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Géométrie algébrique/*Algebraic Geometry*

Remarks on absolute de Rham and absolute Hodge cycles

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Abstract – Let X be a smooth proper variety over a field k of characteristic zero. If $\alpha \in F^i H^{2i}_{DR}(X/k)$ is the class of an algebraic cycle, then it has to be an absolute de Rham cycle of moderate growth, a notion we define and compare to Deligne's notion of an absolute Hodge cycle. In fact, absolute Hodge implies absolute de Rham, while it's not clear why the moderate growth is implied by the absoluteness condition.

Remarques sur les cycles absolument de de Rham et absolument de Hodge

Résumé – *Soit X une variété propre et lisse sur un corps k de caractéristique zéro. Si $\alpha \in H^{2i}_{DR}(X/k)$, est la classe d'un cycle algébrique, alors α doit être une classe absolument de de Rham à croissance modérée, une notion que nous définissons et comparons à celle d'un cycle absolument de Hodge due à Deligne. En fait, absolument de Hodge implique absolument de de Rham, alors qu'il n'est pas clair que la croissance modérée soit impliquée par la condition « absolument ».*

Version française abrégée – Un cycle algébrique sur une variété X propre et lisse sur un corps k de caractéristique zéro a une classe *absolument de de Rham* dans $H^{2i}_{DR}(X/k)$, qui est aussi, suivant Deligne, *absolument de Hodge*. En fait

PROPOSITION (*voir* corollary 6). – *Une classe absolument de Hodge est absolument de de Rham.*

La raison en est la dégénérescence en E_2 de la suite spectrale de Leray pour la cohomologie de Betti, la régularité du système de Gauss-Manin et la comparaison des suites spectrales de Leray pour les cohomologies de Betti et de de Rham.

Inversement il est connu ([3], (2.6)) qu'une classe Gauss-Manin plate, par exemple une classe absolument de de Rham, rationnelle relativement à un plongement $\sigma : k \to \mathbb{C}$, l'est relativement à tout autre plongement coïncidant avec σ sur le corps des constantes de k.

Par ailleurs, un cycle algébrique a aussi une classe dans $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$, relevant les classes dans $H^{2i}_{DR}(X/\mathbb{Q})$ et $F^i H^{2i}_{DR}(X/k)$. En fait, il n'est pas clair qu'une classe absolument de Hodge de $F^i H^{2i}_{DR}(X/k)$ doive être dans l'image de $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$. Nous introduisons un sous-groupe $F^i H^{2i}_{DR}(X/k)^{\log}$ *de croissance modérée* de $F^i H^{2i}_{DR}(X/k)$ et prouvons le

THÉORÈME (*voir* theorem 11). – *Un cycle modéré Gauss-Manin plat provient de* $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Outre les arguments plus haut, on applique la dégénérescence de la suite spectrale Hodge vers de Rham de [6] et [11].

Let X be a smooth proper variety over a field k of characteristic zero. For any embedding σ of k into the field of complex numbers $\mathbb{C}$, the $\mathbb{C}$ valued points of $X \otimes_\sigma \mathbb{C}$ form a complex manifold denoted by X_σ. By base change for the de Rham cohomology $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C} = H^j_{DR}(X \otimes_\sigma \mathbb{C}/\mathbb{C})$ and by the GAGA principle one has an isomorphism I_σ from $H^j_{DR}(X/k) \otimes_\sigma \mathbb{C}$ to the Betti cohomology $H^j_B(X_\sigma, \mathbb{C})$ ([5], p. 96).

An element of the $\mathbb{Q}$ Chow group $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a de Rham class

$$\alpha \in F^i H^{2i}_{DR}(X/k) = \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/k}) \subset H^{2i}_{DR}(X/k)$$

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such that for all embeddings $\sigma : k \to \mathbb{C}$

$$I_\sigma(\alpha) \in I_\sigma(F^i H^{2i}_{DR}(X/k) \otimes_\sigma \mathbb{C}) \cap H^{2i}_B(X_\sigma, \mathbb{Q}).$$

So α is an absolute Hodge cycle, a notion defined by Deligne [3], section 2, which we slightly modify, as we are only interested here in de Rham cohomology (*see* [3], open questions 2.2 and 2.4).

DEFINITION 1. – A class $\alpha \in F^i H^{2i}_{DR}(X/k)$ is said to be an *absolute Hodge cycle* if for all embeddings $\sigma : k \to \mathbb{C}$, $I_\sigma(\alpha)$ lies in $H^{2i}_B(X_\sigma, \mathbb{Q})$.

On the other hand, such an algebraic cycle has an absolute de Rham class in $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$. In fact, there is an absolute differential

$$d\log : \mathcal{O}^*_X \to \Omega^{\geq 1}_{X/\mathbb{Q}}[1]$$

inducing an absolute differential

$$d\log : \mathcal{K}^M_i \to \Omega^{\geq i}_{X/\mathbb{Q}}[i]$$

where $\mathcal{K}^M_i$ is the Zariski sheaf of Milnor K theory. As $CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = H^i(X, \mathcal{K}^M_i)$ ([9], théorème 5), $d\log$ induces the absolute de Rham cycle class map

$$CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\psi} \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}}).$$

One composes this map with

$$\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}}) \to \mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/k}) = F^i H^{2i}_{DR}(X/k)$$

to obtain the de Rham cycle class map. As we don't have a reference for this, we indicate how to prove it. By base change $F^i H^{2i}_{DR}(X/k) \otimes_\sigma \mathbb{C} = F^i H^{2i}_{DR}(X \otimes_\sigma \mathbb{C}/\mathbb{C})$, so it is enough to handle $k = \mathbb{C}$, in which case the compatibility is proven in [2], (2.2.5.1) and (2.2.5.2) for $i = 1$. For $i > 1$, resolving the structure sheaf of an effective cycle by vector bundles, and for a given vector bundle, computing its Chern classes on the Grassmannian bundle $G \xrightarrow{\pi} X$, with $\pi^* : F^i H^{2i}_{DR}(X/\mathbb{C}) \hookrightarrow F^i H^{2i}_{DR}(G/\mathbb{C})$, one reduces the compatibility to the case $i = 1$.

Remark 2. – The existence of the absolute de Rham cycle class is proved in great generality in [10] when X is singular. In fact, this class is convenient to formulate some questions. For example, its injectivity for a surface X over $k = \mathbb{C}$ would imply Bloch's conjecture when $H^2(X, \mathcal{O}_X) = 0$.

At any rate, the existence of ψ motivates the following

DEFINITION 3. – A class $\alpha \in F^i H^{2i}_{DR}(X/k)$ is said to be an *absolute de Rham cycle* if it lies in the image of $H^{2i}_{DR}(X/\mathbb{Q})$ in $H^{2i}_{DR}(X/k)$.

We denote by $\nabla : H^j_{DR}(X/k) \to \Omega^1_{k/\mathbb{Q}} \otimes_k H^j_{DR}(X/k)$ the Gauss-Manin connection for the smooth morphism $X \to \operatorname{Spec} k$ of schemes over $\operatorname{Spec} \mathbb{Q}$.

PROPOSITION 4. – *The sequence*

$$H^j_{DR}(X/\mathbb{Q}) \to H^j_{DR}(X/k) \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes H^j_{DR}(X/k)$$

is exact.

Proof. – The sequence is obviously a complex.

Let $k_0 \subset k$ be the field of definition of X. One has $X = X_0 \otimes_{k_0} k$, where X_0 is smooth proper over k_0, and $k_0 = \mathbb{Q}(S_0)$ for a smooth affine variety S_0 over $\mathbb{Q}$, such that there is a smooth proper map $f_0 : \mathcal{X}_0 \to S_0$ with $\mathcal{X}_0 \otimes_{\mathcal{O}_{S_0}} k_0 = X_0$.

As $H^j_{DR}(X_0/k_0)$ is a finite dimensional k_0 vector space, any

$$\alpha \in H^j_{DR}(X/k) = H^j_{DR}(X_0/k_0) \otimes_{k_0} k$$

lies in $H^j_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S)$, where $k_0 \subset \mathbb{Q}(S) \subset k$ and S is a smooth affine variety mapping to S_0. If $x \in \operatorname{Ker} \nabla$, then x lies in the kernel of

$$H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S)) \to \Omega^1_{\mathbb{Q}(S)/\mathbb{Q}} \otimes H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))$$

and to prove exactness, one has to see that

$$\alpha \in \operatorname{Im}(H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}) \to H^j_{DR}(X_0 \otimes_{k_0} \mathbb{Q}(S)/\mathbb{Q}(S))).$$

Denote by $f\colon \mathcal{X} = \mathcal{X}_0 \times_{S_0} S \to S$ the smooth proper morphism obtained by base change $S \to S_0$ of f_0. Making S smaller, one may assume that there is

$$\beta \in \operatorname{Ker}(H^j_{DR}(\mathcal{X}/S) \xrightarrow{\nabla} \Omega^1_{S/\mathbb{Q}} \otimes H^j_{DR}(\mathcal{X}/S))$$

such that $\beta \otimes_{\mathcal{O}_S} \mathbb{Q}(S) = \alpha$, and one wants to show that $\beta \in \operatorname{Im} H^j_{DR}(\mathcal{X}/\mathbb{Q})$.

On $\Omega^\bullet_{\mathcal{X}/\mathbb{Q}}$ one considers the filtration by the subcomplexes $f^*\Omega^{\geq a}_{S/\mathbb{Q}} \wedge \Omega^{\bullet - a}_{\mathcal{X}/\mathbb{Q}}$. It defines a spectral sequence

$$E_1^{ab} = \Omega^a_{S/\mathbb{Q}} \otimes H^b_{DR}(\mathcal{X}/S)$$

converging to $H^{a+b}_{DR}(\mathcal{X}/\mathbb{Q})$, whose d_1 differential is the Gauss-Manin connection ∇. As S is affine, one has

$$E_2^{ab} = \mathbb{H}^a(S, \Omega^\bullet_{S/\mathbb{Q}} \otimes H^b_{DR}(\mathcal{X}/S)).$$

We now consider the analytic varieties $S_{an} = (S \otimes_{\mathbb{Q}} \mathbb{C})_{an}$, $\mathcal{X}_{an} = (\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C})_{an}$. The corresponding spectral sequence

$$\begin{aligned} E^{ab}_{2,an} &= \mathbb{H}^a(S_{an}, \Omega^\bullet_{S_{an}} \otimes H^b_{DR}(\mathcal{X}_{an}/S_{an})) \\ &= \mathbb{H}^a(S_{an}, \Omega^\bullet_{S_{an}} \otimes R^b f_* \Omega^\bullet_{\mathcal{X}_{an}/S_{an}}) \\ &= H^a(S_{an}, R^b f_* \mathbb{C}) \end{aligned}$$

abuts to $\mathbb{H}^{a+b}(\mathcal{X}_{an}, \Omega^\bullet_{\mathcal{X}_{an}}) = H^{a+b}(\mathcal{X}_{an}, \mathbb{C})$. This spectral sequence is, according to Deligne ([11], (2.77) and (15.6)) the Leray spectral sequence, and by [2], (4.1.1) (i), it degenerates at E_2.

On the other hand, by the regularity of the Gauss-Manin connection, one has

$$\begin{aligned} E^{ab}_{2,an} &= \mathbb{H}^a(S \otimes_{\mathbb{Q}} \mathbb{C}, \Omega^\bullet_{S \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{C}} \otimes H^b_{DR}(\mathcal{X} \otimes_{\mathbb{Q}} \mathbb{C}/S \otimes_{\mathbb{Q}} \mathbb{C})) \\ &= E_2^{ab} \otimes_{\mathbb{Q}} \mathbb{C} \end{aligned}$$

([1], (6.2) and (7.9)).

This implies that $(E_1^{ab}, d_1) \otimes_{\mathbb{Q}} \mathbb{C}$ degenerates at E_2, and so does (E_1^{ab}, d_1). In particular

$$\begin{aligned} H^j_{DR}(\mathcal{X}/\mathbb{Q}) &= \mathbb{H}^0(S, \Omega^\bullet_{S/\mathbb{Q}} \otimes H^j_{DR}(\mathcal{X}/S)) \\ &= \operatorname{Ker}(H^0(S, H^j_{DR}(\mathcal{X}/S)) \to H^0(S, \Omega^1_{S/\mathbb{Q}} \otimes H^j_{DR}(\mathcal{X}/S))). \end{aligned}$$

This proves the required exactness by base change to $\mathbb{Q}(S)$.

Remark 5. – In fact, even if S is not affine, there is a Leray spectral sequence for the de Rham cohomology [7] (3.3), which again degenerates at E_2 by the comparison between the Leray spectral sequences for the Betti and the de Rham cohomologies, and the regularity of Gauss-Manin. For more on this, see [8].

Corollary 6. – *If α is an absolute Hodge cycle, then it is an absolute de Rham cycle.*

Proof. – By [3] (2.5), we know that $\nabla\alpha = 0$, where ∇ is as in (4) for $j = 2i$. Then we apply 4.

Corollary 7. – *If α is an absolute de Rham cycle such that* $I_\sigma(\alpha) \in H^{2i}_B(X_\sigma, \mathbb{Q})$ *for some embedding* $\sigma : k \to \mathbb{C}$, *then for any* $\tau : k \to \mathbb{C}$ *such that* $\sigma_{|K} = \tau_{|K}$, *where* $K = \mathrm{Ker}\ k \to \Omega^1_{k/\mathbb{Q}}$ *is the algebraic closure of* $\mathbb{Q}$ *in k, then* $I_\tau(\alpha) \in H^{2i}_B(X_\tau, \mathbb{Q})$.

Proof. – In fact, this is [3] (2.6). More precisely, choose S as in the proof of 4 and $\beta \in H^{2i}_{DR}(\mathcal{X}/S)$ restricting to α. The embeddings $\mathbb{Q}(S) \to k \xrightarrow{\sigma} \mathbb{C}$ define an embedding $\mathbb{Q}(S) \otimes_{\mathbb{Q}} K' \to \mathbb{C}$, where $K' = K \cap \mathbb{Q}(S)$, that is a generic $\mathbb{C}$ valued point of one component of the (non irreducible) variety $\mathrm{Spec}\, \mathbb{Q}[S] \otimes_{\mathbb{Q}} K'$. Call this component S^0, and $\mathcal{X}^0 \to S^0$ the restriction of $\mathcal{X} \to S$ to S^0. Then similarly, τ defines a generic $\mathbb{C}$ valued point of S^0 as well. Thus the image $\beta(\sigma)$ of β in

$$H^0(S^0_{an}, R^{2i} f_* \mathbb{C}) = H^{2i}((\mathcal{X}^0_{an})_\sigma, \mathbb{C})^{\pi_1(S_{an}, \sigma)}$$

lies in

$$H^0(S^0_{an}, R^{2i} f_* \mathbb{Q}) = H^{2i}((\mathcal{X}^0_{an})_\sigma, \mathbb{Q})^{\pi_1(S_{an}, \sigma)}.$$

Therefore $\beta|_{(\mathcal{X}^0_{an})_s}$ is rational for all s, in particular for those s coming from an embedding $\tau : k \to \mathbb{C}$ defining a point of S^0_{an}.

On the other hand, we have seen that if $\alpha \in F^i H^{2i}_{DR}(X/k)$ is the class of an algebraic cycle, then not only it is an absolute de Rham cycle, but also it is coming from $\mathbb{H}^{2i}(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Let $f : \mathcal{X} \to S$, $\beta \in F^i H^j_{DR}(\mathcal{X}/S) = H^0(S, R^j f_* \Omega^{\geq i}_{\mathcal{X}/S})$, such that

$$\beta \otimes_{\mathbb{Q}(S)} k = \alpha \in F^i H^j_{DR}(X/k)$$

as in the proof of 4. Let $f_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to S_{\mathbb{C}}$ be the smooth proper morphism obtained from f by base change $\mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{C}$, and $\beta_{\mathbb{C}}$ be $\beta \otimes_{\mathbb{Q}} \mathbb{C}$. Let $f'_{\mathbb{C}} : \mathcal{X}'_{\mathbb{C}} \to S'_{\mathbb{C}}$ be a compactification of $f_{\mathbb{C}}$ such that $\Sigma = S'_{\mathbb{C}} - S_{\mathbb{C}}$, $D = f'^{-1}_{\mathbb{C}}(\Sigma)$ are normal crossing divisors and $\mathcal{X}'_{\mathbb{C}}$ is smooth.

Définition 8. – A class $\alpha \in F^i H^j_{DR}(X/k)$ is said to be of *moderate growth* if for some $(\beta, f'_{\mathbb{C}})$ as above, it verifies

$$(\star) \qquad \beta_{\mathbb{C}} \in H^0(S'_{\mathbb{C}}, R^j f'_{\mathbb{C}*} \Omega^{\geq i}_{\mathcal{X}'_{\mathbb{C}}/S'_{\mathbb{C}}}(\log D)) \subset H^0(S_{\mathbb{C}}, R^j f_{\mathbb{C}*} \Omega^{\geq i}_{\mathcal{X}_{\mathbb{C}}/S_{\mathbb{C}}}).$$

Remark 9. – The definition 8 does not depend on the couple $(\beta, f'_{\mathbb{C}})$ choosen. In fact, take (γ, g) with $g : \mathcal{Y} \to T$, $\mathbb{Q}(T) \subset k$, $\mathcal{Y} \otimes_{\mathbb{Q}(T)} k = X$, $\gamma \otimes_{\mathbb{Q}(T)} k = \alpha$. Then considering in k a function field $\mathbb{Q}(U)$ containing $\mathbb{Q}(S)$ and $\mathbb{Q}(T)$, one has base changes $\sigma : U \to S$, $\tau : U \to T$, $f_U : \mathcal{X}_U = \mathcal{X} \times_S U \to U$, $g_U : \mathcal{Y}_U = \mathcal{Y} \times_T U \to U$, such that there is an isomorphism $\iota : \mathcal{X}_U \to \mathcal{Y}_U$, with $g_U \circ \iota = f_U$, $\iota^*(\gamma \otimes_{\mathcal{O}_T} \mathcal{O}_U) = \beta \otimes_{\mathcal{O}_S} \mathcal{O}_U$, for U small enough. As $\beta_{\mathbb{C}}$ fulfills $(\star)$ on $S'_{\mathbb{C}}$, it fulfills $(\star)$ on any blow up $\sigma'_{\mathbb{C}} : U'_{\mathbb{C}} \to S'_{\mathbb{C}}$ such that a commutative diagram exists

$$\begin{array}{ccc} \mathcal{X}'_{U,\mathbb{C}} & \to & \mathcal{X}'_{\mathbb{C}} \\ f'_{U,\mathbb{C}} \downarrow & & \downarrow f'_{\mathbb{C}} \\ U'_{\mathbb{C}} & \xrightarrow{\sigma'_{\mathbb{C}}} & S'_{\mathbb{C}} \end{array}$$

with the properties: $\sigma'^{-1}_{\mathbb{C}} \Sigma$, $\Delta = f'^{-1}_{U,\mathbb{C}} \sigma'^{-1}_{\mathbb{C}} \Sigma$ are normal crossing divisors, $\mathcal{X}'_{U,\mathbb{C}}$ and $U'_{\mathbb{C}}$ are smooth. Choose $U'_{\mathbb{C}}$ such that τ extends to $\tau'_{\mathbb{C}} : U'_{\mathbb{C}} \to T'_{\mathbb{C}}$, with a commutative diagram

$$\begin{array}{ccc} \mathcal{X}'_{U,\mathbb{C}} & \xrightarrow{\iota'_{\mathbb{C}}} & \mathcal{Y}'_{\mathbb{C}} \\ f'_{U,\mathbb{C}} \downarrow & & \downarrow g'_{\mathbb{C}} \\ U'_{\mathbb{C}} & \xrightarrow{\tau'_{\mathbb{C}}} & T'_{\mathbb{C}} \end{array}$$

with the same properties as above. One has now

$$H^0(U'_C, R^j f'_{U,C*}\Omega^{\geq i}_{\mathcal{X}'_{U,C}/U'_C}(\log\Delta)) = H^0(T'_C, R^j g'_{C*}\Omega^{\geq i}_{\mathcal{Y}'_C/T'_C}(\log g'^{-1}_C(T'_C - T'_C))$$

[6], (4.13).

This implies in particular that classes of moderate growth build a k subvectorspace of $F^i H^j_{DR}(X/k)$.

NOTATION 10. – We denote this subvectorspace by $F^i H^j_{DR}(X/k)^{\log}$, and by $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})^{\log}$ its inverse image in $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

THEOREM 11. – *The sequence*

$$\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})^{\log} \to F^i H^j_{DR}(X/k)^{\log} \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes F^{i-1} H^j_{DR}(X/k)$$

is exact.

Proof. – We have to prove that if $\alpha \in \operatorname{Ker}\nabla$, then it lies in the image of $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$. With the notations as above,

$$\beta_C \in H^0(S'_C, \Omega^\bullet_{S'_C}(\log\Sigma) \otimes R^j f'_{C*}\Omega^{\geq i-\bullet}_{\mathcal{X}'_C/S'_C}(\log D)).$$

This group is the E_2^{0j} term of a spectral sequence converging to $\mathbb{H}^j(\mathcal{X}'_C, \Omega^{\geq i}_{\mathcal{X}'_C/S'_C}(\log D))$ and defined as in [7] (3.3) on the complex $\Omega^{\geq i}_{\mathcal{X}'_C/S'_C}(\log D)$. One has

$$E_2^{ab} = \mathbb{H}^a(S'_C, \Omega^\bullet_{S'_C}(\log\Sigma) \otimes R^b f'_{C*}\Omega^{\geq i-\bullet}_{\mathcal{X}'_C/S'_C}(\log D)).$$

By [6] (0.4) and its analogue in characteristic zero [4] (2.7), E_2^{ab} injects into

$$\mathbb{H}^a(S'_C, \Omega^\bullet_{S'_C}(\log\Sigma) \otimes R^b f'_{C*}\Omega^\bullet_{\mathcal{X}'_C/S'_C}(\log D)),$$

which is just $H^a(S_{an}, R^b f_{C*}\mathbb{C})$ by [1] II, paragraph 6.

Thus the spectral sequence degenerates at E_2, and β_C comes from $\mathbb{H}^j(\mathcal{X}'_C, \Omega^{\geq i}_{\mathcal{X}'_C}(\log D))$. In particular β_C comes from $\mathbb{H}^j(\mathcal{X}, \Omega^{\geq i}_{\mathcal{X}/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ and the image of α in

$$\frac{F^i H^j_{DR}(X/k) \otimes_{\mathbb{Q}} \mathbb{C}}{\operatorname{Im}\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}}) \otimes \mathbb{C}} = \left(\frac{F^i H^j_{DR}(X/k)}{\operatorname{Im}\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})}\right) \otimes \mathbb{C}$$

vanishes. Therefore α lies in the image of $\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}})$.

Remark 12. – If the transcendence degree of k is $\leqq 1$, then of course the sequence

$$\mathbb{H}^j(X, \Omega^{\geq i}_{X/\mathbb{Q}}) \to F^i H^j_{DR}(X/k) \xrightarrow{\nabla} \Omega^1_{k/\mathbb{Q}} \otimes F^{i-1} H^j_{DR}(X/k)$$

is trivially exact. But if the transcendence degree of k is higher, it is not clear why an absolute Hodge cycle has to be a moderate absolute de Rham cycle.

More generally, one can consider a k subvectorspace V of $H^j_{DR}(X/k)$, such that $I_\sigma(V \otimes_\sigma \mathbb{C})$ is a Hodge substructure of $H^j_{DR}(X_\sigma, \mathbb{C})$. In the light of the above results, one can examine the following questions.

QUESTION 13. – Is V stable under the Gauss-Manin connection?

For this, one would like $I_\sigma^{-1}[I_\sigma(V \otimes_\sigma \mathbb{C}) \cap H^j_B(X_\sigma, \mathbb{Q})]$ to lie in V and to be independent of σ.

If so, then V defines a vector bundle $\mathcal{W}$ with a flat connection on S, where S is defined as in 4 such that $V = W \otimes_{\mathbb{Q}(S)} k$, $W \subset H^j_{DR}(X_0/k_0) \otimes_{k_0} \mathbb{Q}(S)$. Then $\mathcal{W}_{an}$ on S_{an} is generated by a local system $\mathcal{F}$.

QUESTION 14. – In the above situation, is the monodromy representation associated to $\mathcal{F}$ defined over $\mathbb{Q}$?

The knowledge of 13 does not imply the knowledge of 14.

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