Some Arithmetic Properties of Complex Local Systems

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1 Introduction

A group π is said to be *finitely generated* if it is spanned by finitely many letters, that is, if it is the quotient $F \to \pi$ of a free group F on finitely many letters. It is said to be *finitely presented* if the kernel of such a quotient is itself finitely generated. This does not depend on the choice of generation chosen. For example the trivial group $\pi = \{1\}$ is surely finitely presented as the quotient of the free group in 1 generator by itself (!). The following finitely presented group shall play a role in the note:

Example 1.1 *The group Γ_0 is generated by two elements (a, b) with one relation $b^2 = a^2ba^{-2}$.*

There are groups which are finitely generated but not finitely presented, see the interesting MathOverflow elementary discussion on the topic (`https://tinyurl.com/3cavr69a`).

The finitely presented groups appear naturally in many branches of mathematics. The fundamental group $\pi_1(M, m)$ of a topological space M based at a point m is defined to be the group of homotopy classes of loops centered at m. A group is finitely presented *if and only if* it is the fundamental group $\pi_1(M, m)$ *of a connected finite CW-complex M* based at a point m. This is essentially by definition of a CW (C =closure-finite, W =weak) complex which is a topological space defined by an increasing sequence of topological subspaces, each one obtained by gluing cells of growing dimension to the previous one. So the 1-cells glued to the 0-cell m yield the loops on which we take the free group F, and the relations come from the finitely many 2-cells glued to the loops.

If X is a smooth connected quasi-projective complex variety, its complex points $X(\mathbb{C})$ form a topological manifold which has the homotopy type of a connected finite CW-complex M.

The difference between X and its complex points $X(\mathbb{C})$ is subtle, and crucial for the note. If X is projective for example, when we say X we mean the set of defining homogeneous polynomials in finitely many variables with coefficients in $\mathbb{C}$. This collection of polynomials is called a *scheme*. On the other hand, only finitely many of those polynomials are necessary to describe them all (this is the Noetherian property of the ring of polynomials over a field), so in fact there is a ring R of finite type over $\mathbb{Z}$ which contains all the coefficients. We write $X_{\mathbb{C}}$ to remember $\mathbb{C}$, X_R to remember R. We can then take any maximal ideal $\mathfrak{m}$ in R. The residue field $R/\mathfrak{m}$ is finite, say $\mathbb{F}_q$, and has characteristic $p > 0$. Then we write $X_{\mathbb{F}_q}$ for the scheme defined by this collection of polynomials where the coefficients are taken modulo $\mathfrak{m}$. Fixing an algebraic closure $\mathbb{F}_q \subset \bar{\mathbb{F}}_p$, and thinking of the polynomials as having coefficients in $\bar{\mathbb{F}}_q$ we write $X_{\bar{\mathbb{F}}_p}$ etc.

When we say $X(\mathbb{C})$, we mean the complex solutions of the defining polynomials. (Of course there is the similar notion $X_R(R), X_{\mathbb{F}_q}(\mathbb{F}_q), X_{\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = X_{\mathbb{F}_q}(\bar{\mathbb{F}}_p)$ etc.)

The notion of a quasi-projective complex variety X is easily understood on its complex points $X(\mathbb{C})$. They have to be of the shape $\bar{X}(\mathbb{C}) \backslash Y(\mathbb{C})$ where both $\bar{X}$ and $Y (\subset \bar{X})$ are projective varieties.

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We do not know how to characterize the fundamental groups $\pi_1(X(\mathbb{C}), x)$, where $x \in X(\mathbb{C})$, among all possible $\pi_1(M, m)$. In this small text, we use the following terminology:

Definition 1.2 *A finitely presented group π is said to come from geometry if it is isomorphic to $\pi_1(X(\mathbb{C}), x)$ where X is a smooth connected quasi-projective complex variety and $x \in X(\mathbb{C})$.*

The aim of this note is to describe a few obstructions for a finitely presented group to come from geometry.

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2 Classical obstructions: topology and Hodge theory

A classical example comes from the uniformization theory of complex curves: any *free* group on n letters, where n is a natural number, is the fundamental group of the complement of $(n+1)$-points on the Riemann sphere $\mathbb{P}^1$. This is because we understand exactly $\pi_1(X(\mathbb{C}), x)$ if X has dimension 1, that is if $X(\mathbb{C})$ is a Riemann surface. The simplest possible example is the Riemann sphere $X = \mathbb{P}^1$. Then $\pi_1(X(\mathbb{C}), x) = \{1\}$ as any loop centered at x can be retracted to a point, see Figure 1.
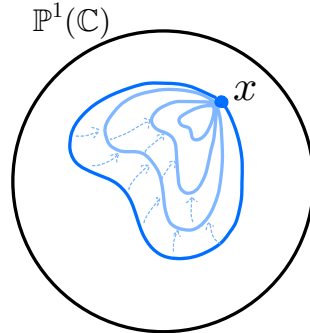

Figure 1: Any loop is retracted on $\mathbb{P}^1(\mathbb{C})$

The same holds true on $X = \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. The first interesting example is $X = \mathbb{P}^1 \setminus \{0, \infty\}$. Then $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, and $\pi_1(X(\mathbb{C}), 1) = \mathbb{Z} \cdot \gamma$ where $\gamma : [0\ 1] \to \mathbb{C} \setminus \{0\}$, $t \mapsto \exp(2\pi\sqrt{-1}t)$ is the circle turning around the origin $\{0\}$, see Figure 2.
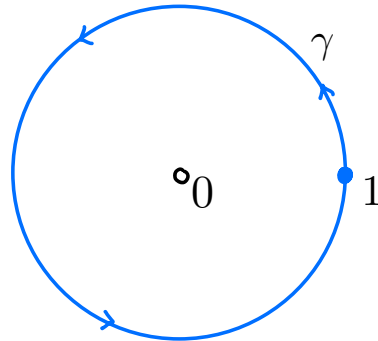

Figure 2: A nontrivial loop γ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$

More generally, if a smooth compactification $\bar{X}$ of X has genus g, topologically $\bar{X}(\mathbb{C})$ is a donut with g holes. Then $\pi_1(\bar{X}(\mathbb{C}), x)$ is spanned by $2g$ elements $(a_i, b_i), i = 1, \ldots, g$ with one relation $\prod_{i=1}^{g}[a_i, b_i] = 1$. If $(\bar{X} \setminus X)(\mathbb{C})$ consists of $(n+1)$ points, $\pi_1(X(\mathbb{C}), x)$ is spanned by $2g + n + 1$ elements $(a_i, b_i), i = 1, \ldots, g$, $c_j, j = 1, \ldots, n+1$ with one relation $\prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n+1} c_j = 1$. The literature is full of beautiful coloured pictures visualizing this

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classical computation.

Beyond Riemann surfaces, that is, for X of dimension ≥ 2, our understanding is very limited.

The 2 in the $2g$ in the previous example is more general: by the fundamental structure theorem on finitely generated $\mathbb{Z}$-modules, the maximal abelian quotient $\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}}$, that is, the abelianization of $\pi_1(X(\mathbb{C}), x)$, is isomorphic to a direct sum of $\mathbb{Z}^{\oplus b}$ for some natural number b and of a finite abelian group T.

Any abelian finitely presented group $\mathbb{Z}^b \oplus T$ comes from geometry: Serre's classical construction [?serre] realizes any finite group as the fundamental group of the quotient Z of a complete intersection of large degree in the projective space of large dimension, while the fundamental group of $(\mathbb{P}^1 \setminus \{0, \infty\})(\mathbb{C})$ is equal to $\mathbb{Z}$, see Figure 2. As the fundamental group of a product variety is the product of the fundamental groups of the factors (Künneth formula), we can take $X = (\mathbb{P}^1 \setminus \{0, \infty\})^b \times Z$ and there is *no* obstruction for $\mathbb{Z}^b \times T$ to be the abelianization of the fundamental group of a smooth connected quasi-projective complex variety. If we require X to be projective, then *Hodge theory*, more precisely, *Hodge duality* implies that b is *even*. This is the only obstruction as we can then take $X = E^{\frac{b}{2}} \times Z$ instead, where E is any elliptic curve, so $E(\mathbb{C})$ is a donut with one hole, so $\pi_1(X(\mathbb{C}), x) = \mathbb{Z}^2$, see Figure 3.
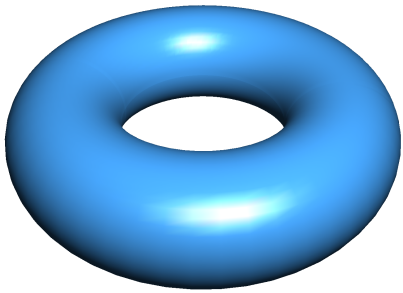

Figure 3: Riemann surface of genus $g = 1$

In the same vein, but *much deeper* is the fact that the pro-nilpotent completion of $\pi_1(X(\mathbb{C}), x)$ (also called Malčev completion) is endowed with a *mixed Hodge structure*. While so far we commented the *topological* structure of $X(\mathbb{C})$, Hodge theory studies in addition the *analysis* stemming from the *complex structure*, and the more refined properties, packaged in the notion of *Kähler geometry and harmonic theory*, which come from the property that X is defined algebraically by complex polynomials. A modern way (due to Beilinson) to think of it is to identify the Malčev completion with the cohomology of an (infinite) simplicial complex scheme and to apply the classical Hodge theory on its truncations. We do not elaborate further.

3 Profinite completion: the étale fundamental group

Thus the difficulty lies in the kernel of the group to its abelianization. To study it, we first introduce the classical notion:

Definition 3.1 *A complex local system $\mathbb{L}_\rho$ is a complex linear representation*

$$\rho : \pi_1(X(\mathbb{C}), x) \to \mathrm{GL}_r(\mathbb{C}),$$

considered modulo conjugacy by $\mathrm{GL}_r(\mathbb{C})$. The local system $\mathbb{L}_\rho$ is said to be irreducible if its underlying representation ρ (thus defined modulo conjugacy) is irreducible.

Why modulo conjugacy? A path γ_{xy} from x to y defines an isomorphism $\gamma_{yx}^{-1} \pi_1(X(\mathbb{C}), y) \gamma_{yx} = \pi_1(X(\mathbb{C}), x)$. This isomorphism is not unique, any other path from x to y differs from this one by left multiplication by a loop $\gamma_x \in \pi_1(X(\mathbb{C}), x)$ centered at x, which thus conjugates the isomorphism by γ_x. Thus *not fixing the base point* forces us to consider representations modulo conjugacy.

As $\pi_1(X(\mathbb{C}), x)$ is finitely presented, thus in particular finitely generated, ρ factors through $\pi_1(X(\mathbb{C}), x) \xrightarrow{\rho_A} \mathrm{GL}_r(A)$ where $A \subset \mathbb{C}$ is a ring of finite type. Any such A can be embedded into the

ring of ℓ-adic integers $\mathbb{Z}_\ell$ for some prime number ℓ, say $\iota : A \subset \mathbb{Z}_\ell$. (For example if $A = \mathbb{Z}$, ι has to be the natural pro-ℓ-completion for any choice of ℓ. If $A = \mathbb{Z}[T]$ we take in $\mathbb{Z}_\ell$ a transcendental element over $\mathbb{Q}$ and send T to it, etc. The main point is that the field of fractions $\mathbb{Q}_\ell$ of $\mathbb{Z}_\ell$, that is the field of ℓ-adic numbers, has infinite transcendence degree over $\mathbb{Q}$). Thus the datum of ρ is equivalent to the one of $\iota \circ \rho_A$ whose range $\mathrm{GL}_r(\mathbb{Z}_\ell)$ is profinite. In particular, $\iota \circ \rho_A$ factors through the profinite completion

$$\mathfrak{prof} : \pi_1(X(\mathbb{C}), x) \to \pi_1(X(\mathbb{C}), x)\widehat{}$$

and induces

$$\widehat{\rho} : \pi_1(X(\mathbb{C}), x)\widehat{} \to \mathrm{GL}_r(\mathbb{Z}_\ell),$$

a representation which is *continuous* for the profinite topology on both sides. Recall that the profinite completion $\mathfrak{prof} : \pi \to \widehat{\pi}$ of an abstract group π is the projective limit over all finite quotients $\pi \to H$. It inherits the profinite topology compatible with the group structure for which a basis of open neighbourhoods of 1 is defined to be the inverse image of $1 \in H$ by one of those projections.

However, Toledo in [?toledo] constructed a smooth connected complex projective variety X with the property that $\mathfrak{prof}$ is *not* injective. It answered a problem posed by Serre. It is an important fact which in particular implies that *the study of complex local systems ignores* $\mathrm{Ker}(\mathfrak{prof})$. This leads us in two different directions.

The invariants b and T of the abelianization are seen on the complex abelian algebraic group

$$\mathrm{Hom}(\pi_1(X(\mathbb{C}), x), \mathrm{GL}_1(\mathbb{C})) \cong (\mathbb{C}^\times)^b \times \mathrm{Hom}(T, \mathbb{C}^\times).$$

Here the notation $\mathbb{C}^\times$ means the set $\mathbb{C} \setminus \{0\}$ endowed with the (abelian) multiplicative group structure.

More generally the finite generation of $\pi_1(X(\mathbb{C}), x)$ enables one to define a "moduli" (parameter) space $M_B^{\mathrm{irr}}(X, r)$ of all its irreducible local systems $\mathbb{L}_\rho$ in a given rank r. It is called the *Betti moduli space* of X of irreducible local systems in rank r or the *character variety* of $\pi_1(X(\mathbb{C}), x)$ of irreducible local systems in rank r. It is a complex quasi-projective scheme of

finite type. Its study is the content of Simpson's *non-abelian Hodge theory* developed in [?simpson]. It is an *analytical theory* relying on harmonic theory, as is classical Hodge theory.

The second direction relies on the profinite completion homomorphism $\mathfrak{prof}$. By the Riemann existence theorem, a finite topological covering is the complexification of a finite étale cover. Thus $\pi_1(X(\mathbb{C}), x)\widehat{}$ is identified with the *étale fundamental group* $\pi_1(X_\mathbb{C}, x)$ of the *scheme* $X_\mathbb{C}$ defined over $\mathbb{C}$, based at the complex point x, as defined by Grothendieck in [?SGA1]:

This profinite group is defined by its representations in finite sets. A representation of $\pi_1(X_\mathbb{C}, x)$ in finite sets is "the same" (in the categorial sense) as a pointed (above x) finite étale cover of X.

We denote by

$$\rho_{\mathbb{C}, \ell} : \pi_1(X_\mathbb{C}, x) \xrightarrow{\widehat{\rho}} \mathrm{GL}_r(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$$

the composite morphism. Here $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$.

The notion of a complex local system (Definition 3.1) generalizes naturally:

Definition 3.2 *An ℓ-adic local system $\mathbb{L}_{\rho_\ell}$ on the variety $X_\mathbb{C}$ is a continuous linear representation*

$$\rho_\ell : \pi_1(X_\mathbb{C}, x) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell),$$

considered modulo conjugacy by $\mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$. The local system $\mathbb{L}_{\rho_\ell}$ is said to be irreducible if its underlying representation ρ_ℓ (thus defined modulo conjugacy) is irreducible.

As the kernel of the projection $\mathrm{GL}_r(\mathbb{Z}_\ell) \to \mathrm{GL}_r(\mathbb{F}_\ell)$ is a pro-ℓ-group (that is all its finite quotients H have order of power of ℓ), Grothendieck's specialization's theory in *loc. cit.* implies that the specialization homomorphism

$$sp_{\mathbb{C}, \bar{\mathbb{F}}_p} : \pi_1(X_\mathbb{C}, x) \to \pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$$

induces an isomorphism on the image of $\rho_{\mathbb{C}, \ell}$ for p larger than the order of $\mathrm{GL}_r(\mathbb{F}_\ell)$. Here $X_{\bar{\mathbb{F}}_p}$ is a reduction of $X_\mathbb{C}$ as explained in the introduction, and is *good*, that is smooth, as well as the stratification of

the boundary divisor if X is not projective. The upper script t refers to the tame quotient of $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$ in case X was not projective. We do *not* detail with precision the tameness concept, for which we refer to [?tame]. This roughly works as follows. Representations in finite sets of the étale fundamental group $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$ which factor through the tame quotient $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$ have base change properties "as if" $X_{\bar{\mathbb{F}}_p}$ were proper. We can contract the fundamental group of X over a p-adic ring R with residue field $\bar{\mathbb{F}}_p$ to the one over $\bar{\mathbb{F}}_p$ in the way we do topologically in order to identify the topological fundamental group of a tubular neighborhoud of a compact manifold to the one of the compact manifold. The natural identification of $\pi_1(X_{\mathbb{C}}, x)$ with $\pi_1(X_K, x)$ where K is an algebraic closure of the field of fractions of R (this is called base change property) enables us to define $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$. Grothendieck computes that $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$ induces an isomorphism on all finite quotients of $\pi_1(X_{\mathbb{C}}, x)$ and $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$ of order prime to p.

The factorization defines the irreducible ℓ-adic local system $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ on $X_{\bar{\mathbb{F}}_p}$ from which $\mathbb{L}_{\mathbb{C}, \ell}$ comes. This leads us to study $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ in order to derive *arithmetic properties* of the initial $\mathbb{L}_\rho$. We can remark that again we know extremely little on the kernel of $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$ and that the study of complex local systems ignores them as well, for a chosen $\iota : A \to \mathbb{Z}_\ell$ and p large as before.

On the other hand, $X_{\mathbb{C}}$ is defined over a field of finite type over $\mathbb{Q}$, thus with a huge Galois group, and $X_{\bar{\mathbb{F}}_p}$ is defined over a finite field $\mathbb{F}_q$ of characteristic $p > 0$, with a very small Galois group isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, topologically spanned by the Frobenius φ of $\mathbb{F}_q$. Nonetheless, we shall see that this small Galois group yields nontrivial information.

Our goal now is twofold. First we shall illustrate how to go back and forth between the Hodge theory side and the arithmetic side on a *particular example*. This by far does not cover the whole deepness of the theory, but we hope that it gives some taste on how it functions. Then we shall mention on the way and at the end more general theorems to the effect that deep arithmetic properties stemming from the Langlands program, notably the "integrality" illustrated

on this particular example, enable one to find a new obstruction for the finitely presented group to come from geometry.

4   An example to study

Let X be a smooth connected quasi-projective complex variety. If X is not projective, we fix a smooth projective compactification $X \hookrightarrow \bar{X}$ so that the divisor at infinity $D = \bar{X} \setminus X = \cup_{i=1}^M D_i$ is a strict normal crossings divisor (so its irreducible components D_i are smooth and meet transversally). For each i we fix r roots of unity $\mu_{ij}, j = 1, \ldots, r$, possibly with multiplicity. They uniquely determine a conjugacy class T_i of a semi-simple matrix of finite order. The normal subgroup spanned by the conjugacy classes of small loops γ_i around the components D_i is identified with the kernel of the surjection $\pi_1(X(\mathbb{C}), x) \to \pi_1(\bar{X}(\mathbb{C}), x)$. We fix an extra natural number $\delta > 0$.

We make the following assumption

Assumption $(\star)_r$:
For a given rank $r \geq 2$, there are finitely many irreducible rank r complex local systems $\mathbb{L}_\rho$ on X such that the determinant of $\mathbb{L}_\rho$ has order dividing δ, and, if X is not projective, such that the semi-simplification of $\rho(\gamma_i)$ falls in T_i.

It is simple to describe $(\star)_r$: in the Betti moduli space $M_B^{irr}(X, r)$ we have the subscheme $M_B^{irr}(X, r, \delta, T_i)$ defined by the conditions $\{\delta, T_i\}$. The condition $(\star)_r$ means precisely that $M_B^{irr}(X, r, \delta, T_i)$ is 0-dimensional, or equivalently that $M_B^{irr}(X, r, \delta, T_i)(\mathbb{C})$ consists of finitely many points.

Note the condition on δ depends only on $\pi_1(X(\mathbb{C}), x)$ so could be expressed on the character variety, not however the condition on T_i. For this we have to know which γ_i in $\pi_1(X(\mathbb{C}), x)$ come from the boundary divisor, so we need the geometry.

If $r = 1$, we drop the condition on the determinant, and assume for simplicity that X is projective. So the assumption becomes that there are finitely many

irreducible rank 1 complex local systems $\mathbb{L}_\rho$ on X. This then forces b to be 0, so $\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}}$ to be finite.

Consequently, those finitely many $\mathbb{L}_\rho$ of rank 1 have finite monodromy (i.e. $\rho(\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}})$ is finite). This implies that the $\mathbb{L}_\rho$ *come from geometry*, that is there is a smooth projective morphism $g : Y \to U \subset X$ where U is a Zariski dense open in X (in our case $U = X$), such that $\mathbb{L}_\rho$ restricted to U is a subquotient of the local system $R^i g_* \mathbb{C}$ coming from the representation of $\pi_1(U(\mathbb{C}), x)$ in $GL(H^i(g^{-1}(x), \mathbb{C}))$ for some i (in our case g is finite étale and $i = 0$).

A different way of thinking of finiteness is using Kronecker's analytic criterion [?eilenberg]: the set of the rank 1 local systems is invariant under the action of the automorphisms of $\mathbb{C}$ acting on $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. Finiteness of the monodromy is then equivalent to the monodromy being *unitary* (i.e. lying in $S^1 \subset \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$) and being *integral* (i.e. lying in $\mathrm{GL}_1(\bar{\mathbb{Z}}) \subset \mathrm{GL}_1(\mathbb{C})$). We now discuss the generalization of these two properties: *unitarity and integrality*.

We first observe that $(\star)_r$ implies that the irreducible rank r complex local systems are *rigid* if we preserve the $\{\delta, T_i\}$ conditions. As the terminology says, it means that we can not "deform" non-trivially the local system $\mathbb{L}_\rho$. Precisely it says that a formal deformation

$$\rho_t : \pi_1(X(\mathbb{C}), x) \to \mathrm{GL}_r(\mathbb{C}[[t]])$$

of $\rho = \rho_{t=0}$ with the same $\{\delta, \mu_{ij}\}$ conditions does not move $\mathbb{L}_\rho$, that is there is a $g \in \mathrm{GL}_r(\mathbb{C}((t)))$ such that in $\mathrm{GL}_r(\mathbb{C}((t)))$ the relation

$$\rho_t = g\rho_{t=0}g^{-1}$$

holds.

A classical example where $(\star)_r$ is fulfilled is provided by Shimura varieties of real rank ≥ 2. Margulis super-rigidity [?margulis] implies that all complex local systems are semi-simple and all irreducible ones are rigid. *While by super-rigidity they all are integral* (i.e. the image of the representations lie in $\mathrm{GL}_r(\bar{\mathbb{Z}})$

up to conjugacy), *we do not know whether they come from geometry.*

Another example is provided by connected smooth projective complex varieties X with the property that all symmetric differential forms, except the functions, are trivial. In this case, non-abelian Hodge theory implies $(\star)_r$ is fulfilled. Indeed, the Betti moduli space of semi-simple rank r complex local systems is affine, while the moduli space of semi-stable Higgs bundles with vanishing Chern classes (which we discuss below) admits a projective morphism to the so-called Hitchin base. The latter consists of one point under our assumption. As by a deep theorem of Simpson [?simpson], both spaces are real analytically isomorphic, they are both affine and compact, thus are 0-dimensional. It is proven in [?BKT], using Hodge theory, the period domain and birational geometry, that all the $\mathbb{L}_\rho$ have then finite monodromy. This yields a positive answer to a conjecture I had formulated. As the proof uses Hodge theory, it is analytic. As of today, there is no arithmetic proof of the theorem.

5 Non-abelian Hodge theory

We first assume that X is projective. We discuss a little more the notion of Higgs bundles mentioned above. Simpson in [?simpson] constructs the moduli space $M^s_{Dol}(X, r, \delta)$ of stable Higgs bundles (V, θ) of degree 0, where V is a vector bundle of rank r, $\theta : V \to \Omega^1_X \otimes V$ is a $\mathcal{O}_X$-linear operator fulfilling the integrality condition $\theta \wedge \theta = 0$, such that $\det(V, \theta)$ has finite order dividing δ. (The integrality notion here is for the Higgs field θ, and is not related to the integrality of a linear representation mentioned in Section 4). The stability condition is defined on the pairs (V, θ), that is one tests it on Higgs subbundles. The finite order of $\det(V, \theta)$ implies that the underlying Higgs field of $\det(V, \theta)$ is equal to 0, so $\det(V, \theta) = (\det(V), 0)$. The moduli space $M^s_{Dol}(X, r, \delta)$ is a complex scheme of finite type. It has several features.

There is a real analytic isomorphism $M^{\mathrm{irr}}_B(X, r, \delta) \overset{\cong}{\longrightarrow} M^s_{Dol}(X, r, \delta)$. So $(\star)_r$ implies that $M^s_{Dol}(X, r, \delta)$ consists of finitely many points.

Simpson defines on Higgs bundles the algebraic $\mathbb{C}^\times$-action which assigns $(V, t\theta)$ to (V, θ) for $t \in \mathbb{C}^\times$. It preserves stability near $1 \in \mathbb{C}^\times$ and semi-stability in general. Thus under the assumption $(\star)_r$, the $\mathbb{C}^\times$-action stabilizes $M_{Dol}^s(X, r, \delta)$ pointwise. Simpson proves in *loc. cit.* that $\mathbb{C}^\times$-fixed points correspond to polarized complex variations of Hodge structure (PCVHS). Mochizuki in [?mochizuki] generalized this part of Simpson's theory to the smooth quasi-projective case so the conclusion remains valid in general.

We summarize this section: *The assumption $(\star)_r$ implies that the irreducible $\mathbb{L}_\rho$ of rank r, with determinant of order diving δ and semi-simplification of $\rho(\gamma_i)$ falling in T_i, underlie a PCVHS. This property is the analog of the unitary property in rank $r = 1$.*

We can not expect more as already on Shimura varieties of real rank ≥ 2, not all local systems are unitary. If they all were, as they are integral, they would have finite monodromy. This is not the case.

6 Arithmeticity

Again we fix r. Once we obtain the finitely many local systems $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ on $X_{\bar{\mathbb{F}}_p}$ by specialization as in Section 3, also taking p large enough so it be prime to the orders of δ (and the μ_{ij} in case X is not projective), we consider Grothendieck's homotopy exact sequence

$$1 \to \pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \pi_1(X_{\mathbb{F}_q}, x) \to \widehat{\mathbb{Z}} \cdot \varphi \to 1$$

([?SGA1]). Here the finite field $\mathbb{F}_q \subset \bar{\mathbb{F}}_p$ is chosen so $X_{\bar{\mathbb{F}}_p}$ is defined over $\mathbb{F}_q$ and φ is the Frobenius endomorphism of $\bar{\mathbb{F}}_p$ sending λ to λ^q.

Let us first discuss the meaning of the sequence in terms of finite étale covers. The surjectivity on the right says that if $\mathbb{F}_q \subset \mathbb{F}_{q'}$ is a finite field extension, then the induced finite étale cover $X_{\mathbb{F}_{q'}} \to X_{\mathbb{F}_q}$ has no section. The injectivity on the left says that any finite étale cover of $X_{\bar{\mathbb{F}}_p}$ can be dominating by one induced by a finite étale cover of $X_{\mathbb{F}_q}$. The exactness in the middle says that if a finite étale cover of $X_{\mathbb{F}_q}$ acquires a section on $X_{\bar{\mathbb{F}}_p}$, then the induced cover of $X_{\bar{\mathbb{F}}_p}$ is completely split.

The kernel of $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$ to its tame quotient $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$ is normal in $\pi_1(X_{\mathbb{F}_q}, x)$. Thus a lift of φ to $\pi_1(X_{\mathbb{F}_q}, x)$, which is well defined up to conjugation by $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$, acts by conjugation on $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$, therefore on ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ and respects tameness.

This action preserves r, irreducibility, δ and the T_i. Thus φ acts as a bijection on the finite set $\{\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}\}$. We conclude that replacing q by some non-trivial finite power q^t, all $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ descend to ℓ-adic local systems $\mathbb{L}_{\mathbb{F}_{q^t}, \ell}$ on $X_{\mathbb{F}_{q^t}}$. We say that the $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ are *arithmetic*. (This argument is adapted from [?integrality]).

We summarize this section: *The assumption $(\star)_r$ implies that the local systems $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$ are arithmetic.*

More generally, without the assumption $(\star)_r$ being fulfilled, Simpson proves in [?simpson] in all generality that the $\mathbb{L}_{\mathbb{C}, \ell}$ *coming from irreducible rigid local systems are arithmetic*, that is they descend to ℓ-adic local systems on X_F where F is a field of finite type over $\mathbb{Q}$.

7 ℓ-adic companions and integrality

Quoted from [?eilenberg], with adapted notation:

"Given a field automorphism τ of $\mathbb{C}$, we can postcompose the underlying monodromy representation of a complex local system $\mathbb{L}_\rho$ by τ to define a *conjugate* complex local system $\mathbb{L}_\rho^\tau$. Given a field automorphism σ of $\bar{\mathbb{Q}}_\ell$, which then can only be continuous if it is the identity on $\mathbb{Q}_\ell$, or more generally given a field isomorphism σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$ for some prime number ℓ', the postcomposition of a *continuous* non-finite monodromy representation is no longer continuous (unless $\ell = \ell'$ and σ is the identity on $\mathbb{Q}_\ell$), so *we cannot define a conjugate $\mathbb{L}_{\mathbb{C}, \ell}^\sigma$ of an ℓ-adic local system by this simple postcomposition procedure.*"

However, when $X_{\mathbb{C}}$ is replaced by $X_{\bar{\mathbb{F}}_p}$, Deligne conjectured in Weil II [?deligne] that we can. Let us first state the conjecture.

By the Čebotarev density theorem, an *irreducible* ℓ-adic sheaf $\mathbb{L}_\ell$ defined by an irreducible continuous

representation $\rho_\ell : \pi_1(X_{\mathbb{F}_q}, x) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ considered modulo conjugacy by $\mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$, is determined *uniquely* by the characteristic polynomials

$$P(\mathbb{L}_\rho, y, T) = \det(T - \rho_\ell(Frob_y))$$

for all closed point y of $X_{\mathbb{F}_q}$, where $Frob_y$ is the arithmetic Frobenius at y. This expression just means that the closed point y has a residue field $\kappa(y) \subset \bar{\mathbb{F}}_p$ which is a finite extension of $\mathbb{F}_q$, say of degree m_y. Then y is a rational point of $X_{\kappa(y)}$, thus the conjugacy class of φ^{m_y} is well defined as a subgroup in $\pi_1(X_{\mathbb{F}_q}, x)$.

The first part of the conjecture predicts that if the determinant of ρ has finite order, then

$$P(\mathbb{L}_\ell, y, T) \in \bar{\mathbb{Q}}[T] \subset \bar{\mathbb{Q}}_\ell[T].$$

In particular, its σ-conjugate

$$P(y, T)^\sigma \in \bar{\mathbb{Q}}[T] \subset \bar{\mathbb{Q}}_{\ell'}[T]$$

is defined.

The second part of the conjecture predicts the existence of an irreducible ℓ'-adic local system $\mathbb{L}_\ell^\sigma$ with the property that

$$P(\mathbb{L}_\ell^\sigma, y, T) = P(\mathbb{L}, y, T)^\sigma.$$

Again Čebotarev density theorem implies unicity up to isomorphism once we know the existence.

The two parts have been proven on smooth curves $X_{\mathbb{F}_q}$ by Drinfeld in rank $r = 2$ [?shtuka] and L. Lafforgue in any rank [?lafforgue] as a corollary of Langlands' conjecture over functions fields. It thus uses automorphic forms. Drinfeld's Shtukas also imply that all $\mathbb{L}_\ell$ on $X_{\bar{\mathbb{F}}_q}$ come from geometry. The existence of companions has been extended to smooth quasi-projective $X_{\bar{\mathbb{F}}_q}$ of any dimension by Drinfeld by arithmetic-geometric methods [?drinfeld], reducing to curves. The reduction method goes back initially to Götz Wiesend.

Deligne's initial conjecture is for normal varieties. To go from smooth to normal varieties is still an open problem.

So coming back to $(\star)_r$, we see that under this assumption, and abusing notation replacing q^t by q, it holds that for any $\ell' \neq \ell, \ell' \neq p$, we have as many $\mathbb{L}_{\mathbb{F}_q, \ell | X_{\bar{\mathbb{F}}_p}}^\sigma$ as $\mathbb{L}_{\bar{\mathbb{F}}_p, \ell}$. (The companion formation preserves the irreducibility on $X_{\bar{\mathbb{F}}_p}$, the T_i and δ as we had taken p prime to all those orders, so in particular it also preserves the tameness).

We now use this fact to prove that *the assumption $(\star)_r$ implies that all irreducible rank r local systems with the conditions (δ, μ_{ij}) are integral.*

The initial irreducible complex $\mathbb{L}_\rho$ being rigid, they are defined over $\mathcal{O}_K[N^{-1}]$ where K is a number field and N is a positive natural number. Precomposing the $\mathbb{L}_{\mathbb{F}_q, \ell | X_{\bar{\mathbb{F}}_p}}^\sigma$ with $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$ and then by $\mathfrak{prof}$ yields as many irreducible local systems as the $\mathbb{L}_\rho$ with the conditions given by δ and μ_{ij}. They all are integral at the place above ℓ' determined by

$$K \subset \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell \xrightarrow{\sigma} \bar{\mathbb{Q}}_{\ell'}.$$

This construction is performed for all $\ell' \neq \ell, \ell' \neq p$, all isomorphisms σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$, with p large enough (larger than N, δ, the order of the μ_{ij}, and such that $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$ is well defined and surjective). This finishes the proof of the integrality under the assumption $(\star)_r$.

This proof is taken from [?integrality]. It is shown in *loc. cit.* that the argument applies for cohomologically rigid local systems (a notion we do not detail here) without the assumption $(\star)_r$. The assumption $(\star)_r$ does not imply cohomological rigidity, and cohomological rigidity does not imply $(\star)_r$.

8   The $(\star)_r$ condition for an abstract finitely presented group

In [?dJE] we report on the example 1.1 constructed by Becker-Breuillard-Varljú. We quote from *loc. cit.* adapting the notation:

"For $r = 2$ and $\delta = 1$, that is for SL_2 representations, the authors compute that Γ_0 has exactly two ir-

reducible complex representations modulo conjugacy. The first one $\mathbb{L}_1$ is defined by

$$\rho_1(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \rho_1(b) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix},$$

where j is a primitive 3-rd root of unity. It is defined over $\mathbb{Q}(j)$. The local system $\mathbb{L}_2$ is Galois conjugate to $\mathbb{L}_1$. The authors compute

$$\rho(ab) = \frac{j}{\sqrt{2}} \begin{pmatrix} 1 & j \\ -1 & j \end{pmatrix}.$$

As $\mathrm{Trace}(\rho(ab)) = -\frac{1}{\sqrt{2}}$, $\mathbb{L}_1$ is not integral at $\ell = 2$, so $\mathbb{L}_2$ is not integral at $\ell = 2$ either. Furthermore, $\rho_1(a)$ does not preserve the eigenvectors of $\rho_1(b)$, so $\mathbb{L}_1$ and thus $\mathbb{L}_2$ are irreducible with dense monodromy in $\mathrm{SL}_2(\mathbb{C})$. They also compute that those representations are cohomologically rigid."

So we see that Γ_0 can not be isomorphic to $\pi_1(X(\mathbb{C}), x)$ for a connected smooth *projective* complex variety X. We conclude that *the integrality property in Section 7 is an obstruction for a finitely presented group to come from projective geometry.*

Jakob Stix remarks that Γ_0^{ab} is isomorphic to $\mathbb{Z}$, which has rank 1, so Γ_0 obeys the Hodge theoretic obstruction mentioned in Section 2 as well.

The rest of the note is devoted to indicating how to extend the obstruction based on integrality to *all* connected quasi-projective varieties. This is the content of [?dJE].

9 de Jong's conjecture

If $X_{\bar{\mathbb{F}}_p}$ is a connected normal quasi-projective variety, and $\ell \neq p$ is a prime number, de Jong conjectured in [?dejong] that an irreducible representation

$$\pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \mathrm{GL}_r(\overline{\mathbb{F}_\ell((t))})$$

which is arithmetic is in fact constant in t, thus in particular has finite monodromy. Here $\mathbb{F}_\ell((t))$ is the Laurent power series field over the finite field $\mathbb{F}_\ell$ and $\overline{\mathbb{F}_\ell((t))}$ is an algebraic closure.

He shows that *assuming the conjecture, irreducible representations $\pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ always lift to arithmetic ℓ-adic local systems if $X_{\bar{\mathbb{F}}_p}$ is a smooth connected curve.*

Drinfeld in [?kashiwara] applied this argument to produce over a connected normal complex quasi-projective variety $X_{\mathbb{C}}$ ℓ-adic local systems with the property that via $sp_{\mathbb{C}, \bar{\mathbb{F}}_p}$ for p large they are arithmetic over $X_{\bar{\mathbb{F}}_p}$.

de Jong's conjecture has been proved by Böckle-Khare in specific cases and Gaitsgory [?gaitsgory] in general for $\ell \geq 3$. The latter proof uses the geometric Langlands program.

10 Weak integrality for groups

Let Γ be a finitely presented group, together with natural numbers $r \geq 1, \delta \geq 1$. We define in [?dJE] the following notion: Γ has the *weak integrality property with respect to* (r, δ) if, assuming there is an irreducible representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ with determinant of order δ, then for any prime number ℓ, there is a representation $\rho_\ell : \Gamma \to \mathrm{GL}_r(\bar{\mathbb{Z}}_\ell)$ which is irreducible over $\bar{\mathbb{Q}}_\ell$ and of determinant of order δ.

The main theorem of *loc. cit.* is that *if X is a connected smooth quasi-projective complex variety, then $\Gamma = \pi_1(X(\mathbb{C}), x)$ does have the weak integrality property with respect to any (r, δ).*

Using now the example by Becker-Breuillard-Varjú presented in Section 8, we see that their Γ_0 *does not come from geometry* at all, whether the desired X is assumed to be projective or quasi-projective.

So we conclude that *the weak integrality property for Γ with respect to all (r, δ) is an obstruction for Γ to come from geometry.* This new kind of obstruction does not rest on analytic methods, but on arithmetic properties, more specifically *the arithmetic Langlands program for the existence of companions* and *the geometric Langlands program for de Jong's conjecture*, as we briefly discuss in the next and last section.

11 Weak arithmeticity and density

The main theorem of *loc. cit.* is proven by combining

1) the method discussed in Section 7 to show integrality once we have ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ which are arithmetic;

2) and the use de Jong's conjecture discussed in Section 9, roughly as Drinfeld did in [?kashiwara], to produce many such arithmetic ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$.

By Grothendieck's classical "quasi-unipotent monodromy at infinity" theorem [?serretate], arithmetic tame ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ have quasi-unipotent monodromies at infinity. So their pull-back to $X(\mathbb{C})$ via $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ and $\mathfrak{prof}$ do as well.

In order to apply the method described in Section 7 involving the existence of ℓ-companions ultimately yielding integrality, we need quasi-unipotent monodromies at infinity. The method developed in [?dejong] shows that those in $M_B^{\mathrm{irr}}(X, r, \mathrm{torsion})$ are Zariski dense, where "torsion" refers to the determinant of $\mathbb{L}_\rho$ being torsion. This is precisely this fact which enables one to "forget" the quasi-unipotent conditions at infinity and to develop the argument.