as well.

We apply the above argument with

$$\eta = TP(dg \cdot g^{-1} + gAg^{-1}) - TP(A)$$

and $T = \mathbb{A}^m \times \mathrm{GL}_N$. As a scheme, $\mathrm{GL}_N \cong \mathbb{G}_m \times \mathrm{SL}_N$, and for some large integer M we can find a surjection $\coprod_{\text{finite}} \mathbb{A}^M \to \mathrm{SL}_N$ by taking products of upper and lower triangular matrices with 1 on the diagonal and then taking a disjoint sum of translates. Pulling back, it suffices to show that a closed form of degree ≥ 2 on $\mathbb{A}^{M+m} \times \mathbb{G}_m$ is exact. This is clear. □

Construction 2.3. Let E be a vector bundle of rank N on a smooth quasi-projective variety X. Let P be an invariant polynomial as above of degree n on the Lie algebra $\mathcal{GL}_N$. Suppose a given connection ∇ on E. (Such a connection exists when X is affine because the Atiyah sequence splits.) Let $X = \bigcup U_i$ be an open affine covering such that $E \mid U_i \cong \mathcal{O}^{\oplus N}$, and let A_i be the matrix of 1-forms corresponding to $\nabla \mid U_i$. The class of $TP(A_i) \in \Gamma(U_i, \Omega^{2n-1}/d\Omega^{2n-2})$ is independent of the choice of basis for $E \mid U_i$ by (2.2.2). It follows that these classes glue to give a global class

$$w_n(E, \nabla, P) \in \Gamma(X, \Omega^{2n-1}/d\Omega^{2n-2}). \tag{2.3.1}$$

PROPOSITION 2.3.1. *Let E be a rank N-vector bundle on a smooth affine variety X. Let ∇ and ∇' be two connections on E. Let P be an invariant polynomial of degree n. Then there exists a form*

$$\eta \in \Gamma(X, \Omega_X^{2n-1})$$

such that

$$w_n(E, \nabla, P) - w_n(E, \nabla', P) \equiv \eta \bmod(d\Omega^{2n-2}).$$

Proof. Because X is affine, any affine space bundle $Y \to X$ admits a section. (An affine space bundle is a torseur under a vector bundle.) Thus, we may replace X by an affine space bundle over X. Since X is affine, E is generated by its global sections, so we may find a Grassmannian G and a map $X \to G$ such that E is pulled back from G. We may find an affine space bundle $Y \to G$ with Y affine. Replacing X with $X \times_G Y$, which is an affine bundle over X, we may assume E pulled back from a bundle F on Y. Since Y is affine, F admits a connection Ψ, and it clearly suffices to prove the proposition for ∇ the pullback of Ψ. Write $\nabla' - \nabla = \gamma$ with $\gamma \in \mathrm{Hom}_{\mathcal{O}_X}(E, E \otimes \Omega^1)$. Let $\iota\colon X \hookrightarrow \mathbb{A}^m$ be a closed immersion. The product map $X \hookrightarrow Y \times \mathbb{A}^m$ is a closed immersion, hence γ lifts to $\varphi \in \mathrm{Hom}_{\mathcal{O}_{Y\times\mathbb{A}^m}}(F, F \otimes \Omega^1_{Y\times\mathbb{A}^m})$. Let $\Psi' := \Psi + \varphi$. We are now reduced to the case $X = Y \times \mathbb{A}^m$. Writing $\mathcal{H}^{2n-1}$ for the Zariski cohomology sheaf of the de Rham complex on X, one knows that $\Gamma(X, \mathcal{H}^{2n-1}) \subset \Gamma(U, \mathcal{H}^{2n-1})$ for any open $U \neq \emptyset$ ([2]). Taking $U = \mathbb{A}^{M+m}$, where $\mathbb{A}^M$ is an affine cell in Y, we may assume $\Gamma(X, \mathcal{H}^{2n-1}) = (0)$. If P corresponds to a polynomial F in the Chern classes, $dw(E, \nabla, P)$ and $dw(E, \nabla', P)$ both represent the same class $F(E)$ in cohomology, so, since X is affine, there exists $\eta \in \Gamma(X, \Omega^{2n-1})$ such that

$$w_n(E, \nabla, P) - w_n(E, \nabla', P) - \eta \in \Gamma(X, \mathcal{H}^{2n-1}) = (0). \qquad \square$$

PROPOSITION 2.3.2. *Let ∇ be an integrable connection on E, and let P be an invariant polynomial of degree n. Let $\mathcal{H}^{2n-1} = \Omega^{2n-1}_{\mathrm{closed}}/d\Omega^{2n-2}$. Then $w_n(E, \nabla, P) \in \Gamma(X, \mathcal{H}^{2n-1})$, i.e. $dw = 0$.*

Proof. $dw = P(F(\nabla)) = 0$ since ∇ integrable implies $F(\nabla) = 0$. $\square$

PROPOSITION 2.3.3. *Let ∇ be an integrable connection on E, and let $P = \lambda P_n + Q$ be an invariant polynomial of degree n, where P_n is the nth elementary symmetric function and Q is a sum with rational coefficients of decomposable polynomials $P_{i_1} \dots P_{i_r}$ with $r \geq 2$ and $i_j \geq 1$. Then*

$$w_n(E, \nabla, P) = \lambda w_n(E, \nabla)$$

(see notation (0.2.3)*).*

Proof. Writing elementary symmetric functions P_i of degree $i \leq n$ as polynomials with rational coefficients in the elementary Newton functions $Q_i(M) = \mathrm{Tr}\ M^i$ of degree $i \leq n$ and vice-versa, it is enough to show $w_n(E, \nabla, Q_I) = 0$ where $Q_I = Q_{i_1} \dots Q_{i_r}$ for $r \geq 2$ and $i_j \geq 1$. For A with $dA - A^2 = 0$ one has $F(tA) = (t - t^2)A^2$, and therefore

$$\begin{aligned} &Q_I(A \wedge F(tA)^{i_1+\cdots+i_r-1}) \\ &\qquad Q_{i_1}(AF(tA)^{i_1-1})((t-t^2)^{i_2+\cdots+i_r})Q_{i_2}(A^{2i_2}) \cdots Q_{i_r}(A^{2i_r}) = 0 \end{aligned}$$

as $\mathrm{Tr}\ A^{2j} = 0$ for $j \geq 1$. ◻

2.4. Rigidity.

THEOREM 2.4.1. *Let $f\colon X \to S$ be a smooth proper morphism between smooth algebraic varieties defined over a field k of characteristic zero. Assume* $\dim S = 1$. *Let $\nabla\colon E \to \Omega^1_{X/S} \otimes E$ be a relative flat connection, and P be an invariant polynomial. Then*

$$w_n(E, \nabla, P) \in H^0(X, \mathcal{H}^{2n-1}(X/S))$$

lifts canonically to a class in $H^0(X, \mathcal{H}^{2n-1})$ for $n \geq 2$.

Proof. Take locally the matrix $A'_i \in H^0(X_i, M(N, \Omega^1_{X/S}))$ of the connection, N being the rank of E. Take liftings $A_i \in H^0(X_i, M(N, \Omega^1_X))$, and define $TP(A_i)$ looking at the Ω^1_X valued connection defined by A_i. Since $F(A_i) \in H^0(X_i, M(N, f^*\Omega^1_{\int} \otimes \Omega^1_X))$, one has $F(A_i)^n = 0$ for $n \geq 2$, and $dTP(A_i) = P(F(A)^n) = 0$. On $X_i \cap X_j$, one has

$$A_j = dg \cdot g^{-1} + gA_i g^{-1} - \Gamma_{ij}$$

where $\Gamma_{ij} \in H^0(X_i \cap X_j, f^*\Omega^1_S)$. Using Proposition 2.2.2, we just have to show that $TP(B) - TP(B + \Gamma)$ is locally exact for some matrix of one forms $B = dg \cdot g^{-1} + gA_i g^{-1}$, verifying $F(B)\omega = 0$ for any $w \in M(N, f^*\Omega^1_S)$, and $\Gamma = \Gamma_{ij} \in f^*\Omega^1_S$. By Proposition 2.3.3 it is enough to consider $P(M) = \mathrm{Tr} M^n$. One has

$$\varphi_t(B + \Gamma) = F(t(B + \Gamma)) = F(tB) + td\Gamma - t^2(\Gamma B + B\Gamma) \tag{2.4.1}$$

and

$$F(tB)\omega = (t - t^2)dB\omega \tag{2.4.2}$$

with ω as above. Thus

$$\begin{aligned} P((B+\Gamma)\wedge\varphi_t^{n-1}(B+\Gamma)) &= \mathrm{Tr}(B+\Gamma) \\ &\quad \times[(tdB - t^2B^2)^{n-1} + (n-1)(t-t^2)^{n-2} \\ &\qquad \times (dB)^{n-2}(td\Gamma - t^2(B\Gamma+\Gamma B)] \\ &= P(B\wedge\varphi_t^{n-1}(B)) + R \end{aligned} \tag{2.4.3}$$

with

$$\begin{aligned} R &= \mathrm{Tr}\Gamma(dB)^{n-1}[(t-t^2)^{n-1} - 2t^2(n-1)(t-t^2)^{n-2}] \\ &\quad + (n-1)(t-t^2)^{n-2}t\mathrm{Tr}B(dB)^{n-2}d\Gamma. \end{aligned} \tag{2.4.4}$$

Write $\mathrm{Tr}\, d(B\Gamma) = \mathrm{Tr}\, dB\Gamma - \mathrm{Tr}\, Bd\Gamma$. Then we have

$$R = F(t)\mathrm{Tr}\,\Gamma(dB)^{n-1} \tag{2.4.5}$$

modulo exact forms, with

$$F(t) = n(t-t^2)^{n-1} - (n-1)t^2(t-t^2)^{n-2} = (t(t-t^2)^{n-1})'. \tag{2.4.6}$$

The assertion now follows from (2.2.10). □

3. Flat Bundles. The following notations will reoccur frequently.

3.1. X will be a smooth variety, and $D = \bigcup D_i \subset X$ will be a normal crossings divisor, with $j\colon X - D \to X$. We will assume unless otherwise specified that the ground field k has characteristic 0.

3.2. (E, ∇) will be a vector bundle E of rank r on X with connection $\nabla\colon E \to E \otimes \Omega^1_{X/k}(\log D)$ having logarithmic poles along D. The Poincaré residue map $\Omega^1(\log; D) \to \mathcal{O}_{D_i}$ is denoted res_{D_i}, and $\Gamma_i := \mathrm{res}_{D_i} \circ \nabla\colon E \to E|_{D_i}$.

3.3. When E is trivialized on the open cover $X = \cup X_i$, with basis $\underline{e}_i$ on X_i, then (E, ∇) is equivalent to the data

$$\begin{aligned} g_{ij} &\in \Gamma(X_i \cap X_j, GL(r, \mathcal{O}_X)) \\ g_{ik} &= g_{ij}g_{jk} \\ A_i &\in \Gamma(X_i, M(r, \Omega^1_X(\log D))) \end{aligned}$$

with $g_{ij}^{-1}dg_{ij} = g_{ij}^{-1}A_ig_{ij} - A_j$.

3.4. The curvature

$$\nabla^2\colon\ E \to \Omega^2_X(\log D) \otimes E$$

is given locally by

$$\nabla^2 = F(A_i) := dA_i - A_i A_i.$$

The connection ∇ is said to be flat, or integrable if $\nabla^2 = 0$.

3.5. For two $r \times r$ matrices A and B of differential forms of weight a and b respectively, one writes Tr AB for the trace of the $r \times r$ matrix AB of weight $a + b$, and one has Tr $AB = (-1)^{ab}$Tr BA. We denote by ${}^tA$ the transpose of A: $({}^tA)_{ij} = A_{ji}$.

3.6. For any cohomology theory H with a localization sequence, the ith level of Grothendieck's coniveau filtration is defined by

$$N^i H^\bullet = \{x \in H^\bullet \mid \exists \text{ subvariety } Z \subset X \text{ of codimension } \geq i \\ \text{such that } 0 = x|_{X-Z} \in H^\bullet(X - Z).\}$$

3.7. For any cohomology theory H defined in a topology finer than the Zariski topology, one defines the Zariski sheaves $\mathcal{H}$ associated to the presheaves $U \mapsto H(U)$ ([2]). When H is the cohomology for the analytic topology with coefficients in a constant sheaf A, we sometimes write $\mathcal{H}(A)$. For example the Betti or de Rham sheaves $\mathcal{H}(\mathbb{C})$ are simply the cohomology sheaves for the complex of algebraic differentials Ω^*_X. For $D \subset X$ as above, we write $\mathcal{H}^\bullet(\log D)$ for the cohomology sheaves of $\Omega^*(\log D)$. It is known that $\mathcal{H}^\bullet(\log D) \cong j_*\mathcal{H}^\cdot_{X-D}$.

3.8. When $k = \mathbb{C}$ we use the same notation Ω^*_X for the analytic and algebraic de Rham complexes. For integers a and b, the analytic Deligne cohomology is defined to be the hypercohomology of the complex of analytic sheaves

$$H^a_{\mathcal{D},\mathrm{an}}(X, \mathbb{Z}(b)) := \mathbb{H}^a(X_{\mathrm{an}}, \mathbb{Z}(b) \to \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{b-1}).$$

(This should be distinguished from the usual Deligne cohomology, which is defined using differentials with at worst log poles at infinity.) One has a cycle class map from the Chow group of algebraic cycles modulo rational equivalence to Deligne cohomology:

$$CH^i(X) \to H^{2i}_{\mathcal{D},\mathrm{an}}(X, \mathbb{Z}(i)).$$

3.9. We continue to assume $k = \mathbb{C}$. Let α: $X_{\mathrm{an}} \to X_{\mathrm{Zar}}$ be the identity map. For a complex $C^\cdot$, let $t_{\geq i}C^\cdot$ be the subcomplex which is zero in degrees $< i$ and coincides with $C^{\cdot}$ in degrees $\geq i$. There is a map of complexes $t_{\geq i}C^{\cdot} \to C^{\cdot}$. The complex

$$\mathbb{Z}(j) \to \mathcal{O}_X \to \Omega^1_X \to \cdots$$

in the analytic topology is quasi-isomorphic to the cone $\mathbb{Z}(j) \to \mathbb{C}$, and hence to $\mathbb{C}/\mathbb{Z}(j)[-1]$. We obtain in this way a map in the derived category

$$(t_{\geq j}\Omega^{\cdot}_X) \to \mathbb{C}/\mathbb{Z}(j).$$

The kernel of the resulting map

$$R^j\alpha_*(t_{\geq j}\Omega^{\cdot}_X) \cong \ker\left(\alpha_*\Omega^j \to \alpha_*\Omega^{j+1}\right) \to R^j\alpha_*(\mathbb{C}/\mathbb{Z}(j))$$

is denoted $\Omega^j_{\mathbb{Z}(j)}$ ([8]). Note $\Omega^j_{\mathbb{Z}(j)}$ is a Zariski sheaf. Writing $\mathcal{K}^m_j$ for the Milnor K-sheaf (subsheaf of the constant sheaf $K^{\mathrm{Milnor}}_j(k(X))$), the $d\log$-map

$$\{f_1, \ldots, f_j\} \mapsto df_1/f_1 \wedge \cdots \wedge df_j/f_j$$

induces a map

$$d\log\colon\ \mathcal{K}^m_j \to \Omega^j_{\mathbb{Z}(j)}. \tag{3.9.1}$$

To see this, note the exponential sequence induces a map

$$\mathcal{O}^*_{X_{\mathrm{Zar}}} \to R^1\alpha_*\mathbb{Z}(1)$$

and we get by cup product a commutative diagram with left-hand vertical arrow surjective

$$\begin{array}{ccc} \mathcal{O}^{*\otimes j}_{X_{\mathrm{Zar}}} & \longrightarrow & R^j\alpha_*\mathbb{Z}(j) \\ \Big\downarrow \text{surj.} & & \Big\downarrow \\ \mathcal{K}^m_j & \xrightarrow{d\log} & R^j\alpha_*\mathbb{C}. \end{array}$$

We shall need some more precise results about the sheaf $\Omega^j_{\mathbb{Z}(j)}$.

LEMMA 3.9.1. (1) *There is a natural map*

$$H^i(X_{\mathrm{Zar}}, \Omega^i_{\mathbb{Z}(i)}) \to H^{2i}_{\mathcal{D},\mathrm{an}}(X, \mathbb{Z}(i)).$$

(2) *Let $D \subset X$ be a normal crossings divisor. Then there is a natural map*

$$\begin{aligned}&\mathbb{H}^i(X_{\mathrm{Zar}}, \Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_X(\log\ D) \to \cdots)\\ &\quad \to \mathbb{H}^{2i}(X_{\mathrm{an}}, \mathbb{Z}(i) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{i-1}_X \to \Omega^i(\log\ D)_X \to \cdots).\end{aligned}$$

(3) *There is a natural map*

$$\begin{aligned}&\varphi\colon\ \mathbb{H}^i(X_{\mathrm{Zar}}, \mathcal{K}^m_i \xrightarrow{d\log} \Omega^i_X(\log\ D) \to \cdots)\\ &\quad \to \mathbb{H}^{2i}(X_{\mathrm{an}}, \mathbb{Z}(i) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{i-1}_X \to \Omega^i(\log\ D)_X \to \cdots)\end{aligned}$$

In particular, for $D = \emptyset$, we get a map

$$\mathbb{H}^i(X_{\mathrm{Zar}}, \mathcal{K}^m_i \xrightarrow{d\log} \Omega^i_X \to \cdots) \to H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)).$$

Proof. We consider the spectral sequence

$$\begin{aligned}R^j &:= R^j\alpha_*(\mathbb{Z}(i) \to \mathcal{O} \to \Omega^1 \to \cdots \to \Omega^{i-1})\\ E_2^{p,q} &= H^p(X_{\mathrm{Zar}}, R^q) \Rightarrow H^{p+q}_{\mathcal{D},\mathrm{an}}(X, \mathbb{Z}(i)).\end{aligned}$$

One checks that

$$\begin{gathered}R^s \cong \mathcal{H}^{s-1}(\mathbb{C}/\mathbb{Z}(i)); \qquad s < i\\ 0 \to \mathcal{H}^{i-1}(\mathbb{C}/\mathbb{Z}(i)) \to R^i \to \Omega^i_{\mathbb{Z}(i)} \to 0\\ 0 \to \mathcal{H}^{i-1}(\mathbb{C}/\mathbb{Z}(i)) \to R^s \to \ker(\mathcal{H}^s(\mathbb{C}) \to \mathcal{H}^s(\mathbb{C}/\mathbb{Z}(i))) \to 0; \qquad s > i.\end{gathered}$$

We have by ([2]) that $H^a(X_{\mathrm{Zar}}, \mathcal{H}^b(A)) = (0)$ for $a > b$ and A any constant sheaf of abelian groups. Applying this to the above, we conclude $E_2^{a,2i-a} = H^a(X_{\mathrm{Zar}}, R^{2i-a}) = (0)$ for $a > i$, and $E_2^{i,i} \cong H^i(X, \Omega^i_{\mathbb{Z}(i)})$. Assertion (3.9.1) follows. The construction of the map in (2) is similar and is left for the reader. Finally, (3) follows by composing the arrow from (2) with the $d\log$ map (3.9.1). □

3.10. Characteristic classes. Let (E, ∇) be a bundle with connection as in 3.2 and assume ∇ is flat. Functorial and additive characteristic classes

$$c_i(E, \nabla) \in \mathbb{H}^i(X_{\mathrm{Zar}}, \mathcal{K}^m_i \to \Omega^i_X(\log D) \to \Omega^{i+1}_X(\log D) \to \cdots)$$

were defined in [7]. These classes have the following compatibilities:

3.10.1. Under the map

$$\mathbb{H}^i(X_{\mathrm{Zar}}, \mathcal{K}_i^m \to \Omega_X^i(\log\ D) \to \Omega_X^{i+1}(\log\ D) \to \cdots) \to H^i(X, \mathcal{K}_i^m) \cong CH^i(X)$$

we have $c_i(E, \nabla) \mapsto c_i^{\mathrm{Chow}}(E) \in CH^i(X)$.

3.10.2. Assume X proper and $D = \phi$. The classes $c_i(E, \nabla)$ lift classes $c_i^{\mathrm{an}}(E, \nabla) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$ defined in [8], via the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^i(X, \mathcal{K}_i^m \to \Omega_X^i \to \Omega_X^{i+1} \to \cdots) & \longrightarrow & CH^i(X) \\ \varphi\ (3.9.1(3)) \big\downarrow & & \big\downarrow \psi\text{=cycle map} \\ H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)) & \longrightarrow & H^{2i}_{\mathcal{D}}(X, i). \end{array}$$

3.10.3. When $D \neq \phi$ and X is proper, classes

$$c_i^{\mathrm{an}}(E, \nabla) \in \mathbb{H}^{2i}(X_{\mathrm{an}}, \mathbb{Z}(i) \to \mathcal{O}_X \to \cdots \to \Omega_X^{i-1} \to \Omega_X^i(\log D) \to \cdots)$$

lifting $c_i^{\mathcal{D}}(E) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ are defined in [8]. In general, for X not proper, these classes lift

$$c_i^{\mathcal{D}}(E|_{X-D}) \in H^{2i}_{\mathcal{D}}(X - D, \mathbb{Z}(i))$$

via the factorization through $H^{2i-1}(X - D, \mathbb{C}/\mathbb{Z}(i))$ ([9], (3.5)).

Proposition 3.10.1. *The map φ from (Lemma* 3.9.1(3)*) carries $c_i(E, \nabla)$ to $c_i^{\mathrm{an}}(E, \nabla)$. For X proper, the diagram*

$$\begin{array}{ccc} \mathbb{H}^i(X, \mathcal{K}^m \to \Omega_X^i(\log D) \to \Omega_X^{i+1}(\log D) \to \cdots) & \longrightarrow & CH^i(X) \\ \varphi \big\downarrow & & \big\downarrow \psi \\ \mathbb{H}^{2i}(X_{\mathrm{an}}, \mathbb{Z}(i) \to \mathcal{O}_X \to \cdots \Omega_X^{i-1} \to \Omega_X^i(\log D) \to \cdots) & \longrightarrow & H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \end{array}$$

commutes. For X not proper, the diagram remains commutative if one replaces the bottom row by

$$H^{2i-1}((X - D)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)) \to H^{2i}_{\mathcal{D}}(X - D, \mathbb{Z}(i))$$

or if one replaces $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ by $H^{2i}_{\mathcal{D},\mathrm{an}}(X, \mathbb{Z}(i))$.

Proof. The central point, for which we refer the reader to ([8]) is the following. Let $\pi\colon\ G \to X$ be the flag bundle of E over which E has a filtration $E_{i-1} \subset E_i$ by $\tau\nabla$ stable subbundles with successive rank 1 quotients $(L_i, \tau\nabla)$ (see [7]). Then

$c_i(E,\nabla)$ and $c_i^{\mathrm{an}}(E,\nabla)$ are both defined on G by products starting from

$$c_1(L_\alpha,\tau\nabla)\in\mathbb{H}^1(G,\mathcal{K}_1\to\pi^*\Omega^1_X(\log D)\to\cdots)$$

$$c_1^{\mathrm{an}}(L_\alpha,\tau\nabla)\in\mathbb{H}^2(G,\mathbb{Z}(i)\to\mathcal{O}_G\to\pi^*\Omega^1_X(\log D)\to\cdots).$$

It suffices to observe that the "algebraic" product

$$\begin{aligned}&\mathbb{H}^1(G,\mathcal{K}_1\to\pi^*\Omega^1_X(\log D)\to\cdots)^{\otimes i}\\ &\qquad\to\mathbb{H}^i(G,\mathcal{K}_i^m\to\pi^*\Omega^i_X(\log D)\to\cdots)\end{aligned}$$

([8], p. 51) is defined compatibily with the "analytic" product

$$\begin{aligned}&\mathbb{H}^2(G,\mathbb{Z}(i)\to\mathcal{O}_G\to\pi^*\Omega^1_X(\log D)\to\cdots)^{\otimes i}\\ &\qquad\to\mathbb{H}^{2i}(G,\mathbb{Z}(i)\to\mathcal{O}_G\to\cdots\to\Omega_G^{i-1}\to\pi^*\Omega^i_X(\log D)\to\cdots).\end{aligned}$$ □

3.11. Let $\tau\colon\ \Omega^*_X\to N^*$ be a map of complexes , with $\mathcal{O}_X=N^0$, such that if a is the smallest degree b for which $B^b:=\mathrm{Ker}\Omega^b_X\to N^b\neq 0$, then $B^b=B^a\wedge\Omega_X^{b-a}$. For example, let $\nabla\colon\ \mathcal{F}\to\Omega^1_X(\log D)\otimes\mathcal{F}$ be a nonintegrable connection. Then the local relation $dF(A)=[A,F(A)]$ shows that one can define N^* by

(3.11.1)
$$\begin{aligned}N^1&=\Omega^1_X(\log D)\\ N^i&=\Omega^i_X(\log D)/B^2\wedge\Omega_X^{i-2}(\log D)\end{aligned}$$

where B^2 is locally generated by the entries of the curvature matrix of ∇.

Let (E,∇) be a flat N^* valued connection, that is a k linear map $\nabla\colon\ E\to N^1\otimes E$ satisfying the Leibniz rule

(3.11.2)
$$\nabla(\lambda e)=\tau d\lambda e+\lambda\nabla(e),$$

the sign convention

(3.11.3)
$$\nabla(\omega\otimes e)=\tau d(\omega)\otimes e+(-1)^o\omega\wedge\nabla(e),$$

where $o=\deg\omega$, and $(\nabla)^2=0$. Then the computations of [7] and [8] allow one to show the existence of functorial and additive classes

(3.11.4)
$$c_i(E,\nabla)\in\mathbb{H}^i(X,\mathcal{K}_i^m\to N^i\to N^{i+1}\cdots)$$

mapping to analytic classes

(3.11.5)
$$c_i^{\mathrm{an}}(E,\nabla)\in\mathbb{H}^{2i}(X_{\mathrm{an}},\mathbb{Z}(i)\to\cdots\to\Omega^i_X\to N^{i+1}\cdots)$$

compatibly with the classes $c_i^{\mathcal{D}}(E)$ and $c_i^{\mathrm{Chow}}(E)$ as before. As we won't need those classes, we don't repeat the construction in detail.

3.12. Finally, the $c_i(E, \nabla)$ map to classes

$$\theta_i(E,\nabla) \in H^0\left(X, \frac{\Omega_X^{2i-1}(\log D)}{d\Omega_X^{2i-2}(\log D)}\right).$$

In the next section these will be related to the classes $w_i(E, \nabla)$.

4. The classes θ_n and w_n. Recall that we had defined $w_n(E,\nabla) = w_n(E,\nabla,P_n)$ in (0.2.3) for the nth elementary symmetric function P_n.

THEOREM 4.0.1. *Let X be a smooth quasi-projective variety over $\mathbb{C}$. Let E, ∇ be a rank d vector bundle on X with integrable connection. For $d \geq n \geq 2$, we have $w_n(E,\nabla) = \theta_n(E,\nabla)$, and $w_n(E,\nabla,P) = \lambda\theta_n(E,\nabla)$ (with the notations of Proposition* 2.3.2*).*

The proof will take up this entire section. We begin with

Remark 4.0.2. We may assume X is affine, and $E \cong \mathcal{O}_X^{\oplus N}$. In this situation, the class $w_n(E,\nabla)$ lifts canonically to a class in

$$H^0(X, \Omega^{2n-1})/dH^0(X, \Omega^{2n-2}).$$

Indeed, one knows from [2] that for $U \subset X$ nonempty open, the restriction map $H^0(X, \mathcal{H}^{2n-1}) \to H^0(U, \mathcal{H}^{2n-1})$ is injective. The assertion about lifting follows from the construction of w_n in Section 2 because the trivialization can be taken globally.

4.1. The connection is now given by a matrix of 1-forms and so can be pulled back in many ways from some (nonintegrable) connection Ψ on the trivial bundle $\mathcal{E} \cong \mathcal{O}_{\mathbb{A}^p}^N$. We will want to assume Ψ "general" in a sense to be specified below. For convenience, write $T = \mathbb{A}^p$ and let $\varphi\colon X \to T$ be the map pulling back the connection. Let $\pi\colon P \to T$ be the flag bundle for $\mathcal{E}$ and let $Q = \varphi^* P$, so we get a diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\varphi} & T. \end{array} \tag{4.1.1}$$

4.2. The curvature $F(\Psi)$ defines an $\mathcal{O}_T$-linear map

$$F(\Psi)\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_T} \Omega_T^2. \tag{4.2.1}$$

In concrete terms, we take $p = 2N^2q$ for some large integer q, and we write $x_{ij}^{(k)}$ and $y_{ij}^{(k)}$ for $1 \leq i,j \leq N$ and $1 \leq k \leq q$ for the coordinates on $\mathbb{A}^p$. The connection Ψ then corresponds to an $N \times N$ matrix of 1-forms $A = (a_{ij})$, and the curvature is given by $F(\Psi) := (f_{ij}) = dA - A^2$. We take

$$a_{ij} = \sum_{\ell=1}^{q} x_{ij}^{(\ell)} dy_{ij}^{(\ell)}; \quad f_{ij} = da_{ij} - \sum_{m=1}^{N} a_{im} \wedge a_{mj}. \tag{4.2.2}$$

Notice that for q large, we can find $\varphi\colon X \to T$ so that $(\mathcal{E}, \Psi)$ pulls back to (E, ∇).

4.3. We want to argue universally by computing characteristic classes for $(\mathcal{E}, \Psi)$, but the curvature gets in the way. We could try to kill the curvature and look for classes in the quotient complex of Ω_T^* modulo the differential ideal generated by the f_{ij} (see 3.11), but this gratuitous violence seems to lead to difficulties. Instead, we will use the notion of τ-connection defined in [7] and [8] and work with a sheaf of differential algebras

$$M^* = \Omega_P^*/\mathcal{I} \tag{4.3.1}$$

on the flag bundle P.

Let

$$\mu\colon \Omega^1_{P/T} \xrightarrow{\iota} \pi^* Hom(\mathcal{E}, \mathcal{E}) \xrightarrow{\pi^* F(\Psi)} \pi^*\Omega_T^2 \tag{4.3.2}$$

be the composition, where ι is the standard inclusion on a flag bundle. An easy way to see ι is to consider the fibration $R \to P = R/B$, where R is the corresponding principal $G = GL(N)$ bundle and B is the Borel subgroup of upper triangular matrices, and to write the surjection $\mathcal{T}(R/T)/B \to \mathcal{T}(P/T)$ dual to ι, where $\mathcal{T}(A/B)$ is the relative tangent space of A with respect to B. There is an induced map of graded $\pi^*\Omega_T^*$-modules, and we define the graded algebra M^* to be the cokernel as indicated:

$$\Omega^1_{P/T} \otimes_{\mathcal{O}_P} \pi^*\Omega_T^*[-2] \xrightarrow{\mu \otimes 1} \pi^*\Omega_T^* \to M^* \to 0. \tag{4.3.3}$$

Note $M^0 = \mathcal{O}_P$ and $M^1 = \pi^*\Omega_T^1$.

PROPOSITION 4.4.1.

(i) *Associated to the connection Ψ on $\mathcal{E}$ there is an $\mathcal{O}_P$-linear splitting $\tau\colon \Omega_P^1 \twoheadrightarrow \pi^*\Omega_T^1$ of the natural inclusion $\pi^*\Omega_T^1 \xrightarrow{i} \Omega_P^1$. The resulting map $\delta := \tau \circ d\colon \mathcal{O}_P \to \pi^*\Omega_T^1$ is a derivation, which coincides with the exterior derivative on $\pi^{-1}\mathcal{O}_T \subset \mathcal{O}_P$. By extension, one defines*

$$\delta\colon \pi^*\Omega_T^n \to \pi^*\Omega_T^{n+1}; \quad \delta(f\pi^{-1}\omega) = f\pi^{-1}d\omega + \delta(f) \wedge \pi^{-1}\omega.$$

(ii) *One has*

$$\delta^2 = \mu \circ d_{P/T}\colon \mathcal{O}_P \xrightarrow{d_{P/T}} \Omega^1_{P/T} \to \pi^*\Omega^2_T,$$

where μ *is as in* (2).

(iii) *There is an induced map* $\delta\colon M^n \to M^{n+1}$ *making* M^* *a differential graded algebra. The quotient map* $\Omega^*_P \twoheadrightarrow \pi^*\Omega^*_T \twoheadrightarrow M^*$ *is a map of differential graded* $\mathcal{O}_P$*-algebras.*

Proof. We will give a somewhat different construction of M^* which we will show coincides with that defined by (4.3.3).

Let Y be a scheme, and let $\mathcal{F}$ be a vector bundle on Y. Let $\pi_1\colon P_1 := \mathbb{P}(\mathcal{F}) \to Y$. Let $\mathcal{I} \subset \Omega^*_Y$ be a differential graded ideal, and write $M^*_0 = \Omega^*_Y/\mathcal{I}$. (All our differential graded ideals will be trivial in degree 0, so $M^0_0 = \mathcal{O}_Y$.) Assume we are given an M_0-connection $\nabla\colon \mathcal{F} \to \mathcal{F} \otimes M^1_0$ in an obvious sense, that is a k linear map fulfilling the "Leibniz" rule $\nabla(\lambda f) = \delta(\lambda)f + \lambda\nabla(f)$ for $\lambda \in \mathcal{O}_Y, f \in \mathcal{F}$. Define $\mathcal{J} := \pi_1^{-1}\mathcal{I}\Omega^*_{P_1} \subset \Omega^*_{P_1}$, and let $\tilde{M}^* := \Omega^*_{P_1}/\mathcal{J}$. As a consequence of the Leibniz rule, the pullback $\pi_1^*\mathcal{F}$ has a $\tilde{M}^*$-connection $\tilde{\nabla}\colon \pi_1^*\mathcal{F} \to \pi_1^*\mathcal{F} \otimes \tilde{M}^1$.

We want to construct a quotient differential graded algebra $\tilde{M}^* \twoheadrightarrow M^*_1$ such that with respect to the quotient M_1-connection, the universal sequence

$$0 \to \Omega^1_{P_1/Y}(1) \xrightarrow{j} \pi_1^*\mathcal{F} \xrightarrow{q} \mathcal{O}_{P_1}(1) \to 0 \tag{4.4.1}$$

is horizontal. The composition

$$\Omega^1_{P_1/Y}(1) \xrightarrow{j} \pi_1^*\mathcal{F} \xrightarrow{\tilde{\nabla}} \pi_1^*\mathcal{F} \otimes \tilde{M}^1 \xrightarrow{q\otimes 1} \tilde{M}^1(1) \tag{4.4.2}$$

is easily checked to be $\mathcal{O}_{P_1}$-linear. Let $\mathcal{K}\tilde{K}_1 \subset \tilde{M}^1$ denote the image of the above map twisted by $\mathcal{O}_{P_1}(-1)$. Define $\tilde{\mathcal{K}}^* \subset \tilde{M}^*$ to be the graded ideal generated by $\tilde{\mathcal{K}}^1$ in degree 1 and $\delta\tilde{\mathcal{K}}^1$ in degree 2. Let $M^*_1 := \tilde{\mathcal{M}}^*/\tilde{\mathcal{K}}^*$. It is immediate that M^*_1 is a differential graded algebra, and that the subbundle $\Omega^1_{P_1/Y}(1) \subset \pi_1^*\mathcal{F}$ is horizontal for the quotient connection $\pi_1^*\mathcal{F} \to \pi_1^*\mathcal{F} \otimes M^1_1$.

Now let P denote the flag bundle for $\mathcal{F}$. Realize P as a tower of projective bundles

$$P = P_{N-1} \to \cdots \to P_2 \to P_1 \to Y$$

where P_i is the projective bundle on the tautological subbundle on P_{i-1}. Starting with an M_0-connection on $\mathcal{F}$ on Y, we can iterate the above construction to get a sheaf of differential graded algebras M^*_i on P_i, and an M_i-connection on $\mathcal{F} \mid P_i$ such that the tautological partial flag is horizontal. Let M'^* be the resulting sheaf of differential graded algebras on P.

Suppose $M_0^1 = \Omega_Y^1$. We will show by induction on i that $M_i^1 \cong \Omega_Y^1 \mid P_i$ in such a way that the surjection $\Omega_{P_i}^1 \twoheadrightarrow M_i^1$ splits the natural inclusion $\Omega_Y^1 \mid P_i \hookrightarrow \Omega_{P_i}^1$, or in other words that the kernel of the former is complementary to the image of the latter. This assertion is local on P_{i-1} (in fact, it is local on P_i), so we may assume $P_i = \mathbb{P}(\mathcal{G})$ where $\mathcal{G}$ is trivial on P_{i-1}. We can then lift the M_{i-1}-connection on $\mathcal{G}$ to an $\Omega_{P_{i-1}}^1$-connection. The analog of (4.4.2) is now

$$\Omega_{P_i/P_{i-1}}^1(1) \to \mathcal{G}|P_i \to \mathcal{G}|P_i \otimes \Omega_{P_i}^1 \to \Omega_{P_i}^1(1).$$

This composition twisted by $\mathcal{O}_{P_i}(-1)$ is shown in [7] (0.6.1) to be (up to sign) a splitting of $\Omega_{P_i}^1 \to \Omega_{P_i/P_{i-1}}^1$. In particular, its image is complementary to $\Omega_{P_{i-1}}^1 \mid P_i$. Factoring out $\Omega_{P_i}^1$ by the image of this map and by the pullback of the kernel of $\Omega_{P_{i-1}}^1 \twoheadrightarrow M_{i-1}^1 \cong \Omega_Y^1 \mid P_{i-1}$, it follows easily that $M_i^1 \cong \Omega_Y^1 \mid P_i$ as claimed.

To show M'^* as constructed here coincides with M^* from (4.3.3) we must prove for $Y = T$ and $\mathcal{F} = \mathcal{E}$ that $M^2 \cong M'^2$. We filter $\Omega_{P/T}^1$ so $\text{fil}_0 = (0)$ and $\text{gr}_i = \Omega_{P_i/P_{i-1}}^1 \mid P$. We will show by induction on i that with reference to (4.3.3) we have

$$\mu(\text{gr}_i\Omega_{P/T}^1) = \delta(\mathcal{K}_i^1) \subset (\Omega_T^2 \mid P)/\mu(\text{fil}_{i-1}\Omega_{P/T}^1) \tag{4.4.3}$$

where $\mathcal{K}_i^1$ is the image of $\Omega_{P_i/P_{i-1}}^1 \mid P$ in $\Omega_T^1 \mid P$ under the map analogous to (4.4.2). Suppose first $i = 1$. Let $e_0, e_1, \ldots$ be a basis of $\mathcal{E}$, and let t_i be the corresponding homogeneous coordinates on $P_1 = \mathbb{P}(\mathcal{E})$ so $q(e_i) = t_i$ in (4.4.1). The inclusion j: $\Omega_{P_1/T}^1(1) \hookrightarrow \pi_1^*\mathcal{E}$ is given by

$$t_0 d(t_i/t_0) \mapsto e_i - (t_i/t_0)e_0. \tag{4.4.4}$$

Consider the diagram

$$\begin{array}{ccccc} \Omega_{P_1/T}^1(1) & \twoheadrightarrow & \mathcal{K}^1(1) \subset \Omega_{P_1}^1(1) & \overset{a}{\dashrightarrow} & (\Omega_{P_1}^2/\mathcal{K}^1 \wedge \Omega^1)(1) \\ j\downarrow & & \uparrow & & \uparrow q \otimes 1 \\ \pi_1^*\mathcal{E} & \xrightarrow{\pi_1^*\Psi} & \pi_1^*\mathcal{E} \otimes \Omega_{P_1}^1 & \xrightarrow{\pi_1^*\Psi} & \pi_1^*\mathcal{E} \otimes \Omega_{P_1}^2 \end{array} \tag{4.4.5}$$

It is straightforward to check that $q \otimes 1 \circ \pi_1^*\Psi$ factorizes through $\Omega_{P_1}^1(1)$, thereby defining the dashed arrow a, and that for $\kappa \in \mathcal{K}^1$ we have $a(\kappa \otimes t_i) = d\kappa \otimes t_i \in (\Omega_{P_1}^2/\mathcal{K}^1 \wedge \Omega^1)(1)$. Thus $M'^2 = \Omega_{P_1}^2/(\mathcal{K}^1 \wedge \Omega^1 + d\mathcal{K}^1)$ is obtained by factoring out on the upper right of (4.4.5) by the image of the composition across the top twisted by $\mathcal{O}_{P_1}(-1)$. Note that the composition across the bottom is the curvature $F(\Psi)$. If we write (f_{ij}) for the curvature matrix with respect to the basis $e_0, e_1, \ldots$, we find using (4.4.4) that, e.g., on the open set $t_0 \neq 0$, $d\mathcal{K}^1$ is generated by elements

$$t_0^{-1}(q\otimes 1)F(\Psi)(e_j-(t_j/t_0)e_0) \\ =\sum_i f_{ij}(t_i/t_0)-(t_j/t_0)\sum_k f_{k0}(t_k/t_0). \tag{4.4.6}$$

On the other hand, the map ι in (4.3.2) is given by

$$\begin{gathered} \mathcal{O}_{P_1}(-1)\hookrightarrow \pi_1^*\mathcal{E}^\vee;\ t_i^{-1}\mapsto \textstyle\sum_j (t_j/t_i)e_j^\vee \\ \Omega^1_{P_1/T}\hookrightarrow \pi_1^*\mathcal{E}(-1)\hookrightarrow \pi_1^*\mathcal{E}\otimes\pi_1^*\mathcal{E}^\vee \\ d(t_j/t_0)\mapsto (e_j-(t_j/t_0)e_0)\otimes \textstyle\sum_i (t_i/t_0)e_i^\vee. \end{gathered} \tag{4.4.7}$$

The map μ from (4.3.2) is given by $\mu(e_i^\vee\otimes e_j)=f_{ij}$ hence by (4.4.7) we get

$$\mu(d(t_j/t_0))=\sum_i (t_i/t_0)f_{ij}-\sum_i (t_it_j/t_0^2)f_{i0}. \tag{4.4.8}$$

Comparing (4.4.6) and (4.4.8), we conclude that (4.4.3) holds for $i=1$. The inductive step is precisely the same. We have $P_{i+1}=\mathbb{P}(\mathcal{G}_i)$ for some subbundle $\mathcal{G}_i\subset\mathcal{G}_{i-1}\mid P_i$. The question is local, so we may assume $\mathcal{G}_i$ is free. We assume inductively that G_{i-1} has a $M_{i-1}=\Omega^*_{P_{i-1}}/\mathcal{I}^*_{i-1}$-connection. Define $\tilde{M_{i-1}}=\Omega^*_{P_i}/\mathcal{I}^*_{i-1}\cdot\Omega^*_{P_i}$, so $\mathcal{G}_{i-1}\mid P_i$ has a $\tilde{M}_{i-1}$-connection. One factors out by the image $\mathcal{K}_i^1$ of $\Omega^1_{P_i/P_{i-1}}$ as in (4.4.2) to define M_i^1 and the writes down a diagram like (4.4.5) to compare $d\mathcal{K}_i^1$ with the image of μ as in (4.3.2). At this point it is good to remark that the curvature F_τ: $\mathcal{O}_Q(1)\to\pi^*\Omega^1_T\otimes\mathcal{O}_Q(1)\to M^2\otimes\mathcal{O}_Q(1)$ does not vanish. For example, for $N=2$, one has $F_\tau(t_0)=(f_{00}+f_{01}(t_1/t_0))t_0$.

The remaining assertions in Proposition 4.4.1 are easily verified. □

Proposition 4.5.1. *We have $\mathbb{R}\pi_*M^i\cong\pi_*M^i$ for $i<q$. The complex $H^0(T,\pi_*M^*)$ has no cohomology in odd degrees $<q$. For $2n<q$, the map*

$$\delta\colon \mathbb{H}^{n-1}(P,M^n\to\cdots\to M^{2n-1})\to H^0(P,M^{2n})$$

is injective.

4.6. We postpone the proof of Proposition 4.5.1 for a while in order to finish the proof of Theorem 4.0.1. Note first that since the curvature of the original bundle E on X is zero, the construction of Proposition 4.4.1 above applied to E and the flag bundle Q yields a structure of differential graded algebra on $\pi^*\Omega^*_X$, and we have (from (4.1.1)) a pullback map of complexes of sheaves on P

$$\varphi^*\colon\ M^* \to R\varphi_*\pi^*\Omega_X^* \tag{4.6.1}$$

coming from $\varphi^* M^i \to \pi^*\Omega_X^i$.

We will construct classes $\tilde{c}$ and $\tilde{w}$ in $\mathbb{H}^{n-1}(P, M^n \to \cdots \to M^{2n-1})$ such that with reference to the maps

(4.6.2)

$$\begin{array}{ccc}
\mathbb{H}^{n-1}(Q, \pi^*\Omega_X^n \to \cdots \to \pi^*\Omega_X^{2n-1}) & \xleftarrow[\cong]{\ \gamma\ } & H^0(X, \Omega_X^{2n-1})/dH^0(X, \Omega_X^{2n-2}) \\
\alpha\big\downarrow & & \\
\mathbb{H}^n(Q, \mathcal{K}_n^m \to \pi^*\Omega_X^n \to \cdots \to \pi^*\Omega_X^{2n-1}) & & \\
\beta\big\uparrow & & \\
\mathbb{H}^n(Q, \mathcal{K}_n^m \to \pi^*\Omega_X^{\geq n}) & &
\end{array}$$

we have

$$\begin{aligned} \beta\pi^*(c_n(E, \nabla)) &= \alpha\varphi^*\tilde{c} \\ w_n(E, \nabla) &= \gamma^{-1}\varphi^*\tilde{w}. \end{aligned} \tag{4.6.3}$$

(Note that to avoid confusion between $H^0(\Omega_X^{2n-1}/d\Omega_X^{2n-2})$ and $H^0(\Omega_X^{2n-1})/dH^0(\Omega_X^{2n-2})$, it is a good idea here to localize more and replace X by its function field $\mathrm{Spec}\,(k(X))$. Note also that β is always injective, and that γ is an isomorphism because X is affine.)

We then show

$$\delta\tilde{c} = \delta\tilde{w} \in H^0(P, M^{2n}), \tag{4.6.4}$$

whence, by Proposition 4.5.1 we have $\tilde{c} = \tilde{w}$. Now consider the analogue of (4.6.2) down on X, with $\pi^*\Omega$ replaced by Ω. Write $\alpha_X, \beta_X, \gamma_X$ for the corresponding maps. The assertion of Theorem 4.0.1 is

$$\gamma_X(w_n(E, \nabla)) = \beta_X(c_n(E, \nabla)). \tag{4.6.5}$$

It follows from (4.6.2) and evident functoriality of π^* that (4.6.5) holds after pullback by π^*. Theorem 4.0.1 then follows from

LEMMA 4.6.1. *The pullback*

$$\begin{aligned} \pi^*\colon\ &\mathbb{H}^n(X, \mathcal{K}_n^m \to \Omega_X^n \to \cdots \to \Omega_X^{2n-1}) \\ &\to \mathbb{H}^n(Q, \mathcal{K}_n^m \to \pi^*\Omega_X^n \to \cdots \to \pi^*\Omega_X^{2n-1}) \end{aligned}$$

is injective.

Proof of lemma. This is central to the splitting principle involved in the construction of characteristic classes in [8]. (See the argument on p. 52 in the proof of Theorem 1.7 of (op. cit.).) An evident diagram chase involving cohomology of the $\mathcal{K}$-sheaves and the sheaves Ω and $\pi^*\Omega$ reduces one to showing the $\mathcal{K}$-cohomology groups $H^{n-1}(X, \mathcal{K}_n^m)$ and $H^{n-1}(Q, \mathcal{K}_n^m)$ have the same image in

$$\mathbb{H}^{n-1}(Q, \pi^*\Omega_X^n \to \cdots \to \pi^*\Omega_X^{2n-1}).$$

This follows because the multiplication map

$$H^{n-2}(Q, \mathcal{K}_{n-1}^m) \otimes H^1(Q, \mathcal{K}_1) \to H^{n-1}(Q, \mathcal{K}_n^m)/\pi^* H^{n-1}(X, \mathcal{K}_n^m)$$

is surjective. The line bundles on Q have integrable τ-connections in the sense of [8], so their classes in $\mathbb{H}^1(Q, \pi^*\Omega_X^1 \to \cdots)$ vanish. □

4.7. We turn now to the construction of the classes $\tilde{c}$ and $\tilde{w}$. One has

$$\begin{aligned} w(\mathcal{E}, \Psi, P_n) &\in H^0(\Omega_T^{2n-1})/dH^0(\Omega_T^{2n-2}) \\ &\cong \mathbb{H}^{n-1}(T, \Omega_T^n \to \cdots \to \Omega_T^{2n-1}). \end{aligned}$$

We define $\tilde{w}$ by the natural pullback

(4.7.1) $$\tilde{w} = \pi^* w(\mathcal{E}, \Psi, P_n) \in \mathbb{H}^{n-1}(P, M^n \to \cdots \to M^{2n-1}).$$

It follows that

(4.7.2) $$\delta(\tilde{w}) = \pi^*(dw(\mathcal{E}, \Psi, P_n)) = \pi^*(P_n(F(\Psi))) \in H^0(P, M^{2n}).$$

To construct $\tilde{c}$, we remark first that the map

$$\mathbb{H}^{n-1}(P, M^n \to \cdots \to M^{2n-1}) \to \mathbb{H}^n(P, \mathcal{K}_n^m \to M^n \to \cdots \to M^{2n-1})$$

is injective, so it suffices to construct

(4.7.3) $$\tilde{c} \in \ker\left(\mathbb{H}^n(P, \mathcal{K}_n^m \to M^n \to \cdots \to M^{2n-1}) \to H^n(P, \mathcal{K}_n^m)\right).$$

This injectivity follows either from Proposition 4.5.1 or, in order to avoid the long proof of that result, from the structure of $H^{n-1}(P, \mathcal{K}_n^m)$, given that P is a flag bundle over affine space. In fact the construction of $\tilde{c}$ as in (4.7.3) would suffice for our purposes anyway, so we won't give the argument in detail.

Let ℓ_i be the rank one subquotients of $\pi^*\mathcal{E}$. A basic result from [7] is that $\pi^*\mathcal{E}$ admits a "connection" with values in M^*,

$$\pi^*\mathcal{E} \to \pi^*\mathcal{E} \otimes_{\mathcal{O}_P} M^1 \tag{4.7.4}$$

and that the filtration defining the ℓ_i is horizontal for this "connection." Thus there exist local transition functions $f^i_{\alpha,\beta}$ and local connection forms $\omega^i_\alpha \in M^1$ verifying

$$d\log f^i = \partial\omega^i, \tag{4.7.5}$$

and thus defining $\ell_i \in \mathbb{H}^1(P, \mathcal{K}_1 \to M^1)$. Here ∂ is the Cech differential. Then $\tilde{c}$ is defined by the cocyle

$$(x', x^n, \ldots, x^{2n-1}) \in (\mathcal{C}^n(\mathcal{K}_n) \times \mathcal{C}^{n-1}(M^n) \ldots \times \mathcal{C}^0(M^{2n-1}))_{d-\partial} \tag{4.7.6}$$

with

$$\begin{aligned} x' &= \sum_{i_1<\cdots<i_n} f^{i_1} \cup \cdots \cup f^{i_n} \\ x^n &= \sum_{i_1<\cdots<i_n} \omega^{i_1} \wedge \partial\omega^{i_2} \wedge \cdots \wedge \partial\omega^{i_n} \\ x^{n+1} &= \sum_{i_1<\cdots<i_n} \delta\omega^{i_1} \wedge \omega^{i_2} \wedge \partial\omega^{i_3} \wedge \cdots \wedge \partial\omega^{i_n} \\ &\cdots \\ x^{2n-1} &= \sum_{i_1<\cdots<i_n} \delta\omega^{i_1} \wedge \cdots \wedge \delta\omega^{i_{n-1}} \wedge \omega^{i_n}. \end{aligned} \tag{4.7.7}$$

The cup products "$\cup$" here are Cech products. By definition ([8], Theorem 1.7, p. 51), one has $\beta\pi^*(c_n(E, \nabla) = \varphi^*\tilde{c}$. Applying δ to the last equation, it follows that the image of $\tilde{c}$ in $H^0(P, M^{2n})$ is

$$\sum_{i_1<\cdots<i_n} F(\ell_{i_1}) \wedge \cdots \wedge F(\ell_{i_n}). \tag{4.7.8}$$

This is exactly $P_n(F(\oplus \ell_i)) = \pi^*P_n(F(\Psi))$. (As M^* is a quotient complex of Ω^*_P by Proposition 4.4.1, (iii), invariance for P_n guarantees independence of the choice of local bases for $\pi^*\mathcal{E}$.) Comparing this with (4.7.2) we conclude $\delta\tilde{c} = \delta\tilde{w}$ so (4.6.4) holds.

4.8. We turn now to proof of Proposition 4.5.1.

PROPOSITION 4.8.1. *The Koszul complex associated to* (4.3.3)

$$\cdots \to \Omega^2_{P/T} \otimes_{\mathcal{O}_P} \pi^*\Omega^*_T[-4] \to \Omega^1_{P/T} \otimes_{\mathcal{O}_P} \pi^*\Omega^*_T[-2] \xrightarrow{\mu\otimes 1} \pi^*\Omega^*_T \to M^* \to 0 \tag{4.8.1}$$

is exact in degrees $< q$.

To clarify and simplify the argument, we will use commutative algebra. Let B be a commutative ring. Let C be a commutative, graded B-algebra, and let S be a graded C-module. Let Z be a finitely generated free B-module with generators ϵ_α, and let $\nu\colon Z \to C_2$, with $\nu(\epsilon_\alpha) = f_\alpha$. Let $\mathcal{I} \subset C$ be the ideal generated by the f_α. Write $\mathrm{gr}_{\mathcal{I}}(S) := \oplus \mathcal{I}^n S/\mathcal{I}^{n+1}S$. Note $\mathrm{gr}_{\mathcal{I}}(S)$ is a graded module for the symmetric algebra $B[Z]$ (with Z in degree 2). The dictionary we have in mind is

$$\begin{aligned} B &= \Gamma(T, \mathcal{O}_T) \\ C &= \Omega_T^{\mathrm{even}} \subset S = \Omega^*_T \\ Z &= Hom(\mathcal{E}, \mathcal{E}) \\ f_\alpha = f_{ij} &= \text{ entries of curvature matrix.} \end{aligned} \tag{4.8.2}$$

LEMMA 4.8.2. *Let* $d \geq 2$ *be given. The following are equivalent.*

(i) *The evident map*

$$\rho\colon (S/\mathcal{I}S)[Z] := (S/\mathcal{I}S) \otimes_B B[Z] \to \mathrm{gr}_{\mathcal{I}}(S)$$

is an isomorphism in degrees $\leq d$.

(ii) *For all* α, *the multiplication map*

$$f_\alpha\colon S/(f_1, \ldots, f_{\alpha-1})S \to S/(f_1, \ldots, f_{\alpha-1})S \tag{4.8.3}$$

is injective in degrees $\leq d$.

Proof. This amounts to redoing the argument in Chapter 0, §(15.1.1)–(15.1.9) of [11] in a graded situation, where the hypotheses and conclusions are asserted to hold only in degrees $\leq d$. The argument may be sketched as follows.

Step 1. Suppose $\alpha = 1$, and write $f = f_1$. Let $\mathrm{gr}(S) = \oplus f^n S/f^{n+1}S$. Suppose the kernel of multiplication by f on S is contained in degrees $> d$. Then the natural map $\varphi\colon (S/fS)[T] \to \mathrm{gr}(S)$ is an isomorphism in degrees $\leq d$. Here, of course, T is given degree = degree(f) = 2.

Indeed, φ is always surjective, and injectivity in degrees $\leq d$ amounts to the assertion that for $x \in S$ of degree $\leq d - 2k$ with $f^k x = f^{k+1} y$, we have $x = fy$. This is clear.

Step 2. Suppose now the condition in (ii) holds. We prove (i) by induction on α. We may assume by Step 1 that $\alpha > 1$. Let $\mathcal{J}$ (resp. $\mathcal{I}$) be the ideal generated by $f_1, \dots, f_{\alpha-1}$ (resp. $f_1, \dots, f_\alpha$). Write $\mathrm{gr}_{\mathcal{J}}(S) = \oplus J^n S/J^{n+1}S$. By induction, we may assume

$$S/\mathcal{J}S[T_1, \dots, T_{\alpha-1}] \to \mathrm{gr}_{\mathcal{J}}(S) \tag{4.8.4}$$

is an isomorphism in degrees $\leq d$. We have to show the same for

$$\psi\colon\ \mathrm{gr}_{\mathcal{J}}(S)/f_\alpha \mathrm{gr}_{\mathcal{J}}(S)[T_\alpha] \to \mathrm{gr}_{\mathcal{I}}(S). \tag{4.8.5}$$

By (4.8.4) we have that multiplication by f_α on $\mathrm{gr}_{\mathcal{J}}(S)$ is injective in degrees $\leq d$ (where the degree grading comes from S, not the $\mathrm{gr}_{\mathcal{J}}$ grading). An easy argument shows the multiplication map

$$f_\alpha\colon\ S/\mathcal{J}^r S \hookrightarrow S/\mathcal{J}^r S \tag{4.8.6}$$

is injective in degrees $\leq d$ for all r. Define

$$(Q_k)_i = \sum_{j \leq k-i} (\mathrm{gr}^{k-j}_{\mathcal{J}}(S)/f_\alpha \mathrm{gr}^{k-j}_{\mathcal{J}}(S))T^j \tag{4.8.7}$$

$$(Q_k)_0 = Q_k \qquad (Q_k)_{k+1} = (0)$$

$$\mathrm{gr}^i(Q_k) = (\mathrm{gr}^{k-i}_{\mathcal{J}}(S)/f_\alpha \mathrm{gr}^{k-i}_{\mathcal{J}}(S))T^i.$$

Define

$$Q'_k = \psi(Q_k) \qquad (Q'_k)_i = \psi((Q_k)_i) \qquad \mathrm{gr}^i(Q'_k) = (Q'_k)_i/(Q'_k)_{i+1}.$$

The map ψ is surjective, so it will suffice to show the maps

$$\mathrm{gr}^i(Q_k) \to \mathrm{gr}^i(Q'_k) \tag{4.8.8}$$

are injective in (S)-degrees $\leq d$. The left-hand side is

$$\mathcal{J}^i S/\left(f_\alpha \mathcal{J}^i S + \mathcal{J}^{i+1}S\right) T^{k-i}.$$

The right-hand side of (8) is the image of

$$\mathcal{J}^k S + f_\alpha \mathcal{J}^{k-1}S + \cdots + f_\alpha^{k-i-1}\mathcal{J}^{i+1}S$$

in $\mathcal{I}^k S/\mathcal{I}^{k+1}S$. What we have to show is that for $x \in \mathcal{J}^i S$ of degree $\leq d - 2(k-i)$, the inclusion

$$f_\alpha^{k-i}x \in \mathcal{J}^k S + f_\alpha \mathcal{J}^{k-1} + \cdots + f_\alpha^{k-i-1}\mathcal{J}^{i+1} + \mathcal{I}^{k+1}S \tag{4.8.9}$$

implies $x \in f_\alpha \mathcal{J}^i S + \mathcal{J}^{i+1}S$. The right side of (4.8.9) is contained in $\mathcal{J}^{i+1}S + \mathcal{I}^{k+1}S \subset \mathcal{J}^{i+1}S + f_\alpha^{k+1-i}S$. Multiplication by f_α on $S/\mathcal{J}^{i+1}S$ is injective in degrees $\leq d$ by (6), so $f_\alpha^{k-i}x \in f_\alpha^{k+1-i}S + \mathcal{J}^{i+1}S$ implies there exists $y \in S$ such that $x - f_\alpha y \in \mathcal{J}^{i+1}S$. Since $x \in \mathcal{J}^i S$, we have $f_\alpha y \in \mathcal{J}^i S$ whence by (4.8.6) again, $y \in \mathcal{J}^i S$ so $x \in f_\alpha \mathcal{J}^i S + \mathcal{J}^{i+1}S$. This completes the verification of Step 2.

Step 3. It remains to show (i) $\Rightarrow$ (ii). Again we argue by induction on α. Suppose first $\alpha = 1$. Given $x \in S$ nonzero of degree $\leq d$ such that $f_1 x = 0$, it would follow from (i) that $x \in f_1^N S$ for all N, which is ridiculous by reason of degree. Now suppose $\alpha \geq 2$ and that (i) implies (ii) for $\alpha - 1$. By assumption the map

$$S/\mathcal{I}S[T_1, \ldots, T_\alpha] \to \mathrm{gr}_{\mathcal{I}}(S) \tag{4.8.10}$$

is an isomorphism in degrees $\leq d$. In particular, multiplication by f_1 is injective in degrees $\leq d$ on $\mathrm{gr}_{\mathcal{I}}S$. Arguing as above, an $x \in S$ of degree $\leq d$ such that $f_1 x = 0$ would lie in $\mathcal{I}^N S$ for all N, a contradiction. Thus the first step in (ii) holds. To finish the argument, we may factor out by f_1, writing $\bar{S} = S/f_1 S$. Let $\mathcal{K}$ be the ideal generated by $f_2, \ldots, f_\alpha$. Factoring out by T_1 on both sides of (4.8.10) yields

$$\bar{S}/\mathcal{K}\bar{S}[T_2, \ldots, T_\alpha] \to \mathrm{gr}_{\mathcal{K}}(\bar{S})$$

injective in degrees $\leq d$. We conclude by induction that (ii) holds for $\bar{S}$. □

Continuing the dictionary from (4.8.2) above, the ring R and the module W in the lemma below correspond to the ring of functions on some affine in P and the module of 1-forms $\Omega^1_{P/T} \subset \pi^* \mathit{Hom}(\mathcal{E}, \mathcal{E})$.

LEMMA 4.8.3. *Let notation be as above, and assume ν: $Z \to C_2$ satisfies the equivalent conditions of Lemma* 4.8.2. *Let R be a flat B-algebra, and let $W \subset Z \otimes_B R$ be a free, split R-submodule with basis g_β. Then the multiplication maps*

$$g_\beta\colon\ S \otimes_B R/(g_1, \ldots, g_{\beta-1})S \otimes_B R \to S \otimes_B R/(g_1, \ldots, g_{\beta-1})S \otimes_B R$$

are injective in degrees $\leq d$.

Proof. Assume not. We can localize at some prime of R contained in the support for some element in the kernel of multiplication by g_β and reduce to the case R local. Then we may extend $\{g_\beta\}$ to a basis of $Z \otimes_B R$ and use the implication (i) $\Longrightarrow$ (ii) from Lemma 4.8.2. Note that $\nu \otimes 1$: $Z \otimes R \to C_2 \otimes R$ satisfies (i) by flatness. □

LEMMA 4.8.4. *With notations as above, assume Z satisfies the conditions of Lemma* 4.8.2 *for some $d \geq 2$. Let $J \subset R \otimes_B C$ be the ideal generated by* $(1 \otimes \nu)(W)$.

Then the Koszul complex

$$\cdots \to \wedge^2 W \otimes_R (R \otimes_B S) \to W \otimes_R (R \otimes_B S) \\ \to R \otimes_B S \to (R \otimes_B S)/J \to 0$$

is exact in degrees $\leq d$.

Proof. To simplify notation, let $A = R \otimes_B C,\ M = R \otimes_B S,\ V = W \otimes_R C$, so the Koszul complex becomes

$$\cdots \wedge^2 V \otimes_A M \to V \otimes_A M \to M \to M/JM \to 0.$$

We argue by induction on the rank of V. If this rank is 1, the assertion is that the sequence

$$0 \to M \xrightarrow{g_1} M \to M/g_1 M \to 0$$

is exact in degrees $\leq d$, which follows from Lemma 4.8.3. In general, if V has an A-basis $g_1, \ldots, g_\beta$, let V' be the span of $g_1, \ldots, g_{\beta-1}$. By induction, the Koszul complex

$$\cdots \wedge^2 V' \otimes M \to V' \otimes M \to M$$

is a resolution of $M/(g_1, \ldots, g_{\beta-1})M$ in degrees $\leq d$. If we tensor this module with the two-term complex $A \xrightarrow{g_\beta} A$ we obtain a complex which by Lemma 4.8.3 is quasi-isomorphic to $M/(g_1, \ldots, g_\beta)M$ in degrees $\leq d$. On the other hand, this complex is quasi-isomorphic to the complex obtained by tensoring $A \xrightarrow{g_\beta} A$ with the above V'-Koszul complex, and this tensor product is identified with the V-Koszul complex. ◻

For our application, $B = \mathbb{C}[x_{ij}^{(k)}, y_{ij}^{(k)}]$ is the polynomial ring in two sets of variables, with $1 \leq i, j \leq N = \dim(E)$ and $1 \leq k \leq q$ for some large integer q. Let Ω be the free B-module on symbols $dx_{ij}^{(k)}$ and $dy_{ij}^{(k)}$. Let $S = \bigwedge_B \Omega$, graded in the obvious way with dx and dy having degree 1, and let $C = S_{\mathrm{even}}$ be the elements of even degree. Define

$$a_{ij} = \sum_{\ell=1}^{q} x_{ij}^{(\ell)} dy_{ij}^{(\ell)}; \quad f_{ij} = da_{ij} - \sum_{m=1}^{N} a_{im} \wedge a_{mj}. \tag{4.8.11}$$

We have

$$f_{ij} = \sum_{\ell=1}^{q} dx_{ij}^{(\ell)} dy_{ij}^{(\ell)} - \sum_{m,\ell,p} x_{im}^{(\ell)} x_{mj}^{(p)} dy_{im}^{(\ell)} \wedge dy_{mj}^{(p)}. \tag{4.8.12}$$

Now give S and C a second grading according to the number of dx's in a monomial. We denote this grading by $z = \sum z(j)$. For example, $f_{ij} = f_{ij}(1) + f_{ij}(0)$ with $f_{ij}(1) = \sum_k dx_{ij}^{(k)} \wedge dy_{ij}^{(k)}$. Let Z be the free B-module on symbols ϵ_{ij}, with $1 \le i, j \le N$. We consider maps

$$\mu, \mu(1)\colon\ Z \to C_2;\ \ \mu(\epsilon_{ij}) = f_{ij};\ \ \mu(1)(\epsilon_{ij}) = f_{ij}(1).$$

LEMMA 4.8.5. *Suppose the map $\mu(1)$ above satisfies the conditions of Lemma* 4.8.2 *above for some $d \ge 2$. Then so does μ.*

Proof. Suppose

$$f_\alpha \ell_\alpha = \sum_{1 \le \beta \le \alpha - 1} f_\beta \ell_\beta$$

with the ℓ_β homogeneous of some degree $< d$. Write

$$\ell_\beta = \sum_{0 \le j \le r} \ell_\beta(j); \quad 1 \le \beta \le \alpha$$

such that $\ell_\beta(r) \ne 0$ for some β. We have

$$f_\alpha(1)\ell_\alpha(r) = \sum_{1 \le \beta \le \alpha - 1} f_\beta(1)\ell_\beta(r). \tag{4.8.13}$$

We want to show ℓ_α belongs to the submodule generated by $f_1, \ldots, f_{\alpha-1}$, and we will argue by double induction on r and on the set

$$\mathcal{A} = \{\beta \le \alpha \mid \ell_\beta(r) \ne 0\}.$$

If $r = 0$ and $\ell_\alpha \ne 0$, we get a contradiction from (1), since we have assumed the ℓ_β have degree $< d$, and $\ell_\alpha(0)$ cannot lie in the ideal generated by the $f_\beta(1)$. Assume now $r \ge 1$.

Case 1. Suppose $\ell_\alpha(r) \ne 0$. From the above, we conclude that we can write

$$\ell_\alpha(r) = \sum_{\beta \in \mathcal{A}, \beta \ne \alpha} m_\beta(r-1) f_\beta(1).$$

Define

$$\begin{aligned} \ell'_\alpha &= \ell_\alpha - \sum_{\beta \in \mathcal{A}, \beta \ne \alpha} m_\beta(r-1) f_\beta \\ \ell'_\beta &= \ell_\beta - m_\beta(r-1) f_\alpha\ ; \quad \beta \in \mathcal{A},\ \beta \ne \alpha. \end{aligned} \tag{4.8.14}$$

We still have (taking $\ell'_\beta = \ell_\beta$ for $\beta \notin \mathcal{A}$)

$$f_\alpha \ell'_\alpha = \sum_{1 \le \beta \le \alpha - 1} f_\beta \ell'_\beta. \tag{4.8.15}$$

Since $\ell'_\beta(r) = \ell_\beta(r) = 0$ for $\beta \notin \mathcal{A}$, $\ell'_\alpha(r) = 0$, and $\ell'_\beta(s) = 0$ for $s > r$ and all β; the inductive hypothesis says ℓ'_α lies in the ideal generated by the f_β for $\beta < \alpha$. It follows from (4.3.2) that ℓ_α lies in this ideal also.

Case 2. $\ell_\alpha(r) = 0$. Choose $\gamma \in \mathcal{A}$. We have

$$\sum_{\beta \in \mathcal{A}} f_\beta(1)\ell_\beta(r) = 0.$$

Since the $f_\beta(1)$ are assumed to satisfy the equivalent hypotheses of Lemma 4.6.1, we can write

$$\ell_\gamma(r) = \sum_{\beta \in \mathcal{A}, \beta \neq \gamma} m_\beta(r-1) f_\beta(1).$$

As in (4.3.2), we write

$$\begin{aligned} \ell'_\gamma &= \ell_\gamma - \sum_{\beta \in \mathcal{A}, \beta \neq \gamma} m_\beta(r-1) f_\beta \\ \ell'_\beta &= \ell_\beta + m_\beta(r-1) f_\gamma \ ; \quad \beta \in \mathcal{A},\ \beta \neq \gamma. \end{aligned} \tag{4.8.16}$$

Again, taking $\ell'_\beta = \ell_\beta$ for $\beta \notin \mathcal{A}$, we get (4.3.3), so we may conclude by induction.
□

LEMMA 4.8.6. *The map* $\mu(1)$ *defined by*

$$\mu(1)(\epsilon_{ij}) = f_{ij}(1) = \sum_{\ell=1}^{q} dx_{ij}^{(\ell)} \wedge dy_{ij}^{(\ell)} \tag{4.8.17}$$

satisfies the hypotheses of Lemma 4.6.1 *with* $d = q - 1$.

Proof. Let V_{ij} be the $\mathbb{C}$-vector space of dimension $2q$ with basis the $dx_{ij}^{(\ell)}$ and the $dy_{ij}^{(\ell)}$. Write $V = \oplus V_{ij}$. We have

$$S = \bigwedge V \otimes B \cong \otimes_{i,j} \left(\bigwedge V_{ij} \right) \otimes B.$$

It is convenient to well-order the pairs ij, writing $f_\alpha(1) = f_{ij}(1) \in \bigwedge V_\alpha$. We have

$$S/(f_1, \dots, f_{\alpha-1})S \cong \bigotimes_{\beta \geq \alpha} \Big(\bigwedge V_\beta\Big) \otimes \bigotimes_{\beta < \alpha} \Big(\Big(\bigwedge V_\beta\Big) / \Big(f_\beta(1) \bigwedge V_\beta\Big)\Big) \otimes B.$$

It is clear from this that multiplication by $f_\alpha(1)$ will be injective in a given degree d if the multiplication map $f_\alpha(1)\colon \bigwedge V_\alpha \to \bigwedge V_\alpha$ is injective in degrees $\leq d$. It is clear from the shape of $f_\alpha(1)$ in (2) that multiplication by $f_\alpha(1)$ will be injective in degrees $\leq q-1$. □

This completes the proof of Proposition 4.8.1 above.

PROPOSITION 4.9.1. *We have for $i < q$*

$$\pi_* M^0 = \mathcal{O}_T; \quad \pi_* M^1 = \Omega^1_T; \quad R^j \pi_* M^i = (0);\ j \geq 1. \tag{4.9.1}$$

The sheaf $\pi_* M^i$ admits an increasing filtration $\mathrm{fil}_\ell(\pi_* M^i)$, $\ell \geq 0$ which is stable under δ and satisfies

$$\mathrm{gr}_j(\pi_* M^i) \cong H^j(P, \Omega^j_{P/T}) \otimes \Omega^{i-2j}_T \cong CH^j(P) \otimes_{\mathbb{Z}} \Omega^{i-2j}_T \tag{4.9.2}$$

for $j \geq 0$. Here $CH^j(P)$ is the Chow group of codimension j algebraic cycles on P. The differential $\mathrm{gr}_j(\pi_* M^i) \to \mathrm{gr}_j(\pi_* M^{i+1})$ is the identity on the Chow group tensored with the exterior derivative on Ω^*_T up to sign.

Note that the last assertion in (1) implies for $i < q$

$$H^*(P, M^i) \cong \begin{cases} H^0(T, \pi_* M^i) & * = 0 \\ 0 & * \geq 1. \end{cases}$$

It follows from (4.9.2) that the complex $H^0(T, M^*)$ has no cohomology in odd degrees $< q-1$. (Recall that T has no higher de Rham cohomology.) These assertions imply Proposition 4.5.1.

Proof of Proposition 4.9.1. The first two assertions in (1) are clear, because $M^0 = \mathcal{O}_P$ and $M^1 = \pi^* \Omega^1_T$. We define

$$G_j = \mathrm{Im} \Omega^j_{P/T} \otimes \pi^* \Omega^{i-2j}_T \to \Omega^{j-1}_{P/T} \otimes \pi^* \Omega^{i-2j+2}_T,\ G_0 = M^i$$

coming from the resolution of M^i in (4.8.1). Then $R^a\pi_*G_j = 0$ for $a \neq j$. This proves $R^j\pi_*M^i = 0$ for $j \geq 1$. One has a short exact sequence

$$0 \to R^j\pi_*\Omega^j_{P/T} \otimes \Omega_T^{i-2j} \to R^j\pi_*G_j \to R^{j+1}\pi_*G_{j+1} \to 0$$

with $R^0\pi_*G_0 = \pi_*M^i$. One defines

$$\begin{aligned} \mathrm{fil}_j(\pi_*M^i) = \text{inverse image of } R^j\pi_*\Omega^j_{P/T} \otimes \Omega_T^{i-2j} \\ \text{via } R^0\pi_*G_0 \to R^j\pi_*G_j. \end{aligned}$$

This proves (4.9.2).

In order to understand the map $\mathrm{gr}_j(\pi_*M^i) \to \mathrm{gr}_j(\pi_*M^{i+1})$, we construct a commutative diagram

$$\text{(4.9.3)}\qquad \begin{array}{ccccccc} \Omega^2_{P/T} \otimes \pi^*\Omega_T^{i-4} & \xrightarrow{\mu\otimes 1} & \Omega^1_{P/T} \otimes \pi^*\Omega_T^{i-2} & \xrightarrow{\mu\otimes 1} & \pi^*\Omega_T^{i} & \xrightarrow{\mu\otimes 1} & M^i \\ \downarrow \nabla_\tau & & \downarrow \nabla_\tau & & \downarrow \nabla_\tau & & \\ \Omega^2_{P/T} \otimes \pi^*\Omega_T^{i-3} & \xrightarrow{\mu\otimes 1} & \Omega^1_{P/T} \otimes \pi^*\Omega_T^{i-1} & \xrightarrow{\mu\otimes 1} & \pi^*\Omega_T^{i+1} & \xrightarrow{\mu\otimes 1} & M^{i+1} \end{array}$$

mapping the resolution of M^i to the resolution of M^{i+1} given by (4.8.1). To this aim recall that one has an exact sequence of complexes

$$\text{(4.9.4)}\qquad 0 \to K^* \to \Omega_P^* \to M^* \to 0$$

with

$$\text{(4.9.5)}\qquad \begin{aligned} K^i = {} & \Omega^i_{P/T} \oplus \Omega^{i-1}_{P/T} \otimes \pi^*\Omega^1_T \cdots \oplus \Omega^1_{P/T} \otimes \pi^*\Omega_T^{i-1} \\ & \oplus \mu(\Omega^1_{P/T}) \wedge \pi^*\Omega_T^{i-2}. \end{aligned}$$

Note that the differential $K^{i-j-1} \to K^{i-j}$ acts as follows

$$\text{(4.9.6)}\qquad \begin{aligned} \Omega^j_{P/T} \otimes \pi^*\Omega_T^{i-1-2j} \to {} & \Omega^{j+1}_{P/T} \otimes \pi^*\Omega_T^{i-1-2j} \oplus \Omega^j_{P/T} \\ & \otimes \pi^*\Omega_T^{i-2j} \oplus \Omega^{j-1}_{P/T} \otimes \pi^*\Omega_T^{i-2j+1}. \end{aligned}$$

To see this, write

$$\text{(4.9.7)}\qquad \Omega^j_{P/T} \otimes \pi^*\Omega_T^{i-1-2j} = \Omega^j_{P/T} \otimes_{\mathcal{O}_T} \pi^{-1}\Omega_T^{i-1-2j}$$

and apply the Leibniz rule with

$$d\Omega^1_{P/T} \subset \Omega^2_{P/T} \oplus \Omega^1_{P/T} \otimes \pi^*\Omega^1_T \oplus \mu(\Omega^1_{P/T}). \tag{4.9.8}$$

The corresponding map $\Omega^1_{P/T} \to \mu(\Omega^1_{P/T})$ is of course μ. We denote by ∇_τ the corresponding map $\Omega^1_{P/T} \to \Omega^1_{P/T} \otimes \pi^*\Omega^1_T$ and also by ∇_τ the induced map $\Omega^j_{P/T} \otimes \pi^*\Omega^{i-1-2j}_T \to \Omega^j_{P/T} \otimes \pi^*\Omega^{i-2j}_T$. For $\gamma \in \Omega^j_{P/T} \otimes \pi^*\Omega^{i-1-2j}_T$, write $d\gamma = \gamma_{j+1} + \nabla_\tau(\gamma) + (\mu \otimes 1)(\gamma)$ with $\gamma_{j+1} \in \Omega^{j+1}_{P/T} \otimes \pi^*\Omega^{i-1-2j}_T$. The integrability condition $d^2(\gamma) = 0$ in Ω^*_P says that $(\mu \otimes 1)\nabla_\tau(\gamma) = \nabla_\tau(\mu \otimes 1)(\gamma) \in \Omega^{j-1}_{P/T} \otimes \pi^*\Omega^{i-j+2}_T$, up to sign.

Thus $\mathrm{gr}_j\pi_*M^i \to \mathrm{gr}_j\pi_*M^{i+1}$ is the map

$$R^j\pi_*\nabla_\tau\colon\ R^j\pi_*\Omega^j_{P/T} \otimes \Omega^{i-2j}_T \to R^j\pi_*\Omega^j_{P/T} \otimes \Omega^{i-2j+1}_T.$$

Now, $\nabla_\tau = d\,|\,K^*$, where d is the differential of Ω^*_P. Let ℓ_i be the rank one subquotients of $\pi^*\mathcal{E}$, with local algebraic transition functions $f^i_{\alpha,\beta}$. Then $R^j\pi_*\Omega^j_{P/T} \otimes \Omega^{i-2j}_T$ is generated over $\mathcal{O}_T$ by elements $\varphi = F \wedge \omega$, with

$$F = d\log f^{i_1}_{\alpha_0,\alpha_1} \wedge \cdots \wedge d\log f^{i_j}_{\alpha_{j-1},\alpha_j}$$

and $\omega \in \Omega^{i-2j}_T$. Thus $d\varphi = (-1)^j F \wedge d\omega$. This finishes the proof of the proposition. □

5. Chern-Simons classes and the Griffiths group.

5.1. Our objective in this section is to investigate the vanishing of the class $w_n(E,\nabla)$ for a flat bundle E on a smooth, projective variety X over $\mathbb{C}$. We will show that $w_n = 0$ if and only if the nth Chern class $c_n(E)$ vanishes in a "generalized Griffiths group" $\mathrm{Griff}^n(X)$.

Let X be a smooth, quasi-projective variety over $\mathbb{C}$. For $Z \subset X$ a closed subvariety and A an abelian group, we write $H^*_Z(X,A)$ for the singular cohomology with supports in Z and values in A. We write

$$H^*_{\mathcal{Z}^n}(X,A) = \varinjlim_{Z \subset X \text{ cod. } n} H^*_Z(X,A).$$

Purity implies that for Z irreducible of codimension n,

$$H^p_Z(X,A) = \begin{cases} 0 & p < 2n \\ A(-n) & p = 2n. \end{cases}$$

Here $\mathbb{Z}(n) = (2\pi i)^n\mathbb{Z}$ and $A(n) := A \otimes \mathbb{Z}(n)$. As a consequence

$$H^p_{\mathcal{Z}^n}(X, \mathbb{Z}(n)) = \begin{cases} 0 & p < 2n \\ \mathcal{Z}^n(X) & p = 2n \end{cases}$$

where $\mathcal{Z}^n(X)$ is the group of codimension n algebraic cycles on X.

For $m < n$, define the Chow group of codimension n algebraic cycles modulo codimension m equivalence by

$$CH^n_m(X) := \text{Image}\left(\mathcal{Z}^n(X) = H^{2n}_{\mathcal{Z}^n}(X, \mathbb{Z}(n)) \to H^{2n}_{\mathcal{Z}^m}(X, \mathbb{Z}(n))\right). \tag{5.1.1}$$

Of course, $CH^*_0(X)$ is the group of cycles modulo homological equivalence. It follows from [2] (7.3) that $CH^n_{n-1}(X)$ is the group of codimension n algebraic cycles modulo *algebraic* equivalence.

Definition 5.1.1. The generalized Griffiths group $\text{Griff}^n(X)$ is defined to be the kernel of the map $CH^n_1(X) \to CH^n_0(X)$. In other words, the generalized Griffiths group consists of cycles homologous to 0 on X modulo those homologous to 0 on some divisor in X.

Example 5.1.2. $\text{Griff}^2(X)$ is the usual Griffiths group of codimension 2 cycles homologous to zero modulo algebraic equivalence.

5.2. With notation as above, let $\mathcal{H}^p(A)$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto H^p(U_{\text{an}}, A)$, cohomology for the classical (analytic) topology with coefficients in A. The principal object of study in [2] was a spectral sequence

$$E_2^{p,q}(A) = H^p(X_{\text{Zar}}, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(X_{\text{an}}, A). \tag{5.2.1}$$

associated to the "continuous" map $X_{\text{an}} \to X_{\text{Zar}}$. This spectral sequence was shown to coincide from E_2 onward with the "coniveau" spectral sequence

$$E_1^{p,q}(A) = \bigoplus_{x \in \mathcal{Z}^p - \mathcal{Z}^{p+1}} H^{q-p}(x, A) \Rightarrow H^{p+q}(X_{\text{an}}, A). \tag{5.2.2}$$

As a consequence of a Gersten resolution for the sheaves $\mathcal{H}^p(A)$, one had

$$\begin{gathered} H^p(X_{\text{Zar}}, \mathcal{H}^q(A)) = (0) \text{ for } p > q \\ H^n(X_{\text{Zar}}, \mathcal{H}^n(\mathbb{Z}(n))) \cong CH^n_{n-1}(X). \end{gathered} \tag{5.2.3}$$

The E_∞-filtration $N^*H^*(X_{\text{an}}, A)$ is the filtration by codimension,

$$N^pH^*(X_{\text{an}}, A) = \text{Image } (H^*_{\mathcal{Z}^p}(X_{\text{an}}, A) \to H^*(X_{\text{an}}, A)).$$

PROPOSITION 5.3.1. *With notation as above, there is an exact sequence*

$$0 \to H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z}(n))/N^1 \to E_n^{0,2n-1} \xrightarrow{d_n} \mathrm{Griff}^n(X) \to 0. \tag{5.3.1}$$

Proof. It follows from (3) that we have

$$\begin{aligned} H^{2n-1}(X_{\mathrm{an}}, \mathbb{Z}(n)) \twoheadrightarrow E_\infty^{0,2n-1} &= E_{n+1}^{0,2n-1} \subset E_n^{0,2n-1} \subset \cdots \subset E_2^{0,2n-1} \\ &= \Gamma(X, \mathcal{H}^{2n-1}(\mathbb{Z}(n))) \end{aligned} \tag{5.3.2}$$

and

$$\begin{aligned} CH^n_{n-1}(X) &= E_2^{n,n} \twoheadrightarrow E_3^{n,n} \twoheadrightarrow \cdots \twoheadrightarrow E_{n+1}^{n,n} \\ &= E_\infty^{n,n} \subset H^{2n}(X_{\mathrm{an}}, \mathbb{Z}(n)). \end{aligned} \tag{5.3.3}$$

In fact, $E_r^{n,n} \cong CH^n_{n+1-r}(X)$. In particular, $E_n^{n,n} \cong CH^n_1(X)$. To see this, one can, for example, use the theory of exact couples ([13] pp. 232 ff). One gets an exact triangle

$$\begin{array}{ccc} D_r & \xrightarrow{i_r} & D_r \\ {\scriptstyle k_r}\nwarrow & & \swarrow{\scriptstyle j_r} \\ & E_r & \end{array}$$

where in the appropriate degree

$$D_r = \mathrm{Image}\,(H^{2n}_{\mathcal{Z}^n}(X, \mathbb{Z}(n)) \to H^{2n}_{\mathcal{Z}^{n-r+1}}(X, \mathbb{Z}(n))) \cong CH^n_{n-r+1}(X),$$

$i_r = 0$ and j_r is an isomorphism.

The spectral sequence (1) now yields a diagram with exact rows, proving the proposition.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2n-1}(X, \mathbb{Z}(n))/N^1 & \longrightarrow & E_n^{0,2n-1} & \xrightarrow{d_n} & \mathrm{Griff}^n(X) & \longrightarrow & 0 \\ \downarrow = & & \downarrow = & & \downarrow \cap & & & & \\ 0 & \longrightarrow & H^{2n-1}(X, \mathbb{Z}(n))/N^1 & \longrightarrow & E_n^{0,2n-1} & \xrightarrow{d_n} & E_n^{n,n} & \longrightarrow & H^{2n}(X, \mathbb{Z}(n)) \end{array} \tag{5.3.4}$$

□

PROPOSITION 5.4.1. *Let X be smooth and quasi-projective over $\mathbb{C}$. Let (E, ∇) be a vector bundle with an integrable connection on X. Let $n \geq 2$ be given, and let d_n be as in (5.3.4). Let $c_n(E)$ be the nth Chern class in* $\mathrm{Griff}^n(X) \otimes \mathbb{Q}$. *Then*

(i) $w_n(E, \nabla) \in E_n^{0,2n-1}(\mathbb{C}) \subset \Gamma(X, \mathcal{H}^{2n-1}(\mathbb{C}))$.

(ii) $d_n(w_n) = c_n(E)$.

Proof. The spectral sequence (5.2.1) in the case $A = \mathbb{C}$ coincides with the "second spectral sequence" of hypercohomology for

$$H^*(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbb{H}^*(X_{\mathrm{Zar}}, \Omega^*_{X/\mathbb{C}}).$$

This is convenient for calculating the differentials in (5.2.1). Namely, we consider the complexes for $m \geq n$

$$(5.4.1)\quad \begin{aligned} \tau_{n,n}\Omega^* &:= \mathcal{H}^n(\mathbb{C})[-n] \\ \tau_{m,n}\Omega^* &:= \left(\Omega^m_X/d\Omega^{m-1}_X \to \cdots \to \Omega^{n-1}_X \to \Omega^n_{\mathrm{closed}}\right)[-m];\ m < n. \end{aligned}$$

We have maps

$$\tau_{0,n}\Omega^* \to \tau_{1,n}\Omega^* \to \cdots \to \tau_{n,n}\Omega^* \to \tau_{n,n+1}\Omega^* \to \cdots \to \tau_{n,\infty}\Omega^*,$$

and

$$(5.4.2)\quad \begin{aligned} E_r^{0,2n-1} &= \mathrm{Image}\,(H^{2n-1}(X, \tau_{2n-r+1,2n-1}\Omega^*) \\ &\quad \to H^{2n-1}(X, \tau_{2n-1,2n-1}\Omega^*) \\ &= \Gamma(X, \mathcal{H}^{2n-1})). \end{aligned}$$

There is a diagram of complexes

$$(5.4.3)\quad \begin{array}{ccc} \left[\mathcal{K}^m_n \to \Omega^n \to \cdots \to \Omega^{2n-2} \to \Omega^{2n-1}_{\mathrm{closed}}\right] & \xrightarrow{a} & \Omega^\infty\mathcal{K}^m_n \\ b\big\downarrow & & c\big\downarrow \\ \tau_{n+1,2n-1}\Omega^*[n-1] & \xrightarrow{e} & \tau_{2n-1,\infty}\Omega^*[n-1], \end{array}$$

where $\Omega^\infty\mathcal{K}^m_n$ is the complex $\mathcal{K}^m_n \to \Omega^n \to \Omega^{n+1} \to \cdots$. We have

$$(5.4.4)\quad \begin{aligned} c_n(E, \nabla) &\in \mathbb{H}^n(X, \Omega^\infty\mathcal{K}^m_n) \\ c(c_n(E, \nabla)) &= w_n(E, \nabla) \in \mathbb{H}^{2n-1}(X, \tau_{2n-1,\infty}\Omega^*) \\ &\cong H^0(X, \mathcal{H}^{2n-1}). \end{aligned}$$

The map a is the inclusion of a subcomplex, and the quotient has no cohomology sheaves in degrees $< n + 1$, so a is an isomorphism on hypercohomology in degree n. It follows that $w_n(E, \nabla)$ lies in the image of the map e in (5.4.3). By (5.4.2), this image is $E_n^{0,2n-1}$.

To verify $d_n(w_n) = c_n(E)$, write $\bar{\Omega}^\infty\mathcal{K}^m_n$ for the complex

$$\mathcal{K}^m_n \to \Omega^n/d\Omega^{n-1} \to \Omega^{n+1} \to \cdots,$$

and let $\bar{c}_n(E,\nabla) \in \mathbb{H}^n(X, \bar{\Omega}^\infty\mathcal{K}_n^m)$ be the image of $c_n(E,\nabla)$. Consider the distinguished triangle of complexes

$$\tau_{n,2n-2}\Omega^*[n-1] \to \bar{\Omega}^\infty\mathcal{K}_n^m \xrightarrow{\alpha} \mathcal{K}_n^m \oplus \tau_{2n-1,\infty}\Omega^*[n-1]. \tag{5.4.5}$$

We have by definition $\alpha(\bar{c}_n(E,\nabla)) = (c_n(E), w_n(E))$, so, writing ∂ for the boundary map,

$$\partial(c_n(E)) = -\partial(w_n(E,\nabla)) \in \mathbb{H}^{2n}(X, \tau_{n,2n-2}\Omega^*). \tag{5.4.6}$$

Note that the boundary map on $\mathcal{K}_n^m$ factors through the dlog map $\mathcal{K}_n^m \to \mathcal{H}^n$. Thus $\partial c_n(E)$ is the image of the Chern class. On the other hand, by (5.2.3) we have

$$\mathbb{H}^{2n}(X, \Omega^*) \twoheadrightarrow \mathbb{H}^{2n}(X, \tau_{n,\infty}\Omega^*),$$

from which it follows by standard spectral sequence theory that the image of the map

$$H^{2n}(X, \tau_{n,n}\Omega^*) \to \mathbb{H}^{2n}(X, \tau_{n,2n-2}\Omega^*)$$

coincides with $E_n^{n,n}$, and that the boundary map

$$\delta\colon\ \Gamma(X, \mathcal{H}^{2n-1}) \cong \mathbb{H}^{2n-1}(X, \tau_{2n-1,\infty}\Omega^*) \to \mathbb{H}^{2n}(X, \tau_{n,2n-2}\Omega^*)$$

coincides with d_n from the statement of the proposition on $\delta^{-1}(E_n^{n,n}) = E_n^{0,2n-1}$. This completes the proof of the proposition. □

Our next objective is to realize the sequence (5.3.1) as an exact sequence of mixed Hodge structures. To avoid complications, we replace $\mathbb{Z}$ with $\mathbb{Q}$ throughout. More precisely, we work with filtering direct limits of finite dimensional $\mathbb{Q}$-mixed Hodge structures, where the transition maps are maps of mixed Hodge structures.

LEMMA 5.4.2. *The spectral sequence* (5.2.1) *with* $A = \mathbb{Q}(n)$ *can be interpreted as a spectral sequence in the category of mixed Hodge structures.*

Proof. The spectral sequence (5.2.2) can be deduced from an exact couple ([2], p. 188)

$$\cdots \to H^{p+q}_{\mathcal{Z}^p}(X, \mathbb{Q}(n)) \to H^{p+q}_{\mathcal{Z}^{p-1}}(X, \mathbb{Q}(n)) \to H^{p+q}_{\mathcal{Z}^{p-1}/\mathcal{Z}^p}(X, \mathbb{Q}(n)) \to \cdots.$$

These groups clearly have infinite dimensional mixed Hodge structures and the maps are morphisms of mixed Hodge structures. The lemma follows easily, since (5.2.1) coincides with the above from E_2 onward. □

Remark 5.4.3. The groups $E_r^{n,n}$ are all quotients of

$$H^{2n}_{\mathcal{Z}^n/\mathcal{Z}^{n+1}}(X, \mathbb{Q}(n)) \cong \bigoplus_{z \in X^n} \mathbb{Q}$$

so these groups all have trivial Hodge structures.

PROPOSITION 5.5.1. *The Chern-Simons class* $w_n(E, \nabla) \in E_n^{0,2n-1}(\mathbb{C})$ *lies in* F^0 *(zeroeth piece of the Hodge filtration) for the Hodge structure defined by* $E_n^{0,2n-1}(\mathbb{Q}(n))$.

Proof. We have $E_n^{0,2n-1} \subset E_2^{0,2n-1} \subset H^{2n-1}(\mathbb{C}(X), \mathbb{C})$, where the group on the right is defined as the limit over Zariski open sets. Thus, it suffices to work "at the generic point." Let $\mathcal{S}$ denote the category of triples (U, Y, π) with Y smooth and projective, $\pi\colon\ Y \to X$ a birational morphism of schemes, and $U \subset Y$ Zariski open such that Y_U is a divisor with normal crossings and $U \to \pi(U)$ is an isomorphism. Using resolution of singularities, one easily sees that

$$H^n(\operatorname{Spec}(\mathbb{C}(X)), \mathbb{C}) \cong \varinjlim_{\mathcal{S}} \mathbb{H}^n(Y, \Omega^*_Y(\log(Y - U))).$$

The Hodge filtration on the left is induced in the usual way from the first spectral sequence of hypercohomology on the right.

LEMMA 5.5.2. *For* $\alpha \in \mathcal{S}$ *let* $j_\alpha\colon\ U_\alpha \hookrightarrow Y_\alpha$ *be the inclusion. Then*

$$\varinjlim_{\mathcal{S}} H^n(Y_\alpha, j_{\alpha *}\mathcal{K}^m_{n,U_\alpha}) = (0),\ n \geq 1.$$

Proof of lemma. Given $j\colon\ U \hookrightarrow Y$ in $\mathcal{S}$ and $z \in H^n(Y, j_*\mathcal{K}^m_{n,U})$, let $k\colon\ \operatorname{Spec}(\mathbb{C}(X)) \to Y$ be the generic point. We have $H^n(Y, k_*\mathcal{K}^m_{n,\mathbb{C}(X)}) = (0)$ since the sheaf is constant, so there exists $V \subset U$ open of finite type such that writing $\ell\colon\ V \to Y$, z dies in $H^n(Y, \ell_*\mathcal{K}^m_{n,V})$. Let $m\colon\ V \to Z$ represent an object of $\mathcal{S}$ with Z dominating Y. We have a triangle

$$\text{(5.5.1)} \qquad \begin{array}{ccc} H^n(Y, j_*\mathcal{K}^m_{n,U}) & \to & H^n(Z, m_*\mathcal{K}^m_{n,V}) \\ & \searrow \qquad \nearrow & \\ & H^n(Y, \ell_*\mathcal{K}^m_{n,V}) & \end{array}$$

from which it follows that $z \mapsto 0$ in $\varinjlim_{\mathcal{S}} H^n(Y_\alpha, j_{\alpha,*}\mathcal{K}^{\Updownarrow}_{\setminus \mathcal{U}_\alpha})$. ◻

Returning to the proof of Proposition 5.5.1, write $D_\alpha = Y_\alpha - U_\alpha$ for $\alpha \in \mathcal{S}$. We see from the lemma that the map labeled a below is surjective:

$$\begin{array}{lcl} & c_n(E,\nabla) \;\in\; \mathbb{H}^n(X,\Omega^\infty\mathcal{K}_n^m) & \\ & \downarrow & \\ (5.5.2) & \varinjlim_{\mathcal{S}} \mathbb{H}^n(Y_\alpha, j_{\alpha,*}\mathcal{K}^m_{n,U_\alpha} \to \Omega^n(\log(D_\alpha)) \to \cdots) & \\ & \nearrow a \qquad\qquad \downarrow & \\ \varinjlim_{\mathcal{S}} \mathbb{H}^{n-1}(Y_\alpha, \Omega^n(\log(D_\alpha)) \to \cdots) & \xrightarrow{b} H^{2n-1}(\operatorname{Spec}(\mathbb{C}(X)),\mathbb{C}). & \end{array}$$

Since the image of b is F^0 for the Hodge filtration on

$$H^{2n-1}(\operatorname{Spec}(\mathbb{C}(X)),\mathbb{C}),$$

and since the composition of vertical arrows maps $c_n(E,\nabla)$ to the restriction of $w_n(E,\nabla)$ at the generic point, the proposition is proved. ◻

PROPOSITION 5.6.1. *The Chern-Simons class* $w_n(E,\nabla) \in E_n^{0,2n-1}(\mathbb{Q}(n))$.

Proof. The following diagram is commutative

$$\begin{array}{lccc} & \mathbb{H}^n(X,\Omega^\infty\mathcal{K}_n^m) & \to H^0(X,\mathcal{H}^{2n-1}(\mathbb{C})) \to & H^{2n-1}(\operatorname{Spec}(\mathbb{C}(X)),\mathbb{C}) \\ (5.6.1) & \downarrow \varphi(3.9.1(3)) & & \downarrow \\ & H^{2n-1}(X_{\mathrm{an}},\mathbb{C}/\mathbb{Z}(n)) & \longrightarrow & H^{2n-1}(\operatorname{Spec}(\mathbb{C}(X)),\mathbb{C}/\mathbb{Z}(n)) \end{array}$$

We know from [7] and Proposition 3.10.1 above that

$$\varphi(c_n(E,\nabla)) = c_n^{\mathrm{an}}(E,\nabla),$$

and, using the deep theorem of Reznikov ([18]), that this class is torsion. In particular, the image of $c_n(E,\nabla)$ on the upper right lies in $H^{2n-1}(\operatorname{Spec}(\mathbb{C}(X)),\mathbb{Q}(n))$. As a matter of fact, in Theorem 5.6.2, we will only use that $w_n(E,\nabla) \in E_n^{0,2n-1}(\mathbb{R}(n))$. For this we don't need the full strength of [18], but only that $c_n^{\mathrm{an}}(E,\nabla) = 0 \in H^{2n-1}(X_{\mathrm{an}},\mathbb{C}/\mathbb{R}(n))$, which is a consequence of Simpson's theorem ([19]) asserting that (E,∇) deforms to a $\mathbb{C}$ variation of Hodge structure. ◻

THEOREM 5.6.2. *Let X be a smooth, projective variety over $\mathbb{C}$. Let E be a vector bundle on X, and let ∇ be an integrable connection on E. Then $w_n(E,\nabla) \in H^0(X,\mathcal{H}^{2n-1}(\mathbb{C}))$ vanishes if and only if the cycle class $c_n(E)$ is trivial in* $\mathrm{Griff}^n(X)\otimes\mathbb{Q}$.

Proof. Consider the exact sequence of mixed Hodge structures

$$(5.6.2) \qquad 0 \to H^{2n-1}(X_{\mathrm{an}},\mathbb{Q}(n))/N^1 \to E_n^{0,2n-1}(\mathbb{Q}(n)) \to \mathrm{Griff}^n(X)\otimes\mathbb{Q} \to 0.$$

Write H for the group on the left. It is pure of weight -1, so $H(\mathbb{Q}) \cap F^0H(\mathbb{C}) = (0)$. It follows that $w_n(E,\nabla) = 0$ if and only if its image $c_n(E)$ in $\mathrm{Griff}^n(X) \otimes \mathbb{Q}$ vanishes. ◻

The following corollary is a simple application of the theorem to the example (0.2) discussed in the introduction.

COROLLARY 5.6.3. *Let E, X, ∇ be as above. Assume E has rank 2, and that the determinant bundle is trivial, with the trivial connection. Let $U \subset X$ be affine open such that $E \mid U$ is trivial, and let $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ be the connection matrix. Then $c_2(E) \otimes \mathbb{Q}$ is algebraically equivalent to 0 on X if and only if there exists a meromorphic 2-form η on X satisfying $d\eta = \alpha \wedge d\alpha = \alpha \wedge \beta \wedge \gamma$.*

6. Logarithmic poles. In this section we consider a normal crossing divisor $D \subset X$ on a smooth variety X, the inclusion $j\colon X - D \to X$, and a bundle E, together with a flat connection $\nabla\colon E \to \Omega^1_X(\log D) \otimes E$ with logarithmic poles along D. The characteristic of the ground field k is still 0. Finally recall from [2] that one has an exact sequence

$$0 \to H^0(X, \mathcal{H}^j) \to H^0(X - D, \mathcal{H}^j) \to \oplus_i \mathrm{res}\, H^0(k(D_i), \mathcal{H}^{j-1}). \tag{6.0.3}$$

THEOREM 6.1.1. *Let (E, ∇, D) be a flat connection with logarithmic poles. Then*

$$\begin{aligned} w_n(E,\nabla) \in H^0(X, \mathcal{H}^{2n-1}) &\subset H^0(X - D, \mathcal{H}^{2n-1}) \\ &= H^0(X, j_*\mathcal{H}^{2n-1}). \end{aligned} \tag{6.1.1}$$

Proof. By (6.0.3), one just has to compute the residues of $w_n(E,\nabla)$ along generic points of D. So one may assume that the local equation of ∇ is $A = B\frac{dx}{x} + C$, where B is a matrix of regular functions, x is the local equation of a smooth component of D, and C is a matrix of regular one forms. Furthermore, as $dA = A^2 = \frac{1}{2}[A,A]$, the formulae of Theorem 2.2.1 say that the local shape of $w_n(E,\nabla)$ is $\mathrm{Tr}\,\lambda A(dA)^{n-1} = \lambda \mathrm{Tr}\,(B\frac{dx}{x} + C)(dB\frac{dx}{x} + dC)^{n-1}$ for some $\lambda \in \mathbb{Q}$. So up to coefficient one has to compute

$$\begin{aligned} &\mathrm{Tr\ Res}\left(B\frac{dx}{x} + C\right)\left((dC)^{n-1} + \sum_{a+b=n-2} (dC)^a dB \frac{dx}{x} (dC)^b\right) \\ &= \mathrm{Tr\ Res}\left[C \sum_{a+b=n-2} (dC)^a dB (dC)^b + B(dC)^{n-1}\right]\frac{dx}{x}. \end{aligned} \tag{6.1.2}$$

On the other hand, the integrability condition reads

$$(dB - (CB - BC))\frac{dx}{x} + dC - C^2 = 0,$$

from which one deduces

$$dC\frac{dx}{x} = C^2\frac{dx}{x} \tag{6.1.3}$$

$$\operatorname{Res}(dB - (CB - BC))\frac{dx}{x} = 0. \tag{6.1.4}$$

Applying (6.1.3) to (6.1.2), we reduce to calculating

$$\operatorname{Tr}\operatorname{Res}\left[\sum_{a+b=n-2}(dC)^a CdB(dC)^b + B(dC)^{n-1}\right]\frac{dx}{x}. \tag{6.1.5}$$

Since we are only interested in calculating (6.1.5) modulo exact forms, we can use $d(CB) = dCB - CdB$ and move copies of dC to the right in (6.1.5) under the trace. The problem becomes to show

$$\operatorname{Tr}\operatorname{Res}B(dC)^{n-1}\frac{dx}{x} \tag{6.1.6}$$

is exact. It follows from (6.1.4) that

$$\operatorname{Tr}\operatorname{Res}C^{2n-3}dB\frac{dx}{x} = \operatorname{Tr}\operatorname{Res}[C^{2n-2}B - C^{2n-3}BC]\frac{dx}{x}. \tag{6.1.7}$$

Bringing the C to the left in the last term changes the sign, so we get by (6.1.3)

$$\begin{aligned}\operatorname{Tr}\operatorname{Res}(dC)^{n-2}CdB\frac{dx}{x} &= \operatorname{Tr}\operatorname{Res}C^{2n-3}dB\frac{dx}{x}\\ &= \operatorname{Tr}\operatorname{Res}2(dC)^{n-1}B\frac{dx}{x}.\end{aligned} \tag{6.1.8}$$

Thus

$$\begin{aligned}\operatorname{Tr}\operatorname{Res}(dC)^{n-1}B\frac{dx}{x} &= \operatorname{Tr}\operatorname{Res}(dC)^{n-2}(CdB - dCB)\frac{dx}{x}\\ &= -\operatorname{Tr}\operatorname{Res}d\left[C(dC)^{n-3}d(CB)\frac{dx}{x}\right].\end{aligned} \tag{6.1.9}$$

This form is exact, so we are done. □

6.2. We now want to understand the image of $w_n(E,\nabla)$ under the map d_n defined in Proposition 5.3.1. Of course Proposition 5.4.1 says that

$$d_n(w_n((E,\nabla) \mid (X-D))) = c_n(E).$$

Definition 6.2.1. (See [10], Appendix B.) Let (E,∇) be a flat connection with logarithmic poles along D, with residue

$$\Gamma = \oplus \Gamma_s \in \oplus_s H^0(D_s, \operatorname{End} E|_{D_s}).$$

One defines

$$N_i^{CH}(\Gamma) = (-1)^i \sum_{\alpha_1+\cdots+\alpha_s=i} \binom{i}{\alpha} \operatorname{Tr}(\Gamma_1^{\alpha_1} \circ \cdot \cdots\cdot \circ \Gamma_s^{\alpha_s}) \cdot [D_1]^{\alpha_1} \cdots [D_s]^{\alpha_s} \in CH^i(X) \otimes \mathbb{C}. \tag{6.2.1}$$

One defines as usual the corresponding symmetric functions $c_i^{CH}(\Gamma) \in CH^i(X) \otimes \mathbb{C}$ as a polynomial with $\mathbb{Q}$ coefficients in the Newton functions $N_i^{CH}(\Gamma)$. For example

$$c_2^{CH}(\Gamma) = \frac{1}{2}\left[\left(\sum_s \operatorname{Tr}(\Gamma_s) \cdot D_s\right)^2 - 2\left(\sum_s \operatorname{Tr}(\Gamma_s \cdot \Gamma_s) \cdot D_s^2 + 2\sum_{s<t} \operatorname{Tr}(\Gamma_s \cdot \Gamma_t) D_s \cdot D_t\right)\right] \in CH^2(X) \otimes \mathbb{C}. \tag{6.2.2}$$

We denote by $c_2(\Gamma)$ its image in $H^2(X, \Omega^2_{X,cl})$ and also by $c_2(\Gamma)$ its image in $H^2(X, \mathcal{H}^2_{DR})$.

Note that these invariants vanish when the connection has nilpotent residues Γ_s. (This condition forces the local monodromies around the components of D to be unipotent (see [5]).)

THEOREM 6.2.2. *Assume k has characteristic zero and X is proper. Then*

$$c_2(E) - c_2(\Gamma) = d_2(w_2(E,\nabla)) \in H^2(X, \mathcal{H}^2). \tag{6.2.3}$$

Proof. In order to simplify the notations, we denote by $c_2(\Gamma)$ the same expression in $CH^2(X) \otimes \mathbb{C}$, $\oplus_s CH^1(D_s) \otimes \mathbb{C}$, $\oplus_s F^1 H^2_{DR}(D_s)$ etc., where we always distribute $2D_s \cdot D_t$ for $s < t$ as one $D_s \cdot D_t$ on D_s and one on D_t.

We denote by $\pi\colon Q \to X$ the flag bundle of E. As π^* induces an isomorphism

$$\frac{H^2(X, d\Omega^1_X)}{H^1(X, \mathcal{H}^2)} = \frac{H^3(X, \mathcal{O}_X \to \Omega^1_X)}{N^1 H^3(X)} \xrightarrow{\sim} \frac{H^3(Q, \mathcal{O}_Q \to \Omega^1_Q)}{N^1 H^3(Q)} \tag{6.2.4}$$

and an injection $H^2(X,\mathcal{H}^2)\to H^2(Q,\mathcal{H}^2)$, it is enough to prove the compatibility on Q via the exact sequence ([2])

$$(6.2.5)\qquad 0\to \frac{H^3(X,\mathcal{O}_X\to\Omega^1_X)}{N^1H^3(X)}\to H^2(X,\Omega^2_{X,\mathrm{clsd}})\to H^2(X,\mathcal{H}^2).$$

Write $D'_s=\pi^*D_s$, and consider $(\mathcal{O}(D'_s),\nabla_s)\in\mathbb{H}^1(Q,\mathcal{K}_1\to\Omega^1_Q(\log D'_s)_{\mathrm{clsd}})$, where ∇_s is the canonical connection with residue -1 along D'_s.

We define a product

$$\begin{aligned}(\mathcal{K}^m_i\to(\pi^*\Omega^i_X(\log D))_{\tau d})\times(\mathcal{K}^m_j\to(\pi^*\Omega^j_X(\log D))_{\tau d})\\ \xrightarrow{\bullet}(\mathcal{K}^m_{i+j}\to(\pi^*\Omega^{i+j}_X(\log D))_{\tau d})\end{aligned}$$

by

$$(6.2.6)\qquad x\cdot x'=\begin{cases}x\cup x' & \text{if } \deg x'=0\\ \tau d\log x\wedge x' & \text{if } \deg x=0 \text{ and } \deg x'=1\\ 0 & \text{otherwise}\end{cases}.$$

(Here τd: $\pi^*\Omega^i_X(\log D)\to\pi^*\Omega^{i+1}_X(\log D)$ comes from the splitting τ: $\Omega^1_Q(\log D')\to\pi^*\Omega^1_X(\log D)$. See Proposition 4.4.1 as well as [7] and [8].) One verifies that

$$(6.2.7)\qquad d(x\cdot x')=dx\cdot x'+(-1)^{\deg x}x\cdot dx',$$

the only nontrivial contribution left and right being for $\deg x=\deg x'=0$.

This product defines elements (W_1 is the weight filtration)

$$(6.2.8)\qquad \begin{aligned}\epsilon_{st} &= (\mathcal{O}(D'_s),\nabla_s)\cdot(\mathcal{O}(D'_t),\nabla_t)\\ &\in \mathbb{H}^2(Q,\mathcal{K}_2\to W_1\Omega^2_Q(\log(D'_s+D'_t))_{cl})\end{aligned}$$

which map to $D'_s\cdot D'_t$ in $CH^2(Q)$. Moreover Res ϵ_{st} is the class of $D'_s\cdot D'_t$ sitting diagonally in

$$F^1H^2_{DR}(D'_s)\oplus F^1H^2_{DR}(D'_t)$$

if $s\neq t$; or in $F^1H^2_{DR}(D'_s)$ if $s=t$.

Next we want to define a cocycle $N_2(\pi^*(E,\nabla))$.

Let $h_{ij}(=h)$ be the upper triangular transition functions of $E|_Q$ adapted to the tautological flag E_i, and write B_i for the local connection matrix in $\Omega^1_Q(\log D')$, $D'=\pi^{-1}D$. Then τB_i is upper triangular, and $\tau dB_i=d\tau B_i$ has zero's on the diagonal ([7], (0.7), (2.7)). Let

$$w_i=\mathrm{Tr}\,(B_idB_i).$$

Using Tr $(dhh^{-1})^3 = 0$, one computes that $w_i - w_j = -3\mathrm{Tr}\ d(h^{-1}dhB_j)$. But

$$\begin{aligned}\text{(6.2.9)}\qquad \mathrm{Tr}\ h^{-1}dhB_j &= \mathrm{Tr}\ h^{-1}B_ihB_j\\ \mathrm{Tr}\ h_{ik}^{-1}dh_{ij}dh_{jk} &= \mathrm{Tr}\ h_{ik}^{-1}(B_ih_{ij} - h_{ij}B_j)(B_jh_{jk} - h_{jk}B_j)\\ &= \delta\mathrm{Tr}\ (B_ihB_jh^{-1}).\end{aligned}$$

Here δ is the Cech coboundary. Writing $\mathcal{C}^i$ for Cech i-cochains, we may define

$$\begin{aligned}\text{(6.2.10)}\qquad 3N_2(\pi^*(E,\nabla)) &= \left(3\sum_{a=0}^{r}\xi_{ij}^a \cup \xi_{jk}^a, -3\mathrm{Tr}\ (h^{-1}dhB_j), w_i\right)\\ &\in (\mathcal{C}^2(Q,\mathcal{K}_2)\times\mathcal{C}^1(Q,\Omega_Q^2(\log D'))\\ &\quad \times\mathcal{C}^0(Q,\Omega_Q^3(\log D')))_{d+\delta}\end{aligned}$$

where $(\xi_{ij}^1,\ldots,\xi_{ij}^r)$ is the diagonal part of h_{ij}. This defines $3N_2(\pi^*(E,\nabla))$ as a class in $\mathbb{H}^2(Q,\mathcal{K}_2 \to \Omega_Q^2(\log D') \to \cdots)$ which maps to

$$\begin{aligned}\text{(6.2.11)}\qquad 3\tau N_2(\pi^*(E,\nabla)) &= \left(3\sum_{a=1}^{r}\xi_{ij}^a\cup\xi_{jk}^a, 3\sum_{a=1}^{r}\omega_i^a\wedge(\delta\omega^a)_{ij}, 0\right)\\ &\in \mathbb{H}^2(Q,\mathcal{K}_2 \to \pi^*\Omega_X^2(\log D')_{\tau d})\end{aligned}$$

where $(\omega_i^1,\ldots,\omega_i^r)$ is the diagonal part of τB_i.

As the image of $\tau N_2(\pi^*(E,\nabla))$ in $H^2(Q,\mathcal{K}_2)$ is just the second Newton class of E, the argument of [8], (1.7) shows that

$$\begin{aligned}\text{(6.2.12)}\qquad N_2(E,\nabla) &:= \tau N_2(\pi^*(E,\nabla))\\ &\in \mathbb{H}^2(X,\mathcal{K}_2 \to \Omega_X^2(\log D) \to \cdots)\\ &\subset \mathbb{H}^2(Q,\mathcal{K}_2 \to \pi^*\Omega_X^2(\log D) \to \cdots).\end{aligned}$$

We observe that $w(B) = \mathrm{Tr}\ BdB \in W_2\Omega_Q^3(\log D')$ (weight filtration) so the cocycle

$$\begin{aligned}\text{(6.2.13)}\qquad 2x &= -N_2(\pi^*(E,\nabla)) + c_1(\pi^*(E,\nabla))^2\\ &= \Big(-\mathrm{Tr}\ (h^{-1}dh)^2 + \mathrm{Tr}\ h^{-1}dh\cdot\mathrm{Tr}\ h^{-1}dh,\\ &\qquad \mathrm{Tr}\ (h^{-1}dhB) - \mathrm{Tr}\ h^{-1}dh\cdot\mathrm{Tr}\ B, -\frac{w(B)}{3}\Big)\end{aligned}$$

defines a class in

$$\mathbb{H}^2(Q,\Omega_{cl}^2 \to W_1\Omega_Q^2(\log D') \to W_2\Omega_Q^3(\log D')_{cl}).$$

One has an exact sequence

$$(6.2.14)\quad \begin{aligned} 0 &\to \mathbb{H}^2(Q, \Omega^2_{cl} \to W_1\Omega^2_Q(\log D') \to W_1\Omega^3_Q(\log D')_{cl}) \\ &\to \mathbb{H}^2(Q, \Omega^2_{cl} \to W_1\Omega^2_Q(\log D') \to W_2\Omega^3_Q(\log D')_{cl}) \\ &\xrightarrow{\text{residue}} \oplus_{s<t} H^0(D'_{st}, \Omega^1_{D'_{st,cl}}). \end{aligned}$$

As D'_{st} is proper smooth, one has

$$H^0(D'_{st}, \Omega^1_{D'_{st,cl}}) \subset H^0(D'_{st}, \mathcal{H}^1) = H^1(D'_{st}).$$

The residue of $2x$ along D'_{st} is just the residue of $-\frac{1}{3}w(B)$ along D'_{st} via

$$(6.2.15)\quad \begin{aligned} H^0(Q, \mathcal{H}^3(\log D')) &= H^0(Q - D', \mathcal{H}^3) \\ &\to \oplus_s H^0(D'_s - \cup_{t\neq s} D'_t, \mathcal{H}^2) \\ &\to \oplus_{s<t} H^0(D'_{st}, \mathcal{H}^1), \end{aligned}$$

which vanishes. Therefore

$$(6.2.16)\quad 2x \in \mathbb{H}^2(Q, \Omega^2_{cl} \to W_1\Omega^2_Q(\log D') \to W_1\Omega^3_Q(\log D')_{cl}).$$

Its residue in $\oplus_s H^1(D'_s, \Omega^1_{D'_s})$ is (Tr $(h^{-1}dh \cdot \Gamma)$ − Tr $h^{-1}dh \cdot$ Tr Γ). By [10], Appendix B, one has $-h^{-1}dh = \sigma(D') \cdot \Gamma$ in $H^1(Q, \Omega^1_Q \otimes \mathrm{End} E)$ where $\sigma(D')$ is the extension

$$0 \to \Omega^1_Q \to \Omega^1_Q(\log D') \to \oplus_s \mathcal{O}_{D'_s} \to 0.$$

One has

(1) $-D'_s \cdot D'_s$ is the push down extension of $\sigma(D'_s)$ by $\Omega^1_Q \to \Omega^1_{D'_s}$ in $H^1(Q, \Omega^1_{D'_s})$

(2) $-D'_s \cdot D'_t$ is the extension

$$0 \to \Omega^1_{D'_t} \to \Omega^1_{D'_t}(\log(D'_s \cap D'_t)) \to \mathcal{O}_{D'_s \cap D'_t} \to 0$$

in $H^1(Q, \Omega^1_{D'_t})$.

It follows that residue $x = c_2(\Gamma)$ in $\oplus_s H^1(D'_s, \Omega^1_{D'_s})$.

For appropriate $\lambda_{st} \in k$ (the coefficients of $c_2(\Gamma)$), $c_2(\Gamma)$ = residue $\sum \lambda_{st}\epsilon_{st}$ in $\oplus_s H^1(D'_s, \Omega^1_{D'_s})$. So one has

$$(6.2.17)\quad \text{residue}\left(x - \sum \lambda_{st}\epsilon_{st}\right) \in \oplus_s F^2H^2(D'_s).$$

Again, since residue $(x - \sum \lambda_{st}\epsilon_{st})$ = residue $x = 0$ in $\oplus_s H^0(D'_s, \mathcal{H}^2) \subset \oplus_s H^2(k(D'_s))$, one has that in $\oplus_s F^1 H^2(D'_s)$

$$\text{(6.2.18)} \qquad \text{residue}\left(x - \sum \lambda_{st}\epsilon_{st}\right) \in \oplus_s F^2 H^2(D'_s) \cap H^1(D'_s, \mathcal{H}^1) = 0.$$

This shows that residue $(x - \sum \lambda_{st}\epsilon_{st}) = 0$ in $\oplus_s F^1 H^2(D'_s)$, that is

$$\text{(6.2.19)} \qquad \begin{aligned} w_2(E, \nabla) &= \left(x - \sum \lambda_{st}\epsilon_{st}\right) \\ &\in \frac{\mathbb{H}^2(Q, \Omega^2_{cl} \to \Omega^2 \to \Omega^3_{cl}) = H^0(Q, \mathcal{H}^3)}{\mathrm{Im} \oplus_s H^1(D'_s)} \end{aligned}$$

and maps to

$$\text{(6.2.20)} \qquad c_2(E) - c_2(\Gamma) \text{ in } H^2(Q, \mathcal{H}^2).$$

□

Question 6.3. We know (see [10], Appendix B) that on X proper, the image of $c_n(\Gamma)$ in the de Rham cohomology $H^{2n}_{DR}(X)$ is the Chern class $c_n^{DR}(E)$. This inclines us to ask whether

$$c_n(E) - c_n(\Gamma) = d_n(w_n(E, \nabla)) \in \mathrm{Griff}^n(X).$$

6.4.

THEOREM 6.4.1. *Let (E, ∇) be a flat connection with logarithmic poles along a normal crossing divisor D on a smooth proper variety X over $\mathbb{C}$. When $(E, \nabla) \mid (X - D)$ is a complex variation of Hodge structure, then $w_2(E, \nabla) = 0$ if and only if $c_2(E) - c_2(\Gamma) = 0 \in H^2(X, \mathcal{H}^2)$. When furthermore $(E, \nabla) \mid (X - D)$ is a Gauss-Manin system, then $w_2(E, \nabla) \in H^0(X, \mathcal{H}^3(\mathbb{Q}(2)))$, and if it has nilpotent residues along the components of D, then $w_2(E, \nabla) = 0$ if and only if $c_2(E) = 0 \in H^2(X, \mathcal{H}^2)$.*

Proof. As in Proposition 5.5.1, one has $w_n(E, \nabla) \in F^0$. In fact, the proof does not use that ∇ is everywhere regular, but only that $w_n(E, \nabla)$ comes from a class in $\mathbb{H}^n(Y, \mathcal{K}^m_n \to \Omega^n_Y(\log(Y - U) \to \cdots)$ on some $(U, Y, \pi) \in \mathcal{S}$. Further, if $(E, \nabla) \mid (X - D)$ is a $\mathbb{C}$ variation of Hodge structure, then $w_n(E, \nabla) \in H^0(X, \mathcal{H}^{2n-1}(\mathbb{R}(n))$ as

$$w_n(E, \nabla) \mid (X - D) \in H^0(X - D, \mathcal{H}^{2n-1}(\mathbb{R}(n)))$$

(see proof of Proposition 5.6.1). When $(E, \nabla) \mid (X - D)$ is a Gauss-Manin system, then again one argues exactly as in the proof of Proposition 5.6.1 using [4] to show $w_2(E, \nabla) \in H^0(X, \mathcal{H}^3(\mathbb{Q}(2)))$. Finally $c_2(\Gamma) = 0$ when the residues of the connection are nilpotent. □

7. Examples.

7.1. Let X be a good compactification of the moduli space of curves of genus g with some level, such that a universal family $\varphi\colon \mathcal{C} \to X$ exists. Let (E, ∇) be the Gauss-Manin system $R^1\varphi_*\Omega^\bullet_{\mathcal{C}/X}(\log\infty)$. Then Mumford ([16], (5.3)) shows that $c_i^{CH}(E) \otimes \mathbb{Q} = 0$ in $CH^i(X) \otimes \mathbb{Q}$ for $i \geq 1$, so a fortiori $c_i(E) \otimes \mathbb{Q} = 0$ in the Griffiths group. As ∇ has nilpotent residues (the local monodromies at infinity of the Gauss-Manin system are unipotent and (E, ∇) is Deligne's extension [5]), one applies Theorem 6.4.1.

In particular, for any semi-stable family of curves $\varphi\colon Y \to X$ over a field k of characteristic 0,

$$w_n(R^1\varphi_*\Omega^\bullet_{Y/X}(\log\infty)) = 0,$$

for $n = 2$ and for $n \geq 2$ if φ is smooth (or if Question 6.3 has a positive response).

7.2. Let X be a level cover of the moduli space of abelian varieties such that a universal family $\varphi\colon \mathcal{A} \to X$ exists. The Riemann-Roch-Grothendieck theorem applied to a principal polarization L on $\mathcal{A}$ together with Mumford's theorem that

$$\varphi_* L^n = M \otimes \text{ trivial}$$

for some rank 1 bundle M, implies that $c_i^{CH}(E) \otimes \mathbb{Q} = 0$ for $E = R^1\varphi_*\Omega^\bullet_{\mathcal{A}/X} \mid X_0$, where X_0 is the smooth locus of φ. This result was communicated to us by G. van der Geer ([12]). In particular, for any smooth family $\varphi\colon Y \to X$ of abelian varieties with X smooth proper over a field of characteristic 0, $w_n(R^1\varphi_*\Omega^\bullet_{Y/X}) = 0$ for all $n \geq 2$.

7.3. Let $\varphi\colon Y \to X$ be a smooth proper family of surfaces over X smooth. The Riemann-Roch-Grothendieck theorem, as applied by Mumford in [16], implies that the Chern character verifies

$$\operatorname{ch}\left(\sum_{i=0}^{i=4} (-1)^i R^i\varphi_*\Omega^\bullet_{Y/X}\right) \in CH^0(X) \otimes \mathbb{Q} \subset CH^\bullet(X) \otimes \mathbb{Q}.$$

As $R^1\varphi_*\Omega^\bullet_{Y/X}$ is dual to $R^3\varphi_*\Omega^\bullet_{Y/X}$, the two previous examples show that $c_i(R^2\varphi_*\Omega^\bullet_{Y/X}) = 0$ in $CH^i(X) \otimes \mathbb{Q}$ for $i \geq 1$. This implies $w_n(R^2\varphi_*\Omega^\bullet_{Y/X}) = 0$ for all $n \geq 2$ when X is proper.

7.4. As shown in [9], $w_n(E, \nabla) = 0$ in characteristic zero when (E, ∇) trivializes on a finite (not necessarily smooth) covering of X.

7.5. Let $\varphi\colon Y \to X$ be a smooth proper family defined over a perfect field k of sufficiently large characteristic. Let (E, ∇) be the Gauss-Manin system $R^a\varphi_*\Omega^\bullet_{Y/X}$. Consider $w_n(E, \nabla) \in H^0(X, \mathcal{H}^{2n-1})$, which is the restriction of the corresponding class in characteristic zero when φ comes from a smooth proper family in characteristic zero. Assume this. As $H^0(X, \mathcal{H}^{2n-1}) \subset H^0(k(X), \mathcal{H}^{2n-1})$, we may assume that $R^a\varphi_*\Omega^\bullet_{Y/X}$ is locally free and compatible with base change.

Via the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{F_{\mathrm{rel}}} & Y^{(p)} & \longrightarrow & Y \\ & \overset{\varphi}{\searrow} & \big\downarrow \varphi^{(p)} & & \big\downarrow \varphi \\ & & X & \xrightarrow{F} & X \end{array} \tag{7.5.1}$$

where F is the absolute Frobenius of X, $\varphi^{(p)} = \varphi \times_X F$, F_{rel} is the relative Frobenius, one knows by [15], (7.4) that the Gauss-Manin system $R^a\varphi_*\Omega^\bullet_{Y/X}$ has a Gauss-Manin stable filtration $R^a\varphi^{(p)}_* F_{\mathrm{rel}}(\tau_{\leq a}\Omega^\bullet_{Y/X})$, such that the restriction of ∇ to the graded pieces $F^*R^{a-i}\varphi_*\Omega^i_{Y/X}$ is the trivial connection.

In particular, the graded pieces are locally generated by flat sections and $A_i = 0$. So by additivity of the classes $c_i(E, \nabla)$, $w_n(E, \nabla) = \theta_n(E, \nabla) = 0$.

In particular, the classes w_n (Gauss-Manin) provide examples of classes $w \in H^0(X, \mathcal{H}^{2n-1})$ whose restriction modulo p vanish for all but finitely many p. This should imply that $w = 0$ according to [17].

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