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Coniveau of Classes of Flat Bundles Trivialized on a Finite Smooth Covering of a Complex Manifold

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Abstract. On a smooth algebraic complex variety X, we show that the classes of a flat bundle, which is trivialized on a finite cover of X, with values in the odd-dimensional cohomology of the underlying complex manifold with $\mathbb{C}/\mathbb{Z}(i)$, are living in the bottom part of Grothendieck's coniveau filtration. This answers positively when the basis is smooth complex a question of Bruno Kahn [K-Theory (1992), conjecture 2].

Key words: Flat bundles, secondary classes, coniveau filtration.

1. Introduction

Let (E, ∇) be an algebraic bundle with a flat connection on a smooth complex variety X. (We know that the flatness condition implies that the bundle is algebraic [2].) Then (E, ∇) has functorial and additive Chern classes

$$c_i(E, \nabla) \in H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(i))$$

mapping to the Chern classes $c_i^{\mathcal{D}}(E)$ of E in the subgroup $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(i))/F^i$ of the Deligne–Beilinson cohomology group $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ [4, (2.24)] and Theorem 3.5. There are also classes in $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(i))$ defined as the inverse image of the universal classes in $H^{2i-1}(\mathrm{GL}_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}(i))$ via locally constant transition functions (Notation 3.2). In fact, the two classes coincide (Theorem 3.4) and they are rigid if $i \geqslant 2$ (3.6).

When (E, ∇) is trivialized on a finite covering $\pi: Y \to X$, then the class $c_i(E, \nabla)$ lies in the torsion subgroup $H^{2i-1}(X_{\text{an}}, \mathbb{Q}(i)/\mathbb{Z}(i))$ of $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(i))$ (Lemma 4.1). The groups $H^{2i-1}(X_{\text{an}}, \mathbb{Q}(i)/\mathbb{Z}(i))$ have a filtration L defined by the Leray spectral sequence associated to the continuous identity map

$$\alpha: X_{\text{an}} \to X_{\text{zar}}.$$

By [1, (6.2)] there is a surjection

$$H^{i-1}(X_{\text{zar}}, \mathcal{H}^i(\mathbb{Q}(i)/\mathbb{Z}(i))) \to L^{i-1}H^{2i-1}(X_{\text{an}}, \mathbb{Q}(i)/\mathbb{Z}(i))$$

and L^{i-1} is the bottom part of the filtration, where $\mathcal{H}$ is the Zariski sheaf associated to the presheaf $U \mapsto H^i(U_{\text{an}})$. We prove that

$$c_i(E, \nabla) \in L^{i-1}H^{2i-1}(X_{\text{an}}, \mathbb{Q}(i)/Z(i))$$

(Theorem 4.2). Comparing $c_i(E, \nabla)$ with Galois Chern classes

$$c_i^{\mathrm{gal}}(E) \in H^{2i-1}(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z})$$

when (E, ∇) is associated to a finite representation of the fundamental group (Theorem 5.1), we obtain that

$$c_i^{\mathrm{gal}}(E) \in L^{i-1}H^{2i-1}(X_{\text{ét}}, \mathbb{Q}(i)/\mathbb{Z}(i)),$$

where L is the filtration induced by the Leray spectral sequence associated to the continuous identity map

$$\beta\colon X_{\text{ét}} \to X_{\mathrm{zar}}$$

(Theorem 5.2). This answers positively in the case $k = \mathbb{C}$ and X smooth a question by B. Kahn [8, conjecture 2], without assuming, however, that Kato's generalized conjecture is true in degree $\leqslant i$, as formulated in *loc. cit.* It also gives another proof of [8, théorème 1] for F a complex function field (in which case [8, théorème 1] is straightforward).

2. Class in the $(2i-1)$th Cohomology of X_{an} with $\mathbb{C}(i)/\mathbb{Z}(i)$ Values

We keep the notations of the introduction. In [5, (1.7), (1.5)], we constructed for any bundle (E, ∇) with a flat connection functorial and additive classes

$$c_i^{\mathrm{alg}}(E, \nabla) \in \mathbb{H}^{2i}(X_{\mathrm{zar}}, \mathcal{K}_i^m \to \Omega_X^i \to \Omega_X^{i+1} \to \cdots)$$

mapping to

$$c_i(E, \nabla) \in H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

defined in [4, (2.24)] (and also mapping to the Chern classes in the Chow group $CH^i(X)$), where

$$\mathcal{K}_i^m = \operatorname{Im} \mathcal{K}_i^M \to K_i^M(\mathbb{C}(X))$$

is the sheaf of modified Milnor K-theory as introduced by O. Gabber [7] and M. Rost [13]. There is a factorization

$$\begin{aligned}
&\mathbb{H}^{2i}(X_{\mathrm{zar}}, \mathcal{K}_i^m \to \Omega_X^i \to \Omega_X^{i+1} \to \cdots)\\
&\xrightarrow{\varphi} \mathbb{H}^{2i}(X_{\mathrm{zar}}, \Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}} \to \cdots)\\
&\xrightarrow{\psi} H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))
\end{aligned}$$

[5, (1.5) Proof], where

$$\Omega^i_{\mathbb{Z}(i)} = \mathrm{Ker}(\alpha_*\Omega^i_{X_{\mathrm{an}}})_{d\mathrm{closed}} \to \mathcal{H}^i(\mathbb{C}/\mathbb{Z}(i)),$$
$$(\Omega^i_{\mathbb{Q}(i)} = \mathrm{Ker}(\alpha_*\Omega^i_{X_{\mathrm{an}}})_{d\mathrm{closed}} \to \mathcal{H}^i(\mathbb{C}/\mathbb{Q}(i))),$$

and $\alpha: X_{\mathrm{an}} \to X_{\mathrm{zar}}$ is as in the introduction. The complex

$$\mathcal{D}^i_X := \Omega^i_{\mathbb{Z}(i)} \to \alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}} \to \cdots$$

is an extension of

$$0 \to 0 \to \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}}/\alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+2}_{X_{\mathrm{an}}} \to \cdots$$

by $\mathcal{H}^i(\mathbb{C})/\mathcal{H}^i(\mathbb{Z}(i))[-1]$ [5, (1.5). Proof]. We denote by

$$c'_i(E, \nabla) \in \mathbb{H}^{i-2}(X_{\mathrm{zar}}, \alpha_*\Omega^{i+1}_{X_{\mathrm{an}}}/\alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+2}_{X_{\mathrm{an}}} \to \cdots)$$

the image of $\varphi c^{\mathrm{alg}}_i(E, \nabla)$ in this group.

PROPOSITION 2.1. *If there is a smooth variety Y covering X via a finite map $\pi: Y \to X$ such that $\pi^*(E, \nabla)$ is trivial, then*

$$\deg \pi \cdot c_i(E, \nabla) = 0,$$

Proof. The trace maps $\pi_*\alpha_*(\Omega^i_{Y_{\mathrm{an}}})_{d\mathrm{closed}}$ to $(\alpha_*\Omega^i_{X_{\mathrm{an}}})_{d\mathrm{closed}}$ and $\pi_*\mathcal{H}^i(Y, \mathbb{Z}(i))$ to $\mathcal{H}^i(X, \mathbb{Z}(i))$. Thus, it maps $\pi_*\Omega^i_{\mathbb{Z}(i)}$ to $\Omega^i_{\mathbb{Z}(i)}$. The composite map

$$\mathcal{D}^i_X \xrightarrow{\pi^*} R\pi_*\mathcal{D}^i_Y = (\pi_*\Omega^i_{\mathbb{Z}(i)} \to \pi_*\alpha_*\Omega^i_{Y_{\mathrm{an}}} \to \pi_*\alpha_*\Omega^{i+1}_{Y_{\mathrm{an}}} \to \cdots) \xrightarrow{\mathrm{trace}} \mathcal{D}^i_X$$

is the multiplication by $\deg \pi$, and

$$\pi^*\varphi c^{\mathrm{alg}}_i(E, \nabla) = \varphi\pi^* c^{\mathrm{alg}}_i(E, \nabla) = 0.$$

THEOREM 2.2. *Let (E, ∇) be a bundle with a flat connection on a smooth complex variety X such that (E, ∇) is trivialized on a variety Y covering X via a finite map $\pi: Y \to X$. Then*

$$c_i(E, \nabla) \in L^{i-1}H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)).$$

Proof. The complex

$$\alpha_*\Omega^{i+1}_{X_{\mathrm{an}}}/\alpha_*\Omega^i_{X_{\mathrm{an}}} \to \alpha_*\Omega^{i+2}_{X_{\mathrm{an}}} \to \cdots$$

is quasi-isomorphic to $R\alpha_*\mathbb{C}/\tau_{\leqslant i}R\alpha_*\mathbb{C}[i+1]$, and by [1, (6.2)]

$$H^{2i-1}(X_{\text{an}},\ \mathbb{C}) = H^{2i-1}(X_{\text{zar}},\ R\alpha_*\mathbb{C}/\tau_{\leqslant(i-1)}R\alpha_*\mathbb{C}).$$

Therefore, the short exact sequence

$$0 \to \mathcal{H}^i(\mathbb{C})[-i] \to R\alpha_*\mathbb{C}/\tau_{\leqslant(i-1)}R\alpha_*\mathbb{C} \to R\alpha_*\mathbb{C}/\tau_{\leqslant i}R\alpha_*\mathbb{C} \to 0$$

gives a long exact sequence

$$\begin{aligned} \to H^{i-1}(X_{\text{zar}},\ \mathcal{H}^i(\mathbb{C})) &\to H^{2i-1}(X_{\text{an}},\ \mathbb{C}) \\ &\to H^{2i-1}(X_{\text{zar}},\ R\alpha_*\mathbb{C}/\tau_{\leqslant i}R\alpha_*\mathbb{C}) \\ &\to H^i(X_{\text{zar}},\ \mathcal{H}^i(\mathbb{C})). \end{aligned}$$

We first assume that Y is smooth. As the class of E in $H^i(X_{\text{zar}},\ \mathcal{K}_i^m)$ is torsion, it is vanishing in $H^i(X_{\text{zar}},\ \mathcal{H}^i(\mathbb{C}))$. Thus, $c_i'(E,\ \nabla)$ is a torsion class in the torsion-free group $H^{2i-1}(X_{\text{an}},\ \mathbb{C})/H^{i-1}(X_{\text{zar}},\ \mathcal{H}^i(\mathbb{C}))$, and $c_i'(E,\ \nabla) = 0$. So $\varphi c_i^{\text{alg}}(E,\ \nabla)$ lies in the image of

$$H^{i-1}(X_{\text{zar}},\ \mathcal{H}^i(\mathbb{C})/\mathcal{H}^i(\mathbb{Z}(i)) \subset \mathbb{H}^{2i}(X_{\text{an}},\ \mathcal{D}_X^i),$$

and $\psi\varphi c_i^{\text{alg}}(E,\ \nabla) = c_i(E,\ \nabla)$ lies in the image of

$$H^{i-1}(X_{\text{zar}}, \mathcal{H}^i(\mathbb{C})/\mathcal{H}^i(\mathbb{Z}(i))) \subset H^{2i-1}(X_{\text{zar}}, \mathbb{C}/\mathbb{Z}(i)).$$

That is,

$$c_i(E,\ \nabla) \in L^{i-1}H^{2i-1}(X_{\text{an}},\ \mathbb{C}/\mathbb{Z}(i))$$

[1, (6.2)].

Now if Y is no longer smooth, we can say the following. Let

$$\begin{array}{ccc} Y' & \xrightarrow{\tau} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\sigma} & X \end{array}$$

be a commutative diagram, with X', Y' smooth, π' finite, σ birational, proper and τ generically finite [9, (19) Proof]. In fact, if σ is any desingularization of the discriminant of π such that it becomes a normal crossing divisor, there is such a π'. Then $\sigma^*\varphi c_i^{\text{alg}}(E,\ \nabla)$ is torsion by Proposition 2.1 applied to $\sigma^*(E,\ \nabla)$ and

$\sigma^* c_i'(E, \nabla) = 0$. This implies that $\sigma^* c_i^{\mathrm{alg}}(E, \nabla) = c_i^{\mathrm{alg}}(\sigma^*(E, \nabla))$ maps to the image of

$$\mathbb{H}^i(X'_{\mathrm{zar}}, \mathcal{K}_i^m \to (\alpha_* \Omega^i_{X_{\mathrm{an}}})_{d\mathrm{closed}})$$

in

$$\mathbb{H}^i(X'_{\mathrm{zar}}, \mathcal{K}_i^m \to (\alpha_* \Omega^i_{X_{\mathrm{an}}}) \to (\alpha_* \Omega^{i+1}_{X_{\mathrm{an}}}) \to \cdots),$$

and, therefore,

$$\sigma^* c_i(E, \nabla) \in L^{i-1} H^{2i-1}(X'_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)).$$

In other words, there is a subscheme Z of X' of dimension $\geqslant (i-1)$ such that

$$\sigma^* c_i(E, \nabla)|_{(X'-Z)} \in H^{2i-1}((X'-Z)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

is zero. Thus, a-fortiori,

$$\sigma^* c_i(E, \nabla)|_{\sigma^{-1}(X-\sigma_* Z)} \in H^{2i-1}(\sigma^{-1}(X - \sigma_* Z)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

is zero. Here $\sigma_* Z$ is a subscheme of X of codimension $\geqslant (i-1)$ as well.

Take σ to be a succession of blow ups with smooth centers. Then one sees, successively on each blow up, that

$$\sigma^* \colon H^{2i-1}((X - \sigma_* Z)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)) \to H^{2i-1}(\sigma^{-1}(X - \sigma_* Z)_{\mathrm{an}}, \mathbb{C}/Z(i))$$

is injective. This shows that the restriction of $c_i(E, \nabla)$ to $(X - \sigma_* Z)$ is zero.

Remark 2.3. In fact, this cumbersome detour comes from the fact that if Y is singular, one can define $\Omega^i_{\mathbb{Z}(i)}$ thanks to [3, (9.3.1) (c), (d)]: there is a splitting

$$\mathcal{H}^i(\mathbb{C}) \to \mathcal{H}^i(\Omega^\bullet_{Y_{\mathrm{an}}}) \to \mathcal{H}^i(\mathbb{C}),$$

and, therefore, a surjection

$$(\alpha_* \Omega^i_{Y_{\mathrm{an}}})_{d\mathrm{closed}} \to \mathcal{H}^i(\mathbb{C}).$$

One defines

$$\Omega^i_{\mathbb{Z}(i)} := \mathrm{Ker}(\alpha_* \Omega^i_{Y_{\mathrm{an}}})_{d\mathrm{closed}} \to \mathcal{H}^i(\mathbb{C}/\mathbb{Z}(i)).$$

However, I don't know whether $\pi^* \varphi c_i^{\mathrm{alg}}(E, \nabla) = 0$, as it is not clear whether Gabber's projective bundle formula [5, (1.2) (c)] holds true when Y is singular, a necessary assumption to 'descent' the class $c_i^{\mathrm{alg}}(E, \nabla)$ from the flag bundle of E to Y.

3. Identification of Classes in the $(2i-1)$th Cohomology of X_{an} with $\mathbb{C}(i)/\mathbb{Z}(i)$ Coefficients

Let $(E,\ \nabla)$ be a bundle with a flat connection on a smooth complex variety X. One considers algebraic transition functions $g_{\alpha\beta} \in \Gamma(U_{\alpha\beta},\ \mathrm{GL}_n(\mathcal{O}))$ on a trivializing Zariski Cech cover $\{U_\alpha\}$, and locally constant transition functions $\lambda_{\alpha\beta} \in \Gamma(V_{\alpha\beta},\ \mathrm{GL}_n(\mathbb{C}))$ on an analytic refinement $(V_\beta)_{\beta\in J}$ of $(U_\alpha)_{\alpha\in I}$, with refinement map $\varphi\colon J \to I$. One has the following diagram

$$\begin{array}{ccc}
X_\bullet := U_\bullet & \xrightarrow{\quad g \quad} & BG_\bullet \\
\uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle e_a} \\
(X_\bullet)_{\mathrm{an}} := V_\bullet & \xrightarrow[e_d\circ\lambda]{g|(X_\bullet)_{\mathrm{an}}} & (BG_\bullet)_{\mathrm{an}} \\
& {\scriptstyle\lambda}\searrow & \uparrow{\scriptstyle e_d} \\
& & (BG_\bullet)_d
\end{array}$$

where $BG_\bullet$ is the simplicial scheme

$$(G^{\Delta_l}/G) \ \overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}}\ (G^{\Delta_{l-1}}/G) \ \overset{\longrightarrow}{\underset{\longrightarrow}{\equiv}}\ G^{\Delta_1}/G \ \rightrightarrows\ \{1\}$$

[3, (6.1.3)], $(BG_\bullet)_{\mathrm{an}}$ is the simplicial analytic manifold where (G^{Δ_l}/G) is viewed in the analytic topology, $(BG_\bullet)_d$ is the simplicial set where (G^{Δ_l}/G) is viewed in the discrete topology, and $G = \mathrm{GL}_n$. The maps are defined by

$$\begin{aligned}
g\colon U_{i_0 0\ldots i_l} &\to G^{\Delta_l}/G \\
x &\mapsto (g_{i_0 i_l(x)}, \ldots, g_{i_{l-1} i_l}(x))
\end{aligned}$$

$$\begin{aligned}
g|_{(X_\bullet)_{\mathrm{an}}}\colon V_{j_0\ldots j_l} &\to G^{\Delta_l}/G \\
x &\mapsto (g_{\varphi(j_0)\varphi(j_l)}(x), \ldots, g_{\varphi(j_{l-1})\varphi(j_l)}(x))
\end{aligned}$$

$$\begin{aligned}
\lambda\colon V_{j_0\ldots j_l} &\to G^{\Delta_l}/G \\
x &\mapsto (\lambda_{j_0 j_l}, \ldots, \lambda_{j_{l-1} j_l})
\end{aligned}$$

We set $e = e_d \circ e_a$. We denote by $H^j_{\mathcal{D}}((Y_\bullet)_{\mathrm{an}},\ \mathbb{Z}(i))$ the 'analytic' Deligne cohomology of the (simplicial) analytic manifold

$$H^j_{\mathcal{D}}((Y_\bullet)_{\mathrm{an}},\ \mathbb{Z}(i)) := \mathbb{H}^j((Y_\bullet)_{\mathrm{an}},\ \mathbb{Z}(i) \to \mathcal{O}_{(Y_\bullet)_{\mathrm{an}}} \to \cdots \to \Omega^{i-1}_{(Y_\bullet)_{\mathrm{an}}}).$$

PROPOSITION 3.1.

(1) $e_a^*\colon H^{2i}_{\mathcal{D}}(BG_\bullet, \mathbb{Z}(j)) \to H^{2i}_{\mathcal{D}}((BG_\bullet)_{\mathrm{an}}, \mathbb{Z}(j))$

is split for $j \leqslant i$.

(2) $$\begin{aligned} H^j_{\mathcal{D}}((BG_\bullet)_d, i) &= H^{j-1}(G, \mathbb{C}/\mathbb{Z}(i)), \quad && i \geqslant 1, \\ &= H^j(G, \mathbb{Z}), && i = 0. \end{aligned}$$

(3) *One has a factorization*

$$\begin{array}{ccc} (\mathbb{Z}(i)) \to \mathcal{O}_{(BG_\bullet)_{\mathrm{an}}} \to \cdots \to \Omega^{i-1}_{(BG_\bullet)_{\mathrm{an}}}) & \xrightarrow{(e_d \circ \lambda)^*} & R\lambda_*(\mathbb{Z}(i) \to \mathbb{C}) \\ \quad\searrow^{(e_d \circ \lambda)^*} & & \downarrow \\ & R\lambda_*(\mathbb{Z}(i)) \to \mathcal{O}_{(X_\bullet)\mathrm{an}} \to \cdots \to & \Omega^{i-1}_{(X_\bullet)_{\mathrm{an}}}) \end{array}$$

defining the commutative diagram

$$\begin{array}{ccc} H^j_{\mathcal{D}}((BG_\bullet)_{\mathrm{an}}, i) & \xrightarrow{(e_d \circ \lambda)^* = g|^*_{(X_\bullet)\mathrm{an}}} & H^j_{\mathcal{D}}((X_\bullet)_{\mathrm{an}}, i) \\ \downarrow e_d^* & & \uparrow \\ H^{j-1}(G, \mathbb{C}/\mathbb{Z}(i)) & \xrightarrow{\lambda^*} & H^{j-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)) \end{array}$$

for $i \geqslant 1$.

Proof.

(1) The cohomology of $BG_\bullet$ is pure of type (i, i) [3, (9.2)]. Therefore

$$H^{2i}_{\mathcal{D}}(BG_\bullet, \mathbb{Z}(j)) = 0$$

if $j > i$, and if $j \leqslant i$

$$H^{2i}_{\mathcal{D}}(BG_\bullet, \mathbb{Z}(j)) = H^{2i}((BG_\bullet), \mathbb{Z}(j))$$

is split in

$$\begin{aligned} &H^{2i}_{\mathcal{D}}((BG_\bullet)_{\mathrm{an}}, \mathbb{Z}(j)) \\ &\quad = \{(\varphi, z) \in \mathbb{H}^{2i}((BG_\bullet)_{\mathrm{an}}, \Omega^{\geqslant j}_{(BG_\bullet)_{\mathrm{an}}}) \times H^{2i}((BG_\bullet), \mathbb{Z}(j)) \\ &\qquad\quad \text{such that } \operatorname{Im} \varphi = \operatorname{Im} z \in H^{2i}((BG_\bullet)_{\mathrm{an}}, \mathbb{C})\}. \end{aligned}$$

(2) As the topology is discrete

$$\mathbb{Z}(i) \to \mathcal{O}_{(BG_\bullet)_d} \to \cdots \to \Omega^{i-1}_{(BG_\bullet)_d}$$

$$\begin{aligned} &= \mathbb{Z}, && i = 0, \\ &= \mathbb{Z}(i) \to \mathbb{C}, && i > 1. \end{aligned}$$

(3) The transition functions $g_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ describe the same bundle $E|X_{\rm an}$. So there are some $P_\alpha \in \Gamma(V_\alpha, \mathrm{GL}_n(\mathcal{O}_{X_{\rm an}}))$ such that $\lambda_{\alpha\beta} = P_\beta^{-1} g_{\alpha\beta} P_\alpha$. For any complexes of analytic sheaves $\mathcal{K}^\bullet$ on $(BG_\bullet)_{\rm an}$ and $\mathcal{L}^\bullet$ on $(X_\bullet)_{\rm an}$, such that

$$\begin{aligned} (e_d \circ \lambda)^*: &\quad \mathcal{K}^\bullet \to R(e_d \circ \lambda)_* \mathcal{L}^\bullet, \\ g|^*_{(X_\bullet)_{\rm an}}: &\quad \mathcal{K}^\bullet \to R g_* \mathcal{L}^\bullet, \end{aligned}$$

one has

$$(e_d \circ \lambda)^* = g|^*_{(X_\bullet)_{\rm an}}: \mathbb{H}^j((BG_\bullet)_{\rm an}, \mathcal{K}^\bullet) \to \mathbb{H}^j((X_\bullet)_{\rm an}, \mathcal{L}^\bullet).$$

In particular for

$$\begin{aligned} \mathcal{K}^\bullet &= \mathbb{Z}(i) \to \mathcal{O}_{(BG_\bullet)_{\rm an}} \to \cdots \to \Omega^{i-1}_{(BG_\bullet)_{\rm an}}, \\ \mathcal{L}^\bullet &= \mathbb{Z}(i) \to \mathcal{O}_{(X_\bullet)_{\rm an}} \to \cdots \to \Omega^{i-1}_{(X_\bullet)_{\rm an}}. \end{aligned}$$

As λ is constant, $(e_d \circ \lambda)^{-1}$ maps $\mathcal{O}_{(BG_\bullet)_{\rm an}}$ to $\mathbb{C}$ and $\Omega^j_{(BG_\bullet)_{\rm an}}$ to zero for $j > 0$. This shows the factorization, and thereby the commutative diagram.

NOTATION 3.2. We denote by

$$c_i \in H^{2i}_{\mathcal{D}}(BG_\bullet, \mathbb{Z}(i)) = H^{2i}(BG_\bullet, \mathbb{Z}(i))$$

the Chern class of the universal bundle $(G^{\Delta_l} \times_G \mathbb{C}^n)_\bullet$ over $(G^{\Delta_l}/G)_\bullet = BG_\bullet$, by $b_i = e^* c_i \in H^{2i-1}(G, \mathbb{C}/\mathbb{Z}(i))$ $(i \geqslant 1)$ the inverse image, by $c_i^{\mathcal{D}}(E) \in H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ the Chern classes of E in the Deligne–Beilinson cohomology, by

$$\lambda^* b_i \in H^{2i-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(i))$$

the inverse image of b_i on $X_{\rm an}$, which we can also view as the inverse image via $(e_d \circ \lambda)^*$ of $e_a^* c_i \in H^{2i}_{\mathcal{D}}((BG_\bullet)_{\rm an}, \mathbb{Z}(i))$ through the factorization 3.1(3). Again λ^* and $(e_d \circ \lambda)^*$ do not depend on the locally constant transition functions chosen.

COROLLARY 3.3.

(1) *The image of* $\lambda^* b_i$ *in* $H^{2i}_{\mathcal{D}}((X_\bullet)_{\rm an}, \mathbb{Z}(i))$ *coincides with the image of* $c_i^{\mathcal{D}}(E)$ *in* $H^{2i}_{\mathcal{D}}((X_\bullet)_{\rm an}, \mathbb{Z}(i))$.

(2) *In particular, if* X *is proper,* $\lambda^* b_i$ *lifts* $c_i^{\mathcal{D}}(E)$.

Proof. (1) follows from Proposition 3.1, (3) and (2), from the fact that if X is proper,

$$\varphi^*: H^{2i}_{\mathcal{D}}(X_\bullet,\ \mathbb{Z}(i)) \to H^{2i}_{\mathcal{D}}((X_\bullet)_{\text{an}},\ \mathbb{Z}(i))$$

is an isomorphism.

THEOREM 3.4. *One has* $c_i(E,\ \nabla) = \lambda^* b_i$.

Proof. Let $\pi: Y \to X$ be the flag bundle of E, endowed with a splitting $\tau: \Omega^1_Y \to \pi^*\Omega^1_X$ of $i: \pi^*\Omega^1_X \hookrightarrow \Omega^1_Y$, such that τ extends to a map of complexes

$$\tau: (\Omega^\bullet_Y,\ d) \to (\pi^*\Omega^\bullet_X.\ \tau \circ d \circ i),$$

and such that E has a $\tau \circ \nabla$ stable filtration $E_{i-1} \subset E_i$, with E_i/E_{i-1} of rank 1 [4, (2.7)]. One considers a Cech cover of Y obtained by taking the standard (Zariski) Cech cover F^α of F = flag bundle of $\mathbb{C}^n$, a trivialization $Y|_{V_j} \simeq F \times V_j$:

$$Y_a = Y^\alpha_j = F^\alpha \times V_j, \quad a = \begin{pmatrix} \alpha \\ j \end{pmatrix}.$$

Set $\pi(a) = j$. This defines $\pi: (Y_\bullet)_{\text{an}} \to (X_\bullet)_{\text{an}} = (V_\bullet)$. On Y one defines the analytic sheaf $L = \text{Ker}\ \tau \circ d: \mathcal{O}_{Y_{\text{an}}} \to \pi^*\Omega^1_{X_{\text{an}}}$ containing $\mathbb{C}$. Then there are

$$Q_a \in \Gamma(Y_a,\ \text{GL}_n(L)),\ \mu_{\alpha\beta} \in \Gamma(Y_{ab},\ B(L))$$

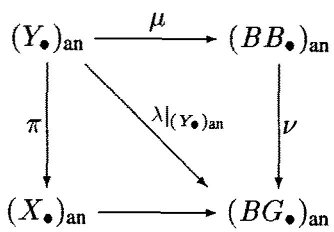
with

$$(\lambda|_{(Y_\bullet)_{\text{an}}})_{ab} = Q_b^{-1}\mu_{ab}Q_a,$$

where

$$(\lambda|_{(Y_\bullet)_{\text{an}}})_{ab} = \lambda_{\pi(a)\pi(b)},$$

and $B \subset \text{GL}_n$ is the Borel subgroup of upper triangular matrices. This defines the diagram



where ν is the natural embedding, and

$$\mu: Y_{a_0 \dots a_l} \to B^{\Delta_l}/B,$$
$$y \mapsto (\mu_{a_0 a_l}(y), \dots, \mu_{a_{l-1} a_l}(y)).$$

Here, to simplify the notations, we dropped e_d.

For any complexes of analytic sheaves $\mathcal{K}^\bullet$ on $(BG_\bullet)_{\text{an}}$ and $\mathcal{L}^\bullet$ on $(Y_\bullet)_{\text{an}}$ such that

$$\lambda|_{(Y_\bullet)_{\text{an}}}\colon \quad \mathcal{K}^\bullet \to R\lambda|_{(Y_\bullet)^*_{\text{an}}}\mathcal{L}^\bullet,$$
$$(\nu \circ \mu)^*\colon \quad \mathcal{K}^\bullet \to R(\nu \circ \mu)_* \mathcal{L}^\bullet,$$

one has

$$\lambda|^*_{(Y_\bullet)_{\text{an}}} = (\nu \circ \mu)^* = \mathbb{H}^j((BG_\bullet)_{\text{an}}, \mathcal{K}^\bullet) \to \mathbb{H}^j((Y_\bullet)_{\text{an}}, \mathcal{L}^\bullet),$$

in particular for

$$\mathcal{K}^\bullet = \mathbb{Z}(i) \to \mathcal{O}_{(BG_\bullet)_{\text{an}}} \to \cdots \to \Omega^{i-1}_{(BG_\bullet)_{\text{an}}}$$

and

$$\mathcal{L}^\bullet = \mathbb{Z}(i) \to L.$$

From the factorization of $\lambda|^*_{(Y_\bullet)_{\text{an}}}$ through $\pi^*\mathbb{H}^j((X_\bullet)_{\text{an}}, \mathbb{Z}(i) \to \mathbb{C})$, one obtains

$$(\nu \circ \mu)^* e_a^* c_i = \text{Image of } \pi^* \lambda^* b_i \quad \text{in} \quad \mathbb{H}^{2i}((Y_\bullet)_{\text{an}}, \mathbb{Z}(i) \to L).$$

Now, one just has to identify $(\nu \circ \mu)^* e_a^* c_i$ with

$$c_i(\pi^*(E, \nabla)) = \text{Image of } \pi^* c_i(E, \nabla) \quad \text{in} \quad \mathbb{H}^{2i}(Y_{\text{an}}, Z(i) \to L),$$

[4, (2.10)] as $\mathbb{H}^{2i}(X_{\text{an}}, \mathbb{Z}(i) \to L)$ injects into $\mathbb{H}^{2i}(Y_{\text{an}}, \mathbb{Z}(i) \to L)$ [4, (2.14)] and [5, (1.7)] for a more precise proof).

The class $(\nu \circ \mu)^* e_a^* c_i$ is given by the ith symmetric product of the classes of E_j/E_{j-1} in $\mathbb{H}^2(Y, \mathbb{Z}(1) \to L)$, where the product is just the Deligne product on the complexes

$$\mathbb{Z}(i) \to \mathcal{O}_{Y_{\text{an}}} \to \pi^*\Omega^1_{Y_{\text{an}}} \to \cdots \to \pi^*\Omega^{i-1}_{Y_{\text{an}}},$$

restricted to the subcomplexes

$$\mathbb{Z}(i) \to L.$$

But the restriction of the Deligne product is just the multiplication by the $\mathbb{Z}(i)$ term, that is the multiplication by the Betti class of E_j/E_{j-1} in $H^2(Y_{\rm an}, \mathbb{Z}(1))$. This is exactly the definition of $(c_i\pi^*(E, \nabla))$ [4, (2.9)].

THEOREM 3.5. *The image of* $c_i(E, \nabla) = \lambda^* b_i$ *3.4 in* $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ *is* $c_i^{\mathcal{D}}(E)$.

Proof. In [4] we constructed $c_i(E, \nabla)$ as a lifting of the image of $c_i^{\mathcal{D}}(E)$ in $H^{2i}_{\mathcal{D}}(X_{\rm an}, \mathbb{Z}(i))$. In particular, if X is proper, Theorem 3.5 is just be construction, and in fact for $\lambda^* b_i$, 3.5 is just Proposition 3.1, (3). If X is not proper, one considers a smooth compactification $\bar{X}$ such that $D = \bar{X} - X$ is a normal crossing divisor, and an extension $(\bar{E}, \bar{\nabla})$ of (E, ∇) as a bundle with an integrable connection with logarithmic poles along D. Then the classes $c_i(\bar{E}, \bar{\nabla})$ in

$$\begin{aligned}\mathbb{H}^{2i}(\bar{X}_{\rm an}, \mathbb{Z}(i) &\to \mathcal{O}_{\bar{X}_{\rm an}} \to \cdots \to \Omega^{i-1}_{\bar{X}_{\rm an}} \to \Omega^{i}_{\bar{X}_{\rm an}}(\log D)\\ &\to \cdots \to \Omega^{\dim X}_{\bar{X}_{\rm an}}(\log D))\end{aligned}$$

constructed in [4, (3.6)] lift $c_i^{\mathcal{D}}(\bar{E}) \in H^{2i}_{\mathcal{D}}(\bar{X}, \mathbb{Z}(i))$. Therefore the restriction $c_i(E, \nabla)$ of $c_i(\bar{E}, \bar{\nabla})$ to X lifts the restriction $c_i^{\mathcal{D}}(E)$ of $c_i^{\mathcal{D}}(\bar{E})$ to X.

We write down a corollary which we do not use in the sequel, but which is related to Reznikov's work [11, 12]. One can compare it to the same statement in which X is proper but ∇ is not integrable [6].

COROLLARY 3.6. *Let* S *be a complex variety, and let* $(\mathcal{E}, \nabla)$ *be a bundle on* $X \times S$ *with an integrable connection with values in* $\Omega^1_{X\times S/S}$. *Then the map*

$$\begin{aligned} S &\to H^{2i-1}(X_{\rm an}, \mathbb{C}/\mathbb{Z}(i)),\\ s &\mapsto c_i((\mathcal{E}, \nabla)|_{X\times\{s\}})\end{aligned}$$

is constant.

Proof. One may assume that S is an affine smooth curve. Let $\pi: X \times S \to S$ be the projection. There is an analytic product Cech cover $V_i \times S_j$ of $X \times S$, trivializing $\mathcal{E}$ such that the transition functions $\lambda_{\alpha\beta} \in \Gamma(V_{ii'} \times S_{jj'}, \mathrm{GL}_n(\pi^{-1}\mathcal{O}_{S_{\rm an}}))$, where $\alpha = (i, j)$, $\beta = (i', j')$. The map $a \mapsto j$ defines the map $\pi: (X \times S)_{\bullet \rm an} \to (S_\bullet)_{\rm an}$, and

$$\begin{aligned}\lambda : U_{\alpha_0\ldots\alpha_l} = V_{i_0\ldots i_l} \times S_{j_0\ldots j_l} &\to G^{\Delta_l}/G,\\ (x, s) &\mapsto (\lambda_{\alpha_0\alpha_l}(s), \ldots, \lambda_{\alpha_{l-1}\alpha_l}(s))\end{aligned}$$

defines a map $\lambda : (X \times S)_{\bullet \rm an} \to (BG_\bullet)_{\rm an}$ with a factorization

$$\begin{array}{ccc}\mathcal{K}^\bullet & \xrightarrow{\lambda^*} & R\lambda_*(\mathbb{Z}(i) \longrightarrow \pi^{-1}\mathcal{O}_{S_{\rm an}} \longrightarrow \pi^{-1}\Omega^1_{S_{\rm an}})\\ & & \big\downarrow \\ & & R\lambda_*\mathcal{L}^\bullet\end{array}$$

for

$$\begin{aligned}\mathcal{K}^\bullet &= \mathbb{Z}(i) \to \mathcal{O}_{(BG_\bullet)_{\mathrm{an}}} \to \cdots \to \Omega^{i-1}_{(BG_\bullet)_{\mathrm{an}}},\\ \mathcal{L}^\bullet &= \mathbb{Z}(i) \to \mathcal{O}_{(X\times S)_{\bullet\mathrm{an}}} \to \cdots \to \Omega^{i-1}_{(X\times S)_{\bullet\mathrm{an}}},\end{aligned}$$

and $i \geqslant 2$, such that

$$\lambda^* e_a^* c_i|_{(X\times\{s\})_{\bullet\mathrm{an}}} \in H^{2i-1}((X\times\{s\})_{\bullet\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

is the class $c_i(E, \nabla)$, where $(E, \nabla) = (\mathcal{E}, \nabla)|_{X\times\{s\}}$ (Notation 3.2, and Theorem 3.4). But

$$\mathbb{Z}(i) \to \pi^{-1}\mathcal{O}_{S_{\mathrm{an}}} \to \pi^{-1}\Omega^1_{S_{\mathrm{an}}}$$

is quasi-isomorphic to $\mathbb{Z}(i) \to \mathbb{C}$. Therefore,

$$\lambda^* e_a^* c_i \in H^{2i-1}((X\times S)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)).$$

This shows that $c_i((\mathcal{E}, \nabla)|_{X\times\{s\}})$ does not depend on s.

4. Classes in the $(2i-1)$th Cohomology of X_{an} with $\mathbb{Q}(i)/\mathbb{Z}(i)$ Coefficients

LEMMA 4.1.

(1) *The natural map*

$$H^j(X_{\mathrm{an}}, \mathbb{Q}(i)/\mathbb{Z}(i)) \to H^j(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$$

is injective and identifies the left group with the torsion of the right group.

(2)

$$L^a H^j(X_{\mathrm{an}}, \mathbb{Q}(i)/\mathbb{Z}(i)) = H^j(X_{\mathrm{an}}, \mathbb{Q}(i)/\mathbb{Z}(i)) \cap L^a H^j(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i)).$$

Proof.

(1) For all j, $H^j(X_{\mathrm{an}}, \mathbb{C})$ surjects onto the torsion free group $H^j(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Q}(i))$, factorizing through $H^j(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(j))$.
(2) Moreover by [1, (6.4)], L^a is the coniveau filtration.

THEOREM 4.2. *Under the assumptions of Theorem 2.2 one has*

$$c_i(E, \nabla) \in L^{i-1}H^{2i-1}(X_{\mathrm{an}}, \mathbb{Q}(i)/\mathbb{Z}(i)).$$

5. Classes in the $(2i-1)$th Cohomology of $X_{\text{ét}}$ with $\mathbb{Q}/\mathbb{Z}$ Coefficients

Let (E, ∇) be a bundle with an integrable connection associated to a finite representation

$$\pi_1(X) \to \bar{\pi} \subset \mathrm{GL}(n, \mathbb{C})$$

of the fundamental group of a smooth complex variety. Then E has Chern classes

$$c_i^{\text{gal}}(E) \in H^{2i-1}(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z})$$

defined as follows [8, § 1]

One considers an étale Cech cover of X trivializing E, and the corresponding map

$$(X_\bullet)_{\text{ét}} \xrightarrow{\lambda_{\text{ét}}} B\bar{\pi}_\bullet \subset (BG_\bullet)_d$$

with the notations as in Section 2. On $B\bar{\pi}_\bullet$, one considers the bundle $(\bar{\pi}^{\Delta_l} \times_{\bar{\pi}} \mathbb{C}^n)_\bullet$, restriction of the bundle $(G^{\Delta_l} \times_G \mathbb{C}^n)_\bullet$ over $(BG_\bullet)_d = (G^{\Delta_l}/G)_\bullet$. It has Chern classes $\bar{\gamma}_i \in H^{2i}(\bar{\pi}, \mathbb{Z})$, restriction of the Chern classes $\gamma_i \in H^{2i}(G, \mathbb{Z})$ of $(G^{\Delta_l} \times_G \mathbb{C}^n)_\bullet$. One has

$$\begin{aligned} c_i^{\text{gal}}(E) &= \lambda_{\text{ét}}^*(\bar{\gamma}_i) \in H^{2i}(X_{\text{ét}}, \mathbb{Z}) = H^{2i-1}(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}) \\ &= \varinjlim_{N} H^{2i-1}\left(X_{\text{ét}}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \\ &= \varinjlim_{N} H^{2i-1}\left(X_{\text{an}}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \\ &= H^{2i-1}(X_{\text{an}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow[(2\pi\sqrt{-1})^i]{\sim} H^{2i-1}(X_{\text{an}}, \mathbb{Q}(i)/\mathbb{Z}(i)), \end{aligned}$$

where the first equality of cohomology groups comes from [1, (4.2.2)], [10, (III, 2.22)] and the third one from [10, (III, 3.12)].

THEOREM 5.1. *One has, via the identifications above*

$$(2\pi\sqrt{-1})^i c_i^{\text{gal}}(E) = c_i(E, \nabla).$$

Proof. One has $\gamma_i = \frac{1}{(2\pi\sqrt{-1})^i} e^* c_i$, viewing

$$c_i \in H_{\mathcal{D}}^{2i}(BG_\bullet, \mathbb{Z}(i)) = H^{2i}(BG_\bullet, \mathbb{Z}(i))$$

as a Betti class, and therefore $\gamma_i = \frac{1}{(2\pi\sqrt{-1})^i} \delta b_i$, where δ is the Bockstein map

$$H^{2i-1}(G, \mathbb{C}/\mathbb{Z}(i)) \to H^{2i}(G, \mathbb{Z}(i))$$

(the notations are as in Section 2). So one applies the commutative diagram

$$\begin{array}{ccccc}
H^{2i-1}(G,\, \mathbb{C}/\mathbb{Z}(i)) & & & & \\
\big\uparrow \cup & & & & \\
(\lambda^*)^{-1}H^{2i-1}(X_{\mathrm{an}},\, \mathbb{Q}(i)/\mathbb{Z}(i)) & \xrightarrow{\quad\lambda^*\quad} & H^{2i-1}(X_{\mathrm{an}},\, \mathbb{Q}(i)/\mathbb{Z}(i)) & & \\
\delta \big\downarrow & & \big\uparrow \cong (2\pi\sqrt{-1})^i & & \\
H^{2i}(G,\, \mathbb{Z}(i)) & \xrightarrow{\frac{1}{(2\pi\sqrt{-1})^i}\lambda^*_{\text{ét}}} & H^{2i}(X_{\text{ét}},\, \mathbb{Z}) \leftarrow H^{2i-1}(X_{\text{ét}},\, \mathbb{Q}/\mathbb{Z}). & &
\end{array}$$

THEOREM 5.2.

$$c_i^{\mathrm{gal}}(E) \in L^{i-1}H^{2i-1}(X_{\text{ét}},\, \mathbb{Q}/\mathbb{Z})$$

Proof. As L^a is the coniveau spectral sequence both on $H^j(X_{\text{ét}})$ and $H^j(X_{\mathrm{an}})$, one has an isomorphism

$$L^{i-1}H^{2i-1}(X_{\text{ét}},\, \mathbb{Q}/\mathbb{Z}) \xrightarrow[(2\pi\sqrt{-1})^i]{\sim} L^{i-1}H^{2i-1}(X_{\mathrm{an}},\, \mathbb{Q}(i)/\mathbb{Z}(i)).$$

One applies (5.1) and (2.2).

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