Rigid Local Systems: Arithmetic Properties

Hélène Esnault
Freie Universität Berlin

Centennial of the **I**nternational **M**athematical **U**nion
September 27, 2021

Acknowledgements

We thank

the committee for the kind invitation, Michael Groechenig for our yearlong collaboration, him and Bertrand Rémy for their helpful comments on our slides.

Rigid Local Systems (RLS)

- X topological space, for us *complex algebraic variety*;
- irreducible *local system* (LS) $\overset{\text{dfn}}{=}$ irreducible
$\rho : \pi_1(X, x) \to GL_r(\mathbb{C})$ up to gauge transformation;
- i.e.: irreducible fibre bundle $\mathcal{V}_\rho \to X$ with locally constant
transition functions.

Rigid Local Systems (RLS)

• X topological space, for us *complex algebraic variety*;
• irreducible *local system* (LS) $\overset{\text{dfn}}{=}$ irreducible
$\rho : \pi_1(X, x) \to GL_r(\mathbb{C})$ up to gauge transformation;
• i.e.: irreducible fibre bundle $\mathcal{V}_\rho \to X$ with locally constant transition functions.

• ρ *rigid*: deformation $\rho_t : \pi_1(X, x) \to GL_r(\mathbb{C}[[t]])$ gauge equivalent to ρ_0, i.e. $\exists g_t \in GL_r(\mathbb{C}[[t]])$ with $\rho_t = g_t \rho_0 g_t^{-1}$, i.e. moduli point $[\mathcal{V}_\rho]$ isolated.
• May fix determinant and conjugacy classes of local monodromy at ∞ for X not proper.

Occurence

dim $(X) = 1$ Katz: $\exists$ RLS $\Longrightarrow$
• $X = \mathbb{P}^1 \setminus \{\text{finitely many points}\}$;
• moduli points have no multiplicity;
• all RLS *come from geometry*, i.e. 'like' (summand of)
$\mathcal{V}_{\rho,\tau\in\mathbb{P}^1\setminus\{\infty,\tau_1,\ldots,\tau_n\}} = H^1_c(Y_\tau)$, $Y_\tau \subset \mathbb{A}^2: \ y^N = (x-\tau)\prod_1^n(x-\tau_i)$.

Occurence

dim $(X) = 1$ Katz: $\exists$ RLS $\Longrightarrow$
- $X = \mathbb{P}^1 \setminus \{\text{finitely many points}\}$;
- moduli points have no multiplicity;
- all RLS *come from geometry*, i.e. 'like' (summand of)

$\mathcal{V}_{\rho,\tau\in\mathbb{P}^1\setminus\{\infty,\tau_1,\ldots,\tau_n\}} = H^1_c(Y_\tau),\ Y_\tau \subset \mathbb{A}^2:\ y^N = (x-\tau)\prod_1^n(x-\tau_i)$.

Shimura varieties of rank ≥ 2: Margulis superrigidity $\Longrightarrow$
- all irreducible LS are rigid;
- moduli points have no multiplicity;
- **but** we do not know whether they come from geometry.

Simpson's Geometricity Conjecture

Simpson's geometricity conjecture (1990)

all RLS come from geometry.

Simpson's Geometricity Conjecture

Simpson's geometricity conjecture (1990)

all RLS come from geometry.

Petrov's theorem (2020) inserts it in a vast arithmetic program

$$[\text{Fontaine-Mazur conj.} \xrightarrow{Scholze+Liu-Zhu} \text{Simpson conj.}]$$

$$/\text{number fields} \xrightarrow{p-adic\ \ fields} /\mathbb{C}.$$

Simpson's Geometricity Conjecture

Simpson's geometricity conjecture (1990)

all RLS come from geometry.

Petrov's theorem (2020) inserts it in a vast arithmetic program

$$[\text{Fontaine-Mazur conj.} \xRightarrow{Scholze+Liu-Zhu} \text{Simpson conj.}]$$

$$/\text{number fields} \quad \xRightarrow{p-adic\ \ fields} /\mathbb{C}.$$

Geometricity conjecture inacessible, except in dim. 1 (Katz).
Instead study consequences.

Consequences of Simpson's geometricity conjecture

Integrality conjecture (Simpson 1990)

RLS are integral, i.e. $\rho : \pi_1(X, x) \to GL_r(\mathcal{O})$, $\mathcal{O}$ number ring.
$\ll$ Betti cohomology is defined over $\mathbb{Z}$. $\gg$

Consequences of Simpson's geometricity conjecture

Integrality conjecture (Simpson 1990)

RLS are integral, i.e. $\rho : \pi_1(X, x) \to GL_r(\mathcal{O})$, $\mathcal{O}$ number ring.
$\ll$ Betti cohomology is defined over $\mathbb{Z}$. $\gg$

Crystalline conjecture (E-Groechenig 2018)

connection corresponding via the Riemann-Hilbert correspondence to a RLS $/X_{\mathbb{Q}_q}$ for a.a. p is
i) an isocrystal with a Frobenius structure;
ii) underlying a p-adic crystalline representation (Fontaine).
$\ll$ Gauß-Manin connections are so by Deligne. $\gg$

The theorems

Integrality true (E-Groechenig 2018)

- for RLS with moduli point with no multiplicity;
- when (in a given rank) all irreducible LS are RLS.

The theorems

Integrality true (E-Groechenig 2018)

- for RLS with moduli point with no multiplicity;
- when (in a given rank) all irreducible LS are RLS.

Crystallinity true (E-Groechenig 2018 and 2021)

- when X proper: i);
- when (in rank $\leq r$) all irreducible LS (of rank $\leq r$) are RLS with moduli point with no multiplicity: i) and ii).

The theorems

Integrality true (E-Groechenig 2018)

- for RLS with moduli point with no multiplicity;
- when (in a given rank) all irreducible LS are RLS.

Crystallinity true (E-Groechenig 2018 and 2021)

- when X proper: i);
- when (in rank $\leq r$) all irreducible LS (of rank $\leq r$) are RLS with moduli point with no multiplicity: i) and ii).

Remark

- Not a single RLS known with moduli point with higher multiplicity.
- Crystallinity applies for all proper or rank ≥ 2 Shimura varieties.

Methods

Integrality

uses Langlands program (Deligne's companions' conjecture, proven by L. Lafforgue and Drinfeld).

Methods

Integrality

uses Langlands program (Deligne's companions' conjecture, proven by L. Lafforgue and Drinfeld).

Crystallinity

uses theory of Higgs-de Rham flows (Lan-Sheng-Zuo).

One consequence of our crystallinity theorem

Pila-Ananth Shankar-Tsimerman (posted on Sept. 21 2021)

André-Oort conjecture for all Shimura varieties.
[i.e: Zariski closure of special points is special $\subset$ Shimura variety.]

One consequence of our crystallinity theorem

Pila-Ananth Shankar-Tsimerman (posted on Sept. 21 2021)

André-Oort conjecture for all Shimura varieties.
[i.e: Zariski closure of special points is special $\subset$ Shimura variety.]

Method

uses (crucially) crystallinity.