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A note on the cycle map

By *Hélène Esnault**) at Bonn

In [B2], theorem (6. 1), S. Bloch proves that on a surface X with $p_g = q = 0$, the degree map on the Chow group $\mathrm{CH}^2(X)$ of points is realized as

$$\deg: \mathrm{CH}^2(X) = H^2_{\mathrm{zar}}(X, \mathscr{K}_2) \to \mathbb{Z} \subset H^2_{\mathrm{an}}(X, \mathscr{K}_{2\mathrm{an}}),$$

where $\mathscr{K}_2$ and $\mathscr{K}_{2\mathrm{an}}$ are the sheaves K_2 in the Zariski and classical topologies, and where the map comes from the change of topologies. This suggests that some natural Zariski cohomology should exist inbetween which describes the cycle map Ψ with values in the Deligne-Beilinson cohomology. This is the purpose of the first chapter of this note. If X is smooth, one defines the Zariski sheaf $\mathscr{F}^{pp}_{\mathbb{Z}}$ (1. 1) as the discrete part of the Deligne-Beilinson sheaf $\mathscr{H}^p_{\mathscr{D}}(p)$ ((1. 3)α)). By forgetting the growth condition at infinity, one obtains an injection $f_{pp}: \mathscr{F}^{pp}_{\mathbb{Z}} \to \Omega^{pp}_{\mathbb{Z}}$. Then $H^p_{\mathrm{zar}}(f_{pp})$ factorizes the cycle map and is exactly the cycle map if $p = \dim X$ (1. 3). Applying this for $p = \dim X$, one proves that the kernel of the Albanese map is exactly

$$H^{n-1}_{\mathrm{zar}}(\Omega^{nn}_{\mathbb{Z}}/\mathscr{F}^{nn}_{\mathbb{Z}})$$

provided $H^{2n-2}(\mathbb{Z})$ is generated by algebraic cycles (2. 5).

Recall that on a smooth projective variety X over $\mathbb{C}$, the Chow group $\mathrm{CH}^1(X)$ of codimension 1 cycles may be described analytically: this is $H^1_{\mathrm{an}}(X, \mathcal{O}^{\times}_{\mathrm{an}})$, the group of holomorphic rank 1 vectorbundles [S]. In the same spirit, S. Bloch [B3] asks whether possibly the Chow group $\mathrm{CH}^2(X)$ of codimension 2 cycles may be described as the analytic cohomology $H^4_{\mathrm{an}}(\mathscr{B})$ of a complex $\mathscr{B}$ (3. 2) he defines in [B1] via the dilogarithm function. We show the existence of a map $\phi: \mathrm{CH}^2(X) \to H^4_{\mathrm{an}}(\mathscr{B})$ factorizing the cycle map Ψ (3. 6). If one knew that ϕ is injective, this would imply Bloch's conjecture on surfaces with $p_g = q = 0$ (3. 8). Moreover this gives a sort of "philosophical" explanation for Mumford's theorem [M], (3. 7).

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Notations. Throughout this paper, we use the following notations:

X	will denote a smooth algebraic scheme over $\mathbb{C}$ of dimension n,
Ω^p_X	will denote the sheaf of holomorphic differential p-forms on X_{an}, the variety X endowed with its classical topology,
A	will denote the ring $\mathbb{Z}$ or $\mathbb{Q}$,
$A(p)$	will denote $(2i\pi)^p A$,
$H^q(A(p))$, $H^q(\mathbb{C}/A(p)), \ldots$	will denote the Betti cohomology groups,
$H^q_{\mathscr{D}}(A(p))$	will denote the Deligne-Beilinson cohomology group (as defined in [Б] or [E.V]).

If $K^\cdot$ is a complex, we denote by $K^{\geqq p}$ its subcomplex defined by: $(K^{\geqq p})^i = 0$ if $i < p$ and $(K^{\geqq p})^i = K^i$ if $i \geqq p$.

If $\bar{X}$ is a smooth proper variety containing X such that $\bar{X} - X$ is a divisor with normal crossings, we call $\bar{X}$ a good compactification of X. On $\bar{X}$ one has the complex $\Omega^\cdot_{\bar{X}}(\log(\bar{X} - X))$ of holomorphic forms with logarithmic poles along $(\bar{X} - X)$. We denote by $F^p H^q(\mathbb{C})$ the Hodge-Deligne F^p-filtration of the de Rham cohomology of X:

$$F^p H^q(\mathbb{C}) = H^q(\bar{X}, \Omega^{\geqq p}_{\bar{X}}(\log(\bar{X} - X))).$$

This is independent of the good compactification choosen [D].

We denote by H_{zar} and H_{an} the sheaf cohomology in the Zariski and classical topologies.

$\mathrm{CH}^p(X) \xrightarrow[\Psi]{} H^{2p}_{\mathscr{D}}(X, \mathbb{Z}(p))$ is the cycle map from the Chow group in the Deligne-Beilinson cohomology [Б], [G], [E.V].

Finally we do not distinguish in our notations between $H^\cdot$ and $\mathbb{H}^\cdot$, the cohomology and the hypercohomology. (See for instance the definition of $F^p H^q(\mathbb{C})$ above.)

§ 1. The cycle map on projective manifolds

1. 1. Let X be as in the notations. For each Zariski open set U of good compactification $\bar{U}$ we consider

$$\begin{aligned} F^{pp}_A(U) := \{\omega \in F^p H^p(U, \mathbb{C}) &= H^p(\bar{U}, \Omega^{\geqq p}(\log(\bar{U} - U))) \\ &= H^0(\bar{U}, \Omega^p(\log(\bar{U} - U)))_{d\,\mathrm{closed}} = H^0(\bar{U}, \Omega^p(\log(\bar{U} - U))), \end{aligned}$$

such that the cohomology class of ω in $H^p(U, \mathbb{C}/A(p))$ vanishes$\}$,

$$\mathrm{Hol}^{pp}_A(U) := \{\omega \in H^p(U, \Omega^{\geqq p}) = H^0(U, \Omega^p)_{d\,\mathrm{closed}},$$

such that the cohomology class of ω in $H^p(U, \mathbb{C}/A(p))$ vanishes$\}$.

Both are presheaves in the Zariski topology. We denote by $\mathscr{F}_A^{pp}$ and Ω_A^{pp} the associated sheaves. Forgetting the logarithmic growth condition at infinity gives a natural injective morphism

$$f_{pp}^A : \mathscr{F}_A^{pp} \longrightarrow \Omega_A^{pp}.$$

We set $f_{pp} := f_{pp}^{\mathbb{Z}}$.

1. 2. We consider the Zariski sheaves

$$\mathscr{H}^q(A(p)),\ \mathscr{H}^q(\mathbb{C}/A(p)),\ \mathscr{H}_{\mathscr{D}}^q(A(p)), \ldots$$

associated to $H^q(A(p))$, $H^q(\mathbb{C}/A(p))$, $\mathscr{H}_{\mathscr{D}}^q(A(p)), \ldots$.

1. 3. Theorem. *Let X be a smooth algebraic scheme over $\mathbb{C}$ of dimension n.*

1) *One has*

$$H_{\mathrm{zar}}^p(X, \mathscr{F}_A^{pp}) = \mathrm{CH}^p(X)_A := \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} A,$$

$$H_{\mathrm{zar}}^{p+l}(X, \mathscr{F}_A^{pp}) = 0 \quad \textit{for} \quad l \geqq 1,$$

$$H_{\mathrm{zar}}^{p-1}(X, \mathscr{F}_{\mathbb{Z}}^{pp}) = H_{\mathrm{zar}}^{p-1}(X, \mathscr{H}_{\mathscr{D}}^p(p)) / H_{\mathrm{zar}}^{p-1}(X, \mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p))).$$

2) *If X is projective there is a factorization of the cycle map*

$$\begin{array}{ccc} \mathrm{CH}^p(X) = H_{\mathrm{zar}}^p(X, \mathscr{F}_{\mathbb{Z}}^{pp}) & \xrightarrow[H_{\mathrm{zar}}^p(f_{pp})]{} & H_{\mathrm{zar}}^p(X, \Omega_{\mathbb{Z}}^{pp}) \\ & \searrow^{\Psi} & \downarrow \\ & & H_{\mathscr{D}}^{2p}(X, \mathbb{Z}(p)). \end{array}$$

3) *If X is projective and $p = n$, then $H_{\mathrm{zar}}^n(X, \Omega_{\mathbb{Z}}^{nn}) = H_{\mathscr{D}}^{2n}(X, \mathbb{Z}(n))$ and $\Psi = H_{\mathrm{zar}}^n(f_{nn})$.*

Proof. 1) For each Zariski open set U one has an exact sequence

$$0 \longrightarrow H^{p-1}(\mathbb{C}/\mathbb{Z}(p)) \longrightarrow H_{\mathscr{D}}^p(\mathbb{Z}(p)) \longrightarrow F_{\mathbb{Z}}^{pp} \longrightarrow 0$$

which gives an exact sequence of Zariski sheaves

$$(1.\,3)\alpha) \qquad 0 \longrightarrow \mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p)) \longrightarrow \mathscr{H}_{\mathscr{D}}^p(p) \longrightarrow \mathscr{F}_{\mathbb{Z}}^{pp} \longrightarrow 0$$

(and describes $\mathscr{F}_{\mathbb{Z}}^{pp}$ as the discrete part of $\mathscr{H}_{\mathscr{D}}^p(p)$).

The Bloch-Orgus theory for the Betti [BO] and the Deligne-Beilinson [Б] cohomologies implies

$$H_{\mathrm{zar}}^{p+1+l}(\mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p))) = H_{\mathrm{zar}}^{p+l}(\mathscr{H}_{\mathscr{D}}^p(\mathbb{Z}(p))) = 0$$

for $l \geqq 1$, and

$$H^p_{\mathrm{zar}}(X, \mathscr{H}^p_{\mathscr{D}}(\mathbb{Z}(p))) = \mathrm{CH}^p(X).$$

This proves 1). The proof of 2) and 3) occupies the rest of this section.

1. 4. Let $\alpha: X_{\mathrm{an}} \to X_{\mathrm{zar}}$ be the continuous identity of X endowed with the analytic and Zariski topologies. Let

$$\mathbb{Z}(p)_{\mathscr{D}} := \mathbb{Z}(p) \to \mathcal{O} \to \cdots \to \Omega^{p-1}$$

be the Deligne complex on X_{an}, (where $\mathbb{Z}(p)$ is in degree 0). Then $\mathbb{Z}(0)_{\mathscr{D}} := \mathbb{Z}$.

Lemma. *One has*

1) $R^0 \alpha_* \mathbb{Z}(p)_{\mathscr{D}} = 0$ *if* $p > 0$ *or* $\mathbb{Z}$ *if* $p = 0$.

2) $R^i \alpha_* \mathbb{Z}(p)_{\mathscr{D}} = \mathscr{H}^{i-1}(\mathbb{C}/\mathbb{Z}(p)),\ 0 < i < p$.

3) *There are exact sequences*

$$0 \to \mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p)) \to R^p \alpha_* \mathbb{Z}(p)_{\mathscr{D}} \to \Omega^{pp}_{\mathbb{Z}} \to 0,$$

$$0 \to \Omega^{pp}_{\mathbb{Z}} \to \alpha_*(\Omega^p \to \cdots \to \Omega^n) \to \mathscr{H}^p(\mathbb{C})/\mathscr{H}^p(\mathbb{Z}(p)) \to 0.$$

4) $R^{p+l} \alpha_* \mathbb{Q}(p)_{\mathscr{D}} = \mathscr{H}^{p+l}(\mathbb{Q}(p))$ *for* $l \geqq 1$.

5) $R^{p+l} \alpha_* \mathbb{Z}(p)_{\mathscr{D}} = 0$ *for* $l > n - p$ *and* $l > 0$.

Proof. Write the exact sequences of complexes

$$0 \to (\mathbb{Z}(p) \to \mathbb{C}) \to \mathbb{Z}(p)_{\mathscr{D}} \to \Omega^{\geqq p} \to 0.$$

As $R^q \alpha_* \Omega^{\geqq p} = 0$ for $q < p$ this proves 1) and 2).

As $\alpha_*(\Omega^p \to \cdots \to \Omega^n)$ surjects onto $\mathscr{H}^p(\mathbb{C})/\mathscr{H}^p(\mathbb{Z}(p))$ with kernel $\Omega^{nn}_{\mathbb{Z}}$, this proves 3). For $l \geqq 1$, $R^{p+l} \alpha_* \Omega^{\geqq p} = \alpha_*(\Omega^{p+l})_{d\,\mathrm{closed}} / d\alpha_* \Omega^{p+l-1}$ surjects onto $\mathscr{H}^{p+l}(\mathbb{C}/\mathbb{Q}(p))$ with kernel $\mathscr{H}^{p+l}(\mathbb{Q}(p))$. This proves 4). If $l > n - p$ then $(\alpha_* \Omega^{p+l-1})_{d\,\mathrm{closed}}$ surjects onto $\mathscr{H}^{p+l-1}(\mathbb{C}/\mathbb{Z}(p))$. This proves 5).

1. 4. 1. Remark. In 5), if $l = 0$, this implies that $p > n$. Then $\mathbb{Z}(p)_{\mathscr{D}} = \mathbb{Z}(p) \to \mathbb{C}$, and

$$R^p \alpha_* \mathbb{Z}(p)_{\mathscr{D}} = \mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p)) = 0 \quad \text{if} \quad p > n + 1.$$

In 4), we have replaced $\mathbb{Z}$ by $\mathbb{Q}$ to ensure that $\alpha_*(\Omega^{p+l})_{d\,\mathrm{closed}}$ surjects onto $\mathscr{H}^{p+l}(\mathbb{C}/\mathbb{Q}(p))$.

1. 5. Corollary. 1) *One has*

$$H^p_{\mathrm{zar}}(X, R^p \alpha_* A(p)_{\mathscr{D}}) = H^p_{\mathrm{zar}}(X, \Omega^{pp}_A).$$

2) *If X is proper there is a natural morphism*

$$H^p_{\mathrm{zar}}(X, \Omega^{pp}_{\mathbb{Z}}) \to H^{2p}_{\mathscr{D}}(X, \mathbb{Z}(p)).$$

3) *If X is proper and $p=n$ one has an isomorphism*

$$H^p_{\mathrm{zar}}(X, \Omega^{nn}_{\mathbb{Z}}) = H^{2n}_{\mathscr{D}}(X, \mathbb{Z}(n)).$$

Proof. Consider the Leray spectral sequence for α applied to $\mathbb{Z}(p)_{\mathscr{D}}$. By 1.4.1) and 2) one has

$$H^{2p-l}_{\mathrm{zar}}(X, R^l\alpha_*\mathbb{Z}(p)_{\mathscr{D}}) = 0 \quad \text{for} \quad l>p$$

and $H^{p+l}_{\mathrm{zar}}(X, R^{p-l+1}\alpha_*\mathbb{Z}(p)_{\mathscr{D}}) = 0$ for $l \geqq 2$. Therefore one has a map

$$H^p_{\mathrm{zar}}(X, R^p\alpha_*\mathbb{Z}(p)_{\mathscr{D}}) \to H^{2p}_{\mathscr{D}}(X, \mathbb{Z}(p)) = H^{2p}_{\mathrm{an}}(X, \mathbb{Z}(p)_{\mathscr{D}}).$$

As $H^{p-1+l}_{\mathrm{zar}}(X, \mathscr{H}^{p-1}(\mathbb{C}/\mathbb{Z}(p))) = 0$ for $l \geqq 1$ this proves 1) and 2). For 3), apply 1.4.5).

1.6. The cup product map

$$\mathscr{H}^p_{\mathscr{D}}(\mathbb{Z}(p)) \times \mathscr{H}^{p'}_{\mathscr{D}}(\mathbb{Z}(p')) \to \mathscr{H}^{p+p'}_{\mathscr{D}}(\mathbb{Z}(p+p'))$$

factorizes over the wedge product

$$\mathscr{F}^{pp}_{\mathbb{Z}} \times \mathscr{F}^{p'p'}_{\mathbb{Z}} \to \mathscr{F}^{p+p',p+p'}_{\mathbb{Z}} \text{ [E.V]},$$
$$(\omega, \omega') \mapsto \omega \wedge \omega'.$$

The intersection of cycles in $\mathrm{CH}^{\bullet}(X)$ is then described as wedge product via 1.3.1):

$$H^p_{\mathrm{zar}}(\mathscr{F}^{pp}_{\mathbb{Z}}) \times H^{p'}_{\mathrm{zar}}(\mathscr{F}^{p'p'}_{\mathbb{Z}}) \to H^{p+p'}_{\mathrm{zar}}(\mathscr{F}^{p+p',p+p'}_{\mathbb{Z}}).$$

Similarly the cup product map

$$R^p\alpha_*(\mathbb{Z}(p)_{\mathscr{D}}) \times R^{p'}\alpha_*(\mathbb{Z}(p')_{\mathscr{D}}) \to R^{p+p'}\alpha_*\mathbb{Z}(p+p')_{\mathscr{D}}$$

factorizes over the wedge product

$$\Omega^{pp}_{\mathbb{Z}} \times \Omega^{p'p'}_{\mathbb{Z}} \to \Omega^{p+p',p+p'}_{\mathbb{Z}},$$
$$(\omega, \omega') \mapsto \omega \wedge \omega'.$$

The intersection of cycles in $H^{2\bullet}_{\mathscr{D}}(X, \cdot)$, when X is proper, is then described as wedge product via 1.5.2):

$$\begin{array}{ccc} H^p_{\mathrm{zar}}(\Omega^{pp}_{\mathbb{Z}}) \times H^{p'}_{\mathrm{zar}}(\Omega^{p'p'}_{\mathbb{Z}}) & \longrightarrow & H^{p+p'}_{\mathrm{zar}}(\Omega^{p+p',p+p'}_{\mathbb{Z}}) \\ \downarrow & & \downarrow \\ H^{2p}_{\mathscr{D}}(\mathbb{Z}(p)) \times H^{2p'}_{\mathscr{D}}(\mathbb{Z}(p')) & \longrightarrow & H^{2(p+p')}_{\mathscr{D}}(\mathbb{Z}(p+p')). \end{array}$$

The wedge products for $\mathscr{F}_{\mathbb{Z}}^{pp}$ and $\Omega_{\mathbb{Z}}^{pp}$ are obviously compatible, and the wedge product for $\Omega_{\mathbb{Z}}^{pp}$ and the cup product for $H_{\mathscr{D}}^{2p}(p)$ are compatible via 1.5.

1.7. Let $f: Y \to X$ be a morphism of smooth schemes. One has obvious maps

$$f^{-1}\mathscr{F}_{\mathbb{Z},X}^{pp} \to \mathscr{F}_{\mathbb{Z},Y}^{pp},$$

$$f^{-1}\Omega_{\mathbb{Z},X}^{pp} \to \Omega_{\mathbb{Z},Y}^{pp},$$

$$f^{-1}R^p\alpha_*\mathbb{Z}(p)_{\mathscr{D},X} \to R^p\alpha_*\mathbb{Z}(p)_{\mathscr{D},Y}.$$

If Y and X are proper, f defines a commutative diagram

(1.7.1)

$$\begin{tikzcd} f^{-1}H_{\mathrm{zar}}^p(X, \mathscr{F}_{\mathbb{Z}}^{pp}) \arrow[r, "f^{-1}H_{\mathrm{zar}}^p(f_{pp})"] \arrow[d] & f^{-1}H_{\mathrm{zar}}^p(X, \Omega_{\mathbb{Z}}^{pp}) \arrow[r, "f^{-1}{[(1.5)2)]}"] \arrow[d] & f^{-1}H_{\mathscr{D}}^{2p}(X, \mathbb{Z}(p)) \arrow[d] \\ H_{\mathrm{zar}}^p(Y, \mathscr{F}_{\mathbb{Z}}^{pp}) \arrow[r] & H_{\mathrm{zar}}^p(Y, \Omega_{\mathbb{Z}}^{pp}) \arrow[r] & H_{\mathscr{D}}^{2p}(Y, \mathbb{Z}(p)). \end{tikzcd}$$

Moreover, the wedge products on $\mathscr{F}_{\mathbb{Z}}^{pp}$ and $\Omega_{\mathbb{Z}}^{pp}$ are obviously functorial.

1.8. Proof of 1.3.2) and 3) for $p=1$. One has $\mathscr{H}_{\mathscr{D}}^1(1) = \mathcal{O}_{\mathrm{alg}}^\times$ the sheaf of algebraic invertible functions and $R^1\alpha_*\mathbb{Z}(1)_{\mathscr{D}} = \alpha_*\mathcal{O}_{\mathrm{an}}^\times$, where $\mathcal{O}_{\mathrm{an}}^\times$ is the sheaf of holomorphic invertible functions.

One has

(1.8.1)

$$\begin{tikzcd} \mathrm{CH}^1(X) = H_{\mathrm{zar}}^1(\mathcal{O}_{\mathrm{alg}}^\times) = H_{\mathrm{zar}}^1(\mathscr{F}_{\mathbb{Z}}^{11}) \arrow[r] \arrow[d] & H_{\mathscr{D}}^2(1) = \mathrm{Pic}\, X \arrow[d, "\wr"] \\ H_{\mathrm{zar}}^1(\alpha_*\mathcal{O}_{\mathrm{an}}^\times) = H_{\mathrm{zar}}^1(\Omega_{\mathbb{Z}}^{11}) \arrow[r] & H_{\mathscr{D}}^2(1) = H_{\mathrm{an}}^1(\mathcal{O}_{\mathrm{an}}^\times). \end{tikzcd}$$

The lower horizontal arrow is always injective and by [S] the composition

(1.8.2)
$$H_{\mathrm{zar}}^1(\mathcal{O}_{\mathrm{alg}}^\times) \to H_{\mathrm{zar}}^1(\alpha_*\mathcal{O}_{\mathrm{an}}^\times) \to H_{\mathrm{an}}^1(\mathcal{O}_{\mathrm{an}}^\times)$$

is an isomorphism. Therefore both maps are isomorphisms. The upper arrow is the canonical isomorphism between $\mathrm{CH}^1(X)$ and $\mathrm{Pic}\, X$.

1.9. *Proof of* 1.3.2) *and* 3). Let $\mathscr{V}$ be a vector bundle. We first prove that

$$\text{image of } H_{\mathrm{zar}}^p(f_{pp})(c_p^{\mathrm{CH}})(\mathscr{V}) \quad \text{in} \quad H_{\mathscr{D}}^{2p}(\mathbb{Z}(p)) = c_p^{\mathscr{D}}(\mathscr{V})$$

where c_p^{CH} and $c_p^{\mathscr{D}}$ are the Chern classes in CH^p and in $H_{\mathscr{D}}^{2p}(\mathbb{Z}(p))$. Let $f: Y \to X$ be the Grassmann bundle of $\mathscr{V}$. One has a commutative diagram

$$\begin{array}{ccccc}
f^{-1}H^p_{\mathrm{zar}}(X, \mathscr{F}_{\mathbb{Z}}^{pp}) & \longrightarrow & f^{-1}H^p_{\mathrm{zar}}(X, \Omega_{\mathbb{Z}}^{pp}) & \longrightarrow & f^{-1}H^{2p}_{\mathscr{D}}(X, \mathbb{Z}(p)) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (1.7.1)} \\
H^p_{\mathrm{zar}}(Y, \mathscr{F}_{\mathbb{Z}}^{pp}) & \longrightarrow & H^p_{\mathrm{zar}}(Y, \Omega_{\mathbb{Z}}^{pp}) & \longrightarrow & H^{2p}_{\mathscr{D}}(Y, \mathbb{Z}(p)) \\
\uparrow & & \uparrow & & \uparrow \\
\overset{p}{\times} H^1_{\mathrm{zar}}(Y, \mathscr{F}_{\mathbb{Z}}^{11}) & \longrightarrow & \overset{p}{\times} H^1_{\mathrm{zar}}(Y, \Omega_{\mathbb{Z}}^{11}) & \longrightarrow & \overset{p}{\times} H^2_{\mathscr{D}}(Y, \mathbb{Z}(1)).
\end{array}$$

By the splitting principle for $\mathrm{CH}^{\cdot}$ and $H_{\mathscr{D}}^{2\cdot}(\cdot)$, the upper left and right vertical arrows are injective. By the splitting principle for $\mathrm{CH}^{\cdot}$ and $H_{\mathscr{D}}^{2\cdot}(\cdot)$ and by 1.6 and 1.8, $f^{-1}c_p^{\mathrm{CH}}(\mathscr{V})$ maps to $c_p^{\mathscr{D}}(\mathscr{V})$ via the bottom horizontal arrow. This proves 1.3. 2) and 3) for Chern classes of vector bundles.

If Z is a codimension p cycle, $\mathcal{O}_Z$ may be resolved by vector bundles. As Chern classes are additive in $\mathrm{CH}^{\cdot}$ and $H_{\mathscr{D}}^{2\cdot}(\cdot)$, and as the wedge product for $\mathscr{F}_{\mathbb{Z}}^{\cdot\cdot}$, $\Omega_{\mathbb{Z}}^{\cdot\cdot}$ and the cup product for $H_{\mathscr{D}}^{2\cdot}(\cdot)$ are compatible (1.6), one reduces the problem to the vector bundle case.

1. 10. Remark. As a trivial consequence of the definition, 1. 3 is compatible with products and is functorial for any morphism $f: Y \to X$, with Y smooth for 1), Y smooth and projective for 2).

1. 11. Remark. The Bloch-Ogus theory for the Deligne-Beilinson cohomology provides us with a coniveau spectral sequence

$$E_2^{kl} = H^k_{\mathrm{zar}}(X, \mathscr{H}_{\mathscr{D}}^l(p))$$

converging to $H_{\mathscr{D}}^{k+l}(X, \mathbb{Z}(p))$. Forgetting the logarithmic growth condition at infinity one obtains a morphism

$$g_{lp}: \mathscr{H}_{\mathscr{D}}^l(p) \to R^l\alpha_* \mathbb{Z}(p)_{\mathscr{D}} \quad \text{(injective for } l=p \quad \text{[E]).}$$

(f_{pp} defined in 1. 1 is just the Hodge theoretical part of g_{pp}.) This defines a map

$$H^k_{\mathrm{zar}}(g_{lp}): E_2^{kl} \to E_{2\mathrm{an}}^{kl} := H^k_{\mathrm{zar}}(X, R^l\alpha_* \mathbb{Z}(p)_{\mathscr{D}}).$$

On the other hand $E_{2\mathrm{an}}^{kl}$ is the E_2 term of the Leray spectral sequence for α applied to $\mathbb{Z}(p)_{\mathscr{D}}$ also converging to $H_{\mathscr{D}}^{k+l}(X, \mathbb{Z}(p))$ (if X is proper). If one knew that the coniveau and the Leray spectral sequences were compatible, one would be able to avoid the cumbersome arguments of 1. 8 and 1. 9 (and prove slightly more than 1. 3).

§ 2. Cycle map for points on projective manifolds

2. 1. In 1. 3. 3) we have seen that the cycle map for points on a projective manifold is just $H^n_{\text{zar}}(f_{nn})$. In particular the kernel of the Albanese mapping (which is the kernel of Ψ) comes from

$$H^{n-1}_{\text{zar}}(X, \Omega^{nn}_{\mathbb{Z}}/\mathcal{F}^{nn}_{\mathbb{Z}}).$$

If X is proper, we denote by alg the image of $\mathrm{CH}^{n-1}(X)$ in $H^{2n-2}(X, \mathbb{Z}(n-1))$. We say that $H^{2n-2}(\mathbb{Z})$ is generated by algebraic cycles if $H^{2n-2}(\mathbb{Z}) = \text{alg}$. This implies $F^n H^{2n-2}(\mathbb{C}) = H^{2n-2}(\Omega^n) = 0$. Conversely, if $F^n H^{2n-2}(\mathbb{C}) = 0$, then $\text{alg} \otimes \mathbb{Q} = H^{2n-2}(\mathbb{Q})$, and $\text{alg} = H^2(\mathbb{Z})$ if $n=2$. Otherwise, Hodge theory implies that the natural map $F^n H^{2n-2}(\mathbb{C}) \to H^{2n-2}(\mathbb{C}/\mathbb{Z}(n))$ is injective. For this reason, we just denote by F^n its image.

2. 2. Lemma. *If X is proper one has*

$$H^n_{\text{zar}}(X, \Omega^{nn}_{\mathbb{Z}}) = H^{2n-2}(X, \mathbb{C}/\mathbb{Z}(n))/F^n + \text{alg} \otimes \mathbb{C}/\mathbb{Z}(n).$$

In particular it is zero (up to torsion) if $H^{2n-2}(\mathbb{Z})$ is generated by algebraic cycles.

Proof. By 1. 4. 2) and $H^q(\mathcal{H}^j(\mathbb{C}/\mathbb{Z}(n))) = 0$ for $q > j$ ([BO]), one has a map

$$H^{n-1}_{\text{zar}}(X, R^n \alpha_* \mathbb{Z}(n)_{\mathcal{D}}) \to H^{2n-1}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

By 1. 4. 5), this is an isomorphism. By 1. 4. 3), one obtains

$$H^{n-1}_{\text{zar}}(X, \Omega^{nn}_{\mathbb{Z}}) = H^{2n-1}_{\mathcal{D}}(X, \mathbb{Z}(n))/\text{im}\,(H^{n-1}_{\text{zar}}(X, \mathcal{H}^{n-1}(\mathbb{C}/\mathbb{Z}(n)))).$$

As $H^{2n-1}_{\mathcal{D}}(X, \mathbb{Z}(n)) = H^{2n-2}(X, \mathbb{C}/\mathbb{Z}(n))/F^n$ and $\text{im}\,(H^{n-1}_{\text{zar}}(X, \mathcal{H}^{n-1}(\mathbb{Z}))) = \text{alg}$ [BO], this proves 2. 2.

2. 3. Lemma. *One has $\mathcal{H}^{n+l}_{\mathcal{D}}(n)$ for $l \geqq 2$.*

The Bloch-Ogus theory applied to the Deligne-Beilinson cohomology gives the exact sequence

$$H^{n-1}_{\text{zar}}(X, \mathcal{H}^n_{\mathcal{D}}(n)) \to H^{2n-1}_{\mathcal{D}}(X, \mathbb{Z}(n))$$
$$\to H^{n-2}_{\text{zar}}(X, \mathcal{H}^{n+1}_{\mathcal{D}}(n)) \to \mathrm{CH}^n(X) \xrightarrow[\Psi]{} H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \to 0.$$

If X is proper and $H^{2n-2}(\mathbb{Z})$ is generated by algebraic cycles then

$$\text{Ker}\,\Psi = H^{n-2}_{\text{zar}}(X, \mathcal{H}^{n+1}_{\mathcal{D}}(n)) \text{ modulo torsion.}$$

Proof. One has an exact sequence

$$0 \to H^{n+l-1}(\mathbb{C}/\mathbb{Z}(n))/F^n \to \mathcal{H}^{n+l}_{\mathcal{D}}(\mathbb{Z}(n)) \to \text{Ker}\,(F^n H^{n+l} \to H^{n+l}(\mathbb{C}/\mathbb{Z}(n))) \to 0.$$

If U is affine one has $H^{n+l}(U, \mathbb{Z})=0$ for $l \geqq 1$ [H]. This proves $\mathscr{H}_{\mathscr{D}}^{n+l}(n)=0$ for $l \geqq 2$. This gives an exact sequence

$$0 \longrightarrow E_{\infty}^{n-1,n} \longrightarrow H_{\mathscr{D}}^{2n-1}(\mathbb{Z}(n)) \longrightarrow E_{\infty}^{n-2,n+1} \longrightarrow 0$$

with

$$E_{\infty}^{n-2,n+1} = E_3^{n-2,n+1} = \operatorname{Ker}(H_{\mathrm{zar}}^{n-2}(\mathscr{H}_{\mathscr{D}}^{n+1}(n)) \longrightarrow \mathrm{CH}^n)$$

and

$$E_{\infty}^{n-1,n} = E_3^{n-1,n} = H_{\mathrm{zar}}^{n-1}(\mathscr{H}_{\mathscr{D}}^{n}(n))/H_{\mathrm{zar}}^{n-3}(\mathscr{H}_{\mathscr{D}}^{n+1}(n)).$$

If $\mathrm{alg} = H^{2n-2}(X, \mathbb{Z})$, then

$$H_{\mathscr{D}}^{2n-1}(X, n) = H^{2n-2}(\mathbb{C}/\mathbb{Z}(n))/F^n = \mathrm{alg} \otimes \mathbb{C}/\mathbb{Z}(n) \text{ modulo torsion.}$$

But the image of

$$H_{\mathrm{zar}}^{n-1}(\mathscr{H}^{n-1}(\mathbb{C}/\mathbb{Z}(n)))$$

in $H_{\mathrm{zar}}^{n-1}(\mathscr{H}_{\mathscr{D}}^{n}(n))$ via (1. 3) α) is $\mathrm{alg} \otimes \mathbb{C}/\mathbb{Z}(n)$. Therefore $E_3^{n-1,n}$ maps surjectively onto $H_{\mathscr{D}}^{2n-1}(\mathbb{Z}(n))$.

2. 4. Lemma. *There is a commutative diagram of exact sequences*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{F}_{\mathbb{Z}}^{nn} \arrow[r] \arrow[d, "f_{nn}"'] & \mathscr{F}_{\mathbb{C}}^{nn} \arrow[r] \arrow[d] & \mathscr{F}_{\mathbb{C}}^{nn}/\mathscr{F}_{\mathbb{Z}}^{nn} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{\mathbb{Z}}^{nn} \arrow[r] \arrow[d] & \alpha_* \Omega^n \arrow[r] \arrow[d] & \mathscr{H}^n(\mathbb{C}/\mathbb{Z}(n)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{\mathbb{Z}}^{nn}/\mathscr{F}_{\mathbb{Z}}^{nn} \arrow[r] \arrow[d] & \alpha_* \Omega^n/\mathscr{F}_{\mathbb{C}}^{nn} \arrow[r] \arrow[d] & \mathscr{H}_{\mathscr{D}}^{n+1}(n) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd} \qquad (cf.\ 1.4.3))$$

Proof. For the middle horizontal arrow one has to know that $\mathscr{H}^{n+1}(\mathbb{Z}(n))=0$ [H]. One has an exact sequence

$$0 \longrightarrow H^n(\mathbb{C}/\mathbb{Z}(n))/F^n \longrightarrow \mathscr{H}_{\mathscr{D}}^{n+1}(n) \longrightarrow \operatorname{Ker}(F^n H^{n+1} \longrightarrow H^{n+1}(\mathbb{C}/\mathbb{Z}(n))) \longrightarrow 0.$$

On each affine open set one has $H^{n+1}=0$, which proves $\mathcal{H}_{\mathcal{D}}^{n+1}=\mathcal{H}^n(\mathbb{C}/\mathbb{Z}(n))/\mathcal{F}_{\mathbb{C}}^{nn}$ and completes the proof.

2.5. Theorem. *If X is a projective manifold of dimension n over $\mathbb{C}$ such that $H^{2n-2}(\mathbb{Z})$ is generated by algebraic cycles one has a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & H_{\mathrm{zar}}^{n-2}(X, \mathcal{H}_{\mathcal{D}}^{n+1}(n)) \arrow[r] \arrow[d, "2.4"', "\wr"] & \mathrm{CH}^n(X) \arrow[r, "\Psi"] \arrow[d, "1.3.1)"', "\wr"] & H_{\mathcal{D}}^{2n}(X, \mathbb{Z}(n)) \arrow[r] \arrow[d, "1.5.3)"', "\wr"] & 0 \\
0 \arrow[r] & H_{\mathrm{zar}}^{n-1}(X, \Omega_{\mathbb{Z}}^{nn}/\mathcal{F}_{\mathbb{Z}}^{nn}) \arrow[r] & H_{\mathrm{zar}}^{n}(X, \mathcal{F}_{\mathbb{Z}}^{nn}) \arrow[r, "H_{\mathrm{zar}}^n(f_{nn})"'] & H_{\mathrm{zar}}^{n}(X, \Omega_{\mathbb{Z}}^{nn}) \arrow[r] & 0
\end{tikzcd}$$

up to torsion.

Proof. The horizontal sequences are exact by 2.3 and 2.2. As $\Psi=H_{\mathrm{zar}}^n(f_{nn})$ (1.3.3)), the vertical left arrow is an isomorphism.

2.6. Remarks. 1) If $H^{2n-2}(\mathbb{Z})$ is not generated by algebraic cycles, one has the same diagram replacing the left vertical arrow by its image in $\mathrm{CH}^n(X)$.

2) As $H_{\mathrm{zar}}^{n+1}(\mathcal{F}_{\mathbb{Z}}^{nn})=0$ (1.3.1) or simply cohomological dimension of abelian Zariski sheaves), one obtains $H_{\mathrm{zar}}^n(\Omega_{\mathbb{Z}}^{nn}/\mathcal{F}_{\mathbb{Z}}^{nn})=0$ when X is projective.

§ 3. Miscellaneous comments on the Bloch complex

3.1. In [B3], S. Bloch (and S. Lichtenbaum) ask whether it is possible that

$$\mathrm{CH}^2(X)=H_{\mathrm{an}}^4(X, \mathcal{B})$$

where $\mathcal{B}$ is a complex of analytic sheaves defined in [B1] which we shall call the Bloch complex. (This is not the Zariski Bloch-Suslin complex, although it is related to it.)

In this section we say a few words about it.

3.2. Recall first the definition of $\mathcal{B}$ (see [B1]).

Let $\Delta/\mathbb{Z}(2)\subset\mathbb{C}/\mathbb{Z}(2)$ be the image of the regulator map $K_3(\mathbb{C})\to\mathbb{C}/\mathbb{Z}(2)$. This is a countable group.

As we shall not use here the sheaf of regular functions, we change the notations:

Let $\mathcal{O}$ be the sheaf of holomorphic functions, $\mathcal{O}^\times$ be the sheaf of holomorphic invertible functions. S. Bloch defines

$$v(\mathcal{O}):=\mathcal{O}\otimes_{\mathbb{Z}}\mathcal{O}^\times/\mathcal{R}$$

where the stalk $\mathcal{R}_z$ of $\mathcal{R}$ in z is the subgroup of $\mathcal{O}_z\otimes\mathcal{O}_z^\times$ generated by

$$-\int_0^{f(x)}\frac{dt}{1-t}\otimes f(x)+2i\pi\otimes\exp\left(-\frac{1}{2i\pi}\int_0^{f(x)}\log(1-t)\,\frac{dt}{t}\right)$$

where f and $(1-f)$ lie in $\mathcal{O}_z^\times$ and x is (analytically) close enough to z to force $(1-f(x))$ to be invertible, and where one has choosen a path of integration from 0 to $f(x)$.

Let $\mathcal{K}_{2\mathrm{an}}$ be the analytic sheaf K_2. There is a surjective map

$$v(\mathcal{O}) \longrightarrow \mathcal{K}_{2\mathrm{an}},$$
$$F \otimes g \mapsto \{\exp F, g\}.$$

Define

$$\mathcal{O}/\mathbb{Z}(2) \longrightarrow v(\mathcal{O})$$

by

$$\exp \frac{1}{2i\pi} G \longrightarrow 2i\pi \otimes \exp \frac{1}{2i\pi} G.$$

This defines a complex $\mathcal{B} := \mathbb{Z}(2) \longrightarrow \mathcal{O} \longrightarrow v(\mathcal{O})$, with $\mathbb{Z}(2)$ in degree 0, together with a map $\mathcal{B} \xrightarrow[\exp\otimes\mathrm{id}]{} \mathcal{K}_{2\mathrm{an}}[-2]$.

On the other hand S. Bloch [B1] computes that the map

$$\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}^\times \longrightarrow \Omega^1,$$
$$F \otimes g \mapsto F \frac{dg}{g}$$

factorizes over $v(\mathcal{O})$ as

$$\log(1-f(x)) \frac{df(x)}{f(x)} + 2i\pi \cdot d\log\left(\exp\left(-\frac{1}{2i\pi} \int_0^{f(x)} \log(1-t) \frac{dt}{t}\right)\right) = 0.$$

This defines a commutative square

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\varphi"] \arrow[d, "\exp\otimes\mathrm{id}"'] & \mathbb{Z}(2)_{\mathcal{D}} \arrow[d] \\
\mathcal{K}_{2\mathrm{an}}[-2] \arrow[r, "d\log\wedge d\log"'] \arrow[d] & \Omega^{\geqq 2} \arrow[d] \\
0 & 0
\end{tikzcd}$$

where $d\log\wedge d\log\{f, g\} = \frac{df}{f} \wedge \frac{dg}{g}$. Therefore $\operatorname{Ker}\exp\otimes\mathrm{id}$ maps to $\operatorname{Ker} d = \mathbb{Z}(2) \longrightarrow \mathbb{C}$. S. Bloch shows that it is exactly $\mathbb{Z}(2) \longrightarrow \Delta$. Here we should warn the reader. In [B1], S. Bloch has a different notation. He considers the complex $\mathcal{B}^{\sim} := \mathbb{Q}(2) \longrightarrow \mathcal{O} \longrightarrow v(\mathcal{O})$

and shows that $\operatorname{Ker}\exp\otimes\mathrm{id}$ in $\mathscr{B}^{\sim}$ is a constant non trivial countable subgroup of $\mathbb{C}/\mathbb{Q}(2)[-1]$ which he denotes by $\Delta^*[-1]$. Therefore, if Δ denotes the preimage of Δ^* in $\mathbb{C}$, one has $\operatorname{Ker}\exp\otimes\mathrm{id}=\Delta/\mathbb{Z}(2)[-1]$. We shall use only that this group is constant, and not its identification with the image of the regulator map

$$K_3(\mathbb{C})\to\mathbb{C}/\mathbb{Z}(2),$$

for which we refer to [B3].

This gives a commutative diagram whose columns are exact sequences.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
{[\mathbb{Z}(2)\to\Delta]} \arrow[r, "i"] \arrow[d] & {[\mathbb{Z}(2)\to\mathbb{C}]} \arrow[d] \\
\mathscr{B} \arrow[r, "\varphi"] \arrow[d, "\exp\otimes\mathrm{id}"'] & \mathbb{Z}(2)_{\mathscr{D}} \arrow[d] \\
\mathscr{K}_{2\mathrm{an}}[-2] \arrow[r, "d\log\wedge d\log"'] \arrow[d] & \Omega^{\geqq 2} \arrow[d] \\
0 & 0.
\end{tikzcd} \tag{3.2.1}$$

3.3. The vertical left sequence of (3.2.1) defines a morphism in the derived category

$$\mathscr{K}_{2\mathrm{an}}\xrightarrow[b]{}\Delta/\mathbb{Z}(2)[2]\xrightarrow[i]{}\mathbb{C}/\mathbb{Z}(2)[2].$$

One has also the morphism in the derived category

$$\begin{tikzcd}
\mathscr{K}_{2\mathrm{an}} \arrow[r, "d\log\wedge d\log"'] \arrow[rrrr, bend right, "c"'] & \Omega^{\geqq 2} \arrow[r] & \Omega^{\bullet} & \mathbb{C} \arrow[l, "\sim"'] \arrow[r] & \mathbb{C}/\mathbb{Z}(2).
\end{tikzcd}$$

Define as in 1.4 $\alpha: X_{\mathrm{an}}\to X_{\mathrm{zar}}$. The commutativity of (3.2.1) proves the

Lemma. *One has*

$$i\circ b=c \textit{ in the derived category.}$$

In particular

$$\mathscr{K}_{2\mathbb{Z}} := \operatorname{Ker} \alpha_* b : \alpha_* \mathscr{K}_{2\mathrm{an}} \to \mathscr{H}^2(\Delta/\mathbb{Z}(2))$$

is the subsheaf of $\alpha_* \mathscr{K}_{2\mathrm{an}}$ *consisting of sections whose de Rham classes via the map* $d\log \wedge d\log$ *vanish in* $\mathscr{H}^2(\mathbb{C}/\mathbb{Z}(2))$.

3. 4. Let $\mathscr{K}_2$ be the Zariski sheaf K_2.

Corollary. *There is a map*

$$\mathscr{K}_2 \longrightarrow \mathscr{K}_{2\mathbb{Z}}.$$

Proof. One has $d\log \wedge d\log \mathscr{K}_2 = \mathscr{F}_{\mathbb{Z}}^{11} \wedge \mathscr{F}_{\mathbb{Z}}^{11}$ (with the notations of 1. 1 and 1. 6), the subsheaf of decomposable forms of $\mathscr{F}_{\mathbb{Z}}^{22}$.

3. 5. Lemma. 1) *One has* $R^0 \alpha_* \mathscr{B} = 0$, $R^1 \alpha_* \mathscr{B} = \Delta/\mathbb{Z}(2)$.

2) *There is a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}^1(\Delta/\mathbb{Z}(2)) \arrow[r] \arrow[d, "i"] & R^2\alpha_*\mathscr{B} \arrow[r] \arrow[d, "\varphi"] & \mathscr{K}_{2\mathbb{Z}} \arrow[r] \arrow[d, "d\log\wedge d\log"] & 0 \\
0 \arrow[r] & \mathscr{H}^1(\mathbb{C}/\mathbb{Z}(2)) \arrow[r] & R^2\alpha_*\mathbb{Z}(2)_{\mathscr{D}} \arrow[r] & \Omega_{\mathbb{Z}}^{22} \arrow[r] & 0.
\end{tikzcd}$$

The *proof* is obvious.

3. 6. In 1. 3 we have used the description

$$\mathrm{CH}^p(X) = H^p_{\mathrm{zar}}(X, \mathscr{H}^p_{\mathscr{D}}(p)) = H^p_{\mathrm{zar}}(X, \mathscr{F}_{\mathbb{Z}}^{pp}).$$

We will use now also the description $\mathrm{CH}^2(X) = H^2_{\mathrm{zar}}(X, \mathscr{K}_2)$ (Bloch-Quillen formula), which is related to $H^2_{\mathrm{zar}}(X, \mathscr{H}^2_{\mathscr{D}}(2))$ through the Bloch-Beilinson regulator map

$$\mathscr{K}_2 \longrightarrow \mathscr{H}^2_{\mathscr{D}}(2)$$

which is simply the cup product in this case:

$$\mathscr{K}_1 \otimes_{\mathbb{Z}} \mathscr{K}_1 = \mathscr{H}^1_{\mathscr{D}}(1) \otimes_{\mathbb{Z}} \mathscr{H}^1_{\mathscr{D}}(1) \to \mathscr{H}^2_{\mathscr{D}}(2) \qquad ([Б], [E.V], [E]).$$

Theorem. *Let X be a projective manifold over* $\mathbb{C}$. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{CH}^2(X) \arrow[r] \arrow[dr, "H^2_{\mathrm{zar}}(f_{22})"'] \arrow[rr, bend left, "\Phi"] \arrow[drr, bend right, "\Psi"'] & H^2_{\mathrm{zar}}(X, \mathscr{K}_{2\mathbb{Z}}) \arrow[r] \arrow[d, "d\log\wedge d\log"] & H^4_{\mathrm{an}}(X, \mathscr{B}) \arrow[d, "\varphi"] \\
& H^2_{\mathrm{zar}}(X, \Omega_{\mathbb{Z}}^{22}) \arrow[r] & H^4_{\mathscr{D}}(X, \mathbb{Z}(2)).
\end{tikzcd}$$

Proof. One has

$$H^l_{\text{zar}}(R^0\alpha_*\mathscr{B}) = H^{1+l}_{\text{zar}}(R^1\alpha_*\mathscr{B}) = 0 \quad \text{for} \quad l \geqq 0.$$

Therefore there is a map

$$H^2_{\text{zar}}(X, R^2\alpha_*\mathscr{B}) \to H^4_{\text{an}}(X, \mathscr{B}).$$

By 3.5.2), one has $H^2_{\text{zar}}(X, R^2\alpha_*\mathscr{B}) = H^2_{\text{zar}}(X, \mathscr{K}_{2\mathbb{Z}})$ as $H^2(\mathscr{H}^1(\Delta/\mathbb{Z}(2))) = 0$ by the Bloch-Ogus theory. One also has a map (3.4)

$$\mathrm{CH}^2(X) = H^2_{\text{zar}}(X, \mathscr{K}_2) \to H^2_{\text{zar}}(X, \mathscr{K}_{2\mathbb{Z}}).$$

Then apply 1.3.2).

3.7. Remark. There are maps

$$\alpha_*\mathcal{O}^\times \otimes_{\mathbb{Z}} \mathbb{C}^\times \to \alpha_*(\mathcal{O}^\times \otimes_{\mathbb{Z}} \mathbb{C}^\times) \to \mathrm{Ker}(\mathscr{K}_{2\mathbb{Z}} \to \Omega^{22}_{\mathbb{Z}})$$

which define maps

$$\begin{aligned} H^2_{\text{zar}}(\alpha_*\mathcal{O}^\times_{\text{an}}) \otimes \mathbb{C}^\times &\to H^2_{\text{zar}}(\alpha_*\mathcal{O}^\times_{\text{an}} \otimes \mathbb{C}^\times) \to H^2_{\text{zar}}(\alpha_*(\mathcal{O}^\times \otimes \mathbb{C}^\times)) \\ &\to \mathrm{Ker}(H^2_{\text{zar}}(\mathscr{K}_{2\mathbb{Z}}) \to H^2_{\text{zar}}(\Omega^{22}_{\mathbb{Z}})) \\ &\to \mathrm{Ker}(H^2_{\text{zar}}(\mathscr{K}_{2\mathbb{Z}}) \to H^4_{\mathscr{D}}(\mathbb{Z}(2))). \end{aligned}$$

On the other hand one computes easily that

$$H^2_{\text{zar}}(\alpha_*\mathcal{O}^\times) = H^2_{\text{an}}(\mathcal{O})$$

via the following exact sequence

$$0 \to \frac{\alpha_*\mathcal{O}}{\mathbb{Z}(1)} \to \alpha_*\mathcal{O}^\times \to \mathscr{H}^1(\mathbb{Z}(1)) \to 0$$

which gives the exact sequence

$$H^1_{\text{zar}}(\alpha_*\mathcal{O}^\times) \to H^1_{\text{zar}}(\mathscr{H}^1(\mathbb{Z}(1))) \to H^2_{\text{zar}}\left(\frac{\alpha_*\mathcal{O}}{\mathbb{Z}(1)}\right) \to H^2_{\text{zar}}(\alpha_*\mathcal{O}^\times) \to 0.$$

As the first map is surjective ([BO], [S], (1.8.2)), one has

$$H^2_{\text{zar}}(\alpha_*\mathcal{O}^\times_{\text{an}}) = H^2_{\text{zar}}\left(\frac{\alpha_*\mathcal{O}}{\mathbb{Z}(1)}\right) = H^2_{\text{zar}}(\alpha_*\mathcal{O}) = H^2_{\text{an}}(\mathcal{O}).$$

In order to recover Mumford's result [M] on $\operatorname{Ker}\Psi$ on a projective smooth complex surface with $H^2_{\mathrm{an}}(\mathcal{O}) \neq 0$, one would have to know that the image of

$$H^2_{\mathrm{zar}}(\alpha_* \mathcal{O}^\times) \otimes_{\mathbb{Z}} \mathbb{C}^\times = H^2_{\mathrm{an}}(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

in $H^2_{\mathrm{zar}}(\mathcal{K}_{2\mathbb{Z}})$ meets the image of $\mathrm{CH}^2(X)$.

3. 8. Proposition. *Let X be a smooth algebraic surface over $\mathbb{C}$ with*

$$H^0(X, \Omega^1) = H^0(X, \Omega^2) = 0.$$

If ϕ (defined in 3. 6) *is injective, then* $\mathrm{CH}^2(X) = \mathbb{Z}$.

Proof. One always has a commutative diagram using 3. 6

$$\begin{tikzcd}
\mathrm{CH}^2(X) = H^2_{\mathrm{zar}}(\mathcal{K}_2) \arrow[r] & H^2_{\mathrm{zar}}(\mathcal{K}_{2\mathbb{Z}}) \arrow[r] \arrow[d, "3.3"] & H^4_{\mathrm{an}}(\mathcal{B}) \arrow[d, "H^4_{\mathrm{an}}(\exp \otimes \mathrm{id})"] \\
& H^2_{\mathrm{zar}}(\alpha_* \mathcal{K}_2) \arrow[r] & H^2_{\mathrm{an}}(\mathcal{K}_{2\mathrm{an}}).
\end{tikzcd} \tag{3. 8. 1}$$

If $H^0(\Omega^1) = 0$, then $H^3_{\mathrm{an}}(\Delta/\mathbb{Z}(2))/\text{torsion} = 0$. As $H^4_{\mathrm{an}}(\mathbb{Z}(2)) = \mathbb{Z}$ is torsion free, one has $H^3_{\mathrm{an}}(\Delta/\mathbb{Z}(2)) = 0$. Therefore $H^4_{\mathrm{an}}(\exp \otimes \mathrm{id})$ is injective. By [B2], theorem (6. 1), this is the degree map.

3. 9. Remark. This theorem of S. Bloch [B2], (6. 1), on the description of the degree map on surface with $H^0(\Omega^1) = H^0(\Omega^2) = 0$ was the motivation for the sections 1 and 2 of this note.

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Max-Planck-Institut für Mathematik, Gottfried-Claren-Straße 26, D-5300 Bonn 3

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