SEMI-STABLE LEFSCHETZ PENCILS

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Abstract. We study the geometry and cohomology of Lefschetz pencils for semi-stable schemes over a discrete valuation ring. We relate the global cohomological properties of the Lefschetz pencil and the monodromy-weight conjecture, in particular we show that if one assumes the monodromy-weight conjecture in smaller dimensions then one can obtain a rather complete understanding of the relative cohomology of the pencil. This reduces the monodromy-weight conjecture to an arithmetic variant of a conjecture of Kashiwara for the projective line.

1. Introduction

1.1. **Background.** The theory of Lefschetz pencils is an important tool in the study of the topology of algebraic varieties which originates in the work of Picard and Lefschetz [Lef24]. The theory has been extended to positive characteristic and étale cohomology by Deligne and Katz [SGA7.2]. The idea of Lefschetz is to fibre a smooth projective variety X of dimension n over a field k after a certain blow-up $\tilde{X} \to X$ into a pencil $\phi\colon \tilde{X} \to \mathbb{P}^1_k$ such that ϕ is smooth except for a finite number of quadratic singular points located in different fibres, i.e. ϕ is like a real Morse function in differential topology. The Leray spectral sequence for the pencil map ϕ allows one to study the cohomology of X. This topological idea was a key ingredient in Deligne's first proof of the Weil conjectures [Del74].

The geometric theory of Lefschetz pencils was generalized by Jannsen–Saito [JS12] to smooth projective schemes over a discrete valuation ring. In this note we study the geometry and cohomology of Lefschetz pencils of (strict) semi-stable schemes over discrete valuation rings.

One motivation for this study is the monodromy-weight conjecture [Del70, Section 8.5], [RZ82, p.23], which is an analog of the Riemann hypothesis part of the Weil conjectures over a p-adic local field. In [RZ82], Rapoport–Zink studied the monodromy-weight conjecture in terms of a monodromy-weight spectral sequence of a semi-stable model, which they define. The key unsolved problem is that the d_1-differential in their spectral sequence is not well-understood in relation to the monodromy operator N. It turns out that, assuming the monodromy-weight conjecture is known in smaller dimensions, the analog of this d_1-differential in the relative cohomology of the semi-stable Lefschetz pencil is easy to control. This allows us to reduce

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the monodromy-weight conjecture to an unsolved problem about the cohomology of a Picard-Lefschetz sheaf on $\mathbb{P}^1$, see 9.3.

1.2. **Geometry of semi-stable Lefschetz pencils.** Let $\mathcal{O}$ be a henselian discrete valuation ring with perfect residue field k of characteristic $\mathrm{ch}(k) \neq 2$. Define K to be the field of fractions $K = \mathrm{frac}(\mathcal{O})$, which we assume to be of characteristic zero. Let $X \subset \mathbb{P}^N_{\mathcal{O}}$ be a (strict) semi-stable projective scheme over $\mathcal{O}$, i.e. its special fibre X_k is a simple normal crossing divisor. We endow X with the usual stratification, see the beginning of Section 4. A sufficiently general linear subspace of codimension two $A \subset \mathbb{P}^N_{\mathcal{O}}$ gives rise to a pencil map

$$\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}},$$

where $\tilde{X} = \mathrm{Bl}_{A\cap X}(X)$, see Section 5.

A point $x \in \tilde{X}$ is called *critical* if either x lives over K and is a non-smooth point of ϕ_K or if x lives over k and $\phi|_Z\colon Z \to \mathbb{P}^1_k$ is non-smooth at x, where $Z \subset \tilde{X}_k$ is the stratum containing x.

Roughly speaking, we say that the pencil $\phi_k\colon \tilde{X}_k \to \mathbb{P}^1_k$ is a *stratified Lefschetz pencil* over k if $\phi_k\colon \tilde{X}_k \to \mathbb{P}^1_k$ is a stratified Morse function in the usual sense, Definition 3.3, with at most one critical point per geometric fibre; for the precise definition of stratified Lefschetz pencil see Definition 5.1. It is not hard to show that after a suitable Veronese embedding a generic choice of A_k gives rise to a stratified Lefschetz pencil ϕ_k.

The following theorem has been observed in [JS12, Theorem 1] in the smooth case.

Theorem 1.1 (see Theorem 5.4)**.** *Assume that* $\mathrm{ch}(k) > \dim(X_K) + 1$ *or* $\mathrm{ch}(k) = 0$. *Then the following properties hold.*

(1) *If ϕ_k is a stratified Lefschetz pencil then ϕ_K is a Lefschetz pencil.*
(2) *The subset S of critical points in X is closed and with the reduced subscheme structure it maps isomorphically onto its image in $\mathbb{P}^1_{\mathcal{O}}$.*
(3) *Each connected component of S is a trait (i.e. the spectrum of a henselian discrete valuation ring) which is finite and of ramification index over $\mathcal{O}$ equal to the number of irreducible components of X_k it meets.*

For $\mathrm{ch}(k) \neq 2$ one only obtains a slightly weaker result which we formulate in Theorem 4.2 for a general stratified Morse function.

1.3. **Cohomology of semi-stable Lefschetz pencils.** Let ℓ be a prime number invertible in $\mathcal{O}$ and assume that the residue field k is finite. Set $\Lambda = \overline{\mathbb{Q}}_\ell$ and $n = \dim(X_K)$. Let $\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}}$ be a semi-stable Lefschetz pencil, i.e. ϕ_k is a stratified Lefschetz pencil and ϕ_K is a Lefschetz pencil.

We endow X with the middle perversity, see A.4, so that $\Lambda[n+1] \in D^b_c(X_K, \Lambda)$ is a perverse sheaf. In order to understand the relative cohomology of a semi-stable Lefschetz pencil one has to study the degeneration of the classical Picard-Lefschetz perverse sheaf $\mathsf{L}_K = {}^{\mathrm{p}}R^0\phi_{K,*}(\Lambda[n+1])$ in terms of its nearby cycle perverse sheaf $\mathsf{L}_k = R\Psi_{X/\mathcal{O}}(\mathsf{L}_K)[-1]$. Note that for $i \neq 0$ the perverse sheaf ${}^{\mathrm{p}}R^i\phi_{K,*}(\Lambda[n+1])$ is geometrically constant and it can be analyzed by the Weak Lefschetz theorem [BBD83, Théorème 4.1.1] by cutting with a hyperplane section.

As a consequence of results of Grothendieck [SGA7.1, Exposé I] and Rapoport–Zink [RZ82] the monodromy action on L_k is unipotent, see Proposition 8.7 and Lemma 9.2, in particular it is given in terms of a nilpotent operator $N\colon \mathsf{L}_k \to \mathsf{L}_k(-1)$. This operator induces a monodromy filtration $\mathrm{fil}^{\mathrm{M}}$ on the perverse sheaf L_k, see (7.1).

In order to understand the structure of L_k we have to assume that the monodromy-weight conjecture holds in dimensions smaller than $\dim(X_K)$, see 9.3. Recall that the monodromy-weight conjecture is known for dimension at most two [RZ82, Satz 2.13].

Theorem 1.2 (see Theorem 9.4). *Assume that the monodromy-weight conjecture is known over K for dimensions smaller than $n = \dim(X_K)$. Then the following properties hold.*

(1) *For any $a \in \mathbb{Z}$ the monodromy graded piece $\mathrm{gr}^{\mathrm{M}}_a \mathsf{L}_k$ is pure of weight $n + a$, in particular it is geometrically semi-simple (*[BBD83, Corollaire 5.4.6]*).*
(2) *The non-constant part of $\mathrm{gr}^{\mathrm{M}}_a \mathsf{L}_{\overline{k}}$ satisfies multiplicity one.*

In Corollary 9.5 we show that in order to prove the monodromy-weight conjecture by induction on $n = \dim(X_K)$, one would have to prove that the monodromy filtration of $H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L}_{\overline{k}})$ agrees with the filtration induced by the spectral sequence of the filtered perverse sheaf $(\mathsf{L}_{\overline{k}}, \mathrm{fil}^{\mathrm{M}})$. We call this the *monodromy property*, see Definition 7.2. The monodromy property is of "purely topological" nature and is known to hold in our setting for $\mathcal{O}$ of equal characteristic. It plays an important role in the theory of Hodge modules [Sai88] and twistor $\mathcal{D}$-modules [Moc07]. In mixed characteristic it fits into what we like to call the arithmetic Kashiwara conjecture, see Conjecture 9.7, motivated by the Kashiwara conjecture in complex geometry [Kas98]. We defer the study to a forthcoming work.

Because the monodromy action on L_k is unipotent, it is tame. This tameness generalizes to all models of $\mathbb{P}^1_K$ in the following sense.

Theorem 1.3 (see Corollary 10.7). *Assume $\mathrm{ch}(k) > n + 1$. For any closed point $x \in \mathbb{P}^1_K$ and any $i \in \mathbb{Z}$, the $\mathrm{Gal}(\overline{x}/x)$-action on $H^i(\phi^{-1}(\overline{x}), \Lambda)$ is tame.*

In fact we prove a slightly weaker result for $\mathrm{ch}(k) \neq 2$ in Theorem 10.6.

For x specializing to a regular value in $\mathbb{P}^1_k$, Theorem 1.3 is an immediate consequence of the aforementioned results of Rapoport–Zink about semi-stable reduction. The proof of Theorem 1.3 uses the Grothendieck-Murre criterion of tameness and Nakayama's generalization [Nak98, Theorem 0.1] of the work of Rapoport–Zink to the log-smooth case applied to a certain blow-up. It also relies on the tameness statement in the Picard-Lefschetz formula, see Proposition 6.3.

In forthcoming work we will study the monodromy property in terms of a tilting to equal characteristic zero, for which the tameness as formulated in Theorem 1.3 plays a role.

1.4. **Content.** As a technical ingredient, which is not well-documented in the literature and which is necessary for our study of semi-stable Lefschetz pencils, we develop the algebraic theory of stratified Morse functions for special algebraic analogs of Whitney stratifications, which we call regular stratifications, in Sections 2 and 3.

In Section 4 we prove the main theorem about Morse functions for semi-stable schemes. In Section 5 we generalize without difficulty the usual geometric theory of Lefschetz pencils to the stratified and semi-stable context.

In Section 6 we recast the classical cohomology theory of Lefschetz pencils in terms of perverse sheaves. The only new result is the generalization of a central observation of Katz [SGA7.2, Exposé XVIII, Théorème 5.7] for type (A) pencils to the general case.

Section 7 summarizes properties of the monodromy filtration in the context of perverse sheaves.

Our presentation of the theory of Rapoport–Zink in Section 8 is novel in that we make full use of the duality theory of the nearby cycle functor. This allows us to give a clear-cut axiomatic description of their construction, and of its perverse formulation in [Sai03, Section 2.2]. In order to give a coordinate free presentation in the case of $\mathbb{Z}/\ell^\nu\mathbb{Z}$-coefficients, we use Beilinson's Iwasawa twist. As this formalism, which is extremely useful for unipotent nearby cycles, is not well-documented in the literature, we summarize it in Appendix A.

Our main cohomological results about semi-stable Lefschetz pencils and the relation to the monodromy-weight conjecture are studied in Section 9.

Section 10 discusses tameness of the Picard-Lefschetz sheaf.

Notation. By k we denote a field of characteristic $\neq 2$, which is assumed to be perfect if not stated otherwise. By $\mathcal{O}$ we denote a henselian discrete valuation ring with residue field k. We always assume that $K = \mathrm{frac}(\mathcal{O})$ has characteristic 0. When mentioning the dimension of a scheme, we assume that it is equidimensional. By ℓ we denote a prime number invertible in k.

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2. Morse morphisms and Morse functions

2.1. **Morse morphisms.** Let X be a scheme locally of finite type over a field F with $\mathrm{ch}(F) \neq 2$. For F algebraically closed we say that $X \to \operatorname{Spec} F$ is a *Morse morphism* or simply that X *is Morse* if for any closed point $x \in X$, either X is regular at x or there is an isomorphism of F-algebras

$$\mathcal{O}^{\mathrm{h}}_{X,x} \cong F[X_0, \dots, X_n]^{\mathrm{h}}/(X_0^2 + \dots + X_n^2)$$

for some $n \geq 0$. Here $^{\mathrm{h}}$ denotes the henselization with respect to the maximal ideal $(X_0, \dots, X_n)$. Over a general field F we say that $X \to \operatorname{Spec} F$ is a *Morse morphism* if $X \otimes_F \overline{F} \to \operatorname{Spec} \overline{F}$ is a Morse morphism, where $\overline{F}/F$ is an algebraic closure. A non-smooth point of a Morse morphisms $X \to \operatorname{Spec} F$ is called a *non-degenerate quadratic singularity.*

Let now X and Y be schemes on which 2 is invertible. A morphism of schemes $\phi\colon X \to Y$ is called a *Morse morphism* if it is locally finitely presented, flat and for any point $y \in Y$ the fibre X_y is Morse over $k(y)$. Clearly, Morse morphisms are preserved by base change. Observe that in [Del80, Section (3.6)], it is called "essentiellement lisse", in [JS12, Section 4] the terminology "almost good reduction" is used.

We recall the deformation theory of non-degenerate quadratic singularities [SGA7.2, Exposé XV, Théorème 1.1.4].

Proposition 2.1. *Let $\phi\colon X \to Y$ be flat and locally of finite presentation such that $\phi^{-1}(y)$ is Morse over $k(y)$ for a $y \in Y$ with $k(y)$ separably closed. Let $x \in \phi^{-1}(y)$ be a singular point of $\phi^{-1}(y)$. Then there exists an isomorphism of $\mathcal{O}^{\mathrm{h}}_{Y,y}$-algebras*

$$\mathcal{O}^{\mathrm{h}}_{X,x} \cong \mathcal{O}_{Y,y}[X_0, \dots, X_n]^{\mathrm{h}}/(X_0^2 + \dots + X_n^2 - \alpha),$$

where α is in the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}^{\mathrm{h}}_{Y,y}$.

Here on the right the henselization is with respect to the maximal ideal generated by $\mathfrak{m}_y$ and $(X_0, \dots, X_n)$.

2.2. **Morse functions.** Let in the following X be a regular noetherian scheme and let D be a one-dimensional, regular, noetherian scheme. We always assume that 2 is invertible on X and on D. Let $\phi\colon X \to D$ be a morphism of finite type. A non-smooth point $x \in X$ of ϕ is called a *critical point* of ϕ, the image $\phi(x)$ of a critical point $x \in X$ is called a *critical value* of ϕ. In particular the closed points x with $\dim_x X = 0$ are critical. We say that $\phi\colon X \to D$ is a *Morse function* if ϕ is a Morse morphism around any point $x \in X$ with $\dim_x(X) > 0$. We call a critical point $x \in X$ *non-degenerate* if ϕ is a Morse function in a neighbourhood of x.

Note that in Morse theory one sometimes asks that additionally a Morse function has at most one critical point per fibre.

Remark 2.2. The set of critical points of a Morse function ϕ consists of finitely many closed points, compare proof of Corollary 2.3 below.

For us k will be a perfect field of characteristic different from two. For the rest of this section we assume that k is algebraically closed, that X and D are of finite type over k and that $\phi\colon X \to D$ is a k-morphism, $n = \dim(X)$. Proposition 2.1 has the following corollary.

Corollary 2.3 (Morse lemma)**.** *For $x \in X$ a critical point of a Morse function $\phi\colon X \to D$ with $n \geq 1$ there exist k-isomorphisms*

$$\mathcal{O}^{\mathrm{h}}_{X,x} \cong k[X_1, \dots, X_n]^{\mathrm{h}}, \quad \mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \cong k[T]^{\mathrm{h}}$$

such that $\phi^{\mathrm{h}}_x\colon \mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \to \mathcal{O}^{\mathrm{h}}_{X,x}$ maps T to $X_1^2 + \ldots + X_n^2$.

If $n = 0$, then $\mathcal{O}^{\mathrm{h}}_{X,x} = k$ and $\phi(x)^{\mathrm{h}}(T) = 0$.

Here the henselizations $^{\mathrm{h}}$ are with respect to the maximal ideals $(X_1, \dots, X_n)$ and (T).

Proof. It suffices to show the corollary for x a closed point as then a posteriori it follows that there are no non-closed critical points, using that the set of critical points is closed. Looking at the henselian local presentation of $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{D,\phi(x)}$-algebra from Proposition 2.1 we see that necessarily $\alpha \in \mathcal{O}_{D,\phi(x)}$ is a uniformizer, as otherwise $\mathcal{O}_{X,x}$ would be singular. □

Let J_ϕ be the Jacobian ideal of ϕ, see [HS06, Section 4.4].

Lemma 2.4.
 (i) *$J_\phi = \mathcal{O}_X \Leftrightarrow \phi$ is smooth;*
 (ii) *$V(J_\phi)$ is a finite disjoint union of copies of* $\operatorname{Spec} k$ *$\Leftrightarrow \phi$ is a Morse function.*

The proof of part (ii) of the lemma is explained in [SGA7.2, Exposé XV, Section 1.2].

3. Regular stratifications

3.1. **Regular stratifications and critical points.** Let X be a noetherian scheme and let D be a one-dimensional, regular, noetherian scheme. We always assume that 2 is invertible on X and on D.

Definition 3.1. A *stratification* of X is a finite set $\mathbf{Z}$ consisting of disjoint locally closed subsets $Z \subset X$ called *strata* such that $\overline{Z}$ is a union of strata for each stratum $Z \in \mathbf{Z}$. If X is a scheme and $\mathbf{Z}$ a stratification of X the pair $(X, \mathbf{Z})$ is called a *stratified scheme.*

For $(X, \mathbf{Z})$ a stratified scheme and $x \in X$ we denote by $Z_x \subset X$ the stratum of x. We usually endow a stratum with its reduced subscheme structure. What we call a stratification is called a good stratification in [StPr, Definition 09XZ]. We call the stratification $\mathbf{Z}$ *regular* if the closure $\overline{Z}$ with the reduced subscheme structure is regular for any stratum $Z \in \mathbf{Z}$.

Note that for a morphism of finite type $f\colon Y \to X$, the set-theoretic pullback $f^{-1}(\mathbf{Z})$ of $\mathbf{Z}$ to Y is in general not a stratification but just a partition of Y into locally closed subsets which we call the *pullback partition.* If f is flat, then $f^{-1}(\mathbf{Z})$ is automatically a stratification.

Now assume that X is endowed with a regular stratification $\mathbf{Z}$. Consider a morphism $\phi\colon X \to D$ of finite type.

A point $x \in X$ is called a *critical point* of ϕ (with respect to the stratification $\mathbf{Z}$) if $\phi|_Z\colon Z \to D$ is non-smooth at x, where Z is the stratum containing x endowed with the reduced subscheme structure. Otherwise x is called *non-critical.* For $x \in X$ a critical point we call $\phi(x) \in D$ a *critical value.*

Lemma 3.2. *The set of critical points of ϕ is closed in X.*

Proof. Let $C \subset X$ be the subset of critical points. As for any stratum $Z \in \mathbf{Z}$ the subset $C \cap Z$ is the non-smooth locus of $\phi|_Z\colon Z \to D$, we see that $C \cap Z \subset Z$ is closed. Therefore C is constructible in X. So by [StPr, Lemma 0903 (2)] we just have to show that C is closed under specialization. So consider points $x_2 \in \overline{\{x_1\}}$ with $x_1 \in Z_1 \in \mathbf{Z}$ and $x_2 \in Z_2 \in \mathbf{Z}$. Assume $x_1 \in C$ and $Z_1 \neq Z_2$.

1st case Z_2 is non-flat over D at x_2.
Then of course Z_2 is not smooth over D at x_2, so $x_2 \in C$.

2nd case: Z_2 is flat over D at x_2.
Then also $\overline{Z}_1$ is flat over D at x_2, because flatness is equivalent to being dominant over D. As $Z_2 \hookrightarrow \overline{Z}_1$ is a closed immersion of regular schemes there exists a regular sequence $\underline{a}$ in $\mathcal{O}_{\overline{Z}_1,x_2}$ with $\mathcal{O}_{\overline{Z}_1,x_2}/(\underline{a}) = \mathcal{O}_{Z_2,x_2}$. Let $\pi \in \mathcal{O}_{D,\phi(x_2)}$ be a uniformizer. As π is a non-zero divisor on $\mathcal{O}_{Z_2,x_2}$ we see that $\underline{a}, \pi$ is also a regular sequence in $\mathcal{O}_{\overline{Z}_1,x_2}$, so the image of $\underline{a}$ in $\mathcal{O}_{\overline{Z}_1,x_2}/(\pi)$ is also a regular sequence and it remains a regular sequence after the base change by the algebraic closure $\overline{k(\phi(x_2))}/k(\phi(x_2))$. So as $\mathcal{O}_{\overline{Z}_1,x_2} \otimes_{\mathcal{O}_{D,\phi(x_2)}} \overline{k(\phi(x_2))}$ is singular, since $x_1 \in C$ and since the non-smooth locus of the map $\overline{Z}_1 \to D$ is closed. Then also $\mathcal{O}_{Z_2,x_2} \otimes_{\mathcal{O}_{D,\phi(x_2)}} \overline{k(\phi(x_2))}$ is singular, because it is a quotient of the ring $\mathcal{O}_{\overline{Z}_1,x_2} \otimes_{\mathcal{O}_{D,\phi(x_2)}}$ modulo a regular sequence. We have shown $x_2 \in C$. This finishes the proof. $\square$

The following definition is copied from stratified Morse theory [GM88, 2.0].

Definition 3.3 (Non-degenerate critical points)**.** A critical point x of ϕ is called *non-degenerate* if for $Z \subset X$ the stratum containing x and for any stratum $Z' \neq Z$ with $Z \subset \overline{Z}'$ the following holds:

(1) $\phi|_Z$ has a non-degenerate critical point at x in the sense of Section 2;
(2) $\phi|_{\overline{Z}'}$ is smooth at x.

If every critical point of ϕ is non-degenerate we say that ϕ is a *stratified Morse function.*

Lemma 3.4. *The set of critical points of a stratified Morse function ϕ consists of finitely many closed points.*

Proof. Any critical point x of ϕ is closed. Indeed, let Z be the stratum of x. Assume x is not closed in X. Then there exists a proper specialization x' of x. If x' lies in Z then this contradicts Remark 2.2 applied to $\phi|_Z\colon Z \to D$. If x' lies in a different stratum Z' then $\phi|_{Z'}\colon Z' \to D$ is non-smooth at x' by Lemma 3.2. As also $\phi|_{\overline{Z}}\colon \overline{Z} \to D$ is non-smooth at x' this would contradict the condition that ϕ is a stratified Morse function around x'.

To conclude use that the set of critical points is closed by Lemma 3.2. □

3.2. **Stratified regular immersions.** Let $(X, \mathbf{Z}_X)$ and $(Y, \mathbf{Z}_Y)$ be two noetherian schemes with regular stratifications. Let $i\colon Y \hookrightarrow X$ be a regular closed immersion of codimension c. We say that i is *stratified regular* if for any point $y \in Y$ after replacing X by an open neighbourhood of $i(y)$ and Y by its preimage it holds:

(1) the schematic pullback $i^{-1}(Z_{i(y)})$ is reduced and equal to Z_y;
(2) $i|_{Z_y}\colon Z_y \to Z_{i(y)}$ has codimension c.

Here Z_y is the stratum of $y \in Y$ and $Z_{i(y)}$ is the stratum of $i(y) \in X$.

For a stratified regular closed immersion $i\colon Y \hookrightarrow X$ and for a stratum $Z \subset X$, each connected component of $i^{-1}(Z)$ is a connected component of a stratum of Y. Note that locally around $x = i(y)$ a regular sequence for i restricts to a regular sequence for $Z_y \hookrightarrow Z_x$ by [Mat86, Theorem 17.4].

If Y is not endowed with a stratification but $(X, \mathbf{Z}_X)$ has a regular stratification, we call the closed immersion $i\colon Y \hookrightarrow X$ *stratified regular* if the partition $i^{-1}(\mathbf{Z}_X)$ is a regular stratification and with this stratification i is stratified regular.

Lemma 3.5. *For a stratified regular immersion $i\colon (Y, \mathbf{Z}_Y) \hookrightarrow (X, \mathbf{Z}_X)$ and a stratum $Z \in \mathbf{Z}_X$ the preimage $i^{-1}(\overline{Z})$ is regular.*

Proof. Consider a point $y \in i^{-1}(\overline{Z})$, $x = i(y)$ and a regular sequence $\underline{a} \in \mathcal{O}_{X,x}$ generating the ideal of i. Then the image of $\underline{a}$ in $\mathcal{O}_{\overline{Z},x}$ is part of a regular parameter system as its image in $\mathcal{O}_{Z_x,x}$ has this property. □

The proof of the following lemma is similar.

Lemma 3.6. *For a regularly stratified scheme $(X, \mathbf{Z}_X)$ and for a regular closed immersion $i\colon Y \hookrightarrow X$ and a point $y \in Y$ the following are equivalent:*

(1) *i is a stratified regular immersion in a neighborhood of y;*
(2) *for a regular sequence $\underline{a} \in \mathcal{O}_{X,i(y)}$ locally generating the ideal of i, the sequence $\underline{a}|_{Z_{i(y)}}$ is part of a regular parameter system in $\mathcal{O}_{Z_{i(y)},i(y)}$.*

Let now k be an infinite perfect field and $X \hookrightarrow \mathbb{P}^N_k$ be an immersion. Recall the following Bertini type theorem.

Proposition 3.7. *For a regularly stratified scheme $(X, \mathbf{Z}_X)$ the following properties are verified.*

(i) *For a generic hypersurface $H \hookrightarrow \mathbb{P}^N_k$ the map $i\colon X \cap H \hookrightarrow X$ is a stratified regular immersion.*

(ii) *Assume given a closed point $x \in X$ contained in the stratum Z. Then there exists a hypersurface $H \hookrightarrow \mathbb{P}^N_k$ of large degree such that $i\colon X \cap H \hookrightarrow X$ is stratified regular away from x and such that the schematic intersection $H \cap Z$ is singular at x, i.e. H contains the tangent space to Z at x.*

Proof. Part (i) holds by classical theorem of Bertini. Indeed, i is a stratified regular immersion if and only if H intersects each stratum transversally by Lemma 3.6.

Part (ii) follows from Bertini theorems for hypersurface sections containing a subscheme, see [AK79, Theorem 1]. □

3.3. **Stratified local complete intersection (lci) morphisms.** Let $(X, \mathbf{Z}_X)$ and $(Y, \mathbf{Z}_Y)$ be two noetherian schemes with regular stratifications. We call a morphism which is of finite type $f\colon Y \to X$ *stratified lci* if for any locally given factorisation

$$f = [W \xrightarrow{g} X] \circ [Y \xrightarrow{i} W] \tag{3.1}$$

with g smooth and i a closed immersion we have that i is stratified regular once W is endowed with the stratification $g^{-1}(\mathbf{Z}_X)$. If Y is not endowed with a stratification a priori but becomes regularly stratified by $f^{-1}(\mathbf{Z}_X)$, and, with this stratification, f is stratified lci, we call f *stratified lci.*

Lemma 3.8. (i) *If for one factorisation* (3.1) *with g smooth and i a closed immersion, $i\colon (Y, \mathbf{Z}_Y) \to (W, g^{-1}(\mathbf{Z}_X))$ is a stratified regular immersion, then f is stratified lci.*

(ii) *The composition of stratified lci morphisms is stratified lci.*

(iii) *Let $(X, \mathbf{Z}_X)$ be a regularly stratified scheme and let $Y \hookrightarrow X$ be a stratified regular immersion. Then the blow-up $\mathrm{Bl}_Y(X) \to X$ is stratified lci.*

Proof. We only prove part (iii), since we use it below. The blow-up along a regular immersed center is lci and commutes with base change which preserves the normal bundle of the regular immersion of the center, see [Mic64, Chapitre I, Théorème 1]. So we just have to observe that the blow-up $\mathrm{Bl}_{Y\cap\overline{Z}}(\overline{Z})$ is regular for any stratum $Z \in \mathbf{Z}_X$. Here we use Lemma 3.5 and that the blow-up of a regular scheme in a regular center is regular [Mic64, Chapitre I, Théorème 1]. □

3.4. **Simple normal crossing varieties.** Consider a reduced, separated, equidimensional scheme X of finite type over a perfect field k of characteristic different from two. Let $X_1, \ldots, X_r$ be the different irreducible components of X, endowed with the reduced subscheme structure. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ a non-empty subset, we denote by

$$X_I = X_{i_1} \cap \ldots \cap X_{i_r}$$

the schematic intersection.

We call X a *simple normal crossing variety* or *strict normal crossing variety* (*snc variety*) of dimension n if $X_1, \dots, X_r$ are smooth over k of dimension n and for an algebraic closure $\overline{k}$ of k and any closed point $x \in X_{\overline{k}}$ there exists an isomorphism of $\overline{k}$-algebras

$$\mathcal{O}^{\mathrm{h}}_{X_{\overline{k}},x} \cong \overline{k}[X_0, \dots, X_n]^{\mathrm{h}}/(X_0 \cdots X_m) \tag{3.2}$$

for some $m \geq 0$ depending on x. Here $^{\mathrm{h}}$ denotes the henselization with respect to the ideal $(X_0, \dots, X_n)$. Note that for an snc variety X the X_I are smooth and equidimensional over k for any $\varnothing \neq I \subset \{1, \dots, n\}$.

For an snc variety X over k let $\kappa\colon X \to \mathbb{Z}$ be defined by

$$\kappa(x) = \#\{i \,|\, x \in X_i\}.$$

Then κ is upper semi-continuous. We let $\mathbf{Z}$ be the set of all connected components of $\kappa^{-1}(j)$ for all $j \in \mathbb{Z}$. Then $\mathbf{Z}$ is a regular stratification of X, which we will always use in the following. For each stratum $Z \in \mathbf{Z}$ there exists a canonical non-empty subset $I_Z \subset \{1, \dots, n\}$ such that $\overline{Z}$ is a connected component of X_{I_Z}. Set

$$X^{(a)} = \kappa^{-1}([a+1, \infty)).$$

In the following let D be smooth of dimension one over k and let $\phi\colon X \to D$ be a k-morphism, where X is an snc variety over k.

Proposition 3.9 (Normal crossing Morse lemma)**.** *Assume k is algebraically closed. For a stratified Morse function $\phi\colon X \to D$ with a critical point $x \in X$, there exists a k-algebra isomorphism as in* (3.2) *and a k-algebra isomorphism $\mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \cong k[T]^{\mathrm{h}}$ such that $\phi^{\mathrm{h}}_x\colon \mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \to \mathcal{O}^{\mathrm{h}}_{X,x}$ maps T to*

$$X_0 + \ldots + X_m + X_{m+1}^2 + \ldots + X_n^2.$$

Here the henselization $^{\mathrm{h}}$ of $k[T]$ is with respect to the ideal (T).

Proof. Set $z = \phi(x)$. Say X is presented by (3.2) at x after an étale localization. Then there exists a closed immersion $X \hookrightarrow \hat{X}$, where $\hat{X}$ is étale over $\operatorname{Spec} k[X_0, \dots, X_n]$, sending x to the origin. By projecting $\hat{X}$ onto the first $(m+1)$ coordinates, we obtain a smooth morphism $\hat{f}\colon \hat{X} \to \hat{Y}$, where $\hat{Y} = \operatorname{Spec} k[X_0, \dots, X_m]$, and which sends x to the origin. Set $Y = k[X_0, \dots, X_m]/(X_0 \cdots X_m)$. We can extend ϕ to a k-morphism $\hat{\phi}\colon \hat{X} \to D$.

We assume $n > m$ and leave the similar but easier case $n = m$ to the reader. We obtain a Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \times_k D \\ \downarrow & & \downarrow \\ \hat{X} & \xrightarrow{\hat{g}} & \hat{Y} \times_k D \end{array}$$

with $\hat{g} = (\hat{f}, \hat{\phi})\colon \hat{X} \to \hat{Y} \times_k D$. By the dimension criterion for flatness [Mat86, Theorem 23.1] the morphism $\hat{g}$ is flat at x, where we use that the stratum of x has dimension $n - m > 0$. Then also g is flat at x.

By assumption the fibre $g^{-1}(y)$ over $y = g(x)$ is Morse over $k = k(y)$. So by Proposition 2.1 there exists an $\mathcal{O}^{\mathrm{h}}_{Y\times D,y}$-algebra isomorphism

$$\mathcal{O}^{\mathrm{h}}_{X,x} \cong k[X_0, \dots, X_n, T]^{\mathrm{h}}/(X_0 \cdots X_m, X_{m+1}^2 + \dots + X_n^2 - \alpha)(=: A)$$

with α a non-unit in $\mathcal{O}^{\mathrm{h}}_{Y\times D,y} \cong k[X_0, \dots, X_m, T]^{\mathrm{h}}/(X_0 \dots X_m)(=: B)$. In A the henselization is with respect to $(X_0, \dots, X_n, T)$, in B with respect to $(X_0, \dots, X_m, T)$. Write $\alpha = c_0X_0 + \dots + c_mX_m + bT$ with $c_0, \dots, c_m, b \in B$.

As $A/(X_0, \dots, X_m)$ is the henselian local ring at x of the stratum containing x, this ring is regular. This means $b \in B^\times$. Reparametrizing $X_i \rightsquigarrow b^{\frac{1}{2}} X_i$ for $m < i \le n$ and replacing c_j by $b^{-1}c_j$, we can assume without loss of generality that $b = 1$. As

$$A/(T, X_1, \dots, X_m)$$

is the henselization at x of $\phi^{-1}(z) \cap \overline{Z}$, where Z is a stratum with $x \in \overline{Z} \setminus Z$, this ring is regular, since ϕ is a stratified Morse function. This means $c_0 \in B^\times$. By a reparametrization we can assume without loss of generality that $c_0 = 1$. Similarly we can assume that $c_1 = \dots = c_m = 1$. This finishes the proof. □

3.5. **Stratified base change for unipotent nearby cycles.** Let the notation be as in A.3. Consider regular schemes $(X, \mathbf{Z}_X)$ and $(Y, \mathbf{Z}_Y)$ of finite type over $\mathcal{O}$ endowed with regular stratifications. We assume that the (connected components of) X_K and Y_K are strata.

Proposition 3.10. *For a stratified lci morphism $f\colon Y \to X$ over $\mathcal{O}$, the base change map of unipotent nearby cycles*

$$f^*\psi(\Lambda_{X_K}) \to \psi(f^*\Lambda_{X_K}) \in D^{\mathrm{nil}}(Y, \Lambda)$$

is an isomorphism.

Proof. By smooth base change we can assume without loss of generality that f is a stratified regular immersion. By Lemma A.2 it is sufficient to show the isomorphism in restriction to the G-invariants. This means we have to show that

$$f^*i^*j_*\Lambda_{X_K} \to i^*j_*f^*\Lambda_{X_K}$$

is an isomorphism. Here i is the immersion of the closed fibre and j the immersion of the generic fibre. By the exact triangle (A.6) we only have to show that the map

$$f^*i^!\Lambda_X \to i^!f^*\Lambda_X$$

is an isomorphism.

Claim 3.11. For any closed immersion $\tilde{i}\colon \tilde{X} \to X$, where $\tilde{X}$ is a union of strata, the map

$$f^*\tilde{i}^!\Lambda_X \to \tilde{i}^! f^*\Lambda_X \tag{3.3}$$

is an isomorphism.

Consider an open stratum $Z \subset \tilde{X}$ and set $Z_1 = \overline{Z}$. Set $Z_2 = \tilde{X}\backslash Z$. Let $i_1\colon Z_1 \hookrightarrow \tilde{X}$, $i_2\colon Z_2 \hookrightarrow \tilde{X}$, $i_{12}\colon Z_1 \cap Z_2 \hookrightarrow \tilde{X}$ be the closed immersions. By the exact triangle

$$i_{12,*}i_{12}^!\tilde{i}^!\Lambda_X \to i_{1,*}i_1^!\tilde{i}^!\Lambda_X \oplus i_{2,*}i_2^!\tilde{i}^!\Lambda_X \to \tilde{i}^!\Lambda_X \to \cdots,$$

the corresponding triangle for Y and noetherian induction, we have to show that

$$f^*(\tilde{i}\,\tilde{i}_1)^!\Lambda_X \to (\tilde{i}\,\tilde{i}_1)^! f^*\Lambda_X$$

is an isomorphism, which follows from Gabber's absolute purity theorem [Fuj02, Theorem 2.1.1] as Z_1 and the schematic pullback $f^{-1}(Z_1)$ are regular by Lemma 3.5. □

4. Semi-stable Morse functions

Let $\mathcal{O}$ be a henselian discrete valuation ring. We assume that $K = \mathrm{frac}(\mathcal{O})$ has characteristic zero and that the residue field $k = \mathcal{O}/(\pi)$ of $\mathcal{O}$ is perfect of characteristic different from two. A regular scheme X separated, flat and of finite type over $\mathcal{O}$ is called *semi-stable* if X_k is an snc variety over k. In this section X is assumed to be semi-stable. Let D be a smooth, separated and quasi-compact scheme over $\mathcal{O}$ of relative dimension one.

Note that our semi-stable concept is sometimes called strictly semi-stable in the literature. We always endow a semi-stable scheme X with the standard stratification with strata being the connected components of X_K and the strata of X_k as in Section 3.4.

4.1. Main geometric theorem.
Recall the local structure of semi-stable schemes.

Lemma 4.1. *If k is algebraically closed and $x \in X_k$ is a closed point, there exists an isomorphism of $\mathcal{O}$-algebras*

$$\mathcal{O}^{\mathrm{h}}_{X,x} \cong \mathcal{O}[X_0, \dots, X_n]^{\mathrm{h}}/(X_0 \cdots X_m - \pi)$$

for some $0 \le m \le n$.

Here the henselization $^{\mathrm{h}}$ is with respect to the ideal $(\pi, X_0, \dots, X_n)$.

Our main theorem about stratified Morse functions for semi-stable schemes is the following.

Theorem 4.2. *Assume that X is proper over $\mathcal{O}$. Let $\phi\colon X \to D$ be an $\mathcal{O}$-morphism such that $\phi_k\colon X_k \to D_k$ is a Morse function. Then*

(i) *$\phi_K\colon X_K \to D_K$ is a Morse function;*
(ii) *the specialisation map $\mathrm{sp}\colon |X_K| \to |X_k|$ on closed points induces a bijection between the critical points $\{x_K\}$ of ϕ_K and the critical points $\{x_k\}$ of ϕ_k;*

(iii) *for a critical point* x_K *of* ϕ_K*, let* m *be* $\operatorname{codim}_{X_k}(Z)$*, where* $Z \subset X_k$ *is the stratum of the point* $x_k = \operatorname{sp}(x_K)$*. Then the schematic closure* S *of* x_K *in* X *is a trait of ramification degree* $m+1$ *over* $\mathcal{O}$*;*
(iv) $\phi|_S \colon S \to D$ *is a closed immersion if* $\operatorname{ch}(k) \nmid m+1$.

Here we use the word *trait* for the spectrum of a henselian discrete valuation ring.

The proof of the theorem relies on the following semi-stable version of the Morse lemma the proof of which is almost verbatim the same as the proof of Proposition 3.9.

Proposition 4.3 (Semi-stable Morse lemma)**.** *Assume that* k *is algebraically closed and that* $\phi\colon X \to D$ *is a* $\mathcal{O}$*-morphism such that* $\phi_k \colon X_k \to D_k$ *is a Morse function. Let* $x \in X_k$ *be a closed point. Then there exist isomorphisms of* $\mathcal{O}$*-algebras*

$$\mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \cong \mathcal{O}[T]^{\mathrm{h}}, \quad \mathcal{O}^{\mathrm{h}}_{X,x} \cong \mathcal{O}[X_0,\dots,X_n]^{\mathrm{h}}/(X_0\cdots X_m - u\pi),$$

where u *is a unit in* $\mathcal{O}[X_0,\dots,X_m]^{\mathrm{h}}$ *such that*

(i) *if* $x \in X_k$ *is non-critical, then* $\phi^{\mathrm{h}}_x \colon \mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \to \mathcal{O}^{\mathrm{h}}_{X,x}$ *sends* T *to* X_{m+1}*, in particular* $n > m$*;*
(ii) *if* $x \in X_k$ *is critical then* $\phi^{\mathrm{h}}_x \colon \mathcal{O}^{\mathrm{h}}_{D,\phi(x)} \to \mathcal{O}^{\mathrm{h}}_{X,x}$ *sends* T *to*

$$X_0 + \ldots + X_m + X_{m+1}^2 + \ldots + X_n^2,$$

where $m = n$ *is allowed.*

Here the henselization $^{\mathrm{h}}$ of $\mathcal{O}[T]$ is with respect to the ideal (π, T) and that of $\mathcal{O}[X_0,\dots,X_n]$ with respect to the ideal $(\pi, X_0,\dots,X_n)$.

4.2. **Local version and proof.** Theorem 4.2 follows immediately from the following local version.

Proposition 4.4. *Let* $\phi\colon X \to D$ *be an* $\mathcal{O}$*-morphism such that* $\phi_k \colon X_k \to D_k$ *is a Morse function with only one critical point* $x_k \in X_k$*. Let* m *be the codimension in* X_k *of the stratum containing* x_k*. Then after replacing* X *by a small Zariski neighbourhood of* x_k*, the morphism* $\phi_K \colon X_K \to D_K$ *has precisely one critical point* $x_K \in X_K$ *and it satisfies the following properties:*

(i) x_K *is non-degenerate;*
(ii) *the schematic closure* S *of* x_K *is a trait which is finite of ramification index* $m+1$ *over* $\mathcal{O}$ *with* $S_k = \{x_k\}$*;*
(iii) *the morphism* $\phi|_S \colon S \to D$ *is a closed immersion if* $\operatorname{ch}(k) \nmid m+1$.

Proof. Case $m = 0$. This is classical and follows from Proposition 2.1.

Case $m > 0$. After base change by $\mathcal{O} \to \mathcal{O}^{\mathrm{sh}}$ we can assume that k is algebraically closed. We can also assume without loss of generality that $\mathcal{O}$ is complete.

Let J_ϕ be the Jacobian ideal sheaf of ϕ and set $J = J_{\phi,x_k}\widehat{\mathcal{O}}_{X,x_k}$. Set $I = (J : \pi^\infty)$, i.e. $I = \{\alpha \in \widehat{\mathcal{O}}_{X,x_k} \mid \pi^N\alpha \in J \text{ for some } N \geq 0\}$. Set

$$A = \widehat{\mathcal{O}}_{X,x_k}/I.$$

Take a presentation as in the semi-stable Morse lemma, Proposition 4.3. By abuse of notation we will identify I and J with their preimage in $\mathcal{O}[\![X_0,\dots,X_n]\!]$. We can assume without loss of generality that $n=m$ by using that the external sum of Morse functions is a Morse function and using case $m=0$ above.

Lemma 4.5. *There exist elements $u_1,\dots,u_m\in\mathcal{O}[\![X_0,\dots,X_m]\!]$ such that the ideal I is generated by the power series*

$$X_0 - X_i - u_i X_0 X_i \quad (1\le i\le m), \tag{4.1}$$

$$X_0\cdots X_m - u\pi. \tag{4.2}$$

Here $u\in\mathcal{O}[\![X_0,\dots,X_m]\!]^\times$ is the unit from the semi-stable Morse lemma (Proposition 4.3). In particular $A\neq 0$ is finite flat over $\mathcal{O}$.

Before proving Lemma 4.5 we observe

Claim 4.6. Let $I'\subset\mathcal{O}[\![X_0,\dots,X_m]\!]$ be the ideal generated by the elements (4.1) and (4.2) for arbitrary $u_1,\dots,u_m\in\mathcal{O}[\![X_0,\dots,X_m]\!]$ and for an arbitrary unit $u\in\mathcal{O}[\![X_0,\dots,X_m]\!]$. Then $A'=\mathcal{O}[\![X_0,\dots,X_m]\!]/I'$ is a discrete valuation ring which is totally ramified of degree $m+1$ over $\mathcal{O}$.

Proof. The implicit function theorem implies the existence of formal power series $f_1,\dots,f_n\in\mathcal{O}[\![X]\!]$ with the property $f_i\in X+X^2\mathcal{O}[\![X]\!]$, such that $X_i=f_i(X_0)\in A'$. Then the formal power series

$$h = Xf_1(X)\cdots f_m(X) - u(X,f_1(X),\dots,f_m(X))\pi\in\mathcal{O}[\![X]\!]$$

satisfies

$$h\in X^{m+1}+\pi\mathcal{O}[\![X]\!]+X^{m+2}\mathcal{O}[\![X]\!]$$

with constant coefficient indivisible by π^2. So by the Weierstraß preparation theorem $h=v\tilde h$ with $v\in\mathcal{O}[\![X]\!]^\times$ and $\tilde h\in\mathcal{O}[X]$ an Eisenstein polynomial of degree $m+1$.

As $\tilde h(X_0)=v^{-1}(X_0)h(X_0)=0\in A'$, we obtain a surjective $\mathcal{O}$-algebra homomorphism

$$\theta\colon \mathcal{O}[X]/(\tilde h)\to A',\quad X\mapsto X_0,$$

This implies that A' is finite over $\mathcal{O}$, but then by [Mat86, Theorem 22.6] A' is also flat over $\mathcal{O}$. As $K[X]/(\tilde h)$ is a field by the Eisenstein irreducibility criterion, and as $A'\otimes_{\mathcal{O}}K\neq 0$, we deduce that θ is an isomorphism. □

Proof of Lemma 4.5. Note that J is the (continuous) Jacobian ideal of the local ring homomorphism

$$\mathcal{O}[\![T]\!]\to\mathcal{O}[\![T,X_1,\dots,X_m]\!]/(g),$$

where $g=X_1\cdots X_m(T-X_1-\ldots-X_m)-u\pi$ is the element from the semi-stable Morse lemma in which we made the substitution $X_0=T-X_1-\ldots-X_m$. So J is the ideal generated by the $m+1$ elements

(4.3) $(1 \le i \le m):\ \partial_{X_i} g =$

$$-2X_1\cdots X_m + X_1\cdots \widehat{X}_i\cdots X_m(T - \sum_{i\neq j\geq 1} X_j) - \pi u u_i = (X_0 - X_i)X_1\cdots\widehat{X}_i\cdots X_m - \pi u u_i,$$

$$X_0\cdots X_m - u\pi.$$

in the coordinates $X_0,\dots,X_m$. Here the partial derivative ∂_{X_i} is the one killing T, X_j for $i \neq j \geq 1$ and $u_i = u^{-1}\partial_{X_i} u$. Multiplying the expression on the right side of the equation for $\partial_{X_i} g$ by X_0X_i and substituting $u\pi$ for $X_0\cdots X_m$ we obtain the element (4.1) times $u\pi$.

Let $I' \subset \mathcal{O}[\![X_0,\dots,X_m]\!]$ be the ideal generated by the elements (4.1) and (4.2) with the u_i as above. We have just shown $\pi I' \subset J$. A simple calculation shows that $J \subset I'$. Therefore $I = (I' : \pi) = I'$, where the second equality follows from Claim 4.6. We have shown Lemma 4.5. □

Now part (i) and part (ii) of Proposition 4.4 follow by combining Claim 4.6 and Lemma 4.5.

For part (iii) observe that $\phi|_S\colon S \to D$ is given locally by the ring homomorphism $\mathcal{O}[\![T]\!] \to A$ which sends T to

$$X_0 + f_1(X_0) + \ldots + f_m(X_0) \in (m+1)X_0 + X_0^2 A$$

with the notation of the proof of Claim 4.6. So this ring homomorphism is surjective if $m+1$ is invertible in $\mathcal{O}$. □

5. Geometry of Lefschetz pencils

5.1. **Stratified Lefschetz pencils.** Let X be a projective scheme over k together with a regular stratification. Fix a closed immersion $\iota\colon X \hookrightarrow \mathbb{P}^N_k$.

A k-rational point V of the Grassmannian $\mathrm{Gr}(2, H^0(\mathbb{P}^N_k, \mathcal{O}(d))$ is called a *pencil of hypersurfaces of degree d*. The *base* A of the pencil is the intersection of two different hypersurfaces of the pencil. The *pencil map* is the canonical map $\mathbb{P}^N_k \setminus A \to \mathbb{P}(V) \cong \mathbb{P}^1_k$ sending a point to the unique hypersurface of the pencil containing it. The *compactified pencil map* (or for short just *pencil map*) is the induced morphism $\mathrm{Bl}_A(\mathbb{P}^N_k) \to \mathbb{P}^1_k$.

Definition 5.1 (Lefschetz pencil)**.** A pencil of degree d hypersurfaces is called a *Lefschetz pencil* for X if

(i) the base A of the pencil intersects each stratum $Z \hookrightarrow X$ transversally, i.e. $A \cap X \to X$ is a stratified regular immersion of codimension two;
(ii) the pencil map $X \setminus (A \cap X) \to \mathbb{P}^1_k$ is a stratified Morse function and has at most one critical point per geometric fibre.

If for a Lefschetz pencil for X we set $\tilde{X} = \iota^*\mathrm{Bl}_A(\mathbb{P}^N_k)$ then we obtain the pencil map $\phi\colon \tilde{X} \to \mathbb{P}^1_k$. Note that as $A \hookrightarrow \mathbb{P}^N_k$ and $A \cap X \hookrightarrow X$ are regular closed immersions of the same codimension ($= 2$) we have an isomorphism $\mathrm{Bl}_{A\cap X}(X) \cong \iota^*\mathrm{Bl}_A(\mathbb{P}^N_k)$ [Mic64, Chapitre I, Théorème 1]. The pullback of the stratification of X to $\tilde{X}$ is a regular stratification by Lemma 3.8. The compactified pencil map $\phi\colon \tilde{X} \to \mathbb{P}^1_k$ is a Morse function which has no critical points over $A \cap X$.

Proposition 5.2. *For d large there is an open dense subset of Lefschetz pencils of X in* $\mathrm{Gr}(2, H^0(\mathbb{P}^N_k, \mathcal{O}(d))$.

Proof. For simplicity of notation we replace ι by its composition with the Veronese embedding

$$\mathbb{P}^N_k \hookrightarrow \mathbb{P}_k^{\binom{N+d}{d}-1}$$

of degree d, so now $\mathbb{P}(V)$ is a line in the dual space $\check{\mathbb{P}}^N_k$. We also assume for simplicity that k is algebraically closed.

For a smooth subscheme $Z \subset X$ let $\check{Z} \subset \check{\mathbb{P}}^N_k$ be its closed dual variety, i.e. the closure of the set of hyperplanes H with $H \cap Z$ singular. For $d > 1$ the reference [SGA7.2, Exposé XVII, Proposition 3.5] implies that $\check{Z}$ is a hypersurface and a line $\mathbb{P}(V)$ in $\check{\mathbb{P}}^N_k$ with associated base $A \subset \mathbb{P}^N_k$ is a Lefschetz pencil for $\overline{Z}$ if and only if

(I) A intersects $\overline{Z}$ transversally and
(II) $\mathbb{P}(V)$ does not intersect $\check{Z}^{\mathrm{sing}}$.

The critical values are $\mathbb{P}(V) \cap \check{Z}$. By Proposition 3.7 we obtain that for $d \gg 0$ the dual varieties $\check{Z}$ of all strata $Z \in \mathbf{Z}$ are different. Then any pencil V which satisfies (I) and such that

$$(\cup_{Z\in\mathbf{Z}}\check{Z})^{\mathrm{sing}} \cap \mathbb{P}(V) = \varnothing$$

is a stratified Lefschetz pencil. The set of those V is clearly open and non-empty by the theorem of Bertini. ☐

5.2. **Semi-stable Lefschetz pencils.** Let X be a projective semi-stable scheme over $\mathcal{O}$ of relative dimension n. Fix a closed immersion $\iota\colon X \hookrightarrow \mathbb{P}^N_{\mathcal{O}}$. Recall that we endow X with the standard stratification as in Section 4.

An $\mathcal{O}$-point V of the Grassmannian $\mathrm{Gr}(2, H^0(\mathbb{P}^N_{\mathcal{O}}, \mathcal{O}(d))$ is called a *pencil of hypersurfaces of degree d.* This is the same as a rank 2 free $\mathcal{O}$-submodule of $H^0(\mathbb{P}^N_{\mathcal{O}}, \mathcal{O}(d))$ which has a free complementary submodule.

Definition 5.3. If V_k defines a stratified Lefschetz pencil for X_k and V_K defines a Lefschetz pencil for X_K, then we say that V defines a *semi-stable Lefschetz pencil for X*. The set of *critical points* resp. *critical values* is the union of the set of critical points resp. critical values of the pencil maps ϕ_K and ϕ_k.

Note that if V_k defines a stratified Lefschetz pencil, the base A has the property that $A \cap X \to X$ is a stratified regular immersion by Lemma 3.6. So if under this assumption we set $\tilde{X} = \mathrm{Bl}_{A\cap X}(X) = \iota^*\mathrm{Bl}_A(\mathbb{P}^N_{\mathcal{O}})$ then we get the pencil map $\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}}$ which is stratified lci by Lemma 3.8.

Let us recall the properties we showed in Theorem 4.2 in the case of large residue characteristic.

Theorem 5.4. *If* $\mathrm{ch}(k) > n+1$ *or* $\mathrm{ch}(k) = 0$ *and* V_k *defines a Lefschetz pencil for* X_k *then* V *defines a Lefschetz pencil for* X*. Moreover, the set of critical points on* X *is closed and maps isomorphically (as a scheme) onto the set of critical values. Each connected component of the critical points is a trait which is finite and of ramification index over* $\mathcal{O}$ *equal to the number of irreducible components of* X_k *it meets.*

In case $0 < \mathrm{ch}(k) \leq n+1$ we cannot combine Proposition 5.2 and Theorem 4.2 to deduce the existence of a Lefschetz pencil over $\mathcal{O}$, since using them it is not clear whether ϕ_K has at most one critical point per geometric fibre. However, one easily proves the following proposition.

Proposition 5.5. *For* d *large there is an open dense subset in* $\mathrm{Gr}(2, H^0(\mathbb{P}^N_k, \mathcal{O}(d))$ *such that if* V_k *lies in this subset, then* V *defines a Lefschetz pencils for* X*.*

Proof. Let $B_F \subset \mathrm{Gr}(2, H^0(\mathbb{P}^N_F, \mathcal{O}(d))$ be the Zariski closure of all non-Lefschetz pencils over F for $F = K$ or $F = k$. Then B_F is not dense in $\mathrm{Gr}(2, H^0(\mathbb{P}^N_F, \mathcal{O}(d))$ for $F = K$ and $F = k$. Thus $\overline{B_K} \otimes_{\mathcal{O}} k$ is a proper closed subset of $\mathrm{Gr}(2, H^0(\mathbb{P}^N_k, \mathcal{O}(d))$, so the open subset

$$\mathrm{Gr}(2, H^0(\mathbb{P}^N_k, \mathcal{O}(d)) \setminus (\overline{B_K} \otimes_{\mathcal{O}} k \cup B_k)$$

has the required properties. □

6. Reminder on the cohomology of Lefschetz pencils

In this section we reformulate the classical results about the cohomology of Lefschetz pencils in terms of perverse sheaves. This has the advantage that one gets rid of the unpleasant dichotomy of cases (A) and (B) in [SGA7.2, Exposé XVII].

6.1. **Picard-Lefschetz theory.** Let ℓ be a prime number invertible in k and let $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ or let Λ be an algebraic field extension of $\mathbb{Q}_\ell$. Let $f\colon X \to \operatorname{Spec} k$ be a smooth projective morphism of schemes, $n = \dim X$. In this section we use the perversity associated to $\delta_X\colon X \to \mathbb{Z}$, $\delta_X(x) = \dim \overline{\{x\}}$, see A.4.

For $\Lambda = \overline{\mathbb{Q}}_\ell$, a perverse sheaf $\mathsf{F} \in D^b_c(X, \Lambda)$ is called *geometrically semi-simple* if $\mathsf{F}_{\overline{k}} \in D^b_c(X_{\overline{k}}, \Lambda)$ is semi-simple. For X geometrically connected a geometrically semi-simple perverse sheaf $\mathsf{F} \in D^b_c(X, \Lambda)$ has a canonical decomposition

$$\mathsf{F} = \mathsf{F}^{\mathrm{c}} \oplus \mathsf{F}^{\mathrm{nc}}$$

into a *geometrically constant* part $\mathsf{F}^{\mathrm{c}} = f^*(\mathsf{F}')[n]$, where $\mathsf{F}' \in D^b_c(\operatorname{Spec} k, \Lambda)$ is situated in degree 0, and a part F^{nc} which has no non-trivial geometrically constant subsheaves, see [BBD83, Corollaire 4.2.6.2 and following comment]. We say that a geometrically semi-simple perverse sheaf F on X satisfies *multiplicity one* if $\mathsf{F}_{\overline{k}}$ is a direct sum of pairwise non-isomorphic irreducible perverse sheaves. Let $\mathsf{C} = \Lambda[n]$ be the constant perverse sheaf on X.

Fix an immersion $X \hookrightarrow \mathbb{P}^N_k$. Consider a Lefschetz pencil with center A as above and $\tilde{X} = \mathrm{Bl}_{X\cap A}(X)$. Let $\phi\colon \tilde{X} \to \mathbb{P}^1_k$ be the pencil map. We call the perverse sheaf

$$\mathsf{L} = {}^{\mathrm{p}}R^0\phi_*\mathsf{C} \text{ on } \mathbb{P}^1_k$$

the associated *Picard-Lefschetz sheaf.*

Theorem 6.1. *Assume $\Lambda = \overline{\mathbb{Q}}_\ell$. Then the following properties are verified.*

(i) ${}^{\mathrm{p}}R^i\phi_*\mathsf{C}$ *is a geometrically semi-simple perverse sheaf for all $i \in \mathbb{Z}$;*
(ii) ${}^{\mathrm{p}}R^i\phi_*\mathsf{C}$ *is geometrically constant for $i \neq 0$;*
(iii) $\mathsf{L}^{\mathrm{nc}} = ({}^{\mathrm{p}}R^0\phi_*\mathsf{C})^{\mathrm{nc}}$ *satisfies multiplicity one.*

More precisely, if X is geometrically connected then either

(A) *there is no skyscraper in L and L^{nc} is geometrically irreducible or*
(B) L^{nc} *is a sum of one-dimensional skyscrapers at the critical values of ϕ.*

Remark 6.2. By [SGA7.2, Théorème 6.3] the Lefschetz pencil is of type (A) if the degree of the pencil is sufficiently large. For n odd type (B) does not occur.

From a modern point of view Theorem 6.1(i) is simply the decomposition theorem for proper morphisms [BBD83, Théorème 6.2.5], while Theorem 6.1(ii) is a simple consequence of the Weak Lefschetz theorem. The proof of Theorem 6.1(iii) uses the local Picard-Lefschetz formula, see [SGA7.2, Exposé XV]. In the following we recall the part of the local theory which we use explicitly in this paper.

Consider a point $x \in \mathbb{P}^1_k$. Fix a geometric point $\overline{\eta}_x$ over the generic point $\eta_x \in \operatorname{Spec}\mathcal{O}^{\mathrm{h}}_{\mathbb{P}^1_k,x}$. Set $G_x = \mathrm{Gal}(\overline{\eta}_x/\eta_x)$. Consider $\overline{\jmath}_x\colon \overline{\eta}_x \to \mathbb{P}^1_k$, $i_x\colon \operatorname{Spec} k(x) \to \mathbb{P}^1_k$. The Λ-module of pre-vanishing cycles at x is defined by

$$V_x := H^{-1}(x, \mathrm{cone}(i_x^*\mathsf{L} \to i_x^*\overline{\jmath}_{x,*}\overline{\jmath}_x^*\mathsf{L})).$$

Note that all other cohomology groups in degree $\neq -1$ vanish [Ill94, Corollaire 4.6].

The following proposition is a consequence of the Picard-Lefschetz formula [SGA7.2, Exposé XV].

Proposition 6.3. *Let x be a critical value of ϕ.*

- *If n is even there is a canonical isomorphism $V_x \cong \Lambda(-\frac{n}{2})$ up to sign.*
- *If n is odd there is a canonical isomorphism $V_x \cong \Lambda(-\frac{n-1}{2})$ up to sign.*

The proposition means that the homomorphism of $G_x \to \mathrm{Aut}(V_x)/\{\pm\mathrm{id}\}$ is given in terms of a power of the cyclotomic character. In particular the order of $\mathrm{im}(G_x) \subset \mathrm{Aut}(V_x)$ divides 2 if G_x acts trivially on $\Lambda(1)$.

6.2. **The middle primitive cohomology.** The results of this section are due to [SGA7.2, Exposé XVIII, Théorème 5.7] for Lefschetz pencils of type (A). In Proposition 6.4 we formulate our result and and in Remark 6.5 we explain what this means in the novel case of Lefschetz pencils of type (B).

Let Λ be an algebraic field extension of $\mathbb{Q}_\ell$, $\mathsf{C} = \Lambda_X[n]$, $n = \dim(X)$. Consider the Lefschetz operator

$$L\colon H^i(X_{\overline{k}}, \mathsf{C}) \to H^{i+2}(X_{\overline{k}}, \mathsf{C}(1))$$

associated to the given polarization. By the Hard Lefschetz theorem [Del80, Théorème 6.2.13] the canonical map from the kernel of L to its (Tate twisted) cokernel is an isomorphism in degree 0. We write this group

$$H_{\mathrm{prim}} = H^0(X_{\overline{k}}, \mathsf{C})^L \xrightarrow{\sim} H^0(X_{\overline{k}}, \mathsf{C})_L.$$

Let $\mathsf{L} = {}^{\mathrm{p}}R^0\phi_*\mathsf{C}$ be a Picard-Lefschetz sheaf as in Theorem 6.1.

Proposition 6.4. *The* $\mathrm{Gal}(\overline{k}/k)$*-module* H_{prim} *is canonically a direct summand of* $H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L})$.

Proof. Set $X^\circ = X \setminus (X \cap A)$, $\phi^\circ = \phi|_{X^\circ}$. Let $Y \hookrightarrow X$ be a generic hypersurface section in the fixed Lefschetz pencil, in particular Y is smooth over k. Observe that $Y^\circ = Y \setminus (Y \cap A)$ and $X^\circ \setminus Y^\circ = X \setminus Y$ are affine. We study the commutative diagram

$$\begin{array}{ccccc}
H^0(\mathbb{P}^1_{\overline{k}}, {}^{\mathrm{p}}R^0\phi^\circ_!\mathsf{C}) & \xrightarrow[\sim]{\Upsilon_c} & H^0_c(X^\circ_{\overline{k}}, \mathsf{C})^L & \xrightarrow[\sim]{\Omega_c} & H^0(X_{\overline{k}}, \mathsf{C})^L \\
\Xi \big\downarrow & & & & \\
H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L}) & & & & \big\downarrow \wr \\
\Delta \big\downarrow & & & & \\
H^0(\mathbb{P}^1_{\overline{k}}, {}^{\mathrm{p}}R^0\phi^\circ_*\mathsf{C}) & \xleftarrow[\sim]{\Upsilon} & H^0(X^\circ_{\overline{k}}, \mathsf{C})_L & \xleftarrow[\sim]{\Omega} & H^0(X_{\overline{k}}, \mathsf{C})_L
\end{array} \tag{6.1}$$

where Υ and Υ_c are induced by the Leray spectral. Below we will show that the horizontal maps are isomorphisms. We deduce that Ξ is injective and Δ is surjective. So we see that the upper part of the diagram induces an injection $H_{\mathrm{prim}} \to H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L})$ with splitting $H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L}) \to H_{\mathrm{prim}}$ induced by the lower part of the diagram.

In order to show that the upper and the lower horizontal maps in (6.1) are isomorphisms, it suffices by duality to consider the upper part of the diagram. In the sequel the cohomology groups are meant to be with coefficients in C and we assume that $k = \overline{k}$. We now proceed by a series of claims (I)–(VI).

(I) $H^0_c(X^\circ) \to H^0(X)$ *is injective.*

This follows from a diagram chase in the following commutative diagram with exact rows

$$\begin{array}{ccccccc} H^{-1}(X) & \longrightarrow & H^{-1}(A\cap X) & \longrightarrow & H^0_c(X^\circ) & \longrightarrow & H^0(X) \\ \uparrow L & & \wr\uparrow L & & & & \\ H^{-3}(X) & \longrightarrow & H^{-3}(A\cap X) & \longrightarrow & H^{-2}_c(X^\circ)=0. & & \end{array}$$

The right vertical Lefschetz arrow is an isomorphism by the Hard Lefschetz theorem. The vanishing of the right bottom group follows from the exact sequence

$$0 = H^{-2}_c(X\setminus Y_1)\oplus H^{-2}_c(X\setminus Y_2)\to H^{-2}_c(X^\circ)\to H^{-1}_c(X\setminus (Y_1\cup Y_2)) = 0$$

where Y_1 and Y_2 are two different hypersurfaces of the fixed Lefschetz pencil. Indeed, the Weak Lefschetz theorem implies that the groups on the right and on the left vanish in this exact sequence.

(II) *It holds*

$$H^0_c(X^\circ)^L \supset \ker(H^0_c(X^\circ)\to H^0_c(Y^\circ)),\quad H^0(X)^L = \mathrm{im}(H^0_c(X\setminus Y)\to H^0(X)).$$

For the first inclusion use that $L\colon H^0_c(X^\circ)\to H^2_c(X^\circ)(1)$ is the composition of

$$H^0_c(X^\circ)\to H^0_c(Y^\circ)\to H^2_c(X^\circ)(1).$$

For the second equality use the exact sequence

$$H^0_c(X\setminus Y)\to H^0(X)\to H^0(Y)$$

and the description of $L\colon H^0(X)\to H^0(X)$ as the composition

$$H^0(X)\to H^0(Y)\to H^2(X)(1).$$

in which the second arrow is injective by the Weak Lefschetz theorem as $H^1(X\setminus Y) = 0$.

(III) *The maps* $\ker(H^0_c(X^\circ)\to H^0_c(Y^\circ))\xrightarrow{\sim} H^0_c(X^\circ)^L\xrightarrow{\sim} H^0(X)^L$ *are isomorphisms.*

The first (injective) map is well-defined by (II). The second map is injective by (I). From (II) we get a surjection $H^0_c(X\setminus Y)\to H^0(X)^L$, which factors through these maps as $X\setminus Y = X^\circ\setminus Y^\circ$.

In particular (III) implies that Ω_c is an isomorphism.

(IV) $0\to H^0(\mathbb{P}^1_k, {}^{\mathrm{p}}R^0\phi^\circ_!\mathsf{C})\to H^0_c(X^\circ)\xrightarrow{\beta} H^{-1}(\mathbb{P}^1_k, {}^{\mathrm{p}}R^1\phi^\circ_!\mathsf{C})$ *is exact.*

This sequence is induced by the perverse Leray spectral sequence and the exactness follows since $H^a(\mathbb{P}^1_k,\mathsf{G}) = 0$ for any perverse sheaf G and $a<-1$ and since ${}^{\mathrm{p}}R^a\phi^\circ_!\mathsf{C} = 0$ for $a<0$ by the Weak Lefschetz theorem.

(V) ${}^{\mathrm{p}}R^1\phi^\circ_!\mathsf{C}$ *is smooth.*

Let $E \subset \tilde{X}$ be the exceptional divisor of the blow-up $\tilde{X} \to X$. Then $E \cong (A \cap X) \times \mathbb{P}^1_k$ and $\phi^E := \phi|_E : E \to \mathbb{P}^1_k$ is given by the projection, see [SGA7.2, Exposé XVIII, 2.]. We consider the exact sequence

$$ {}^{\mathrm{p}}R^0\phi^E_* \mathsf{C}|_E \to {}^{\mathrm{p}}R^1\phi^\circ_! \mathsf{C} \to {}^{\mathrm{p}}R^1\phi_* \mathsf{C}. $$

Then the perverse sheaf on the left is smooth by smooth base change and the one on the right is smooth by Theorem 6.1.

(VI) *It holds* $\ker(\beta) = \ker(H^0_c(X^\circ) \to H^0_c(Y^\circ))$.

By (V) and proper base change the restriction map

$$ H^{-1}(\mathbb{P}^1_k, {}^{\mathrm{p}}R^1\phi^\circ_! \mathsf{C}) \to H^0_c(Y^\circ) $$

in injective.

The combination of (III), (IV) and (VI) gives the isomorphism Υ_c. □

Remark 6.5. If a geometrically connected X has a Lefschetz pencil of type (B), Proposition 6.4 combined with Proposition 6.3 yields an isomorphism

$$ H_{\mathrm{prim}} \cong \Lambda(-\frac{n}{2})^{\oplus r} $$

where r is the number of critical points, which is canonical up to signs on each summand on the right. For Lefschetz pencils of type (A) the complement in the direct sum decomposition in Proposition 6.4 is calculated in [SGA7.2, Exposé XVIII, Théorème 5.7].

7. Monodromy filtration and weight filtration

7.1. **Monodromy filtration for perverse sheaves.** Let k be a field. Let X, Y be separated schemes of finite type over k. Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or let Λ be an algebraic field extension of $\mathbb{Q}_\ell$. We use the notation of A.3.

Consider a perverse sheaf $\mathsf{F} \in D^{\mathrm{nil}}(X, \Lambda)$ with its nilpotent endomorphism

$$ N : \mathsf{F} \to \mathsf{F}(-1)^{\mathrm{Iw}}. $$

Note that if $\mathbb{Q}_\ell \subset \Lambda$ there is a canonical isomorphism between Iwasawa twist and Tate twist $\mathsf{F}(-1)^{\mathrm{Iw}} \cong \mathsf{F}(-1)$, so in this case one can also write $N : \mathsf{F} \to \mathsf{F}(-1)$, see A.2.

We use the perverse t-structure from Section 6. Consider the two filtrations by perverse subsheaves of F

$$ \mathrm{fil}_a\mathsf{F} = \ker[N^{a+1} : \mathsf{F} \to \mathsf{F}(-a-1)^{\mathrm{Iw}}], $$
$$ \mathrm{fil}^a\mathsf{F} = \mathrm{im}[(N(1)^{\mathrm{Iw}})^a : \mathsf{F}(a)^{\mathrm{Iw}} \to \mathsf{F}] $$

for $a \in \mathbb{Z}$. We denote by $\mathrm{gr}_a\mathsf{F}, \mathrm{gr}^a\mathsf{F}$ the corresponding graded subquotients. Set

$$ \mathrm{gr}^b_a\mathsf{F} = \mathrm{fil}_a\mathsf{F} \cap \mathrm{fil}^b\mathsf{F}/(\mathrm{fil}_{a-1}\mathsf{F} + \mathrm{fil}^{b+1}\mathsf{F}) \cap \mathrm{fil}_a\mathsf{F} \cap \mathrm{fil}^b\mathsf{F}. $$

The *monodromy filtration* is the increasing convolution

$$\mathrm{fil}^{\mathrm{M}}_a \mathsf{F} = \sum_{b-c=a} \mathrm{fil}_b \mathsf{F} \cap \mathrm{fil}^c \mathsf{F}, \tag{7.1}$$

filtration. See [Sai03, Section. 2.1], where one finds a proof of the following lemma.

Lemma 7.1. (i) *The canonical map*

$$\bigoplus_{b-c=a} \mathrm{gr}^c_b \mathsf{F} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{M}}_a \mathsf{F}$$

is an isomorphism.

(ii) N *sends* $\mathrm{fil}^{\mathrm{M}}_a \mathsf{F}$ *to* $\mathrm{fil}^{\mathrm{M}}_{a-2} \mathsf{F}(-a)^{\mathrm{Iw}}$ *for all* $a \in \mathbb{Z}$ *and it induces an isomorphism*

$$\mathrm{gr}^{\mathrm{M}}_a \mathsf{F} \xrightarrow{N^a} \mathrm{gr}^{\mathrm{M}}_{-a} \mathsf{F}(-a)^{\mathrm{Iw}}$$

for all $a \geq 0$.

(iii) *The monodromy filtration is the only finite filtration of* F *by perverse subsheaves which satisfies* (ii).

Definition 7.2. For $f\colon X \to Y$ a proper k-morphism and for $\mathsf{F} \in D^{\mathrm{nil}}(X, \Lambda)$ perverse we say that the *monodromy property* holds (for F and f) if

$$\mathrm{fil}^{\mathrm{M}}_a \, {}^{\mathrm{p}}R^i f_* \mathsf{F} = \mathrm{im}[{}^{\mathrm{p}}R^i f_* \mathrm{fil}^{\mathrm{M}}_a \mathsf{F} \to {}^{\mathrm{p}}R^i f_* \mathsf{F}] \tag{7.2}$$

in the category of perverse sheaves for all $a, i \in \mathbb{Z}$.

Remark 7.3. A variant of a conjecture of Kashiwara [Kas98] says that for k separably closed and Λ an algebraic extension of $\overline{\mathbb{Q}}_\ell$, any $\mathsf{F} = \psi(\mathsf{G}) \in D^{\mathrm{nil}}(X, \Lambda)$ produced by the unipotent nearby cycle functor ψ for a morphism $X \to \mathbb{A}^1_k$ and a semi-simple perverse $\mathsf{G} \in D^b_c(X, \Lambda)$ satisfies the monodromy property for any proper morphism $f\colon X \to Y$. If $\mathrm{ch}(k) = 0$, one can deduce this conjecture from the work by T. Mochizuki [Moc07] or by applying the method of Drinfeld [Dri01]. If $\mathrm{ch}(k) > 0$ and G is arithmetic in the sense of [EK20, Definition 1.4], one can deduce it from the work of [Laf02] and Gabber [BB93, Section 5].

7.2. **mw pure sheaves.** Assume now that k is a finite field, that Λ is an algebraic extension of $\mathbb{Q}_\ell$ and that $\mathsf{F} \in D^{\mathrm{nil}}(X, \Lambda)$ is mixed and perverse. Then there is a canonical weight filtration $\mathrm{fil}^{\mathrm{W}}_a \mathsf{F} \subset \mathsf{F}$, see [BBD83, Théorème 5.3.5].

Definition 7.4. We call the mixed perverse sheaf F *monodromy-weight pure* (or *mw pure*) of weight $w \in \mathbb{Z}$ if

$$\mathrm{gr}^{\mathrm{M}}_a \mathsf{F} \text{ is pure of weight } w + a \text{ for all } a \in \mathbb{Z}$$

i.e. if

$$\mathrm{fil}^{\mathrm{M}}_a \mathsf{F} = \mathrm{fil}^{\mathrm{W}}_{a+w} \mathsf{F}.$$

Recall the following well-known result, see e.g. [FO22, Theorem 2.49] for the case of $X = \mathrm{Spec}(\mathbb{F}_q)$.

Proposition 7.5. *For fixed $w \in \mathbb{Z}$ the mw pure perverse sheaves $\mathsf{F} \in D^{\mathrm{nil}}(X, \Lambda)$ of weight w form an abelian subcategory of all perverse sheaves closed under extensions.*

Proof. Mixed perverse sheaves form an abelian category for which every morphism is strict with respect to the weight filtration [BBD83, Théroème 5.3.5]. On the other hand for a morphism of perverse sheaves $\mathsf{F} \to \mathsf{G}$ in $D^{\mathrm{nil}}(X, \Lambda)$ which is strict with respect to the monodromy filtration, the monodromy filtration on the kernel and cokernel is the induced subspace and quotient filtration. This shows that the mw pure perverse sheaves in $D^{\mathrm{nil}}(X, \Lambda)$ form an abelian subcategory. Similarly, one sees that this category is closed under extensions. □

Assume in the following that $f : X \to Y$ is a proper k-morphism.

Proposition 7.6. *If F is mw pure the monodromy spectral sequence*

$$E_1^{p,q} = {}^{\mathrm{p}}R^{p+q} f_* \mathrm{gr}^{\mathrm{M}}_{-p} \mathsf{F} \Rightarrow {}^{\mathrm{p}}R^{p+q} f_* \mathsf{F}$$

degenerates at the E_2-page.

Proof. The perverse sheaf ${}^{\mathrm{p}}R^{p+q} f_* \mathrm{gr}^{\mathrm{M}}_{-p} \mathsf{F}$ is pure of weight $w + q$ if F is mw pure of weight w, see [BBD83, Corollaire 5.4.2]. So the differentials $d_2, d_3, \ldots$ vanish by weight reasons. □

Proposition 7.7. *If F is mw pure of weight w the following are equivalent:*

(i) *The monodromy property holds for F and f.*
(ii) ${}^{\mathrm{p}}R^i f_* \mathsf{F}$ *is mw pure of weight $w + i$ for all i.*
(iii) *The map*

$$N^a : E_2^{-a,i+a} \to E_2^{a,i-a}(-a)$$

is an isomorphism for all $a \geq 0$ and $i \in \mathbb{Z}$.

Proof. (i) ⇒ (ii): By [BBD83, Corollaire 5.4.2] the filtration on the right of (7.2) is the shifted weight filtration.
(ii) ⇒ (iii): As in the proof of Proposition 7.6 the perverse sheaf $E_2^{-a,i+a}$ is the weight $w + i + a$ graded piece of ${}^{\mathrm{p}}R^i f_* \mathsf{F}$, so if the weight filtration agrees up to shift with the monodromy filtration on ${}^{\mathrm{p}}R^i f_* \mathsf{F}$ we obtain part (iii) by Lemma 7.1(ii).
(iii) ⇒ (i): Similarly, this follows from Lemma 7.1(iii). □

8. Rapoport-Zink sheaves

The study of the nearby cycle functor for the constant sheaf on a semi-stable scheme originated from [SGA7.1, Exposé I]. In the complex analytic framework a more precise calculation was envisioned in [Ste76], which was made precise in the étale setting by [RZ82, Abschnitt 2]. However the latter approach is not formulated in terms of perverse sheaves, which was done later in [Sai90, Theorem 3.3] in the setting of $\mathcal{D}$-modules, and by [Sai03], [Car12] in the étale setting. In this section we give a summary of the theory based on the duality of the nearby cycle functor.

We make systematic use of the notion of Iwasawa twist developed by Beilinson, which is necessary in order to give a coordinate free description, see Appendix A.

8.1. **Constant sheaf.** Let k be a field. Let $f\colon X \to \operatorname{Spec} k$ be a simple normal crossing variety of dimension n. Let Λ be $\mathbb{Z}/\ell^\nu\mathbb{Z}$ or an algebraic extension of $\mathbb{Q}_\ell$. Set $\varpi = f^!\Lambda$.

We consider the perverse t-structure on $D^b_c(X,\Lambda)$ as in 6.1. For a perverse sheaf $\mathsf{F} \in D^b_c(X,\Lambda)$ we denote by $\mathrm{fil}^a_{\mathrm{S}}\mathsf{F}$ the largest perverse subsheaf of F supported in codimension $\geq a$. Set $\mathsf{C} = \Lambda_X[n]$. Note that C is perverse as f is a local complete intersection (lci) [KW01, Lemma 6.5]. In this subsection we recall the description of its support filtration $\mathrm{fil}_{\mathrm{S}}\mathsf{C}$ from [Sai03, Section 1.1].

Let $\pi\colon \tilde{X} \to X$ be the normalization and set

$$\Lambda_X^{(0)} = \pi_*\Lambda_{\tilde{X}} \text{ and } \Lambda_X^{(a)} = \bigwedge_{\Lambda}^{a+1} \Lambda^{(0)} \text{ for } a > 0.$$

Note that if one chooses an ordering $X_1, \dots, X_r$ of the irreducible components of X then

$$\Lambda_X^{(a)} \cong \bigoplus_{1\leq i_0<\cdots<i_a\leq r} \Lambda_{X_{i_0}\cap\cdots\cap X_{i_a}},$$

so $\Lambda_X^{(a)}[n-a]$ is perverse.

The canonical map $\Lambda_X \to \pi_*\pi^*\Lambda_X = \Lambda_X^{(0)}$ defines a Koszul complex

$$\mathsf{Ko}_X = [\Lambda_X^{(0)} \to \Lambda_X^{(1)} \to \Lambda_X^{(2)} \to \cdots],$$

where $\Lambda_X^{(0)}$ is put in degree 0, which resolves Λ_X, i.e. we have a quasi-isomorphism $\mathsf{C} \xrightarrow{\sim} \mathsf{Ko}_X[n]$. We consider the truncation filtration

$$\mathrm{fil}^a\mathsf{Ko}_X = [0 \to \Lambda_X^{(a)} \to \Lambda_X^{(a+1)} \to \cdots].$$

By descending induction on a and the exact triangle

$$\mathrm{fil}^{a+1}\mathsf{Ko}_X \to \mathrm{fil}^a\mathsf{Ko}_X \to \Lambda_X^{(a)}[-a] \to \mathrm{fil}^{a+1}\mathsf{Ko}_X[1]$$

we deduce

Lemma 8.1. (i) $\mathrm{fil}^a\mathsf{Ko}_X[n]$ *is perverse for all* $a \in \mathbb{Z}$.

(ii) *The isomorphism* $\mathsf{C} \xrightarrow{\sim} \mathsf{Ko}_X[n]$ *induces an isomorphism of perverse subsheaves* $\mathrm{fil}^a_{\mathrm{S}}\mathsf{C} \cong \mathrm{fil}^a\mathsf{Ko}_X[n]$ *for all* $a \in \mathbb{Z}$.

(iii) *There is a canonical isomorphism*

$$\kappa\colon \Lambda_X^{(a)}[n-a] \xrightarrow{\sim} \mathrm{gr}^a_{\mathrm{S}}\mathsf{C}$$

which induces a canonical perfect pairing

$$\mathsf{c}^a\colon \mathrm{gr}^a_{\mathrm{S}}\mathsf{C} \otimes \mathrm{gr}^a_{\mathrm{S}}\mathsf{C}(n-a) \to \varpi.$$

Following the sign convention of M. Saito in [Sai88, 5.4] one should incorporate the sign $(-1)^{b(b-1)/2}$ in the perfect pairing

$$\Lambda_Y[b] \otimes \Lambda_Y[b](b) \to g^! \Lambda$$

for a smooth scheme $g\colon Y \to \operatorname{Spec} k$ of dimension b. The reason is that we have a shift -1 in the unipotent nearby cycle functor ψ, so we need this sign for Proposition 8.6 below to hold.

Proposition 8.2. *There are canonical isomorphisms of étale sheaves*

$$\mathcal{H}om_\Lambda(\mathrm{fil}^a_{\mathrm{S}}\mathsf{C}, \mathrm{fil}^b_{\mathrm{S}}\mathsf{C}) \cong \begin{cases} 0 & \text{for } a < b \\ \Lambda_{X^{(a)}} & \text{for } a \geq b \end{cases}$$

and

$$\mathcal{E}xt^1_\Lambda(\mathrm{fil}^a_{\mathrm{S}}\mathsf{C}, \mathrm{fil}^b_{\mathrm{S}}\mathsf{C}) \cong \begin{cases} 0 & \text{for } a \leq b \\ \Lambda_{X^{(a)}}(-1)^{\oplus(a+1)} & \text{for } a > b. \end{cases}$$

One can deduce the Proposition 8.2 for example from the combinatorial description of perverse sheaves [Sai90, Theorem 3.3], which in the étale setting is not available in the literature to the best of our knowledge. We skip the argument.

8.2. **Rapoport-Zink sheaves.** We use the notation of 7.1, 8.1 and of A.3. We consider the corresponding derived category of Λ-sheaves with a continuous unipotent $\mathbb{Z}_\ell(1)$-action $D^{\mathrm{nil}}(X,\Lambda)$, see A.2. Any object F in $D^{\mathrm{nil}}(X,\Lambda)$ has a canonical Λ-linear nilpotent morphism $N^{\mathrm{Iw}}\colon \mathsf{F} \to \mathsf{F}(-1)^{\mathrm{Iw}}$. For simplicity of notation we also write N for N^{Iw}.

We define the category $\mathrm{RZ}^{\mathrm{pre}}(X,\Lambda)$ of *pre-Rapoport-Zink sheaves.* Its objects are pairs (RZ,ι), where $\mathsf{RZ} \in D^{\mathrm{nil}}(X,\Lambda)$ is a perverse sheaf and

$$\iota\colon \mathsf{C} \xrightarrow{\sim} \ker[\mathsf{RZ} \xrightarrow{N} \mathsf{RZ}(-1)^{\mathrm{Iw}}]$$

is an isomorphism. A morphism $\phi\colon (\mathsf{RZ},\iota_{\mathsf{RZ}}) \to (\mathsf{RZ}',\iota_{\mathsf{RZ}'})$ in $\mathrm{RZ}^{\mathrm{pre}}(X,\Lambda)$ is a morphism $\phi\colon \mathsf{RZ} \to \mathsf{RZ}'$ in $D^{\mathrm{nil}}(X,\Lambda)$ such that $\phi \circ \iota_{\mathsf{RZ}} = \iota_{\mathsf{RZ}'}$. Note that $\mathrm{RZ}^{\mathrm{pre}}(X,\Lambda)$ is a *groupoid* by Lemma A.2.

The category of *Rapoport-Zink sheaves* $\mathrm{RZ}(X,\Lambda)$ is now defined as a full subcategory of $\mathrm{RZ}^{\mathrm{pre}}(X,\Lambda)$.

Definition 8.3. We call $(\mathsf{RZ},\iota) \in \mathrm{RZ}^{\mathrm{pre}}(X,\Lambda)$ a *Rapoport-Zink sheaf* (*RZ-sheaf*) if

(i) [compatibility of filtrations] $\iota^{-1}(\mathrm{fil}^a\mathsf{RZ}) = \mathrm{fil}^a_{\mathrm{S}}\mathsf{C}$ for all $a \in \mathbb{Z}$;

(ii) [polarizability] there exists a perfect pairing

$$\mathsf{p}\colon \mathsf{RZ} \otimes_{\Lambda^{\mathrm{Iw}}} \mathsf{RZ}^- \to \varpi(-n)$$

in $D^{\mathrm{nil}}(X,\Lambda)$ such that the induced pairing

$$\mathrm{gr}^a_0\mathsf{RZ} \otimes \mathrm{gr}^0_a\mathsf{RZ}^- \to \varpi(-n)$$

coincides with the pairing

$$\mathrm{gr}_0^a\mathsf{RZ}\otimes\mathrm{gr}_a^0\mathsf{RZ}^- \xrightarrow{\mathrm{id}\otimes N^a} \mathrm{gr}_0^a\mathsf{RZ}\otimes\mathrm{gr}_0^a\mathsf{RZ}^-(-a)^{\mathrm{Iw}}$$
$$\xrightarrow[\sim]{\iota^{-1}\otimes\iota^{-1}} \mathrm{gr}_{\mathrm{S}}^a\mathsf{C}\otimes\mathrm{gr}_{\mathrm{S}}^a\mathsf{C}(-a) \xrightarrow{\mathsf{c}^a} \varpi(-n)$$

for all $a>0$.

Here we use the notation as in A.2 for Iwasawa module sheaves and $\varpi = f^!\Lambda$.

Remark 8.4. (i) For a RZ-sheaf (RZ,ι) we have canonical isomorphisms

$$\mathrm{gr}_b^a\mathsf{RZ} \xrightarrow[\sim]{N^b} \mathrm{gr}_0^{a+b}\mathsf{RZ}(-b) \xleftarrow[\sim]{\iota} \mathrm{gr}_{\mathrm{S}}^{a+b}\mathsf{C}(-b),$$
$$\mathrm{gr}_a^{\mathrm{M}}\mathsf{RZ} \cong \bigoplus_{p-q=a} \Lambda_X^{(p+q)}(-p)[n-p-q],$$
$$\mathrm{gr}_a\mathsf{RZ} \xrightarrow[\sim]{N^a} \mathrm{fil}^a\mathsf{RZ}\cap\mathrm{fil}_0\mathsf{RZ}(-1)^a \xleftarrow[\sim]{\iota} \mathrm{fil}_{\mathrm{S}}^a\mathsf{C}(-a).$$

(ii) For $a>0$ the local extension class of $\mathrm{gr}_a\mathsf{RZ}$ by $\mathrm{gr}_{a-1}\mathsf{RZ}$ is given as $(1,1,\dots,1)\in\Lambda_{X^{(a)}}^{\oplus(a+1)}$ via Proposition 8.2. One deduces this from the polarizability in Definition 8.3.

Proposition 8.5. *For Λ an algebraic extension of $\mathbb{Q}_\ell$ the following properties hold.*

(i) *For k a finite field a RZ-sheaf in $\mathrm{RZ}(X,\Lambda)$ is mw pure of weight n.*
(ii) *If additionally X is proper over k, then the groupoid $\mathrm{RZ}(X,\Lambda)$ is either empty or contractible. So in this case our axioms uniquely determine an RZ-sheaf if it exists.*

Recall that a groupoid (or to be precise its nerve) is contractible if it is non-empty and there is exactly one morphism between any two objects.

Proof. For part (i) observe that $\Lambda_X^{(a+b)}[n-a-b](-1)^b$ is pure of weight $n+b-a$. Then the statement follows from Remark 8.4 and Lemma 7.1.

For part (ii) let (RZ,ι) be in $\mathrm{RZ}(X,\Lambda)$. We first show that the only automorphism of this RZ-sheaf is the identity. By Remark 8.4 and Proposition 8.2 we obtain an isomorphism of sheaves

$$\mathcal{E}nd_{\mathrm{gr}\Lambda^{\mathrm{Iw}}}(\oplus_{a\in\mathbb{Z}}\ \mathrm{gr}_a\mathsf{RZ}) \cong \bigoplus_{a\geq 0}\Lambda_{X^{(a)}}(a).$$

Here we use the filtration of Λ^{Iw} by powers of the augmentation ideal. By [BBD83, Proposition 3.2.2] and weight reasons, we deduce from this an injection

$$\mathrm{End}_{\Lambda^{\mathrm{Iw}}}(\mathsf{RZ}) \to \mathrm{End}_{\mathrm{gr}\Lambda^{\mathrm{Iw}}}(\oplus_{a\in\mathbb{Z}}\mathrm{gr}_a\mathsf{RZ}) = H^0(X,\Lambda_{X^{(0)}}) = \Lambda^{\pi_0(X)}.$$

So the identity is the only endomorphism of RZ which restricts to the identity on the image of ι.

Now we show that up to isomorphism there is at most one RZ-sheaf (RZ, ι) on X. Note that the extension class of $\mathrm{gr}_a\mathsf{RZ}$ by $\mathrm{gr}_{a-1}\mathsf{RZ}$ is determined by Remark 8.4 for $a > 0$. So it suffices to observe that the the group in the middle of the exact sequence

$$H^1(X, Hom_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{gr}_{a-i}\mathsf{RZ})) \to \mathrm{Ext}^1_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{gr}_{a-i}\mathsf{RZ}) \to H^0(X, Ext^1_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{gr}_{a-i}\mathsf{RZ}))$$

vanishes for $a \geq i$, $i > 1$, because the groups on the left and on the right vanish. Indeed,

$$H^1(X, Hom_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{gr}_{a-i}\mathsf{RZ})) = H^1(X^{(a)}, \Lambda(i)) = 0$$
$$H^0(X, Ext^1_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{gr}_{a-i}\mathsf{RZ})) = H^0(X^{(a)}, \Lambda(i-1))^{\oplus a+1} = 0$$

by weight reasons, Remark 8.4 and Proposition 8.2.

In fact this shows that the extension class of $\mathrm{fil}_a\mathsf{RZ}$ in $\mathrm{Ext}^1_\Lambda(\mathrm{gr}_a\mathsf{RZ}, \mathrm{fil}_{a-1}\mathsf{RZ})$ is uniquely determined. So by induction on a this shows that the isomorphism class of $(\mathrm{fil}_a\mathsf{RZ}, \iota)$ in $\mathrm{RZ}^{\mathrm{pre}}(X, \Lambda)$ as an iterated extension is unique. □

Let $f\colon X \to Y$ be a proper k-morphism. For a RZ-sheaf (RZ, ι) we can write the monodromy spectral sequence from Proposition 7.6 explicitly as

$$E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} {}^{\mathrm{p}}R^{p+q} f_*(\Lambda_X^{(p+2i)}(-i)[n-p-2i]) \Rightarrow {}^{\mathrm{p}}R^{p+q} f_*(\mathsf{RZ}) \tag{8.1}$$

Note that for Λ an algebraic extension of $\mathbb{Q}_\ell$ and k finite, $E_1^{p,q}$ is a pure perverse sheaf of weight $n+q$ by [BBD83, Corollaire 5.4.2].

8.3. **Construction via nearby cycles.** Let X be a semi-stable scheme over $\mathcal{O}$ with generic fibre $j\colon X_K \to X$ and special fibre $i\colon X_k \to X$, $n = \dim(X_K)$. Note that ${}^{\mathrm{p}}\mathcal{H}^{-1}(i^*j_*\Lambda[n+1]) = \Lambda[n]$ is the constant perverse sheaf which we also denote C, see Remark A.3. Set $\mathsf{RZ} = \psi(\Lambda[n+1])$ and let $\iota\colon \mathsf{C} \to \mathsf{RZ}^N$ be the isomorphism induced by the fundamental exact triangle (A.5). Let $\mathsf{p}\colon \mathsf{RZ} \otimes_{\Lambda^{\mathrm{Iw}}} \mathsf{RZ}^- \to \varpi(-n)$ be the perfect pairing defined in A.6.

The following proposition is essentially shown in [Sai03], so we provide only a sketch of a proof. Another argument in the setting of $\mathcal{D}$-modules is given in [Sai90, Theorem 3.3].

Proposition 8.6. *The above* (RZ, ι) *together with the pairing* p *satisfy the properties of a Rapoport-Zink sheaf from Definition 8.3.*

Proof sketch. It is shown in [SGA7.1, Exposé I] that N acts trivially on $\mathcal{H}^a(\mathsf{RZ})$ for all $a \in \mathbb{Z}$. In [Sai03, Lemma 2.5] it is shown that the fundamental exact triangle (A.5) induces the horizontal quasi-isomorphisms in the commutative diagram for $a \geq -n$

$$\begin{array}{ccccccc}
\mathcal{H}^{a+1}(\mathsf{RZ})[-a-1] & \xrightarrow{\sim} & [0 & \longrightarrow & \mathcal{H}^{a+1}(i^*j_*\mathsf{C}(1)) & \xrightarrow{\{\pi\}} & \cdots] \\
\downarrow{\scriptstyle N} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathcal{H}^{a}(\mathsf{RZ})[-a](-1) & \xrightarrow{\sim} & [\mathcal{H}^a(i^*j_*\mathsf{C}) & \xrightarrow{\{\pi\}} & \mathcal{H}^{a+1}(i^*j_*\mathsf{C}(1)) & \xrightarrow{\{\pi\}} & \cdots]
\end{array}$$

Here $\{\pi\} \in H^1(\operatorname{Spec} K, \Lambda(1))$ is the tame symbol of π. Note that by the purity isomorphism

$$\mathcal{H}^{a-n}(i^* j_* \mathsf{C})(1) \cong \Lambda_X^{(a)}(-a) \tag{8.2}$$

the objects in the diagram are perverse sheaves and the right vertical map is a monomorphism. So by definition of the filtration fil_a we obtain that $\mathrm{fil}_a \mathsf{RZ} = \tau^{\leq a-n} \mathsf{RZ}$.

The isomorphisms

$$\mathrm{gr}_a^0 \mathsf{RZ} \xrightarrow{\sim} \mathcal{H}^{a-n}(i^* j_* \mathsf{C})(1)[n-a] \overset{(8.2)}{\cong} \Lambda_X^{(a)}(-a)[n-a]$$

are dual to the isomorphisms

$$\Lambda_X^{(a)}(n)[n-a] \overset{\kappa}{\cong} \mathrm{gr}^a_{\mathrm{S}} \mathsf{C}(n) \overset{\iota}{\xrightarrow{\sim}} \mathrm{gr}_0^a \mathsf{RZ}(n)$$

in view of (A.11) and the definition of the purity isomorphism.

Finally, the purity isomorphism (8.2) also induces the right vertical isomorphism of complexes in the commutative diagram in $D^b_c(X_k, \Lambda)$

$$\begin{array}{ccccccccc}
\mathcal{H}^{-n}(\mathsf{RZ})[n] & \xleftarrow[\iota]{\sim} & [& & \mathsf{Ko}_X[n] & & & &] \\
\| & & & & \downarrow \wr & & & & \\
\mathcal{H}^{-n}(\mathsf{RZ})[n] & \xrightarrow{\sim} & [0 & \longrightarrow & \mathcal{H}^{-n}(i^* j_* \mathsf{C}(1)) & \xrightarrow{\{\pi\}} & \mathcal{H}^{1-n}(i^* j_* \mathsf{C}(2)) & \xrightarrow{\{\pi\}} & \cdots]
\end{array}$$

This proves part (i) and (ii) of Definition 8.3 for (RZ, ι) and the polarization p. ☐

Proposition 8.7. *The nearby cycles $R\Psi_{X/\mathcal{O}}(\Lambda)$ are unipotent and in particular tame, i.e. the canonical map $\psi(\Lambda) \to R\Psi_{X/\mathcal{O}}(\Lambda)[-1]$ is an isomorphism.*

Tameness is shown in [RZ82, Satz 2.23], for a simple proof see [Ill04, Theorem 1.2]. Under the assumption of tameness and a semi-stable case of absolute purity [RZ82, Korollar 3.7], the unipotence was shown much earlier by Grothendieck, see [SGA7.1, Exposé I].

9. Cohomology of semi-stable Lefschetz pencils

In this section we establish a close connection between the monodromy-weight conjecture and properties of the cohomology of semi-stable Lefschetz pencils. We assume that k is a finite field.

9.1. **The monodromy-weight conjecture.** For X a projective semi-stable scheme over $\mathcal{O}$ we know by Proposition 8.7 that the inertia subgroup of $\mathrm{Gal}(\overline{K}/K)$ acts unipotently on

$$H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \cong H^i(X_{\overline{k}}, R\Psi_{X/\mathcal{O}} \mathbb{Q}_\ell)$$

for all $i \in \mathbb{Z}$, which means that it is given in terms of a nilpotent operator

$$N \colon H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i(X_{\overline{K}}, \mathbb{Q}_\ell(-1)),$$

see [SGA7.1, Exposé I] and A.2. The operator N induces the monodromy filtration $\mathrm{fil}^{\mathrm{M}} H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ as in (7.1).

We say that $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is *mw pure of weight i* if $\mathrm{gr}^{\mathrm{M}}_a H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is pure of weight $a+i$ as a $\mathrm{Gal}(\overline{k}/k)$-module. It is equivalent to request that its associated perverse sheaf in $D^{\mathrm{nil}}(\mathrm{Spec}\, k, \mathbb{Q}_\ell)$ is mw pure of weight i in the sense of Definition 7.4.

Consider the following property depending on an integer $n \geq 0$.

$(\mathbf{mw})_n$ For all projective, semi-stable X over $\mathcal{O}$ with $\dim(X_K) \leq n$ the cohomology groups $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ are mw pure of weight i for all $i \in \mathbb{Z}$.

Remark 9.1.
- Deligne conjectures that $(\mathbf{mw})_n$ holds for all n [Del70, Section 8.5] (so called *monodromy-weight conjecture* in the semi-stable case). In fact the general monodromy-weight conjecture follows from the semi-stable case by de Jong's alteration theorem. For a general exposition, see [Ill94, Section 3].
- Grothendieck's degeneration theory of abelian varieties essentially implies $(\mathbf{mw})_1$, a simplified argument was given by Deligne [SGA7.1, Exposé I, Section 6, Appendice].
- Using their spectral sequence (8.1) Rapoport–Zink showed $(\mathbf{mw})_2$ [RZ82, Satz 2.13].
- In equal positive characteristic the conjecture is a theorem which is essentially due to Deligne, see [Del80, Théorème (1.8.4)].
- The monodromy-weight conjecture was shown by Scholze for set theoretic complete intersections in $\mathbb{P}^N_K$ or more generally in a projective toric variety in [Sch12, Theorem 9.6]. See [Sch12, after Conjecture 1.13] for a summary of further results for special varieties.

9.2. **A Lefschetz pencil approach.** Set $\Lambda = \overline{\mathbb{Q}}_\ell$. For schemes of finite type over $\mathcal{O}$ we use the perversity as in A.4. In particular for X regular and flat over $\mathcal{O}$ with $\dim(X_K) = n$ the sheaf $\mathsf{C} = \Lambda_X[n+1]$ is perverse. Recall that the unipotent nearby cycle functor $\psi\colon D^b_c(X_K, \Lambda) \to D^{\mathrm{nil}}(X_k, \Lambda)$ maps perverse sheaves to perverse sheaves, A.3.

Let $\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}}$ be a semi-stable Lefschetz pencil as in Definition 5.3. In this section we study the cohomological degeneration over $\mathcal{O}$ of the perverse Picard-Lefschetz sheaf $\mathsf{L}_K = {}^{\mathrm{p}}R^0\phi_{K,*}\mathsf{C} \in D^b_c(\mathbb{P}^1_K, \Lambda)$. In Section 10 we study the tameness of L_K. Let $\mathsf{L}_k := \psi(\mathsf{L}_K)$ be the unipotent nearby cycles of L_K, see A.3.

Lemma 9.2. *The perverse sheaf L_K is unipotent along $\mathbb{P}^1_k$, i.e.*

$$\psi(\mathsf{L}_K) \xrightarrow{\sim} R\Psi_{\mathbb{P}^1_{\mathcal{O}}/\mathcal{O}}(\mathsf{L}_K)[-1]$$

is an isomorphism.

Proof. Consider the RZ-sheaf (RZ, ι) on $\tilde{X}_k$ as in 8.3 with $\mathsf{RZ} = \psi(\mathsf{C}) \xrightarrow{\sim} R\Psi_{\tilde{X}/\mathcal{O}}(\mathsf{C})[-1]$, where the last isomorphism is Proposition 8.7. So by proper base change

$$R\Psi_{\mathbb{P}^1_{\mathcal{O}}/\mathcal{O}}(\mathsf{L}_K)[-1] \cong R\phi_{k,*}\mathsf{RZ} \tag{9.1}$$

has a unipotent inertia action. □

In [Del74] Deligne uses Lefschetz theory to prove the Weil conjectures over a finite field, so it is natural to try to use the following proposition to proceed analogously over the local field K.

Proposition 9.3. *Assume that n is fixed and that $(\mathbf{mw})_{n-1}$ holds. Let X be a projective, semi-stable scheme over $\mathcal{O}$ with $\dim(X_K) = n$. Then $H^a(X_{\overline{K}}, \Lambda)$ is mw pure of weight a for all $a \in \mathbb{Z}$ if there exists a semi-stable Lefschetz pencil $\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}}$ such that*

$$H^{-1}(\mathbb{P}^1_{\overline{K}}, \mathsf{L}_K) \cong H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L}_k) \tag{9.2}$$

is mw pure of weight n for the Picard-Lefschetz sheaves $\mathsf{L}_K = {}^{\mathrm{p}}R^0\phi_{K,*}\mathsf{C}$ on $\mathbb{P}^1_K$ and $\mathsf{L}_k = \psi(\mathsf{L}_K)$ on $\mathbb{P}^1_k$.

Proof. Recall that a direct summand of a an mw pure structure is again mw pure, Proposition 7.5. The isomorphism (9.2) follows from proper base change and Lemma 9.2. Fix a semi-stable Lefschetz pencil as in the proposition with associated Lefschetz operator $L\colon H^i(X_{\overline{K}}, \mathsf{C}) \to H^{i+2}(X_{\overline{K}}, \mathsf{C}(1))$. Let $Y_K \hookrightarrow X_K$ be a smooth fibre of the pencil over K. By the Hard and Weak Lefschetz theorems the restriction map

$$H^i(X_{\overline{K}}, \mathsf{C}) \to H^i(Y_{\overline{K}}, \mathsf{C})$$

is an isomorphism for $i < -2$ and a split injection for $i = -2$ with inverse the composition of

$$H^{-2}(Y_{\overline{K}}, \mathsf{C}) \to H^0(X_{\overline{K}}, \mathsf{C}(1)) \overset{L}{\xleftarrow{\sim}} H^{-2}(X_{\overline{K}}, \mathsf{C}).$$

So we conclude by the Lefschetz decomposition and by Proposition 6.4 (C has a different shift there). □

9.3. **Main cohomological theorem.** By Proposition 8.5 the Rapoport-Zink sheaf RZ as in the proof of Lemma 9.2 is mw pure of weight n, so in order to check whether (9.2) is mw pure of weight n it would suffice to answer positively the following two questions in view of Proposition 7.7. Recall that the monodromy property is defined in Definition 7.2.

(I) Is L_k mw pure of weight n?
(II) Does the monodromy property hold for $\mathsf{L}_{\overline{k}}$ and $\mathbb{P}^1_{\overline{k}} \to \operatorname{Spec} \overline{k}$?

In Theorem 9.4 we give a positive answer to (I) assuming the monodromy-weight conjecture is known in smaller dimensions. Question (II) can be seen as an arithmetic variant of a conjecture of Kashiwara [Kas98], see 9.4, about the topology of complex varieties. The complex version has been proved by T. Mochizuki [Moc07]. We defer the study of the arithmetic variant to a forthcoming work.

Theorem 9.4. *Assume* $(\mathbf{mw})_{n-1}$ *holds. Then the following properties are satisfied.*

(i) *The sheaf* RZ *satisfies the monodromy property for* ϕ_k *in the sense of Definition 7.2. In particular* L_k *as in Proposition 9.3 is mw pure of weight* n *and the monodromy graded pieces* $\mathrm{gr}^{\mathrm{M}}_a \mathsf{L}_{\overline{k}}$ *are semi-simple for all* $a \in \mathbb{Z}$.

(ii) *The non-constant part of* $\mathrm{gr}^{\mathrm{M}}_a \mathsf{L}_{\overline{k}}$ *satisfies multiplicity one in the sense of 6.1 for all* $a \in \mathbb{Z}$.

Corollary 9.5. *Assume* $(\mathbf{mw})_{n-1}$ *holds. In order to show* $(\mathbf{mw})_n$ *it would "suffice" to show the monodromy property for* $\mathsf{L}_{\overline{k}}$ *and the morphism* $\mathbb{P}^1_{\overline{k}} \to \operatorname{Spec} \overline{k}$.

Proof of Corollary 9.5. By Theorem 9.4 the perverse sheaf L_k is mw pure of weight n. Then by Proposition 7.7 the monodromy property for $\mathsf{L}_{\overline{k}}$ and for the morphism $\mathbb{P}^1_{\overline{k}} \to \operatorname{Spec} \overline{k}$ implies that $H^0(\mathbb{P}^1_{\overline{k}}, \mathsf{L}_{\overline{k}})$ is mw pure of weight n. We finish the proof using Proposition 9.3. □

Proof of Theorem 9.4. We check the mw property for RZ and ϕ_k via the criterion of Proposition 7.7(iii) for $f = \phi_k$ and $\mathsf{F} = \mathsf{RZ}$. By the characterizing property of the monodromy filtration, Lemma 7.1(ii),

$$N^a \colon E_1^{-a,i+a} \xrightarrow{\sim} E_1^{a,i-a}(-a) \tag{9.3}$$

is an isomorphism for $a > 0$ in view of the definition of the spectral sequence. By Remark 8.4 and by Theorem 6.1 $E_1^{p,q}$ is geometrically constant for $p + q \neq 0$. So the non-constant part of the d_1-differential vanishes, which means

$$E_2^{p,q} = (E_1^{p,q})^{\mathrm{nc}} \oplus H\left((E_1^{p-1,q})^{\mathrm{c}} \xrightarrow{d_1} (E_1^{p,q})^{\mathrm{c}} \xrightarrow{d_1} (E_1^{p+1,q})^{\mathrm{c}} \right). \tag{9.4}$$

Recall that the upper indices 'c' and 'nc' stand for the geometrically constant part and the geometrically non-constant part as in 6.1.

On the first summand on the right of (9.4) the map N^a is an isomorphism in the bidegrees as in (9.3) as observed above. The second summand is geometrically constant, so we only need to check that the map of perverse sheaves in $D^{\mathrm{nil}}(x, \Lambda)$

$$N^a \colon E_2^{-a,i+a}|_x[-1] \to E_2^{a,i-a}(-a)^{\mathrm{Iw}}|_x[-1] \tag{9.5}$$

is an isomorphism for $a > 0$, where $x \in \mathbb{P}^1_k$ is a closed non-critical value, which we will assume to be k-rational after possibly replacing k by a finite extension.

Lift x arbitrarily to an $\mathcal{O}$-point $s \colon \operatorname{Spec} \mathcal{O} \to \mathbb{P}^1_{\mathcal{O}}$. By Lemma 3.6

$$Y := \phi^{-1}(s) \hookrightarrow \tilde{X}$$

is a stratified regular immersion of semi-stable schemes, see 3.2, where $\tilde{X}$ and Y have the standard stratification as in Section 4. Then by Proposition 3.5 we get the base change isomorphism of perverse sheaves

$$\mathsf{RZ}|_{Y_k}[-1] \cong \psi(\mathsf{C}|_{Y_K}[-1])$$

Note that

$$(\mathrm{fil}_a^{\mathrm{M}}\mathsf{RZ})|_{Y_k}[-1] = \mathrm{fil}_a^{\mathrm{M}}(\mathsf{RZ}|_{Y_k}[-1]),$$

because no irreducible perverse constituent of RZ is supported on Y_k. In particular the spectral sequence in (9.5) is nothing but the monodromy spectral sequence for $H^*(Y_{\overline{K}}, \Lambda)$ with respect to the model Y. Finally, by our assumption $(\mathbf{mw})_{n-1}$ we get the isomorphy of (9.5). This proves part (i).

Part (ii) is then clear as the critical values of ϕ_k coming from different strata Z of $\tilde{X}_k$ are disjoint, but these critical values are just the non-smooth loci of ${}^{\mathrm{p}}R^0(\phi_k|_{\overline{Z}})_*\Lambda[\dim Z]$. As the non-constant parts of these perverse sheaves satisfy multiplicity one by Theorem 6.1 individually, so does their direct sum. □

Example 9.6. If case (B) in Theorem 6.1 holds for the Lefschetz pencil $\phi_{\overline{K}}\colon X_{\overline{K}} \to \mathbb{P}^1_{\overline{K}}$, the monodromy property for $\mathsf{L}_{\overline{k}}$ is clear, because this perverse sheaf is then a direct sum of a constant perverse sheaf and of skyscraper sheaves.

9.4. **Arithmetic Kashiwara conjecture.** Motivated by the Kashiwara conjecture in complex geometry [Kas98] we suggest that the following arithmetic Kashiwara conjecture holds.

Let $\mathcal{O}$ be a strictly henselian discrete valuation ring. We do not have to assume that $K = \mathrm{frac}(\mathcal{O})$ has characteristic zero or that the residue field k is perfect in this subsection. Let X be a proper scheme over $\mathcal{O}$.

Conjecture 9.7. *For a perverse sheaf $\mathsf{F}_K \in D^b_c(X_K, \overline{\mathbb{Q}}_\ell)$ such that its base change $\mathsf{F}_{\overline{K}}$ to a separable closure $\overline{K}$ of K is semi-simple and arithmetic (in the sense of* [EK20, Definition 1.4]*) the following holds for the perverse sheaf $\mathsf{F}_k = \psi(\mathsf{F}_K)$*

(i) $\mathrm{gr}_a^{\mathrm{M}}\mathsf{F}_k$ *is semi-simple for all $a \in \mathbb{Z}$,*
(ii) F_k *satisfies the monodromy property with respect to the morphism $X_k \to \mathrm{Spec}\, k$, see Definition 7.2.*

The condition of being arithmetic in the conjecture cannot be omitted by a counterexample of T. Mochizuki. In equal characteristic one can establish many cases of the conjecture by a spreading argument and the techniques mentioned in Remark 7.3. We defer the study to a forthcoming work.

As a corollary to the results in 9.3 we get

Corollary 9.8. *The arithmetic Kashiwara conjecture for $X = \mathbb{P}^1_{\mathcal{O}}$ and for mixed characteristic $\mathcal{O}$ would imply the monodromy-weight conjecture.*

10. Tameness of the cohomology of semi-stable Lefschetz pencils

10.1. **Reminder on log structures.** For a background on log geometry see [Ogu18]. In this subsection we do not assume that the residue field of $\mathcal{O}$ is perfect. Let X be a scheme flat and of finite type over $\mathcal{O}$. Let $j\colon X_K \to X$ be the open immersion of the generic fibre. Let $M \to \mathcal{O}_X$ be a log structure. We always endow $\operatorname{Spec}\mathcal{O}$ with the *canonical log structure* defined by $\mathbb{N} \to \mathcal{O}$, $1 \mapsto \pi$. If the fine and saturated (fs) log scheme (X, M) is log smooth over $\mathcal{O}$, then it is log regular, in particular X is normal. If additionally the trivial locus of the log structure is X_K then $M = \mathcal{O}_X \cap j_*\mathcal{O}_{X_K}^\times$, so the log structure M is not an extra datum. If no explicit log structure is given on X we say that it is *log smooth* over $\mathcal{O}$ if it is fs and log smooth over $\mathcal{O}$ with respect to the log structure $M = \mathcal{O}_X \cap j_*\mathcal{O}_{X_K}^\times$. By abuse of notation we also call a scheme log smooth if it is pro étale over a log smooth scheme over $\mathcal{O}$.

Our basic source of log smooth schemes over $\mathcal{O}$ stems from the following lemma, which is a consequence of [Ogu18, Theorem III.2.5.5, Example IV.3.1.17].

Lemma 10.1. *The morphism of fs log schemes* $\operatorname{Spec}\mathbb{Z}[\mathbb{N}^m] \to \operatorname{Spec}\mathbb{Z}[\mathbb{N}]$ *induced by the monoid homomorphism* $1 \mapsto (1, \dots, 1)$ *is log smooth and saturated.*

Recall that for a saturated morphism of fs log schemes the base change in the category of log schemes and in the category of fs log schemes coincide [Ogu18, Proposition 2.5.3].

Example 10.2. Let $X = \operatorname{Spec}\mathcal{O}[X_1, \dots, X_n]/(X_1 \cdots X_m - \pi^e)$ with $1 \le m \le n$ and $e > 0$. Then X is log smooth over $\mathcal{O}$. This follows from Lemma 10.1 since X is smooth and strict over the log smooth schemes $\operatorname{Spec}(\mathbb{Z}[\mathbb{N}^m] \otimes_{\mathbb{Z}[\mathbb{N}]} \mathcal{O})$, where we use from left to right the monoid homomorphisms $\mathbb{N} \to \mathbb{N}^m$, $1 \mapsto (1, \dots, 1)$, $\mathbb{N} \to \mathcal{O}$, $1 \mapsto \pi^e$.

Example 10.3. Let

$$X = \operatorname{Spec}\mathcal{O}[X_1, \dots, X_n, T_1, T_2]/(T_1T_2 - \pi, X_1 \cdots X_m - T_1^d T_2^e)$$

with $1 \le m \le n$ and $e, d > 0$. Then X is log smooth over $\mathcal{O}$. This follows from Lemma 10.1 since X is smooth and strict over the log smooth scheme

$$\operatorname{Spec}(\mathbb{Z}[\mathbb{N}^m] \otimes_{\mathbb{Z}[\mathbb{N}]} (\mathbb{Z}[\mathbb{N}^2] \otimes_{\mathbb{Z}[\mathbb{N}]} \mathcal{O})).$$

where we use from left to right the monoid homomorphisms $\mathbb{N} \to \mathbb{N}^m$, $1 \mapsto (1, \dots, 1)$, $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (d, e)$, $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1, 1)$ and $\mathbb{N} \to \mathcal{O}$, $1 \mapsto \pi$.

10.2. **Reminder on tame coverings.** In this subsection we do not assume that the residue field of $\mathcal{O}$ is perfect. Tame étale coverings have been studied systematically for the first time in [SGA1, Exposé XIII] in the form of the so called *Grothendieck-Murre tameness*. The notion of *curve tameness* was first studied by [Wie08] and later in [KS10].

Consider an excellent discrete valuation ring A, $K = \mathrm{frac}(A)$, and a finite field extension $K \subset L$. Let $B \subset L$ be the integral closure of A. Then L is a *tame extension* of the discretely valued field K if the residue field extensions of $A \subset B$ are separable and the ramification indices at all maximal ideals of B are coprime to the residue characteristic.

Let now X be a regular, separated scheme of finite type over $K = \mathrm{frac}(\mathcal{O})$. Consider a proper normal scheme $\overline{X}$ over $\mathcal{O}$ which contains X as an open subscheme. Consider an étale covering $f\colon X' \to X$. For a point $x \in \overline{X}$ of codimension one we call $X' \to X$ tame over $\overline{\{x\}}$ if the normalization $\overline{X}'$ of $\overline{X}$ in X' is tame in the above sense with respect to the discrete valuation defined by x.

We say that $X' \to \overline{X}$ is *curve tame* (or for short just *tame*) if for any closed point $x \in X$ the finite morphism $(X' \times_X x) \to x$ is tame with respect to the unique extension of the discrete valuation from K to $k(x)$. The following lemmas are a consequence of [KS10, Theorem. 4.4].

Lemma 10.4. *If $\mathcal{O} \subset \mathcal{O}'$ is a finite extension of henselian discrete valuation rings and $U \subset X$ is a dense open subscheme then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) with*

(i) *$X' \to X$ is curve tame;*
(ii) *$U' = X' \times_X U \to U$ is curve tame;*
(iii) *the base change $X'_{\mathcal{O}'} \to X_{\mathcal{O}'}$ is curve tame.*

If moreover $\mathcal{O}'$ is a tame extension of $\mathcal{O}$ then all conditions are equivalent.

Assume moreover that $\overline{X} \setminus X$ is a simple normal crossing divisor. We say that $X' \to X$ is *Grothendieck-Murre tame* if is is tame over all irreducible components of $\overline{X} \setminus X$.

Lemma 10.5. *Under the above assumptions an étale covering $X' \to X$ is curve tame if and only if it is Grothendieck-Murre tame.*

A constructible sheaf F on X is called tame if for any closed point $x \in X$ the $\mathrm{Gal}(\overline{x}/x)$-representation $\mathsf{F}_{\overline{x}}$ is tame.

10.3. **Tameness of Picard-Lefschetz sheaves.** Consider a semi-stable Lefschetz pencil of X and let $\phi\colon \tilde{X} \to \mathbb{P}^1_{\mathcal{O}}$ be the pencil map. Let $U \subset X_K$ be the set of regular values over K. Consider the constructible sheaves $\mathsf{L}^i = R^i\phi_{K,*}\Lambda$ on $\mathbb{P}^1_K$ for $i \in \mathbb{Z}$ and set $\mathsf{L} = \oplus_i \mathsf{L}^i$.

Note that L is tame over the prime divisor $\mathbb{P}^1_k$ of $\mathbb{P}^1_{\mathcal{O}}$ by Proposition 8.7, since X has semi-stable reduction over the maximal point of $\mathbb{P}^1_k$. We do not know whether L is tame over $\mathcal{O}$. The main result of this section is the potential tameness of L. In order to formulate it let K'/K be a finite splitting field of all the field extensions $k(x)/K$, for $x \in \mathbb{P}^1_K \setminus U$ and let $\mathcal{O}' \subset K'$ be its ring of integers. Recall that we can bound K' by Theorem 4.2. In the following $p = \mathrm{ch}(k)$.

Theorem 10.6. *The sheaf $\mathsf{L}_{K'}$ on $\mathbb{P}^1_{K'}$ is tame if $p \neq 2$.*

From Theorem 10.6 and Lemma 10.4 we deduce the following corollary. In fact we can choose K'/K to be a tame extension under the assumption $p > n+1$.

Corollary 10.7. *For $p > n+1$ or $p = 0$ the sheaf L is tame.*

Proof of Theorem 10.6. We will assume without loss of generality that $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ and that $\mathcal{O}$ is strictly henselian. The major part of the proof consists in showing the following claim.

Claim 10.8. $\mathsf{L}_{U_{K'}}$ is a tame local system.

Assuming Claim 10.8 let us prove Theorem 10.6. The sheaf L^i is unramified over $\mathbb{P}^1_K$ for $i \notin \{n-1, n\}$ by Theorem 6.1, so Lemma 10.4 and Claim 10.8 imply that L^i is tame in this case. With the notation of 6.1 consider a critical value $x \in \mathbb{P}^1_K$. We have the exact sequence

$$0 \to H^{n-1}(\tilde{X}_{\overline{x}}) \to H^{n-1}(\tilde{X}_{\overline{\eta}_x}) \to V_{\overline{x}} \to H^n(\tilde{X}_{\overline{x}}) \to H^n(\tilde{X}_{\overline{\eta}_x}) \to 0 \tag{10.1}$$

where we omit the coefficients Λ for simplicity. We consider the action of $G_x = \mathrm{Gal}(\overline{\eta}_x/\eta_x)$ on this sequence. The order of $\mathrm{im}(G_x) \subset \mathrm{Aut}(H^*(\tilde{X}_{\overline{\eta}_x}))$ is coprime to p by Lemma 10.5, Claim 10.8 and the Abhyankar lemma [SGA7.1, Exposé XIII, Proposition 5.2] applied to a semi-stable model as in the proof of Claim 10.8 below. The order of $\mathrm{im}(G_x) \subset \mathrm{Aut}(V_{\overline{x}})$ is coprime to p by Proposition 6.3.

So from the exact sequence (10.1) we deduce that the order of $\mathrm{im}(G_x) \subset \mathrm{Aut}(H^i(\tilde{X}_{\overline{x}}))$ is coprime to p for $i \in \{n-1, n\}$. This means that the action of $\mathrm{Gal}(\overline{x}/x)$ on $\mathsf{L}^i_{\overline{x}} = H^i(\tilde{X}_{\overline{x}})$ is tame. □

Proof of Claim 10.8. By replacing $\mathcal{O}'$ by a further tame extension we can in the following assume that the ramification index e of $\mathcal{O}'/\mathcal{O}$ satisfies $e > n+1$, see Lemma 10.4. Let $\pi' \in \mathcal{O}'$ be a uniformizer. Let $S \hookrightarrow \mathbb{P}^1_{\mathcal{O}'}$ be the set of critical values. We construct a semi-stable model of $\mathbb{P}^1_{K'}$ and check Grothendieck-Murre tameness over this model, which is sufficient by Lemma 10.5.

By our assumption on $\mathcal{O}'$ we deduce that S is the union of the images of finitely many sections of $\mathbb{P}^1_{\mathcal{O}'} \to \operatorname{Spec} \mathcal{O}'$. There is an iterated blow-up $\theta\colon Z \to \mathbb{P}^1_{\mathcal{O}'}$ of closed points in the smooth locus over $\mathcal{O}'$ such that the strict transform $S^{\mathrm{st}} \hookrightarrow Z$ of S is a finite disjoint union of sections of $Z \to \operatorname{Spec} \mathcal{O}'$; this is a special case of Néron desingularization, see [Art69, Corollary 4.6]. As in the process we only blow up smooth closed points the scheme Z is semi-stable over $\mathcal{O}'$.

Let z be a maximal point of Z_k and consider the henselian discrete valuation ring $R = \mathcal{O}^{\mathrm{h}}_{Z,z}$. Then $\theta_z\colon \mathcal{O}^{\mathrm{h}}_{\mathbb{P}^1_{\mathcal{O}'},\theta(z)} \to R$ sends both π' and T to uniformizers, where $\pi', T \in \mathcal{O}^{\mathrm{h}}_{\mathbb{P}^1_{\mathcal{O}'},\theta(z)}$ is a regular parameter system. Say $T = -v\pi'$ with $v \in R^\times$.

Let $x \in X_k$ be a non-critical point of ϕ_k and denote by abuse of notation its preimage in X_R by the same symbol. Then by Proposition 4.3 we obtain an isomorphism of R-algebras

$$\mathcal{O}^{\mathrm{h}}_{X_R,x} \cong R[X_0,\dots,X_n]^{\mathrm{h}}/(X_0\cdots X_m - u\pi, X_{m+1}+v\pi')$$

which after a coordinate transformation can be rewritten as

$$\mathcal{O}^{\mathrm{h}}_{X_R,x} \cong R[X_0,\dots,X_{n-1}]^{\mathrm{h}}/(X_0\cdots X_m - \pi'^e).$$

Here the henselization $^{\mathrm{h}}$ is with respect to the maximal ideal generated by π' and the X_i.

In particular X_R is log smooth over R around x by Example 10.2. By a similar calculation one sees that at a critical point x, the scheme X_R is at least nearly semi-stable over R as defined in 10.4 below. By Proposition 10.9 applied over R we see that $\mathsf{L}_{K'}$ is tame over the divisor $\overline{\{z\}}$ for all $i \in \mathbb{Z}$. So $\mathsf{L}_{K'}$ is Grothendieck-Murre tame over $\mathcal{O}'$. □

10.4. **Nearly semi-stable reduction.** In this subsection we do not assume that the residue field k of $\mathcal{O}$ is perfect, but we assume that k is separably closed for simplicity. A scheme X which is flat and of finite type over $\mathcal{O}$ is said to have *nearly semi-stable reduction* at $x \in X$ if either it is log smooth at x over $\mathcal{O}$ with the standard log structure $1 \mapsto \pi$, a uniformizer of $\mathcal{O}$, or if there is an isomorphism of $\mathcal{O}$-algebras $\mathcal{O}^{\mathrm{h}}_{X,x} \cong A$. Here A is of the following form: consider an $\mathcal{O}$-algebra

$$B = \mathcal{O}[X_0,\dots,X_m]^{\mathrm{h}}/(X_0\cdots X_m - u\pi^e)$$

where $m \geq 0$, $u \in (\mathcal{O}[X_0,\dots,X_m]^{\mathrm{h}})^\times$ and where $e > m+1$. Here the henselization $^{\mathrm{h}}$ is with respect to the ideal $(\pi, X_0,\dots,X_m)$. Set $\alpha = X_0 + \ldots + X_m + \pi \in B$. Then A is of the form

$$A = B[X_{m+1},\dots,X_n]^{\mathrm{h}}/(X_{m+1}^2 + \ldots + X_n^2 - \alpha)$$

for some $n \geq m$.

Proposition 10.9. *Assume that X is nearly semi-stable at all points of its closed fibre. Then the following properties are satisfied.*

(i) *If X is proper over $\mathcal{O}$ then the action of* $\mathrm{Gal}(\bar{K}/K)$ *on $H^*(X_{\bar{K}},\Lambda)$ is tame.*
(ii) *The action of of* $\mathrm{Gal}(\bar{K}/K)$ *on $R\Psi_{X/\mathcal{O}}(\Lambda)$ is tame.*

Proof. Without loss of generality $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. Part (i) follows from part (ii) and proper base change. If X is log smooth around $x \in X_k$ then part (ii) follows from [Nak98, Theorem 0.1]. So for part (ii) we have to consider a closed point $x \in X_k$ with $\mathcal{O}^{\mathrm{h}}_{X,x} \cong A$. Write $g\colon X_x \to Y_y$ for $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B)$, $y = g(x)$. Observe that the critical values of g are given by $Y^{\mathrm{crit}} = V(\alpha)$. Set $\tilde{Y} = \mathrm{Bl}_{\{y\}}(Y_y) \xrightarrow{\sigma} Y_y$ and $\tilde{X} = X_x \times_{Y_y} \tilde{Y} \xrightarrow{\sigma} X_x$. Let $\tilde{x} \in \tilde{X}$ be a closed point over x with image $\tilde{y} \in \tilde{Y}$.

Consider the morphisms of henselian local schemes $\tilde{g}_{\tilde{x}}\colon \tilde{X}_{\tilde{x}} \to \tilde{Y}_{\tilde{y}}$ and $\tilde{h}_{\tilde{y}}\colon \tilde{Y}_{\tilde{y}} \to \mathrm{Spec}\,\mathcal{O}$. We consider the object

$$\mathsf{F} = \mathrm{cone}[\Lambda_{\tilde{Y}_{\tilde{y}}} \to R\tilde{g}_{\tilde{x},*}\Lambda_{\tilde{X}_{\tilde{x}}}]$$

of $D^b_c(\tilde{Y}_{\tilde{y}}, \Lambda)$. We need to show that $R\Psi_{\tilde{X}/\mathcal{O}}(\Lambda)_{\tilde{x}} = [R\tilde{h}_{\tilde{y},*}R\tilde{g}_{\tilde{x},*}\Lambda_{\tilde{X}_{\tilde{x}}}]_{\bar{K}}$ is tame, since then by proper base change

$$R\sigma_* R\Psi_{\tilde{X}/\mathcal{O}}(\Lambda)_{\tilde{x}} \cong R\Psi_{X/\mathcal{O}}(\Lambda)_x$$

is also tame. By [Nak98, Theorem 0.1] and Lemma 10.11 we obtain that

$$R\Psi_{\tilde{Y}/\mathcal{O}}(\Lambda)_{\tilde{y}} = R\tilde{h}_{\tilde{y},*}(\Lambda)_{\bar{K}}$$

is tame. So it remains to show that $R\Psi_{\tilde{Y}/\mathcal{O}}(\mathsf{F})_{\tilde{y}} = R\tilde{h}_{\tilde{y},*}(\mathsf{F})_{\bar{K}}$ is tame.

Lemma 10.10. *$\mathcal{H}^i(\mathsf{F})$ vanishes for $i \neq n-m-1$ and $\mathcal{H}^{n-m-1}(\mathsf{F}) = j_!\mathsf{L}$, where $j\colon \tilde{Y}_{\tilde{y}} \setminus \sigma^{-1}Y^{\mathrm{crit}} \to \tilde{Y}_{\tilde{y}}$. Here L is a Λ-rank one local system of order 1 or 2. In the latter case L becomes trivial after taking a square root of $\sigma^{-1}(\alpha)$.*

Proof. Combine [SGA7.2, Exposé XV, Section 2.2] and [Org06, Proposition 4.1]. □

We can locally factor $\sigma^{-1}(\alpha)$ in the ring $\mathcal{O}_{\tilde{Y},\tilde{y}}$ as $\alpha^{\mathrm{st}}\beta$, where the vanishing locus of β is contained in the exceptional divisor of the blow-up, i.e. β is invertible over K while α^{st} does not vanish on the the exceptional divisor.

Lemma 10.11. *Étale locally on the exceptional divisor, $\tilde{Y}$ looks like Example 10.2 or Example 10.3 with α^{st} invertible or corresponding to the variable X_{m+1}. In particular, $\tilde{Y}_{\tilde{y}}$ and its closed subscheme $V(\alpha^{\mathrm{st}})$ are log smooth over $\mathcal{O}$ and the immersion is strict.*

Proof. It suffices to consider without loss of generality the following two blow-up charts of $\tilde{Y}$ over Y. For simplicity of notation we assume that k is algebraically closed.

1st chart.
$\tilde{Y}$ contains as an open subscheme the spectrum of the ring

$$B[\tilde{X}_0, \ldots, \tilde{X}_m]/(\tilde{X}_i\pi - X_i, \tilde{X}_0 \cdots \tilde{X}_m - u\pi^{e-m-1})$$

where $1 \leq i \leq m$. On this chart $\alpha^{\mathrm{st}} = \tilde{X}_0 + \ldots + \tilde{X}_m + 1$. Consider a closed point $\tilde{y}$ of the exceptional divisor in this chart which satisfies without loss of generality $\tilde{X}_0(\tilde{y}) = \cdots = \tilde{X}_{\tilde{m}}(\tilde{y}) = 0$ and $\tilde{X}_{\tilde{m}+1}(\tilde{y}), \ldots, \tilde{X}_m(\tilde{y}) \neq 0$. We obtain

$$\mathcal{O}^{\mathrm{h}}_{\tilde{Y},\tilde{y}} \cong \mathcal{O}[X_0, \ldots, X_m]^{\mathrm{h}}/(X_0 \cdots X_{\tilde{m}} - \pi^{e-m-1})$$

in which α^{st} is a unit or $\alpha^{\mathrm{st}} = X_{\tilde{m}+1}$. In this case $\mathcal{O}^{\mathrm{h}}_{\tilde{Y},\tilde{y}}$ is as in Example 10.2.

2nd chart.
$\tilde{Y}$ contains as an open subscheme the spectrum of the ring

$$B[\tilde{X}_1, \ldots, \tilde{X}_m, \tilde{\pi}]/(\tilde{X}_iX_0 - X_i, \tilde{\pi}X_0 - \pi, \tilde{X}_1 \cdots \tilde{X}_m - u\tilde{\pi}^{m+1}\pi^{e-m-1})$$

where $1 \leq i \leq m$. On this chart $\alpha^{\mathrm{st}} = 1 + \tilde{X}_1 + \ldots + \tilde{X}_m + \tilde{\pi}$. Consider a closed point $\tilde{y}$ of the exceptional divisor in this chart. We can assume without loss of generality

$\tilde{\pi}(\tilde{y}) = 0$ as other points are already in the 1st chart above. Say for simplicity that $\tilde{X}_1(\tilde{y}) = \cdots = \tilde{X}_{\tilde{m}}(\tilde{y}) = 0$ and $\tilde{X}_{\tilde{m}+1}(\tilde{y}), \ldots, \tilde{X}_m(\tilde{y}) \neq 0$. Then we get

$$\mathcal{O}^{\mathrm{h}}_{\tilde{Y},\tilde{y}} \cong \mathcal{O}[X_0, \ldots, X_m, \tilde{\pi}]^{\mathrm{h}}/(\tilde{\pi}X_0 - \pi, X_1 \cdots X_{\tilde{m}} - \tilde{\pi}^{m+1}X_0^{e-m-1}).$$

in which α^{st} is a unit or $\alpha^{\mathrm{st}} = X_{\tilde{m}+1}$. In this case $\mathcal{O}^{\mathrm{h}}_{\tilde{Y},\tilde{y}}$ is as in Example 10.3. □

We resume the proof of Proposition 10.9. We argue case by case.

1st case: L *is constant.*

Consider the exact sequence of sheaves on $\tilde{Y}_{\tilde{y}}$

$$0 \to j_! j^* \Lambda \to \Lambda \to i_* i^* \Lambda \to 0$$

where $i\colon \sigma^{-1}(Y^{\mathrm{crit}}) \to \tilde{Y}_{\tilde{y}}$ is the closed immersion and j the complementary open immersion. The corresponding long exact sequence for $R\tilde{h}_{\tilde{y},*}(-)_{\bar{K}}$, [Nak98, Theorem 0.1] and Lemma 10.11 imply the requested tameness for $R\tilde{h}_{\tilde{y},*}(j_!\mathsf{L})_{\bar{K}}$.

2nd case: L *is non-constant (in particular* $\ell \neq 2$*).*

Let $\hat{Y}_{\tilde{y}} \to \tilde{Y}_{\tilde{y}}$ be the Kummer covering corresponding to adjoining the square root of α^{st}. Then by Lemma 10.11 $\hat{Y}_{\tilde{y}}$ and its closed subscheme $V(\alpha^{\mathrm{st}})$ are log smooth over $\mathcal{O}$ and the immersion is strict.

Consider the morphism of log schemes

$$\hat{Y}_{\tilde{y}} \to \operatorname{Spec} \mathbb{Z}[T], \quad T \mapsto \beta.$$

Consider the Kummer log étale covering

$$\operatorname{Spec} \mathbb{Z}[T'] \to \operatorname{Spec} \mathbb{Z}[T], \quad T \mapsto (T')^2$$

and the corresponding Kummer log étale covering of fs log schemes

$$Y^{\mathsf{L}}_{\tilde{y}} := \hat{Y}_{\tilde{y}} \otimes^{\mathrm{fs}}_{\mathcal{O}[T]} \mathcal{O}[T'] \to \hat{Y}_{\tilde{y}}.$$

Note that this fs log base change agrees with the ordinary base change over K. Then $Y^{\mathsf{L}}_{\tilde{y}}$ and its closed subscheme $V(\alpha^{\mathrm{st}})$ are log smooth over $\mathcal{O}$.

Consider the finite morphism

$$\mu\colon Y^{\mathsf{L}}_{\tilde{y}} \to \tilde{Y}_{\tilde{y}}$$

of degree 4. Then the pullback of the local system L along μ is trivial and F is a direct summand of $R\mu_*\mu^*\mathsf{F}$ as $\ell \neq 2$ and as F vanishes on the ramification locus of μ. We can now argue for $\mu^*(\mathsf{F})$ on $Y^{\mathsf{L}}_{\tilde{y}}$ as we did in the first case for F on $\tilde{Y}_{\tilde{y}}$. □

Appendix A. Nearby cycle functor

In this appendix we describe basic properties of the étale nearby cycle functor in a coordinate free fashion using so called Iwasawa twists. We hope that this coordinate free presentation makes the Verdier duality theory and the discussion of Rapoport-Zink sheaves in Section 8 more transparent. Nothing we present was not known to people in the early 1980s. We recast the theory using the pro-étale topology.

A.1. **Reminder on constructible sheaves.** Let Y be a noetherian scheme. For any ring Λ, let $\mathrm{Sh}(Y,\Lambda)$ be the category of pro-étale sheaves of Λ-modules and $D(Y,\Lambda)$ be its derived category, see [BS15] or [Ker16]. Assume in the following that Λ is a complete local noetherian ring with maximal ideal $\mathfrak{m}$. We define a *constructible* Λ-*sheaf* F as a pro-étale sheaf of Λ-modules such that $\mathsf{F} = \lim_n \mathsf{F}/\mathfrak{m}^n\mathsf{F}$ and such that for all $n > 0$ the pro-étale sheaf of $\Lambda/\mathfrak{m}^n$-modules $\mathsf{F}/\mathfrak{m}^n\mathsf{F}$ comes from an étale constructible sheaf of $\Lambda/\mathfrak{m}^n$-modules, see [StPr, 09BS]. Let $\mathrm{Sh}_c(Y,\Lambda)$ be the category of constructible Λ-sheaves.

One can show [Kri23] that $\mathrm{Sh}_c(Y,\Lambda)$ forms a noetherian abelian subcategory closed under extensions of the category of all pro-étale sheaves of Λ-modules. One also shows [StPr, 09BS] that for $\mathsf{F} \in \mathrm{Sh}_c(Y,\Lambda)$ there exists a stratification $\mathbf{Z}$ of X such that $\mathsf{F}|_Z$ is smooth for all $Z \in \mathbf{Z}$.

Let $\Lambda_\circ$ be a localization of Λ. We set $\mathrm{Sh}_c(Y,\Lambda_\circ) = \mathrm{Sh}_c(Y,\Lambda) \otimes_\Lambda \Lambda_\circ$. Let t be an endomorphism of $\mathsf{F} \in \mathrm{Sh}_c(Y,\Lambda_\circ)$ and assume that $\Lambda_\circ$ is artinian. Then there exists a unique decomposition $\mathsf{F} = \mathsf{F}^{\mathrm{nil}} \oplus \mathsf{F}^{\mathrm{inv}}$ stable under t such that t is nilpotent on $\mathsf{F}^{\mathrm{nil}}$ and invertible on $\mathsf{F}^{\mathrm{inv}}$. Uniqueness is clear while for existence we define $\mathsf{F}^{\mathrm{nil}} = \ker(t^n \colon \mathsf{F} \to \mathsf{F})$ and $\mathsf{F}^{\mathrm{inv}} = \mathrm{im}(t^n \colon \mathsf{F} \to \mathsf{F})$ for $n \gg 0$. One can check the property on a stratification as above on which it reduces to the case of finite $\Lambda_\circ$-modules where it is the classical Weyr-Fitting decomposition [Bou22, Proposition 2.2].

Let $D_c(Y,\Lambda)$ be the triangulated subcategory of complexes with constructible cohomology sheaves inside the derived category of pro-étale sheaves of Λ-modules $D(Y,\Lambda)$. Let $D^b_c(Y,\Lambda)$ be the triangulated subcategory which additionally has bounded cohomology sheaves. Define $D^b_c(Y,\Lambda_\circ)$ as the Verdier localization "up to isogeny" $D^b_c(Y,\Lambda) \otimes_\Lambda \Lambda_\circ$, for a ring $\Lambda_\circ$ which is a localization of Λ.

For a morphism of noetherian schemes $f\colon Y_1 \to Y_2$ there are natural functors $f^*\colon \mathrm{Sh}_c(Y_2,\Lambda) \to \mathrm{Sh}_c(Y_1,\Lambda)$ and $f^*\colon D^b_c(Y_2,\Lambda) \to D^b_c(Y_1,\Lambda)$. From this f^* one deduces the derived pushforwards $Rf_!, Rf_*\colon D^b_c(Y_1,\Lambda) \to D^b_c(Y_2,\Lambda)$ by the usual adjunctions whenever they have a chance to exist [ILO14, Introduction, Théorème 1]. One also gets the exceptional pullback $f^!\colon D^b_c(Y_2,\Lambda) \to D^b_c(Y_1,\Lambda)$ by adjunction.

If Λ is more generally a filtered colimit $\Lambda = \mathrm{colim}_j \Lambda_j$ of complete local noetherian rings Λ_j with flat, finite transition homomorphisms we set

$$D_c(X,\Lambda) = \operatorname*{colim}_j D_c(X,\Lambda_j)$$

and similarly for a localization $\Lambda_\circ$ of Λ. This might depend on the system of the Λ_j.

A.2. **Iwasawa twists.** In this subsection we collect some results from [Bei87], [LZ19].

Let G be a profinite group which is isomorphic to $\mathbb{Z}_\ell$. Let Λ be a noetherian complete local ring with residue characteristic ℓ. Consider the "Iwasawa algebra" $\Lambda^{\mathrm{Iw}} = \Lambda[\![G]\!]$. The augmentation ideal $\mathfrak{I} = \ker(\Lambda^{\mathrm{Iw}} \to \Lambda)$ is generated by a non-zero divisor $[\xi] - 1$, where $\xi \in G$ is a topological generator. Indeed such a choice induces an isomorphism

$$\Lambda[\![t]\!] \xrightarrow{\sim} \Lambda^{\mathrm{Iw}}, \quad t \mapsto [\xi] - 1.$$

Let Y be a noetherian scheme. Consider the derived category of sheaves of torsion Iwasawa modules "with vanishing μ-invariant" up to isogeny

$$D^{\mathrm{Iw}}(Y, \Lambda_\circ) := [D(Y, \Lambda^{\mathrm{Iw}}) \cap R^{-1}(D^b_c(Y, \Lambda))] \otimes_\Lambda \Lambda_\circ.$$

Here $R\colon D(Y, \Lambda^{\mathrm{Iw}}) \to D(Y, \Lambda)$ is induced by the homomorphism $\Lambda \to \Lambda^{\mathrm{Iw}}$. Let $D^{\mathrm{nil}}(Y, \Lambda_\circ)$ resp. $D^{\mathrm{inv}}(Y, \Lambda_\circ)$ be the full subcategory of $D^{\mathrm{Iw}}(Y, \Lambda_\circ)$ on which t is nilpotent resp. invertible. For the rest of this subsection we assume that $\Lambda_\circ$ is artinian.

Lemma A.1. *The have the decomposition*

$$D^{\mathrm{Iw}}(Y, \Lambda_\circ) = D^{\mathrm{nil}}(Y, \Lambda_\circ) \oplus D^{\mathrm{inv}}(Y, \Lambda_\circ)$$

Proof. In order to see this use the exact triangle

$$\tau^{<a}\mathsf{F} \to \tau^{\leq a}\mathsf{F} \to \mathcal{H}^a(\mathsf{F})[-a] \to \tau^{<a}\mathsf{F}[1],$$

induction on a and the above decomposition

$$\mathcal{H}^a(\mathsf{F}) = \mathcal{H}^a(\mathsf{F})^{\mathrm{nil}} \oplus \mathcal{H}^a(\mathsf{F})^{\mathrm{inv}}.$$

□

If we work with an arbitrary profinite group H in place of $G \cong \mathbb{Z}_\ell$, we have to slightly reformulate. Let $\mathrm{Sh}_c(Y \times BH, \Lambda)$, $D^b_c(Y \times BH, \Lambda)$ etc. be the H-equivariant versions of the definitions from A.1, for example for $\mathsf{F} \in \mathrm{Sh}_c(Y, \Lambda)$ the H-action factors through a finite discrete quotient of H on $\mathsf{F}/\mathfrak{m}^n$ for all $n > 0$. We obviously have an equivalence $D^b_c(Y \times BG, \Lambda_\circ) = D^{\mathrm{Iw}}(Y, \Lambda_\circ)$.

Consider an epimorphism $H \to G$ of pro-finite groups whose kernel W has pro-order coprime to ℓ then we denote by

$$\mathrm{Nil}\colon D^b_c(Y \times BH, \Lambda_\circ) \to D^{\mathrm{nil}}(Y, \Lambda_\circ), \quad \mathsf{F} \mapsto (\mathsf{F}^W)^{\mathrm{nil}}$$

the projection to the nilpotent part of the W-invariants, see Lemma A.1.

For $\mathsf{F} \in D^{\mathrm{Iw}}(Y, \Lambda_\circ)$ and $a \in \mathbb{Z}$ we call

$$\mathsf{F}(a)^{\mathrm{Iw}} = \mathsf{F} \otimes_{\Lambda^{\mathrm{Iw}}} \mathfrak{I}^a$$

the *Iwasawa twist* of F. Note that $\mathfrak{I} \subset \Lambda^{\mathrm{Iw}}$ is an invertible ideal. Clearly, if G acts trivially on F there is a canonical isomorphism between $\mathsf{F}(a)^{\mathrm{Iw}}$ and the *Tate twist*

$$\mathsf{F}(a) = \mathsf{F} \otimes_{\mathbb{Z}_\ell} G^{\otimes a},$$

since $\mathfrak{I}^a/\mathfrak{I}^{a+1} \cong \Lambda \otimes_{\mathbb{Z}_\ell} G^{\otimes a}$.

For $\mathsf{F} \in D^{\mathrm{Iw}}(Y, \Lambda_\circ)$ let $\mathsf{F}^G \in D^b_c(Y, \Lambda_\circ)$ be the derived G-invariants. The canonical action map $\mathsf{F} \otimes_{\Lambda^{\mathrm{Iw}}} \mathfrak{I} \to \mathsf{F}$ can be written as $N^{\mathrm{Iw}}\colon \mathsf{F} \to \mathsf{F}(-1)^{\mathrm{Iw}}$. Then we obtain the fundamental exact triangle

$$\mathsf{F}^G \to \mathsf{F} \xrightarrow{N^{\mathrm{Iw}}} \mathsf{F}(-1)^{\mathrm{Iw}} \to \mathsf{F}^G[1] \tag{A.1}$$

For the derived G-coinvariants F_G we have a canonical isomorphism $\mathsf{F}_G \cong \mathsf{F}^G(1)[1]$.

Any $\mathsf{F} \in D^{\mathrm{nil}}(Y, \Lambda_\circ)$ is derived t-complete and there is an isomorphism $\mathsf{F} \otimes_{\Lambda^{\mathrm{Iw}}} \Lambda = \mathsf{F}_G$. Together with the derived Nakayama Lemma [StPr, Lemma 0G1U] we obtain the following lemma.

Lemma A.2. *A morphism $\mathsf{F} \to \mathsf{G}$ in $D^{\mathrm{nil}}(Y, \Lambda_\circ)$ is an isomorphism if $\mathsf{F}^G \to \mathsf{G}^G$ is an isomorphism.*

The group inverse $G \to G, g \mapsto -g$ induces an involutive ring isomorphism $\mathrm{inv}\colon \Lambda^{\mathrm{Iw}} \to \Lambda^{\mathrm{Iw}}$ preserving $\mathfrak{I}$. We denote the sheaf $\mathsf{F} \in D^{\mathrm{Iw}}(Y, \Lambda)$ with the inv-twisted action by F^-. The involution applied to the Iwasawa twist induces an isomorphism in $D^{\mathrm{Iw}}(Y, \Lambda)$

$$\mathrm{inv}(a) = \mathrm{id}_{\mathsf{F}} \otimes \mathrm{inv}^{\otimes a} \colon (\mathsf{F}(a)^{\mathrm{Iw}})^- \xrightarrow{\sim} (\mathsf{F}^-)(a)^{\mathrm{Iw}}. \tag{A.2}$$

Let $\varpi \in D^b_c(Y, \Lambda)$ be a dualizing sheaf and let

$$\mathrm{D}(\mathsf{F}) = \mathit{Hom}_\Lambda(\mathsf{F}^-, \varpi)$$

be the Verdier dual of $\mathsf{F} \in D^{\mathrm{Iw}}(Y, \Lambda)$, where the action of Λ^{Iw} on ϖ is trivial.

We obtain a commutative diagram, which clarifies in which sense N^{Iw} is compatible with duality

$$\begin{array}{ccc} \mathrm{D}(F(-1)^{\mathrm{Iw}}) & \xrightarrow{\mathrm{D}(N^{\mathrm{Iw}})} & \mathrm{D}(F) \\ \uparrow{\scriptstyle \mathrm{inv}(-1)} & & \uparrow{\scriptstyle N^{\mathrm{Iw}}(1)^{\mathrm{Iw}}} \\ \mathit{Hom}(F^-(-1)^{\mathrm{Iw}}, \varpi) & = & \mathrm{D}(F)(1)^{\mathrm{Iw}} \end{array} \tag{A.3}$$

Note that in this duality statement the strength of Iwasawa twists becomes obvious, as previous attempts to write down this compatibility looked quite ad hoc, see Deligne's letter to MacPherson [Del82].

For $\mathbb{Q} \subset \Lambda_\circ$ and $\mathsf{F} \in D^{\mathrm{nil}}(Y, \Lambda_\circ)$ there is a canonical isomorphism between Tate and Iwasawa twists $c_{\mathrm{TIw}}\colon \mathsf{F}(a) \xrightarrow{\sim} \mathsf{F}(a)^{\mathrm{Iw}}$ for $a \in \mathbb{Z}$, which for $a = 1$ is induced by the homomorphism

$$G \to (\mathfrak{I}/\mathfrak{I}^\nu) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad g \mapsto \log([g]),$$

where $\nu > 0$ is chosen such that $\mathfrak{I}^\nu$ acts trivially on F. This gives us a canonical nilpotent map $N\colon \mathsf{F} \to \mathsf{F}(-1)$ such that

$$\begin{array}{ccc} & & \mathsf{F}(-1) \\ & \overset{N}{\nearrow} & \\ \mathsf{F} & & \wr\downarrow{\scriptstyle c_{\mathrm{TIw}}} \\ & \underset{N^{\mathrm{Iw}}}{\searrow} & \\ & & \mathsf{F}(-1)^{\mathrm{Iw}} \end{array}$$

commutes. This means that $g \in G$ acts by $\exp(gN)\colon \mathsf{F} \to \mathsf{F}$, where $gN\colon \mathsf{F} \to \mathsf{F}$ is the contraction of the Tate twist. So we see that in this case the fundamental exact triangle reads

$$\mathsf{F}^G \to \mathsf{F} \xrightarrow{N} \mathsf{F}(-1) \to \mathsf{F}^G[1].$$

Via the identification c_{TIw} the isomorphism (A.2) becomes multiplication by $(-1)^a$, so by abuse of notation we can write the commutativity of (A.3) simply as $\mathrm{D}(N) = -N(1)$.

A.3. **Unipotent nearby cycles.** Let $\mathcal{O}$ be a strictly henselian discrete valuation ring such that the prime number ℓ is invertible in $\mathcal{O}$. Let K be the fraction field of $\mathcal{O}$ and k be the residue field. In this appendix we do not need to assume that k is perfect or that $\mathrm{ch}(K) = 0$. Let $\overline{K}$ be a separable algebraic closure of K. For simplicity of notation we just write Λ for the coefficient ring, which could be a localization of a noetherian complete local ring or a filtered colimit of such as in A.1 and we drop the notation $\Lambda_\circ$ used above.

Let $f\colon X \to \operatorname{Spec}\mathcal{O}$ be a scheme of finite type. Set $H = \mathrm{Gal}(\overline{K}/K)$ and let $H \to G \cong \mathbb{Z}_\ell(1)_{\overline{k}}$ be the maximal ℓ-adic quotient of G with kernel W. In the following we use the notation of A.2. Let $j\colon X_K \to X$ and $i\colon X_k \to X$ be the immersions of fibres and consider the morphism of topoi $\bar{\jmath}\colon X_{\overline{K}} \to X \times BH$ and $\bar{\jmath}_K\colon X_{\overline{K}} \to X_K$.

Recall [SGA7.2, Exposé XIII] that the *nearby cycle functor* is defined as

$$R\Psi_{X/\mathcal{O}}\colon D(X_K,\Lambda) \to D(X_k \times BH,\Lambda), \quad \mathsf{F} \mapsto i^* R\bar{\jmath}_* \bar{\jmath}_K^*(\mathsf{F}).$$

Deligne showed [SGA4.5, Chapitre 7] that $R\Psi_{X/\mathcal{O}}$ preserves bounded constructible complexes. Note that we have the identity

$$(R\Psi_{X/\mathcal{O}})^W = i^* R\tilde{\jmath}_* \tilde{\jmath}_K^*\colon D(X_K,\Lambda) \to D^{\mathrm{Iw}}(X_k,\Lambda), \tag{A.4}$$

with the morphisms of topoi $\tilde{\jmath}\colon X_{\overline{K}^W} \to X \times BG$ and $\tilde{\jmath}_K\colon X_{\overline{K}^W} \to X_K$, which is useful for the general residue field case in A.7.

From now on we assume that Λ is artinian. We write

$$\psi = \psi_{X/S} = \operatorname{Nil} R\Psi_{X/\mathcal{O}}[-1]\colon D^b_c(X_K,\Lambda) \to D^{\mathrm{nil}}(X_k,\Lambda)$$

for the shifted unipotent nearby cycle functor, see Lemma A.1.

The fundamental exact triangle (A.1) applied to $\psi(\mathsf{F})$ now reads

$$\psi(\mathsf{F}) \xrightarrow{N^{\mathrm{Iw}}} \psi(\mathsf{F})(-1)^{\mathrm{Iw}} \to i^* j_* \mathsf{F} \to \psi(\mathsf{F})[1] \tag{A.5}$$

as $[R\bar{\jmath}_*\bar{\jmath}^*(\mathsf{F})]^H = j_*(\mathsf{F})$. Here we omit the right derived sign for j_* as j_* is perverse t-exact for the t-structure in A.4. This fundamental triangle gives a dévissage for $\psi(\mathsf{F})$ in terms of $i^*j_*(\mathsf{F})$. A further dévissage of the latter is accomplished by the exact triangle

$$i^*\mathsf{G} \to i^* j_* j^* \mathsf{G} \to i^!\mathsf{G}[1] \to i^*\mathsf{G}[1] \tag{A.6}$$

for $\mathsf{G} \in D^b_c(X,\Lambda)$.

A.4. **Perverse sheaves.** Assume in the following that $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ or that Λ is an algebraic field extension of $\mathbb{Q}_\ell$. For $f\colon X \to \mathcal{O}$ separated and of finite type $\varpi = f^!\Lambda(1)[2] \in D^b_c(X,\Lambda)$ is a dualizing sheaf, i.e. for $\mathsf{F} \in D^b_c(X,\Lambda)$ and $\mathrm{D}(\mathsf{F}) := RHom_X(\mathsf{F},\varpi) \in D^b_c(X,\Lambda)$ the canonical map $\mathsf{F} \xrightarrow{\sim} \mathrm{D}(\mathrm{D}(\mathsf{F}))$ is an isomorphism.

We always work with the (middle) perverse t-structure on $D^b_c(X,\Lambda)$ induced by the dimension function

$$\delta_X(x) = \operatorname{trdeg}(k(x)/k(f(x))) + \dim_{\operatorname{Spec}\mathcal{O}}(\overline{\{f(x)\}})$$

as in [Gab04]. This means that

$$\mathsf{F} \in {}^{\mathrm{p}}D^{\leq 0}_c(X,\Lambda) \ \Leftrightarrow\ \mathcal{H}^i(i^*_{\bar{x}}\mathsf{F}) = 0$$

for all $x \in X$ and $i > \delta_X(x)$ and $\mathsf{F} \in {}^{\mathrm{p}}D^{\geq 0}_c(X,\Lambda) \Leftrightarrow \mathrm{D}(F) \in {}^{\mathrm{p}}D^{\leq 0}_c(X,\Lambda)$.

Concretely, this means that for X_K regular, connected of dimension n the sheaf $\Lambda[n+1] \in D^b_c(X_K,\Lambda)$ is perverse and for X_k regular connected of dimension n the sheaf $\Lambda[n] \in D^b_c(X_k,\Lambda)$ is perverse.

Gabber showed, see also [Bei87], [Del82], that $R\Psi[-1]$ and therefore ψ are perverse t-exact with respect to this perverse t-structure, see [Ill94, Corollaire 4.5].

Remark A.3. If $\mathsf{G} \in D^b_c(X,\Lambda)$ is perverse with no non-trivial subobjects or quotient objects supported on X_k, i.e. $\mathsf{G} = j_{!*}j^*\mathsf{G}$, then $i^*\mathsf{G}[-1] = {}^p\mathcal{H}^{-1}(i^*j_*j^*\mathsf{G})$ and $i^!\mathsf{G}[1] = {}^p\mathcal{H}^0(i^*j_*j^*\mathsf{G})$ are the (shifted) perverse constituents of $i^*j_*j^*\mathsf{G} \in {}^pD^{[-1,0]}_c(X_s,\Lambda)$ and (A.6) is the corresponding truncation exact triangle.

A.5. **Base change.** Let the notation be as in A.3 and let $f\colon Y \to X$ be a morphism of finite type.

For f proper and K in $D^b_c(Y_\eta)$. The canonical proper base change map

$$\psi \circ f_* \xrightarrow{\sim} f_* \circ \psi$$

is an equivalence of functors from $D^b_c(Y_\eta,\Lambda)$ to $D^{\mathrm{nil}}(X_s,\Lambda)$.

There exists also a canonical base change map

$$f^* \circ \psi \to \psi \circ f^*$$

of functors from $D^b_c(X_\eta,\Lambda)$ to $D^{\mathrm{nil}}(Y_s,\Lambda)$ which is an equivalence if f is smooth. Note that we get an induced morphism of exact triangles (A.5)

$$\begin{array}{ccccccc}
f^*\psi(\mathsf{F}) & \xrightarrow{N^{\mathrm{Iw}}} & f^*\psi(\mathsf{F})(-1)^{\mathrm{Iw}} & \longrightarrow & f^*i^*j_*\mathsf{F} & \longrightarrow & f^*\psi(\mathsf{F})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\psi(f^*\mathsf{F}) & \xrightarrow{N^{\mathrm{Iw}}} & \psi(f^*\mathsf{F})(-1)^{\mathrm{Iw}} & \longrightarrow & i^*j_*f^*\mathsf{F} & \longrightarrow & \psi(f^*\mathsf{F})[1]
\end{array} \tag{A.7}$$

for $\mathsf{F} \in D_c^b(X_\eta, \Lambda)$. Similarly, we have a morphism of exact triangles (A.6)

$$\begin{array}{ccccccc} f^*i^*\mathsf{G} & \longrightarrow & f^*i^*j_*j^*\mathsf{G} & \longrightarrow & f^*i^!\mathsf{G}[1] & \longrightarrow & f^*i^*\mathsf{G}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ i^*f^*\mathsf{G} & \longrightarrow & i^*j_*j^*f^*\mathsf{G} & \longrightarrow & i^!f^*\mathsf{G}[1] & \longrightarrow & i^*f^*\mathsf{G}[1] \end{array} \tag{A.8}$$

for $\mathsf{G} \in D_c^b(X, \Lambda)$.

A.6. **Compatibility with Verdier duality.** Write $\varpi_{X_s} = f_s^!(\Lambda)$, $\varpi_X = f^!(\Lambda(1)[2])$ and $\varpi_{X_\eta} = \varpi_X|_{X_\eta}$. Consider $\mathsf{F}, \mathsf{F}' \in D_c^b(X_\eta, \Lambda)$ and assume they are endowed with a pairing

$$\mathsf{p} \colon \mathsf{F} \otimes_\Lambda \mathsf{F}' \to \varpi_{X_\eta}.$$

As ψ is lax symmetric monoidal up to a shift, the composition of

$$\psi(\mathsf{F}) \otimes \psi(\mathsf{F}') \to \psi(\mathsf{F} \otimes \mathsf{F}')[-1] \xrightarrow{p} \psi(f^!\Lambda(1)[1]) \to \\ f_s^!\psi(\Lambda(1)[1]) \cong f_s^!\Lambda(1) = \varpi_{X_s}(1) \tag{A.9}$$

induces a pairing

$$\psi(\mathsf{p}) \colon \psi(\mathsf{F}) \otimes_{\Lambda^{\mathrm{Iw}}} \psi(\mathsf{F}')^- \to \varpi_{X_s}(1)$$

in $D^{\mathrm{Iw}}(X, \Lambda)$, where G acts trivially on the codomain.

By an analogous construction one gets a pairing

$$i^*j_*\mathsf{F} \otimes_\Lambda i^*j_*\mathsf{F}' \to \varpi_{X_s}[1]. \tag{A.10}$$

Via these pairing the fundamental exact triangle (A.5) becomes self-dual in the sense that we get a commutative diagram

$$\begin{array}{ccccccc} \psi(\mathsf{F}) & \xrightarrow{N^{\mathrm{Iw}}} & \psi(\mathsf{F})(-1)^{\mathrm{Iw}} & \longrightarrow & i^*j_*\mathsf{F} & \longrightarrow & \psi(\mathsf{F})[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{D}(\psi(\mathsf{F}'))(1)^{\mathrm{Iw}} & \xrightarrow{\mathrm{D}(N^{\mathrm{Iw}})} & \mathrm{D}(\psi(\mathsf{F}')) & \longrightarrow & \mathrm{D}(i^*j_*\mathsf{F}')[1] & \longrightarrow & \mathrm{D}(\psi(\mathsf{F}'))(1)^{\mathrm{Iw}}[1] \end{array} \tag{A.11}$$

in $D_c^b(X_s, \Lambda)$.

We say that p is a perfect pairing if it induces an isomorphism $\mathsf{F} \xrightarrow{\sim} \mathrm{D}(\mathsf{F}')$. Gabber [Ill94, Théorème 4.2] showed the following property.

Lemma A.4. *If* p *is a perfect pairing then*

(i) *the pairing* $\psi(\mathsf{p})$ *is perfect and*
(ii) *the pairing* (A.10) *is perfect.*

A.7. **Non-separably closed residue field.** If the residue field k of the henselian discrete valuation ring $\mathcal{O}$ is not assumed to be separably closed we proceed as follows. For k finite the theory is layed out in detail in [HZ23], see also [SGA7.2, Exposé XIII] for background on the Deligne topos.

Let $\tilde{K} \subset \overline{K}$ be the subfield generated by all ℓ^n-th roots of unity and by all π^{1/ℓ^n} for $n > 0$. Let $\tilde{k}$ be the residue field of $\tilde{K}$. Fix a splitting σ of the exact sequence of profinite groups

$$0 \to G = \mathbb{Z}_\ell(1) \to \tilde{G} \to \tilde{g} \to 0$$

where $\tilde{G} = \mathrm{Gal}(\tilde{K}/K)$, $\tilde{g} = \mathrm{Gal}(\tilde{k}/k)$. Such a splitting is for example induced by a choice of compatible ℓ^n-th roots of π for $n > 0$. Consider the morphisms of topoi $\tilde{\jmath}\colon X_{\tilde{K}} \to X \times_{B\tilde{g}} B\tilde{G}$ and $\tilde{\jmath}_K\colon X_{\tilde{K}} \to X_K$. The splitting σ induces an isomorphism

$$D^{\mathrm{Iw}}(X_k, \Lambda) \cong D^b_c(X_k \times_{B\tilde{g}} B\tilde{G}, \Lambda), \tag{A.12}$$

where Iwasawa module sheaves on X_k are defined as before with Λ^{Iw} now a ring in the category of pro-étale sheaves on X_k. We define the unipotent nearby cycle functor

$$\psi = \mathrm{Nil}\, i^* \tilde{\jmath}_* \tilde{\jmath}_K^*[-1]\colon D^b_c(X_K, \Lambda) \to D^{\mathrm{nil}}(X_k, \Lambda)$$

using the identification (A.12).

This construction clearly depends on the the splitting σ. However, for $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ or for Λ an algebraic extension of $\mathbb{Q}_\ell$ and for $\mathsf{F} \in D^b_c(X_K, \Lambda)$ perverse, the monodromy graded pieces $\mathrm{gr}^{\mathrm{M}}_a \psi(\mathsf{F})$, see Section 7, do not depend on the splitting σ in view of the proof of [BBD83, Proposition 5.1.2].

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