Survey on rigid local systems and related arithmetic questions

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- $X \hookrightarrow \bar{X}$ good comp, $D = \cup_i D_i = \bar{X} \setminus X$, $r \in \mathbb{N}_{>0}$, $\mathcal{K}_i \subset GL_r(\mathbb{C})$ quasi-unip conj cl $/K \subset \mathbb{C}$ number field, $\mathcal{L}$ a rk 1 local system $/\mathbb{C}$ with finite monodromy $/K$; (L, ∇), Riemann-Hilbert-Simpson $\rightsquigarrow$ $\mathcal{L}_{dR} = (L, \nabla)$, $\mathcal{L}_{Dol} = (L, 0)$.

Moduli (Simpson; $\mathbb{C} \rightsquigarrow R$ Langer)

- $\exists$ coarse quasi-proj moduli sp $M_B/\mathcal{O}_K[1/N]$ with: $M_B(\mathbb{C}) = \{$iso cl of irred rk r $\mathbb{C}$ loc syst $\mathcal{V}$, $\det(\mathcal{V}) \cong \mathcal{L}$ and local monodromies $/D_i$ in $\mathcal{K}_i\}$. There is an étale μ_r-gerbe $\mathcal{M} \to M_B$ where $\mathcal{M}$ is a algebraic stack, so M_B is étally fine.

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• X proj: $\exists$ coarse quasi-proj moduli sp M_{Dol}/R with: $M_{Dol}(k) = \{$iso cl of stable for Higgs bdls rk r Higgs bundles (V, θ), $\det(V, \theta) = \mathcal{L}_{Dol}$, Hilbert pol $=$ Hilbert pol of $\oplus^r \mathcal{O}_X\}$.

Geometric local systems

Example

Main ex: $\mathcal{V}$ (resp. (E, ∇)) *geometric* i.e. $\exists f : Y \to U$ sm proj, $U \subset X$ dense, so $\mathcal{V}$ (*resp.*(E, ∇)) subq of $R^i f_* \mathbb{C}$, $i \in \mathbb{N}_{>0}$ (resp. $R^i f_* \Omega^\bullet_{Y/U}$) ($\leftrightsquigarrow$ summand by Deligne's ss thm). e.g. f finite étale, $i = 0$.)

Some properties of geometric local systems

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Property (dR: crystalline)

(E,∇) geometric $\Rightarrow \forall \mathbb{Z}_q \to \mathrm{Spec}(R)$, R some ring of defn of $(X,(E,\nabla))$, $\mathrm{Spec}(\mathbb{F}_q) \in \mathrm{Spec}(R)$ cl pt of good reduction, then $(E,\nabla)_{\mathbb{Q}_q}$ on $\hat{X}_{\mathbb{Q}_q}$ is an *isocrystal with Frobenius structure*. We say for short: (E,∇) is *crystalline*.

Riemann-Hilbert-Simpson correspondence $/\mathbb{C}$

$$\begin{array}{ccc} M_B(\mathbb{C}) & \xrightarrow{\;RH\cong\mathbb{C}-an\;} & M_{dR}(\mathbb{C}) \\ & {\scriptstyle S\circ RH\cong\mathbb{R}-an}\searrow \qquad \swarrow{\scriptstyle S\cong\mathbb{R}-an} & \\ & M_{Dol}(\mathbb{C}) & \end{array}$$

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In particular 0-diml comp called *rigid*:

$$M_B^{\mathrm{rig}}(\mathbb{C}) \xrightarrow{RH \cong \mathbb{C}-an} M_{dR}^{\mathrm{rig}}(\mathbb{C})$$

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- $\forall \lambda \in \mathrm{Spec}(\mathcal{O}_L[1/D])$, $\mathcal{V} \in M_B^{\mathrm{rig}}$ is *étale*, i.e. factors through $\pi_1^{\mathrm{top}}(X(\mathbb{C})) \to \widehat{\pi_1^{\mathrm{top}}(X(\mathbb{C}))} = \pi_1^{\text{ét}}(X_{\mathbb{C}}) = \pi_1^{\text{ét}}(X_{\overline{\mathrm{Frac}(R)}})$.

Katz' and Simpson's theorems; Simpson's conjecture

Theorem

(Katz): $\dim(X) = 1 \Rightarrow M_B^{\mathrm{rig}}$ *geometric;*
(Simpson): M_B^{rig} *factors through* $\pi_1^{\text{ét}}(X_F)$, $F \supset \mathrm{Frac}(R)$ *finite, i.e.*

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Conjecture (subconjectures)

$$\mathcal{V} \text{ rigid } \Rightarrow \mathcal{V} \text{ integral}$$
$$(E, \nabla) \text{ rigid } \Rightarrow (E, \nabla) \text{ crystalline}$$

Main theorem (E-Groechenig)

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Remark

- *coh. rigid*: *smooth* isolated point, i.e.

$$T_{[\mathcal{V}]} M_B = H^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V})) = 0.$$

- Katz: $\dim(X) = 1$: rigid $\Rightarrow$ coh. rigid.
- So far: not a single example of a rigid $\mathcal{V}$ which is not coh. rigid.

Corollary 1

Corollary ($\Leftarrow$ Thm + L.Lafforgue-Abe)

(E, ∇) *rigid* $\Rightarrow \forall C \hookrightarrow X$ *curve c.i. of smooth ample divisors*

$$(E, \nabla)|_{\hat{C}_{\mathbb{Q}_q}} \text{ geometric.}$$

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Remark

L. Lafforgue: irred arithmetic ℓ-adic local syst with fin det are geometric. Abe: companions $\ell \leftrightsquigarrow$ isocrystals with Frobenius structure. Big problem: extend geometricity on curves to geometricity on X and then geometricity on X over a finite field to geometricity on X over $\mathbb{C}$.

Corollary 2

Corollary ($\Leftarrow$ method of proof)

(E, ∇) *rigid with p-curvature* 0 *for all* $\mathrm{Spec}(\mathbb{F}_q) \in$ *dense open of* $\mathrm{Spec}(R) \Rightarrow$ *unitary monodromy.*

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Remark

So p-curvature conjecture for coh rigid connections $\Leftarrow$ $[(E, \nabla)$ rigid with p-curvature 0 for all $\mathrm{Spec}(\mathbb{F}_q) \in$ dense open of $\mathrm{Spec}(R)$ $\Rightarrow (E, \nabla)^\sigma$ rigid with p-curvature 0 for all $\mathrm{Spec}(\mathbb{F}_q) \in$ dense open of $\mathrm{Spec}(R)$ $\forall \sigma \in \mathrm{Aut}(\mathbb{C}),\ (E, \nabla)^\sigma \leftrightarrow \mathcal{V}^\sigma]$.

Corollary 3

Corollary ($\Leftarrow$ method of proof)

Deligne's conjectural companion correspondence for $\mathcal{V}$ arithm ℓ-adic irred with fin det on $X_{\bar{\mathbb{F}}_p}$ such that

$$H^1(\bar{X}_{\bar{\mathbb{F}}_p}, j_{!*}\mathcal{E}nd^0(\mathcal{V})) = 0$$

including the crystalline components: $\forall \iota : \bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_{\ell'}$ including $\ell' = p$, $\exists\ \mathcal{V}^\iota$ with the same local char. pol.

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Remark

Recall Deligne's conjecture is known on curves (L. Lafforgue, Abe), and $\ell \rightsquigarrow \ell' \neq p$ on X smooth in any dim (Drinfeld), and with Drinfeld's method $p \rightsquigarrow \ell$ (E-Abe, Kedlaya).

On proof of the integrality theorem

$\mathcal{V}$ coh. rigid, $\lambda \in \mathrm{Spec}(\mathcal{O}_L[1/D])$, L number field $\rightsquigarrow \mathcal{V}_\lambda$ ℓ-adic on $X/\mathbb{C}$ thus on $X_{\bar{\mathbb{F}}_p}$, $p >> 0$ prime to the order of $\mathcal{L}$, r, the order of the res. representation.

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$\mathcal{V}_\lambda$ arithmetic (Simpson) $\rightsquigarrow$ companions $\mathcal{V}_{\lambda'}$. As

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$\mathcal{V} \rightsquigarrow \mathcal{V}'$ bijection. □

On proof of crystalline theorem: crystalline part

$M_{dR}^{\mathrm{rig}} \subset M_{dR}$ has certain mult. $\mu \Rightarrow \forall \mathrm{Spec}(\mathbb{F}_q) \to \mathrm{Spec}(R)$, $p >> 0$, $\mu << p$.

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$X/$ perfeckt field k: p-curv $\in H^0(X, \mathcal{E}nd(E) \otimes F^*\Omega^1_{X'})^\nabla \Rightarrow$ $\chi_{dR} = p$th root of char. pol. of p-curv: $M_{dR} \to \mathbb{A}$ with $\chi^{-1}_{dR}(0) =$ conn. with nilp. p-curv.

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[(existence $(E,\nabla)_{\mathbb{Z}_q/p^2} + \chi_{dR}((E,\nabla)_{\mathbb{F}_q}) \neq 0) \Rightarrow (E,\nabla)_{\mathbb{F}_q}$ deforms to order $(p-1)$] $\Leftarrow$ Ogus-Vologodsky. $\square$