Special loci of Betti Moduli

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Stony Brook, October 24, 2020

Betti Moduli

X sm q-proj var over $\mathbb{C}$, $\pi_1^{\rm top}(X,x)$ top fund gr based at $x \in X(\mathbb{C})$, $1 \le r \in \mathbb{N} \rightsquigarrow \exists$ moduli space $M(X,r) =: M$ of conj cl of rk r ss lin rep of $\pi_1^{\rm top}(X,x)$.

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M is an aff sch$/\mathbb{Z}$. $M(\mathbb{C})$ is the set of iso cl of ss $\mathbb{C}$-loc syst of rk r.

Quasi-unipotency at infinity

Fix a normal comp $j : X \hookrightarrow \bar{X}$. The conj cl of the gen T_s of local fund gr around a comp D_s at ∞ is well defined in $\pi_1^{\mathrm{top}}(X, x) \rightsquigarrow$ notion of loc mon at ∞ of loc syst $\mathcal{V} \in M(\mathbb{C})$.

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Remark

Same definition fixing the determinant $\mathcal{L}$ of $\mathcal{V}$ which is torsion $\rightsquigarrow$ $M(\mathcal{L})$ and $M(\mathcal{L})(\mathbb{C})^{\mathrm{qu}} \subset M(\mathcal{L})(\mathbb{C})$.

Density

Theorem A (E-Kerz '20)

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Heuristic

• Simpson: in each irr comp of M, *exists* $\mathbb{C}$-VHS (viewed as a fixpoint under the $\mathbb{C}^\times$-rescaling on the Higgs field).
• Hope: those coming from geometry are even Zariski dense.
• Their monodromies at ∞ are quasi-unipotent (Brieskorn, Griffiths, Grothendieck).

ℓ-adic points

$\exists$ framed moduli space $M^{\square}(X,r) =: M^{\square}$ defined by the functor $\pi_1^{\mathrm{top}}(X,x) \to GL_r(A)$ on aff $\mathbb{Z}$-alg A. $M^{\square}$ fine, aff sch $/\mathbb{Z}$, $M^{\square}(\mathbb{C}) \xrightarrow{q} M(\mathbb{C})$ cat quotient by GL_r action on frames $\rightsquigarrow$ $M^{\square}(\mathbb{C})^{\mathrm{qu}} = q^{-1}M(\mathbb{C})^{\mathrm{qu}}$.

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ℓ prime given $\rightsquigarrow$ $M^\square(\bar{\mathbb{Z}}_\ell) \subset M^\square(\bar{\mathbb{Q}}_\ell)$ = set of $\rho : \pi_1^{\rm top}(X,x) \to GL_r(\bar{\mathbb{Q}}_\ell)$ which factor through cont rep $\rho^{\rm ét} : \pi_1^{\rm ét}(X,x) \to GL_r(\bar{\mathbb{Z}}_\ell)$.

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ℓ-adic points of M

$$M^{\text{ét},\ell} := q(M^{\square}(\bar{\mathbb{Z}}_\ell)) \subset M(\bar{\mathbb{Q}}_\ell)$$

is the set of étale $\bar{\mathbb{Q}}_\ell$-loc syst.

Galois action on ℓ-adic points

$x \in X(F)$ (if $\emptyset$ enlarge F) $\rightsquigarrow G \circlearrowleft \pi_1^{\text{ét}}(X_{\bar F}, x)$ by conj $\rightsquigarrow$

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Special loci

Definition

A closed subset $S \subset M^{\square}(\bar{\mathbb{Q}}_\ell)$ is *special* if 1) $\forall$ each irred comp S_i $\emptyset \neq S_i \cap M^{\square}(\bar{\mathbb{Z}}_\ell)$; 2) $U \circlearrowleft S \cap M^{\square}(\bar{\mathbb{Z}}_\ell)$ for some $U \subset G$ open.

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Theorem B (E-Kerz '20)

S special $\Rightarrow S \cap M^{\square}(\mathbb{C})^{\mathrm{qu}} \subset S$ Zariski dense.

Thm B $\Rightarrow$ Thm A

- $\bar{\mathbb{Q}}_\ell$ alg. cl. $\Rightarrow$ may replace $\mathbb{C}$ by $\bar{\mathbb{Q}}_\ell$ in Thm A (top loc syst).
- $M^\square(\bar{\mathbb{Z}}_\ell)$ G-invariant;
- so enough to find ℓ so all irr comp of $M^\square(\bar{\mathbb{Q}}_\ell)$ meet $M^\square(\bar{\mathbb{Z}}_\ell)$;
- fin many irr comp $M_i/\bar{\mathbb{Q}}$, pick $\rho_i \in M_i(\bar{\mathbb{Q}})$; as $\pi_1^{\rm top}(X,x)$ fin gen, all ρ_i defined over $\mathcal{O}_{E,\Sigma}$, Σ fin many places of $\mathcal{O}_E$; so $\ell \notin$ char Σ does it.

Proof of Thm B

- $G \rightsquigarrow$ open subgr $\Rightarrow$ may assume S irr;
- $N := \prod_s \mathbb{A}^{r-1} \times \mathbb{G}_m$;
- $$\psi : M^{\square}(\bar{\mathbb{Q}}_\ell)(\to M(\bar{\mathbb{Q}}_\ell)) \to N(\bar{\mathbb{Q}}_\ell)$$
$$\rho \mapsto \prod_s (\text{coeff of } \det(X - \rho(T_s) \cdot \mathrm{Id}))$$
- ψ defined $/\mathbb{Z}_\ell$, G-equivariant on $M^{\square}(\bar{\mathbb{Z}}_\ell) \to N(\bar{\mathbb{Z}}_\ell)$;
- define $N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset N(\bar{\mathbb{Q}}_\ell)$ as the pol with torsion zeroes;
- $S \cap M^{\square}(\bar{\mathbb{Q}}_\ell)^{\mathrm{qu}} = \psi^{-1}(\psi(S) \cap N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}})$;
- Chevalley thm $\Rightarrow \psi(S)$ constructible;
- Zariski cl Z irr and $Z \cap N(\bar{\mathbb{Z}}_\ell) \neq \emptyset$;
- so enough to prove $Z \cap N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset Z$ Zariski dense;
- $h : T = \prod_s \mathbb{G}_m^r \to N$ separates the roots; h fin surj, so enough to prove $h^{-1}Z \cap T(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset h^{-1}Z$ Zariski dense.

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Method of proof of Thm B comes from our proof of density thm of tame arithm $\bar{\mathbb{Q}}_\ell$-loc syst of rk 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}/\bar{\mathbb{F}}_p$.
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Thank you for your attention!