Contemporary Mathematics
Volume **131**, 1992 (Part 3)

SOME EXAMPLES OF COMPUTATION OF A REGULATOR MAP ON SINGULAR VARIETIES

by

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Let X be a complex algebraic variety. In [E2] we have defined a regulator map $c_{nn} : \mathcal{K}_n^M \to \mathcal{H}^n(n)$ from the Zariski sheaf of Milnor K-theory to some sheaf $\mathcal{H}^n(n)$, which coincides with Bloch-Beilinson's regulator if X is smooth. In this little note, we compute examples for which c_{nn} helps to detect elements in the kernel of $\mathcal{K}_n^M \to K_n^M(\mathbb{C}(X))$, where $\mathbb{C}(X)$ is the function field of X, as well as in the cokernel of $\mathcal{K}_{nX}^M \to \pi_* \mathcal{K}_{nY}^M$, where $\pi : Y \to X$ is a desingularization of X. It turns out that in the two cases, those elements are generalized (or "Loday") symbols as defined in [б]. In [E2] we have computed explicitely the image of generalized symbols in $H^n_{\mathcal{D}}(Y,E; \mathbb{Z}(n))$, the Deligne-Beilinson cohomology, relative to some subvariety E. As we may relate $\mathcal{H}^n(n)$ on X and $H^n_{\mathcal{D}}(Y,E; \mathbb{Z}(n))$ on Y for some E, we basically make the computation in the later group.

Except for (2.2) 1), where we slightly improve the sheaf $\mathcal{H}^n(n)$, the main facts used in this note are proved in [E1] and [E2] : we emphasize how to use the methods developed there to compute examples.

I would like to thank M. Levine for useful discussions, as well as the organizing committee of the conference in honor of Prof. A.I. Maltsev for giving me the possibility to participate in it.

1980 *Mathematics Subject Classification* (1985 *Revision*. Primary 19D45, 19E20.

1.1 Let Y be an algebraic variety over $\mathbb{C}$, the field of complex numbers. We denote by $H^q_{\mathcal{D}}(p)$ the Deligne-Beilinson cohomology groups [б] , [E.V]. Let $a_1,\ldots,a_n$ be regular functions on Y , f be an invertible regular function on Y , whose value is 1 along the subvariety T defined by the reduced ideal associated to $a_1 \ldots a_n$. Define S to be the subvariety of Y such that S + T is the subvariety of Y associated to (f-1). One has $f \in H^1_{\mathcal{D}}(Y, S+T\,;\,\mathbb{Z}(1))$, $a_i \in H^0(Y, \mathcal{O}_Y)$. In [E1], we give explicit formulae for the generalized symbol $\{f,a_1,\ldots,a_n\}_S \in H^{n+1}_{\mathcal{D}}(Y, S\,;\,\mathbb{Z}(n+1))$ mapping to the cup product $(f \cup a_1 \cup \ldots \cup a_n)_S$ in $H^{n+1}_{\mathcal{D}}(Y-T, S \cap Y\text{-}T;\, \mathbb{Z}(n+1))$. (To be precise , we define an element $\{f,a_1,\ldots,a_n\}_{S+T} \in H^{n+1}_{\mathcal{D}}(Y, S+T\,;\,\mathbb{Z}(n+1))$ whose image in $H^{n+1}_{\mathcal{D}}(Y, S\,;\,\mathbb{Z}(n+1))$ is the generalized symbol as defined by A. Beilinson in [б]).

1.2 If $a_i \in H^1_{\mathcal{D}}(Y, \mathbb{Z}(1))$, that is if a_i is invertible, then

$$\{f,a_1,\ldots,a_n\}_S = (f \cup a_1 \cup \ldots \cup a_n)_S \in H^{n+1}_{\mathcal{D}}(Y, S\,;\,\mathbb{Z}(n+1))$$

maps to the cup product

$$(f \cup a_1 \cup \ldots \cup a_n) \in H^{n+1}_{\mathcal{D}}(Y, \mathbb{Z}(n+1))$$

So whenever the map

$$H^{n+1}_{\mathcal{D}}(Y, S\,;\,\mathbb{Z}(n+1)) \to H^{n+1}_{\mathcal{D}}(Y, \mathbb{Z}(n+1))$$

is not injective , $\{f,a_1,\ldots,a_n\}_S$ will contain a priori more information than $(f\cup a_1 \cup \ldots \cup a_n)$.

1.3 Recall briefly how to define $\{f,a_1,\ldots,a_n\}_{S+T}$.

We choose an analytic open cover Y_i of Y such that $\log_i f$ is single valued on Y_i, vanishes along S+T (which implies that $z^{n-1}_{i_0 i_1} := (\delta \log f)_{i_0 i_1} := \log_{i_1} f - \log_{i_0} f$ is identically zero on $Y_{i_0 i_1}$ whenever $Y_{i_0 i_1}$ meets S+T), and $\log_{i_0 \ldots i_k} a_k$ is single valued on $Y_{i_0 \ldots i_k}$ whenever $Y_{i_0 \ldots i_k}$ does not meet S+T ([E1], (1.4)). Then we define a "product" ([E1], (1.5)), show that its restriction to Y-T is homotop to the Deligne-Beilinson product ([E1], §2), and that the element so defined in the cohomology $H^{n+1}_{\mathcal{D}}(Y, S+T; \mathbb{Z}(n+1))$ does not depend on the choices made above ([E1], (3.8)).

Then $\{f,a_1,\ldots,a_n\}_{S+T}$ is represented by a Čech cocycle

$$(-1)^{\ell n} \, z^{n-\ell}_{i_0 \ldots i_\ell} \log_{i_0 \ldots i_\ell} a_\ell \frac{da_{\ell+1}}{a_{\ell+1}} \wedge \ldots \wedge \frac{da_n}{a_n}$$
$$\in H^0(Y_{i_0 \ldots i_\ell}, \Omega^{n-\ell}_{Y,S+T})$$

where $\Omega^k_{Y,S+T}$ is the sheaf of Kähler k-forms vanishing along S+T , $z^{n-\ell}$ is defined inductively by $z^{n-\ell} = \delta(z^{n-(\ell-1)}_{i_0 \ldots i_{\ell-1}} \log_{i_0 \ldots i_{\ell-1}} a_{\ell-1})$, and $z^{n-\ell}$ is identically zero if $Y_{i_0 \ldots i_\ell}$ meets $S+T$ and lies in $\mathbb{Z}(\ell)$ otherwise.

1.4 To be honest, we were considering in [E1] only smooth varieties Y . The formulae in (1.3) define a class in $H^{n+1}_{\mathcal{D}}(Y, j_! \, \mathbb{Z}(n+1) \to \Omega^0_{Y,S+T} \to \ldots \to \Omega^n_{Y,S+T})$, where j is the open embedding $Y\text{-}S\text{-}T \to Y$. If Y is smooth, then this group is $H^{n+1}_{\mathcal{D}}(Y, S+T; \mathbb{Z}(n+1))_{an}$ which contains $H^{n+1}_{\mathcal{D}}(Y, S+T; \mathbb{Z}(n+1))$ as the subgroup of classes x whose curvature dx has logarithmic growth at infinity. Recall that if Y is smooth, then $H^n_{\mathcal{D}}(Y, j_! \, \mathbb{C}/\mathbb{Z}(n+1))$ is the subgroup of $H^{n+1}_{\mathcal{D}}(Y, S+T; \mathbb{Z}(n+1))$ of curvature zero, and that $d\{f,a_1,\ldots,a_n\}_{S+T} = \frac{df}{f} \wedge \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$ ([E1], (1.2) (1.3)).

1.5 Consider $b_i \in H^0(Y, \mathcal{O}_Y)$, and assume moreover that $f = 1$ on T_{b_i} defined by the reduced ideal associated to $b_i = 0$. As $\{f,a_1,\dots,a_n\}_{S+T}$ does not depend on the cover choosen in (1.3) with the properties explained there, one obtains

Proposition

$$\{f,a_1,\dots,a_{i-1}, a_ib_i, a_{i+1}, \dots,a_n\}_{S+T+T_{b_i}}$$
$$= \{f,a_1,\dots,a_n\}_{S+T+T_{b_i}} + \{f,a_1,\dots, a_{i-1}, b_i, a_{i+1},\dots,a_n\}_{S+T+T_{b_i}}$$

in

$$H^{n+1}_{\mathcal{D}}(Y, S + T + T_{b_i} ; \mathbb{Z}(n + 1))$$

1.6 Similarly, let $g \in H^1_{\mathcal{D}}(Y, S + T ; \mathbb{Z}(1))$. Then one has

Proposition

$$\{fg,a_1,\dots,a_n\}_{S+T} = \{f,a_1,\dots,a_n\}_{S+T} + \{g,a_1,\dots,a_n\}_{S+T}$$

in

$$H^{n+1}_{\mathcal{D}}(Y, S + T ; \mathbb{Z}(n + 1))$$

1.7 One has also obviously

$$\{f^{-1},a_1,\dots,a_n\}_{S+T} = - \{f,a_1,\dots,a_n\}_{S+T}$$
$$\{f,a_1,\dots,a_{i-1}, a_i^{-1}, a_{i+1}, \dots,a_n\}_{S+T} = - \{f,a_1,\dots,a_n\}_{S+T}$$

if a_i is invertible.

1.8 Let us compute a very simple example.

Set $Y = \mathbb{C} - \{0\} = \operatorname{Spec} \mathbb{C}\, [t, \frac{1}{t}]$

$S = \{1, - 1\}$

$f = t^2$, $a_1 = \varepsilon t$ with $\varepsilon = +1$ or -1

$n = 2$.

One has a commutative diagram

$$\begin{array}{ccc} K_2(Y,S) & \rightarrow & K_2(Y) \\ c_{22}\downarrow & & c_{22}\downarrow \\ H^1(Y, j_! \mathbb{C}/\mathbb{Z}(2)) = H^2_{\mathcal{D}}(Y, S ; \mathbb{Z}(2)) & \rightarrow & H^2_{\mathcal{D}}(Y, \mathbb{Z}(2)) = H^1(Y ; \mathbb{C}/\mathbb{Z}(2)) \end{array}$$

We denote by $< , >_S$ the generalized symbols in $K_2(Y,S)$ and by $\{ \ , \ \}$ the Steinberg symbols in $K_2(Y)$.

We consider $<t^2, \varepsilon t>_S$ in $K_2(Y,S)$.
Its image $\{t^2, \varepsilon t\} = 2\{-\varepsilon t, \varepsilon t\}$ in $K_2(Y)$ vanishes. Therefore $c_{22} <t^2, \varepsilon t>_S = \{t^2, \varepsilon t\}_S$ lies in

$$K := \mathrm{Ker}\left(H^1(Y, j_! \mathbb{C}/\mathbb{Z}(2)) \rightarrow H^1(Y, \mathbb{C}/\mathbb{Z}(2))\right) = \mathbb{C}/\mathbb{Z}(2)$$

Let $[\gamma] \in H_1(Y, S ; \mathbb{Z})$ be the homology cycle such that $<[\bar{\gamma}], K>$ generates $\mathbb{C}/\mathbb{Z}(2)$. We may take a representative γ of the following shape :

$\gamma : [0, \pi] \rightarrow Y$
$\theta \rightarrow e^{i\theta}$

We want to compute $x := <[\gamma], \{t^2, \varepsilon t\}_S>$ in $\mathbb{C}/\mathbb{Z}(2)$.
Cover a tubular neighbourhood $\mathcal{U}$ of γ by two open sets U_{-1}, U_1, with

$$\{1\} \in U_1 - U_{-1}, \{-1\} \in U_{-1} - U_1,$$
$$\gamma \cap U_1 = \{\theta \in [0, \tfrac{3\pi}{4}[\}$$
$$\gamma \cap U_{-1} = \{\theta \in]\tfrac{\pi}{4}, \pi]\} ;$$

Choose $\log_i t^2$ with

$$\log_1 t^2 = \log_{-1} t^2 + 2i\pi \quad \text{on} \quad U_{-11}$$

and

$\log_{-11} \varepsilon t$ on U_{-11}.

Then $\{t^2, \varepsilon t\}_S$ is given as a Čech cocycle by

$$(0, -(\delta \log t^2)_{-11} \log_{-11} \varepsilon t, \log_i t^2 \frac{d\varepsilon t}{\varepsilon t})$$

in

$$\mathcal{C}^2(\mathfrak{U}, j_! \mathbb{C}/\mathbb{Z}(2)) \times \mathcal{C}^1(\mathfrak{U}, \Omega^0_{Y,S}) \times \mathcal{C}^0(\mathfrak{U}, \Omega^1_{Y,S})$$

One has

$$\log_i t^2 \frac{d\varepsilon t}{\varepsilon t} = \frac{1}{4} d((\log_i t^2)^2).$$

Therefore $\{t^2, \varepsilon t\}_S$ is given by the Čech cocycle

$$(0, \bar{x} := -(\delta \log t^2)_{-11} \log_{-11} \varepsilon t + \frac{1}{4} \delta ((\log_i t^2)^2), 0),$$

and one has $x = \bar{x}$ modulo $\mathbb{Z}(2)$.

One has

$$\begin{aligned} \bar{x} &= (\delta \log t^2)_{-11} (- \delta \log_{-11} \varepsilon t + \frac{1}{4} \log_1 t^2 + \frac{1}{4} \log_{-1} t^2) \\ &= (2i\pi) (- \delta \log_{-11} \varepsilon t + \frac{1}{2} \log_{-1} t^2 + \frac{i\pi}{2}). \end{aligned}$$

Therefore

$$\begin{aligned} 0 \neq x &= (2i\pi).\frac{i\pi}{2} \in \mathbb{C}/\mathbb{Z}(2) \quad \text{for } \varepsilon = 1 \\ &= -(2i\pi).\frac{i\pi}{2} \in \mathbb{C}/\mathbb{Z}(2) \quad \text{for } \varepsilon = -1 \end{aligned}$$

1.9 Remark.

Let V be any Zariski open set in Y containing S. Then the restriction map $K \to K_V$ where

$$K_V := \mathrm{Ker}\left(H^1(V, j_! \mathbb{C}/\mathbb{Z}(2)) \to H^1(V, \mathbb{C}/\mathbb{Z}(2))\right)$$

is obviously an isomorphism.

Therefore the restriction of $\{t^2, \varepsilon t\}_S$ to V does not die in $H^1(V, j_! \mathbb{C}/\mathbb{Z}(2))$. We will use this remark in (2.3) in order to construct an element in

$$\mathrm{Ker}\left((K_2(R) \to K_2(Q(R))\right),$$

where R is a local domain and Q(R) is its field of fractions.

2.1 Let X be a reduced algebraic variety over $\mathbb{C}$, whose singular locus Σ is of dimension d. Fix an integer n with $n \geq d+1$ and $n \geq 2$. In [E2], we construct a Zariski sheaf $\mathcal{H}^n(n)$ on X, together with a regulator map $c_{nn} : \mathcal{K}_n^M \to \mathcal{H}^n(n)$, which is functorial and coincides with Bloch-Beilinson's regulator map when X is smooth. (Here $\mathcal{K}_n^M$ is the Zariski sheaf of Milnor K-theory).

Roughly, the construction goes as follows.

Let $\pi : Y \to X$ be a desingularization such that $E := (\pi^{-1}\Sigma)_{red}$ is a divisor with normal crossings, and such that $\mathcal{F} = \pi^* \Omega^n_X/\text{torsion}$ is a locally free sheaf, where Ω^n_X are the Kähler differentials. Define $j : Y - E \to Y$ and $i : X - \Sigma \to X$.

One observes that $\mathcal{F}$ embeds into $\Omega^n_Y(\log E)(-E)$, and therefore that $\mathcal{F}^{\geq n}$ maps to $j_! \mathbb{C}/\mathbb{Z}(n)$, where

$$(\mathcal{F}^{\geq n})^n = \mathcal{F}, \quad (\mathcal{F}^{\geq n})^{\ell} = \Omega^{\ell}_Y(\log E)(-E), \quad \text{for } \ell > n$$
$$= 0 \qquad\qquad \text{for } \ell < n.$$

This gives a map

$$\varphi_i : R\pi_* (\mathcal{F}^{\geq n}) \to i_! \mathbb{C}/\mathbb{Z}(n)$$

and one defines $\mathcal{H}^n(n)_{an,i}$ to be the Zariski sheaf in X associated to $\mathbb{H}^n$ (cone φ_i [-1]). It does not depend on the desingularization π choosen. Then one defines $\mathcal{H}^n(n)_i$ by taking in $\mathcal{F}^{\geq n}$ those sections which have logarithmic growth at infinity (see (2.2), 1)). Finally, there is a subvariety $\Sigma' \subset \Sigma$ of the shape Sing (Sing ... (Sing Σ) ...)), in such a way that if $\mathcal{H}^n(n)$ is the sheaf (with logarithmic growth condition at infinity) associated to $\mathbb{H}^n$ (cone $\varphi_{i'}$ [-1]), where $i' : X - \Sigma' \to X$ and $\varphi_{i'} : R\pi_* (\mathcal{F}^{\geq n}) \to i'_! \mathbb{C}/\mathbb{Z}(n)$, the natural cup product of elements of $\mathcal{K}_1$ lands in ([E2], (1.4)). This defines at the same time c_{nn} ([E2], (2.2)).

2.2 Remarks

1. Let us be more precise on the logarithmic growth at infinity. Let U be an open set in X. Take a good compactification of $V := \pi^{-1}(U)$:

$$\begin{array}{ccc} V & \xrightarrow{\ell} & \bar{V} \\ \pi\downarrow & & \downarrow\bar{\pi} \\ U & \xrightarrow[k]{} & \bar{X} \end{array}$$

such that $\bar{X}$ is any compactification of X, $\bar{V}$ is smooth and $(\bar{V} - V)$ is a normal crossing divisor. The one defines

$$\mathcal{G}^k := \ell_* \mathcal{F}^k \cap \Omega^k_{\bar{V}} (\log (\bar{V} - V)),$$

and $\mathcal{H}^n(n)_i$ is the sheaf associated to

$$\mathbb{H}^n (\bar{X}, \text{cone } (R\, \bar{\pi}_* \mathcal{G}^{\geq n} \to Rk_* i_! \mathbb{C}/\mathbb{Z}(n)) [-1]).$$

Once again, it does not depend on the choices of $\bar{X}$, V, $\bar{V}$.

One defines similarly $\mathcal{H}^n(n)$ by replacing i by i'.

One has for degree reasons

$$\begin{aligned} &\mathbb{H}^n (\bar{X}, \text{cone } (R\, \bar{\pi}_* \mathcal{G}^{\geq n} \to Rk_* i_! \mathbb{C}/\mathbb{Z}(n)) [-1]) \\ = &\mathbb{H}^n (\bar{X}, \text{cone } (R^n \bar{\pi}_* \mathcal{G}^{\geq n} \to Rk_* i_! \mathbb{C}/\mathbb{Z}(n)) [-1]). \end{aligned}$$

One has maps of sheaves

$$\begin{array}{ccc} & & R^n\bar{\pi}_* \mathcal{G}^{\geq n} \\ & & \downarrow \\ k_* \operatorname{Ker}(\Omega^n_U \to \Omega^{n+1}_U) & \to & k_* R^n \pi_* \mathcal{F}^{\geq n} \end{array}$$

Define $\Omega^n_{U,\bar{X}}$ to be the fiber product. As the vertical arrow is injective, $\Omega^n_{U,\bar{X}}$ is a subsheaf of $k_* \Omega^n_U$.

As $H^0(\bar{X}, R^n \bar{\pi}_* \mathcal{G}^{\geq n})$ and $H^0(U, R^n \bar{\pi}_* \mathcal{F}^{\geq n})$ do not depend on $\bar{X}$, V, $\bar{V}$, $H^0(\bar{X}, \Omega^n_{U,\bar{X}})$ does not depend on $\bar{X}$, V, $\bar{V}$ either. Define $\tilde{\mathcal{H}}^n(n)_i$ to be the Zariski sheaf associated to

$$\mathbb{H}^n(\bar{X}, \operatorname{cone}(\Omega^n_{U,\bar{X}} \to Rk_*\, i_!\, \mathbb{C}/\mathbb{Z}(n))\,[-1]),$$

and similarly for $\tilde{\mathcal{H}}^n(n)$ by replacing i by i'. One obtains natural maps

$$\begin{array}{ccc} \tilde{\mathcal{H}}^n(n) & \to & \mathcal{H}^n(n) \\ \tilde{\mathcal{H}}^n(n)_i & \to & \mathcal{H}^n(n)_i \,. \end{array}$$

The point is doing that is that one does not lose the torsion in the Kähler differentials.

One can prove along the same line as in [E2] that this definition is functorial and leads to a regulator

$$\tilde{c}_{nn} = \mathcal{K}^M_n \to \tilde{\mathcal{H}}^n(n)$$

lifting c_{nn}.

We will not use this in the rest of this article.

2. M. Levine [L] defines another Zariski sheaf on X. Roughly speaking, he takes the sheaf associated to

$$\mathbb{H}^n(\bar{U}, \Omega^{\geq n}_{\bar{U}}(\log(\bar{U} - U)) \to Rk_* \ \mathrm{cone}(\mathbb{Z}(n) \to \Omega^\bullet_V))$$

where $k : U \to \bar{U}$ is a compactification such that $(\bar{U} - U)$ is supported by a Cartier divisor and $\Omega^{\geq n}_{\bar{U}}(\log(\bar{U} - U))$ consists of those Kähler forms which have logarithmic growth along the normal crossing divisor $(\bar{V} - V)$ where

$$\begin{array}{ccc} V & \to & \bar{V} \\ \downarrow & & \downarrow \\ U & \to & \bar{U} \end{array}$$

is a diagram of desingularization. Of course $\tilde{\mathcal{H}}^n(n)$ maps to M. Levine's sheaf, whereas $\mathcal{H}^n(n)$ does not : "my" Betti part lifts "his", but I lose the torsion in the forms.

2.3. We will now compute a simple example of c_{22} : X will be a rational curve with a double point.

Set $R := \mathbb{C}\,[1 - t^2, t(1 - t^2), \frac{1}{t^2}] \to A := \mathbb{C}\,[1, \frac{1}{t}]$

$\quad = \mathbb{C}[x, y, \frac{1}{1 - x}]/(x^2 - y^2 - x^3)$

Define

$$\begin{array}{ccc} Y := \mathrm{Spec}\, A & \stackrel{\pi}{\to} & X := \mathrm{Spec}\, R \\ j\uparrow & & \uparrow i \\ Y\text{-}S & \cong & X - 0 \end{array}$$

where $0 := (x = 0, y = 0)$, $S = \{t = -1, t = 1\}$.

We consider the commutative diagram

$$\begin{array}{ccc} K_2(X, \{0\}) & \stackrel{i^*}{\to} & K_2(X) \\ \pi^*\downarrow & & \downarrow\pi^* \\ K_2(Y, S) & \stackrel{j^*}{\to} & K_2(Y) \\ c_{22}\downarrow & & \downarrow c_{22} \\ H^1(Y, j_!\mathbb{C}/\mathbb{Z}(2)) & \to & H^1(Y, \mathbb{C}/\mathbb{Z}(2)). \end{array}$$

In $K_2(X, \{0\})$ one has the generalized symbol

$z := <t^2, \varepsilon t(1 - t^2)>_{\{0\}}$, with $\varepsilon = +1$ or -1.

One has

$$\pi^* i^* <t^2, \varepsilon t(1 - t^2)>_{\{0\}} = j^* <t^2, \varepsilon t(1 - t^2)>_S$$

where $<t^2, \varepsilon t(1 - t^2)>_S$ is the generalized symbol in $K_2(Y, S)$.

By (1.5), one has

$$j^* <t^2, \varepsilon t(1 - t^2)>_S = j^* <t^2, \varepsilon t>_S + j^* <t^2, (1 - t^2)>_S \text{ in } K_2(Y).$$

One has $\quad j^* <t^2, \varepsilon t>_S = \{t^2, \varepsilon t\} = 0$ in $K_2(Y)$.

Let $\sigma :\ Y \rightarrow \mathbb{C}^*$
$\qquad t \rightarrow t^2 =: \tau$
Let $j' :\ \mathbb{C}^* - \{1\} \rightarrow \mathbb{C}^*$.

Then $\qquad <t^2, (1 - t^2)>_S = \sigma^* <\tau, 1 - \tau>_{\{1\}}$
where $\qquad <\tau, 1 - \tau>_{\{1\}} \in K_2(\mathbb{C}^*, \{1\})$.

By functoriality, one has

$$c_{22} <t^2, =(1 - t^2)>_S = c_{22}\, \sigma^* <\tau, 1 - \tau>_{\{1\}}$$
$$= \{t^2, (1 - t^2)\}_S = \sigma^* \{\tau, 1 - \tau\}_{\{1\}}$$

But one has injections :

$$H^1(\mathbb{C}^*, j_! \mathbb{C}/\mathbb{Z}(2)) \xrightarrow{\sim} H^1(\mathbb{C}^*, \mathbb{C}/\mathbb{Z}(2))$$
$$\downarrow$$
$$H^1(\mathbb{C}^* - \{1\}, \mathbb{C}/\mathbb{Z}(2))$$

Therefore $<\tau, 1-\tau>_{\{1\}} = 0$ as its image in $H^1(\mathbb{C}^* - \{1\}, \mathbb{C}/\mathbb{Z}(2))$ vanishes (by Bloch's construction of the regulator !).

One obtains :

$$c_{22}\,(<t^2, \varepsilon t(1-t^2)>_S) = \{t^2, \varepsilon t\}_S$$

By (1.6), it does not die in

$$K = \mathrm{Ker}\,(H^1(Y, j_!\mathbb{C}/\mathbb{Z}(2)) \to H^1(Y, \mathbb{C}/\mathbb{Z}(2)))$$

Finally, $\pi^* z = \{t^2, \varepsilon t\}_S + \{t^2, 1-t^2\} = 0$ in $K_2(\mathbb{C}(t))$.

So we have constructed an element $z \in K_2(X)$, whose image in $K_2(\mathbb{C}(X)) = K_2(\mathbb{C}(t))$ vanishes, and which is non zero. Let $\mathfrak{m}$ be the maximum ideal of $\{0\}$ in R, and $R_{\mathfrak{m}}$ be the localization of R in $\mathfrak{m}$. It remains to show that the image $\bar{z}$ of z in $K_2(R_{\mathfrak{m}})$ does not vanish.

Apply c_{22} ; one has

$$c_{22}\,(\bar{z}) \in \mathcal{H}^2(2)_0 = \mathcal{H}^1(i_!\mathbb{C}/\mathbb{Z}(2)) \text{ ([E2] (1.4)),}$$

where

$$\begin{aligned}\mathcal{H}^1(i_!\mathbb{C}/\mathbb{Z}(2)) &= \varinjlim_{0 \in U \text{ Zariski}} H^1(U, i_!\mathbb{C}/\mathbb{Z}(2)) \\ &= \varinjlim_{0 \in U \text{ Zariski}} H^1(\pi^{-1} U, j_!\mathbb{C}/\mathbb{Z}(2))\end{aligned}$$

By (1.9) $c_{22}\,(\bar{z}) \neq 0$.

<u>Conclusion</u>. We have used the regulator c_{22} to detect an explicit element $\bar{z}$ in $K_2(R_{\mathfrak{m}})$, whose image in $K_2(\mathbb{C}(t))$ vanishes.

In [G], the case of a semi-normal curve singularity is treated in general, without use of a regulator.

2.4 Let us now take M. Levine's definition of c_{22} in the example (2.3). One has maps

$$H^1(i_!\mathbb{C}/\mathbb{Z}(2)) \to H^1(\mathbb{C}/\mathbb{Z}(2)) \to H^2(\mathbb{Z}(2)) \to \mathcal{O}_X \to \Omega^1_X).$$

where the first map in an isomorphism and the second one is injective. Therefore one can also see that $z \neq 0$.

2.5 Remark.

Let X, Σ, π, Y, i, i' etc... be as in (2.1).

Consider $n = 2$.

The map $\mathcal{H}^2(2) \to \pi_* \mathcal{H}^2_{\mathcal{D}}(2)$ [E2], (1.7), has more precisely the following shape at the presheaf level [E2], (1.4), proof of 1).

There is a commutative diagram of exact sequences :

$$\begin{array}{ccccccccc} 0 \to & H^1(U, i'_!\mathbb{C}/\mathbb{Z}(2)) & \to & H^2(U,2) & \to & \mathrm{Ker}\,(H^0(\bar{V}, \mathcal{G})_{c\ell} & \to & H^2(i'_!\mathbb{C}/\mathbb{Z}(2)) & \to 0 \\ (*) & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & H^1(V, \mathbb{C}/\mathbb{Z}(2)) & \to & H^2_{\mathcal{D}}(V,2) & \to & \mathrm{Ker}\,\big(H^0(\bar{V}, \Omega^2(\log(\bar{V}-V)) & \to & H^2(V, \mathbb{C}/\mathbb{Z}(2))\big) & \to 0 \end{array}$$

As $H^1(U, i'_!\mathbb{C}/\mathbb{Z}(2)) = H^1(U, j'_!\mathbb{C}/\mathbb{Z}(2))$ with $j' : Y - E' \to Y$ where $E' := \pi^{-1}\Sigma'$, one sees that the map $H^2(U,2) \to H^2_{\mathcal{D}}(V,2)$ is injective if and only if E' is connected.

As $H^2_{\mathcal{D}}(V,2) = H^0(V, \mathcal{H}^2_{\mathcal{D}}(V,2))$, one obtains that *the map* $\mathcal{H}^2(2) \to \pi_* \mathcal{H}^2_{\mathcal{D}}(2)$ *is injective if and only if E' is connected*.

In particular, if $\Sigma = \Sigma'$ and X is normal (e.g a normal surface singularity), the regulator c_{22} will never detect elements in $\mathrm{Ker}\,(\mathcal{K}_{2X} \to K_2(\mathbb{C}(X)))$.

Consider now $\tilde{\mathcal{H}}^2(2)$ as defined in (2.2) 1). Then in the diagram (*) one has to replace $\mathcal{G}$ by $\Omega^2_{U,\bar{X}}$, and one sees that $\mathrm{Ker}\,(\tilde{\mathcal{H}}^2(2) \to \pi_* \mathcal{H}^2_{\mathcal{D}}(2))$ is contained in the torsion of Ω^2_X.

Then $\tilde{c}_{22}$ will detect elements in $\mathrm{Ker}\,(\mathcal{K}_{2X} \to K_2(\mathbb{C}(X)))$ if one can find $x \in \mathcal{K}_{2X}$ such that dlogx is torsion, where dlog : $\mathcal{K}_{2X} \to \Omega^2_X$ is the map $\mathrm{dlog}\{f, g\} = \frac{df}{f} \wedge \frac{dg}{g}$. Of course we knew that already without complicated regulator !

3.1 Keeping the notations of (2.1), we will now be interested in

$$\mathcal{C} := \pi_* \mathcal{K}^M_{nY}/\mathcal{K}^M_{nX}\,.$$

There is a map

$$\mathcal{C} \to \pi_* R^n \alpha_* (\Omega^{\geq n}_Y/\mathcal{F}^{\geq n})$$

where $\alpha : X_{an} \to X_{zar}$ is the continuous map from the classical to the Zariski topology ([E2], (2.2)), simply defined by

$$\mathrm{dlog} : \mathcal{K}^M_{nY} \to \Omega^{\geq n}_Y [n]$$

$$\{f_1, \dots, f_n\} \to \frac{df_1}{f_1} \wedge \dots \wedge \frac{df_n}{f_n}\,.$$

3.2 We compute a singularity of type A_1. Set $X := \mathrm{Spec}\, \mathbb{C}[x, y, t, \frac{1}{1-t}]/(t^2 - xy)$; $\pi : Y \to X$ is the blow up of $\{0\} := (x = 0, y = 0, t = 0)$, with exceptional line E.

A) Cover Y be three Zariski open sets Y_0, Y_1, Y_2 of coordinates and equations

$Y_0 : (a, b, t), x = at, y = bt ; 1 - ab$
$Y_1 : (x, b', T), y = b'x, t = Tx ; T^2 - b'$
$Y_2 : (a', y, T'), x = a'y, t = T'y ; T'^2 - a'$

We consider in $K_2(Y_0)$ the generalized symbol

$$\alpha_0 := \langle 1 - t, at \rangle_E\,.$$

One has

$$\alpha_{0|Y_0 \cap Y_1)} = \langle 1 - Tx, x\rangle_E = \alpha_{1|Y_0 \cap Y_1)} \text{ with } \alpha_1 := \langle 1 - Tx, x\rangle_E \in K_2(Y_1)$$

$$\begin{aligned} \alpha_{0|Y_0 \cap Y_2)} &= \langle 1 - T'y, T'^2 y\rangle_E \\ &= \langle 1 - T'y, T'\rangle_E + \langle 1 - T'y, T'y\rangle_E \qquad (1.5) \end{aligned}$$

Consider
$$\sigma : Y_0 \cap Y_2 \to \mathbb{C}^*$$
$$(a, y, T') \to \tau := 1 - T'y.$$

One has $\langle 1 - T'y, T'y\rangle_E = \sigma^* \langle \tau, 1 - \tau\rangle_{\{1\}}$

As $\langle \tau, 1 - \tau\rangle_{\{1\}} \in K_2(\mathbb{C}^*, \{1\})$ is uniquely determined by its restriction to $K_2(\mathbb{C}^* - \{1\})$, it is zero.

Therefore $\alpha_{0|Y_0 \cap Y_2)} = \alpha_{2|Y_0 \cap Y_2)}$ with $\alpha_2 := \langle 1 - T'y, T'\rangle$ in $K_2(Y_2)$.

Similarly, one has $\alpha_{1|Y_1 \cap Y_2)} = \alpha_{2|Y_1 \cap Y_2)} \in K_2(Y_1 \cap Y_2)$.
Define $\alpha \in H^0(Y, \mathcal{K}_2)$ to be α_i on Y_i.

B) Now one easily computes that
$$\mathfrak{F} := \pi^* \Omega^2_X / \text{torsion} = \Omega^2_Y(-E).$$

As $\mathfrak{F}$ is generated by global sections and $(X, 0)$ is rational singularity, one has $\pi_* \Omega^2_Y/\mathfrak{F} = \mathbb{C}$. It is generated by the image in $\pi_* \Omega^2_Y/\mathfrak{F}$ of

$$\text{dlog } \alpha = -\frac{dt \wedge da}{(1-t).a} = -\frac{dT \wedge dx}{1-Tx} = -\frac{dy \wedge dT'}{1 - T'y}$$

3.3 We compute a singularity of type A_2 ([E2], 2.12), 2))
Set $X := \text{Spec } \mathbb{C}[x, y, t, \frac{1}{1 - t^2}]/(t^3 - xy)$; $\pi : Y \to X$ is the blow up of $\{0\} := (x = 0, y = 0, t = 0)$, with exceptional line E. One has $E = E_1 + E_2$, $E_i^2 = -2$, $E_1 \cap E_2 =: p$.

A) Cover Y be three Zariski open sets Y_0, Y_1, Y_2 of coordinates as in (3.2), and equations :

$Y_0 : t - ab, E_1 : \langle a = 0\rangle, E_2 : \langle b = 0\rangle$
$Y_1 : T^3 x - b'$
$Y_2 : T'^3 y - a'$.

We consider in $K_2(Y_0)$ the two generalized symbols

$$\alpha_0 := \langle 1 - ab, b\rangle_{E_2}, \beta_0 := \langle 1 - (ab)^2, b^2\rangle_{E_2}.$$

One has

$$\begin{aligned}\alpha_{0|Y_0 \cap Y_1)} &= \langle 1 - Tx, T^2x\rangle \\ &= \langle 1 - Tx, T\rangle_{E_2} + \langle 1 - Tx, Tx\rangle_{E_2}.\end{aligned}$$

As in 3.2, one has $\langle 1 - Tx, Tx\rangle_{E_2} = 0$, and $\alpha_{0|Y_0 \cap Y_1)} = \alpha_{1|Y_0 \cap Y_1)}$ where $\alpha_1 = \langle 1 - Tx, T\rangle_{E_2}$. Similarly, one has $\alpha_{0|Y_0 \cap Y_2)} = \alpha_{2|Y_0 \cap Y_2)}$ where $\alpha_2 = - \langle 1 - T'y, T'\rangle_{E_2}$ $\in K_2(Y_2)$. One computes in the same way that $\alpha_{1|Y_1 \cap Y_2)} = \alpha_{2|Y_1 \cap Y_2)}$ in $K_2(Y_1 \cap Y_2)$.

Define $\alpha \in H^0(Y, \mathcal{K}_2)$ to be α_i on Y_i.
Similarly, $\beta_0 \in K_2(Y_0)$,

$$\beta_1 := \langle 1 - (Tx)^2, T\rangle_{E_2} \in K_2(Y_1)$$
$$\beta_2 := \langle 1 - (T'y)^2, T'^2\rangle_{E_2} \in K_2(Y_2)$$

define a global section in $H^0(Y, \mathcal{K}_2)$.

B) One has $\pi_* \Omega^2_Y/\text{torsion} = \mathfrak{m}\, \Omega^2_Y(-E)$ where $\mathfrak{m}$ the maximal ideal of p. As $\pi_* \Omega^2_Y/\text{torsion}$ is generated by global sections and $(X, 0)$ is a rational singularity, one has

$$R^1\pi_* (\pi^* \Omega^2_X/\text{torsion}) = 0.$$

Let $\sigma: Z \to Y$ be the blow up of p with exceptional line F. Then one has

$$\mathcal{F} = \sigma^* \pi^* \Omega^2_X/\text{torsion} = \sigma^* \Omega^2_Y (- E) \otimes \mathcal{O}_Z (- F).$$

As $R^1 \sigma_* \mathcal{O}_Z(-F) = 0$, one has

$$\pi_* \sigma_* (\Omega^2_Z/\mathfrak{F}) = \pi_* (\Omega^2_Y/\mathfrak{m}\, \Omega^2_Y(-E))$$

$$= \mathbb{C}_p \oplus \mathbb{C}$$

where $\mathbb{C}_p$ is $\Omega^2_Y(-E)/\mathfrak{m}\, \Omega^2_Y(-E))$

and $\mathbb{C}$ maps isomorphically to $H^0(\omega_E(-E))$. It is obvioulsy generated by the image of

$$d\log\alpha = -\frac{da \wedge db}{1 - ab} = -\frac{dx \wedge dT}{1 - xT} = \frac{dy \wedge dT'}{1 - yT'}$$

$$\frac{1}{4}\, d\log\beta = -ab\frac{da \wedge db}{1 - (ab)^2} = -xT\frac{dx \wedge dT}{1 - (xT)^2} = yT'\frac{dy \wedge dT'}{1 - (yT')^2}$$

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