FINITE PRESENTATION OF THE TAME FUNDAMENTAL GROUP

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ABSTRACT. Let p be a prime number, and let k be an algebraically closed field of characteristic p. We show that the tame fundamental group of a smooth affine curve over k is a projective profinite group. We prove that the fundamental group of a smooth projective variety over k is finitely presented; more generally, the tame fundamental group of a smooth quasi-projective variety over k, which admits a good compactification, is finitely presented.

1. INTRODUCTION

The aim of this work is to study analogs in characteristic $p > 0$ of some classical properties of the topological and étale fundamental groups of varieties over $\mathbb{C}$. Our main object of study for fundamental groups is the property of being finitely presented.

If X is a smooth connected algebraic variety over $\mathbb{C}$, the topological space on its complex points $X(\mathbb{C})$ is homotopy equivalent to a finite CW-complex, see [Mor78]. As a result, the fundamental group $\pi_1(X(\mathbb{C}))$ (based at some point which is irrelevant for the discussion) admits a finite presentation. Consequently the étale fundamental group $\pi_1^{\text{ét}}(X)$, which by the Rieman existence theorem is continuously isomorphic to the profinite completion

$$\widehat{\pi_1(X(\mathbb{C}))} = \varprojlim_{H \lhd \pi_1(X(\mathbb{C})), \ [\pi_1(X(\mathbb{C})):H] < \infty} \pi_1(X(\mathbb{C}))/H$$

of $\pi_1(X(\mathbb{C}))$, is a finitely presented profinite group. The existence of a finite presentation for étale fundamental groups of smooth connected varieties over other algebraically closed fields of characteristic 0 is deduced from this.

In this work, we consider the possibility of obtaining analogous finite presentability results for a smooth connected variety X over an algebraically closed field k of positive characteristic p.

We work throughout with profinite groups, so all group-theoretic notions should be understood in a topological sense. For instance, all subgroups are closed, and all homomorphism are continuous. Our notation for a closed (respectively, open) subgroup U of a profinite group π is $U \leq_c \pi$ (respectively, $U \leq_o \pi$). In this vein, a generating set S of a profinite group π is a subset that generates a dense subgroup, so π is (topologically) finitely generated if it admits a finite generating set S in this (topological) sense.

We use free profinite groups of finite rank, namely profinite completions of free groups on finitely many letters. The aforementioned finite generation of a profinite group π can then be restated as the existence of an epimorphism $\varphi \colon F \to \pi$ from a free profinite group F of finite rank onto π. If we can find such a surjection φ with $\mathrm{Ker}(\varphi)$ finitely generated as a normal subgroup of F, we say that π is (topologically) finitely presented (see [Lub01, Introduction]).

On a smooth connected variety X over an algebraically closed field k of characteristic $p > 0$, the choice of a geometric point $x \in X$ enables one to define the étale fundamental group $\pi_1^{\text{ét}}(X, x)$, which is a profinite group, see [SGA1, Exposé V]. The isomorphism class of this group does not depend on the choice of x, so we will omit x from our notation. Furthermore, since we only work with étale fundamental groups, we shall abbreviate $\pi_1^{\text{ét}}(X, x)$ by $\pi_1(X)$.

As is well known, if X is not proper, $\pi_1(X)$ is not necessarily finitely generated. For example, for each natural number s, and a choice of s natural numbers m_i prime to p and pairwise different, the Artin-Schreier covers $y^p - y = z^{m_i}$ of the affine line $\mathbb{A}_k^1 = \mathrm{Spec}(k[z])$ yield a surjection $\pi_1(\mathbb{A}_k^1) \twoheadrightarrow \oplus_{i=1}^s \mathbb{Z}/p\mathbb{Z}$. In order to have finite generation one has to restrict (in some sense) the ramification at points on a compactification of X, so to restrict attention to an appropriate quotient of the fundamental group.

We consider several quotients/completions of our profinite groups. For a profinite group π, and a prime number ℓ, we denote by

$$\pi^{(\ell)} = \varprojlim_{U \triangleleft_o \pi,\ [\pi:U] \text{ is a power of } \ell} \pi/U$$

the maximal pro-ℓ quotient of π, which is also called the pro-ℓ completion of π. Similarly, we denote by

$$\pi^{(\ell')} = \varprojlim_{U \triangleleft_o \pi,\ [\pi:U] \text{ is not divisible by } \ell} \pi/U$$

the maximal pro-prime-to-ℓ quotient of π, which is also called the prime-to-ℓ completion of π.

In this work, our main way of restricting ramification is to consider tame fundamental groups. For simplicity, we discuss here the case of curves, and refer to Section 3 for the general case. Let $X \hookrightarrow \bar{X}$ be a smooth projective compactification of a smooth connected quasi-projective curve. Recall that we have fixed a geometric point x, thus a universal cover $\widetilde{X} \to X$ based at x which identifies $\pi_1(X, x)$ with $\mathrm{Aut}(\widetilde{X}/X)$. We drop now x from the notation. The tame fundamental group $\pi_1^t(X)$ is the quotient of $\pi_1(X)$ defined by

$$\pi_1^t(X) = \varprojlim_Y \mathrm{Aut}(Y/X)$$

where $\widetilde{X} \to Y \to X$ is an intermediate cover, Y is connected and $Y \to X$ is finite tame. Tameness is a condition on each point on a normal compactification Y: an extension of discretely valued rings is tame if the ramification index is prime to p and the residue field extension is separable ([GM71, Definition 2.1.2, p.30]). By [SGA1, XIII, Corollaire 2.12], the (profinite) group $\pi_1^t(X)$ is finitely generated. We have the surjections

$$\pi_1(X) \twoheadrightarrow \pi_1^t(X) \twoheadrightarrow \pi_1(X)^{(p')}$$

and the prime-to-p-completion of $\pi_1^t(X)$ is $\pi_1(X)^{(p')}$. Moreover, if X is proper, the surjection $\pi(X) \to \pi_1^t(X)$ is an isomorphism (by definition).

In higher dimension, the Lefschetz theorem and Grothendieck's specialization theorem are used to show that $\pi_1^t(X)$ is finitely generated (see e.g. [Esn17, Corollary 7.2], [EK16, Theorem 1.1 (a)]).

The main theorem of our note is the following.

Theorem 1.1. *Let X be a smooth connected quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. If X admits a smooth projective compactification $j : X \hookrightarrow \bar{X}$ such that $\bar{X} \setminus X$ is a normal crossings divisor, then $\pi_1^t(X)$ is finitely presented.*

If X is smooth projective then Theorem 1.1 says that $\pi_1(X)$ is finitely presented. In case X is a smooth projective curve, Theorem 1.1 is a consequence of [Shu18, Corollary 1.2], which also gives a similar result over finite fields, including a determination of the number of relations in a presentation. Obtaining (a good bound on) the number of relations in higher dimension remains a challenge.

If X is a smooth connected variety defined over an algebraically closed field, we know by Michèle Raynaud ([Ray72, Théorème 2.3.1, Remarque 2.3.2]) under the assumption of strong desingularization

([SGA5, I 3.1.5]) and by Orgogozo in general ([Org03, Théorème 4.11]) that $\pi_1(X)^{(p')}$ is finitely presented. The proof relies on étale descent ([SGA1, Exposé IX, Théorème 4.10]) for $\pi_1(X)^{(p')}$ to reduce to a smooth projective normal crossings divisor compactification, then in this situation on the Lefschetz theorem for $\pi_1(X)^{(p')}$ ([Ray74, Remarque 4.3]) and for smooth surfaces on a Lefschetz pencil method. As a descended finite étale cover of a tame finite étale cover is not necessarily tame, we can not make the reduction to a good compactification case in Theorem 1.1. Our method also gives another proof of the finite presentation of $\pi_1(X)^{(p')}$, see Corollary 3.6.

For X a curve, by [SGA1, Exposé XIII, Corollaire 2.12], the group $\pi_1^{(p')}(X)$ is described by the presentation

$$\langle x_1, \ldots, x_{2g}, y_1, \ldots, y_n : [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] y_1 \cdots y_n = 1 \rangle$$

where g is the genus of X, n is the number of points on a compactification of X that are not on X, and $[a, b] = aba^{-1}b^{-1}$ is the commutator. This presentation is obtained by lifting X to characteristic 0. Theorem 1.1 cannot be proven in this way since certain quasi-projective varieties do not lift to characteristic 0.

Recall that a profinite group π is called projective if for every diagram

$$\begin{array}{ccc} & & \pi \\ & \overset{\exists\theta}{\underset{\swarrow}{\cdots}} & \big\downarrow \lambda \\ A & \underset{\varphi}{\longrightarrow} & B \end{array}$$

where φ and λ are surjective homomorphisms of profinite groups, there exists a homomorphism $\theta\colon \pi \to A$ making the diagram commute. A diagram as above is called an embedding problem, and the map θ is called a weak solution. By [RZ10, Proposition 7.6.7], originally proven by Tate, see [Gru67], we know that π is projective if and only if its cohomological dimension is at most 1. By [RZ10, Lemma 7.6.3], projectivity is also equivalent to being isomorphic to a subgroup of a free profinite group. For a prime number ℓ, being a projective pro-ℓ group is equivalent to being a free pro-ℓ group by [RZ10, Theorem 7.7.4]. Every finitely generated projective profinite group is finitely presented by [Lub01, Proposition 1.1].

If X is a smooth curve, the group $\pi_1^t(X)$ is finitely presented by Theorem 1.1. We strengthen this in the affine case.

Theorem 1.2. *Let X be a smooth connected affine curve defined over an algebraically closed field k of characteristic p. Then $\pi_1^t(X)$ is projective.*

This result is a variant of the folklore fact that $\pi_1(X)$ is projective, see for instance [NSW13, Theorems 10.1.2, 10.1.12]. Our proof proceeds along similar lines, one difference being the need to invoke a result of Shafarevich, saying that the maximal pro-p quotient of the fundamental group of a smooth compactification $\bar{X}$ of X is a free pro-p group.

We enhance Theorem 1.2 by introducing the notion due to Drinfeld of ramification bounded by a given finite étale connected cover $h : X' \to X$. Under the assumption that h is saturated (see Definition 2.4), there is a naturally defined quotient $\pi_1^h(X)$ of $\pi_1(X)$ with the property that a Galois cover $Y \to X$ has ramification bounded by h precisely when the surjection $\pi_1(X) \to \mathrm{Aut}(Y/X)$ factors through $\pi_1^h(X)$. The quotient $\pi_1(X) \to \pi_1^t(X)$ factors through $\pi_1^h(X)$ and is equal to it if h is an isomorphism. We prove in Theorem 2.5 that for a smooth connected affine curve X over an algebraically closed field of positive characteristic, the group $\pi_1^h(X)$ is finitely presented.

We shall now discuss the methods we use. The starting point of our proof of Theorem 1.1 is the following characterization of finite presentability taken from [Lub01, Theorem 0.3].

Theorem 1.3 (Lubotzky). *Let π be a finitely generated profinite group. Then π is finitely presented if and only if there is a constant $\gamma \in \mathbb{R}_{\geq 0}$ such that for any r-dimensional linear representation M of π over a prime finite field $\mathbb{F}$, one has $\dim_{\mathbb{F}} H^2(\pi, M) \leq \gamma \cdot r$.*

In [Lub01, Theorem. 0.2] there is a (more refined) cohomological expression for the number of relations of π, but we do not use it here. In order to obtain a bound on the growth of $H^2(\pi, M)$, linear in r, for $\pi = \pi_1^t(X)$ we use different methods according to the characteristic of $\mathbb{F}$.

If $\mathbb{F}$ has characteristic prime to p, we could rely on Lubotzky's theorem together with the finite generation of $\pi(X)^{(p')}$ to obtain one part of the necessary information. We do not do this, in particular to show how efficient is Lubotzky's theorem: it proves everything at once. In fact, the linear bound can be shown to exist without assuming the existence of a smooth normal crossings compactification, see Proposition 3.5. So assuming $\mathbb{F}$ has characteristic prime to p, we first compute that the natural map $H^2(\pi_1^t(X), M) \to H^2(X, M)$ is injective (see Corollary 3.4). Using this and an alteration à la de Jong-Gabber-Temkin, we can reduce the problem to the case where X has a smooth projective compactification with a normal crossings divisor at infinity. We can then apply the Lefschetz theorem for the tame fundamental group

[EK16, Theorem 1.1], thus reducing the problem to surfaces (varieties X of dimension 2).

We use purity and Deligne's theorem [Ill81, Corollaire 2.7] according to which the Euler characteristic of our tame M depends only on the geometry of X and on r. This enables us to obtain the linear bound for each $\mathbb{F}_\ell$ separately. Showing that the bound can be made independent of ℓ relies on the cohomological representation of the zeta function in case k is the algebraic closure of $\mathbb{F}_p$.

If $\mathbb{F} = \mathbb{F}_p$, we need numerical tameness for $\pi_1^t(X)$ (rather than just tameness), a notion introduced in [KS10, Section 5], which says that the local inertial groups of the finite quotients of $\pi_1^t(X)$ have order prime to p. This condition is fulfilled when X has a smooth compactification $j :$ $X \hookrightarrow \bar{X}$ with a normal crossings divisor at infinity ([KS10, Theorem 5.4 (a)]). This is the reason why we need this assumption. This allows us to conclude that Grothendieck's spectral sequence with computes equivariant cohomology in the tower defined by the finite quotients of $\pi^t(X)$ degenerates, see (4.2), yielding an inclusion $H^2(\pi_1^t(X), M) \subset$ $H^2(\bar{X}, j_*M)$, see Proposition 4.1. Again the Lefschetz theorem *loc.cit.* reduces the problem to dimension 2.

In dimension 2, we construct a locally free sheaf $\mathcal{M}$ on $\bar{X}$ of rank r which is acted on by the Frobenius F and admits an Artin-Schreier exact sequence which describes j_*M as the F-invariants of $\mathcal{M}$. If X is projective, then $\mathcal{M}$ is semi-stable of degree 0. This enables us to find an ample curve which does not depend on M which contains the necessary numerical information. If X is not projective, in order for our construction of $\mathcal{M}$ to be helpful, we need that the degree of $\mathcal{M}$, which is not positive by construction, is bounded below by a quantity which depends on M only via its rank r. We deduce this from Abhyankar's lemma.

For Theorem 1.2, the inclusion $H^2(\pi_1^t(X), M) \to H^2(X, M)$ above implies $H^2(\pi_1^t(X), M) = 0$ by Artin's affine theorem if $\mathbb{F}$ has characteristic not equal to p and by [Sha47, Theorem 2] if $\mathbb{F} = \mathbb{F}_p$.

2. THE TAME FUNDAMENTAL GROUP OF A SMOOTH AFFINE CURVE

Proposition 2.1. *Let Y be a smooth connected affine curve over an algebraically closed field k of characteristic $p > 0$, and let ℓ be a prime*

number. *Then the maximal pro-ℓ quotient $\pi_1^t(Y)^{(\ell)}$ of the tame funda-mental group $\pi_1^t(Y)$ of Y is a free pro-ℓ group.*

Proof. Choose a smooth compactification $Y \hookrightarrow \bar{Y}$. In case $\ell \neq p$, the proposition follows from [SGA1, XIII, Corollaire 2.12]. In general, the continuous homomorphism $\pi_1^t(Y) \to \pi_1(\bar{Y})$ is surjective as if $\bar{Z} \to \bar{Y}$ corresponds to a finite quotient H of $\pi_1(\bar{Y})$, and admits an intermediate finite cover $\bar{Z}|_Y \to X \to Y$ in restriction to Y, then the normalization $\bar{X}$ of X in the function field of $\bar{Z}$ is unramified along $\bar{Y} \setminus Y$.

We assume next that $\ell = p$. Since the tame local inertia at a point in $\bar{Y} \setminus Y$ is a pro-p'-group, every finite étale tame Galois cover of Y of degree a power of p is unramified along $\bar{Y} \setminus Y$. We conclude that the surjection $\pi_1^t(Y) \to \pi_1(\bar{Y})$ induces an isomorphism on $\pi_1^t(Y)^{(p)} \xrightarrow{\cong} \pi_1(\bar{Y})^{(p)}$. To finish the proof, we apply [Sha47, Theorem 2] which says that $\pi_1(\bar{Y})^{(p)}$ is a free pro-p group. $\qquad\square$

For a modern proof of the theorem of Shafarevich referred to above, see [Cre84, pp. 37-38].

Proposition 2.2. *Let X be a smooth connected affine curve over an algebraically closed field k of characteristic $p > 0$, and let ℓ be a prime number. For every open subgroup U of $\pi_1^t(X)$, the group $U^{(\ell)}$ is free pro-ℓ.*

Proof. Let $X \hookrightarrow \bar{X}$ be a smooth projective compactification. An open subgroup $U \subset \pi_1^t(X)$ corresponds to a finite étale cover Y of X, tamely ramified along $\bar{X} \setminus X$. The following claim reduces Proposition 2.2 to Proposition 2.1. $\qquad\square$

Claim 2.3. $U \cong \pi_1^t(Y)$.

Proof. Since U is an open subgroup of $\pi_1^t(X)$, we have

$$U \cong \varprojlim_{N \leq U,\ N \lhd_o \pi_1^t(X)} U/N,$$

as the open subgroups N of U which are normal subgroups of $\pi_1^t(X)$ form a cofinal family of open normal subgroups of U, namely they intersect trivially. On the other hand the finite étale Galois cover $Z \to Y$ corresponding to $V \leq U$ is tame if and only if the composite $Z \to Y \to X$, which corresponds to $V \leq \pi_1^t(X)$ is. Indeed the corresponding statement is true for extensions of discretely valued rings, that is, if $R \hookrightarrow S$ is tame, then $R \hookrightarrow S \hookrightarrow T$ is tame if and only if $S \hookrightarrow T$ is tame. Thus $U = \pi_1^t(Y)$.

$\qquad\square$

Proof of Theorem 1.2. We need to show that the cohomological dimension of $\pi_1^t(X)$ is at most 1. By definition this amounts to showing that, for every prime number ℓ, the ℓ-cohomological dimension of $\pi_1^t(X)$ is at most 1. By [NSW13, Corollary 3.3.6], this is equivalent to the cohomological dimension of an ℓ-Sylow subgroup $\mathcal{S}_\ell$ of $\pi_1^t(X)$ being at most 1. Recall from [NSW13, page 209] that

$$\mathcal{S}_\ell = \varprojlim_{\mathcal{S}_\ell \leq U \leq_o \pi_1^t(X)} U^{(\ell)}$$

which in view of Proposition 2.2 is an inverse limit of free pro-ℓ groups. We conclude from [RZ10, Theorem 3.5.15] that $\mathcal{S}_\ell$ is a free pro-ℓ group. In other words, the cohomological dimension of $\mathcal{S}_\ell$ is at most 1.   □

Let X be as in Theorem 1.2. Fix a finite étale cover $h : X' \to X$ with X' connected. Following Drinfeld [Dri12, Theorem 2.5 (ii)] we pose the following definition.

Definition 2.4.   1) *A finite étale cover $Y \to X$ has ramification bounded by h if $Y \times_X X' \to X$ is tame.*
   2) *We say that h is saturated if $\mathcal{K} = \mathrm{Ker}(\pi_1(X') \to \pi_1^t(X'))$ is normal in $\pi_1(X)$.*
   If h is saturated we set $\pi_1^h(X) = \pi_1(X)/\mathcal{K}$.

Of course any h is dominated by $h' : X'' \to X' \xrightarrow{h} X$ such that X'' is connected, h' is finite étale Galois and saturated. With this definition, we see that a finite Galois étale cover $Y \to X$ with Galois group H has ramification bounded by h if and only if the quotient $\pi_1(X) \to H$ factors through $\pi_1^h(X) \to H$. The quotient $\pi_1(X) \to \pi_1^t(X)$ factors through $\pi_1^h(X)$ and $\pi_1^h(X) \to \pi_1^t(X)$ is an isomorphism if h is tame.

Theorem 2.5 (Corollary of Theorem 1.2). *We assume that h is saturated. Then the group $\pi_1^h(X)$ is finitely presented.*

Before giving the proof of Theorem 2.5 we need to recall a couple of facts. We shall make constant use of the following elementary bounds throughout the article.

Claim 2.6. For any profinite group π, let M be a linear representation of π of rank r defined over the prime field $\mathbb{F}$. Then

   (i) $\dim_\mathbb{F} H^m(\pi, M) \leq |\pi|^m \cdot r$ for all $m \in \mathbb{N}$ if π is finite;
   (ii) $\dim_\mathbb{F} H^1(\pi, M) \leq \delta \cdot r$ if π is generated by δ elements;
   (iii) $\dim_\mathbb{F} H^0(\pi, M) \leq r$.

Proof. (i) comes from the description of $H^m(\pi, M)$ with cocycles viewed as maps $\varphi : \pi^m \to M$, (ii) as well together continuity of φ and the

cocycle condition $\varphi(a \cdot b) = a \cdot \varphi(b) + \varphi(a)$, and (iii) follows from the identification of cohomology in degree 0 with the fixed points of π in M. $\qquad\square$

The following claim will give us practice in applying Theorem 1.3 and the accompanying cohomological machinery. Another approach would be to follow the arguments establishing the analogous facts in combinatorial (discrete) group theory.

Claim 2.7. (i) If $1 \to S \to G \to Q \to 1$ is an extension of profinite groups where S and Q are finitely presented, then G is finitely presented.

(ii) If $U \subset G$ is a finitely presented open subgroup of a profinite group G, then G is finitely presented.

Proof. We prove (i). A union of a generating set for S with a lift of a generating set for Q is a generating set for G, so G is finitely generated. We can therefore apply Lubotzky's Theorem 1.3. Let M be a finite-dimensional representation of G of rank r defined over a prime field $\mathbb{F}$. We use the spectal sequence

$$E_2^{a,b} = H^a(Q, H^b(S, M)) \Rightarrow H^{a+b}(G, M).$$

One uses Claim 2.6 for the groups $H^b(S, M)$ with $b = 0, 1, 2$ then again *loc.cit.* for $E_2^{a,b}$ with $a = 0, 1, 2$. This finishes the proof of (i).

We prove (ii). Clearly G is generated by the generators of U and lifts of the finite set G/U. For M a rank r representation of G defined over $\mathbb{F}$, note that $N = (\mathrm{Cor} \circ \mathrm{Res}(M))/M$ is a representation of G of rank $(e-1)r$ where $e = [G : U]$. We apply Claim 2.6 (ii) to $H^1(G, N)$ and the assumption to $H^2(U, M)$ in the exact sequence $H^1(G, N) \to H^2(G, M) \to H^2(U, M)$ to finish the proof of (ii).
$\qquad\square$

Proof of Theorem 2.5. By definition, $\pi_1(X') \subset \pi_1(X)$ is open, thus $\pi_1^t(X') = \pi_1(X')/\mathcal{K} \subset \pi_1(X)/\mathcal{K}$ is open as well. We apply Theorem 1.2 to $\pi_1^t(X')$, which together with Claim 2.7 (ii) concludes the proof.
$\qquad\square$

3. THE PRIME TO p PART OF THEOREM 1.1

The aim of this section is to prove Proposition 3.5 and to deduce from it the finite presentation of $\pi_1(X)^{(p')}$.

Let X be a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. The choice of a geometric point $x \in X$ enables one to define the tame fundamental group

$\pi_1^t(X, x)$ as the automorphism group of the fiber functor to Sets defined by x from the finite étale covers which are tame on all curves, [KS10, Sec. 4, Definitions, Theorem. 4.4]. In the case X has a smooth projective compactification $j : X \hookrightarrow \bar{X}$ such that $\bar{X} \setminus X$ is a normal crossings divisor, this is the same definition as [GM71, Section 2.2].

As the isomorphism class of this group does not depend on the choice of x, we omit x in the notation. The restriction functor from the category of tame finite étale covers of X to the finite étale covers defines a surjection $\pi_1(X) \twoheadrightarrow \pi_1^t(X)$.

Remark 3.1. Let $\bar{X}$ be a normal projective compactification of X as above, with boundary $D = \bar{X} \setminus X$. Let $h : Y \to X$ be a tame étale covering, and let $\bar{h} : \bar{Y} \to \bar{X}$ be the branched covering obtained by normalization of $\bar{X}$ in the function field $k(Y)$. Let $x \in \bar{X}$ be a smooth point, which is also a smooth point of D.

Then by definition, the tameness of h implies that the branched covering $\bar{h} : \bar{Y} \to \bar{X}$ has the following structure in a neighbourhood of x: if D has codimension 2 in $\bar{X}$ near x, then $\bar{h}$ is étale over x, while if D has codimension 1, then for any $y \in \bar{h}^{-1}(x)$, the complete local rings $\hat{\mathcal{O}}_{x,\bar{X}}$, $\hat{\mathcal{O}}_{y,\bar{Y}}$ satisfy $\hat{\mathcal{O}}_{y,\bar{Y}} \cong \hat{\mathcal{O}}_{x,\bar{X}}[[t]]/(t^n - s)$, where $s \in \hat{\mathcal{O}}_{x,\bar{X}}$ is a generator of the ideal of D, and n is an integer relatively prime to p, which is the ramification index of $\bar{h}$ at y. We express this by saying that $\bar{h}$ is "locally Kummer" near x. Clearly if $\bar{C} \subset \bar{X}$ is a projective smooth curve containing x, and if $\bar{C}$ meets D transversally at x, then the restricted covering $\bar{h}^{-1}(\bar{C}) \to \bar{C}$ is also locally Kummer near x, with a similar description of the extension of complete local rings.

We recall the following fact.

Proposition 3.2. *Let X be a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. Then its tame fundamental group $\pi_1^t(X)$ and its prime-to-p fundamental group $\pi_1(X)^{(p')}$ are topologically finitely generated.*

Proof. Let $j : X \hookrightarrow \bar{X}$ be a normal projective compactification of X, with boundary $D = \bar{X} \setminus X$. Let $\bar{C} \subset \bar{X}$ be a complete intersection of dimension 1 of ample divisors in good position with respect to D (this means here that $\bar{C}$ is disjoint from the singular locus of $\bar{X}$, as well as the singular locus of D, and further $\bar{C}$ intersects D transversally). Set $C = \bar{C} \cap X$. We now prove that $\pi_1^t(C) \to \pi_1^t(X)$ is surjective.

Of course, if j is a good compactification, this follows from [EK16, Theorem 1]. In general, If $h : Y \to X$ is a connected étale cover, let us denote by $\bar{h} : \bar{Y} \to \bar{X}$ the normalization of $\bar{X}$ in the field of functions $k(Y)$. Then $\bar{h}|_{\bar{C}} : \bar{B} := \bar{f}^{-1}(\bar{C})$ is connected. On the other hand, if

in addition h is tame, then as $\bar{C}$ is transversal to the smooth locus of D, $\bar{h}|_{\bar{C}}$ is locally Kummer (see Remark 3.1 above), thus $\bar{B}$ is smooth. Thus $B = Y \cap \bar{B}$ is connected as well. Since h was any connected tame étale cover, this proves the claimed surjectivity on tame fundamental groups.

We then apply Theorem 1.2, in particular its consequence that $\pi_1^t(C)$ is finitely presented. This concludes the proof for $\pi_1^t(X)$, thus for its quotient $\pi_1(X)^{(p')}$ as well.

$\square$

We simplify the notation and set $\pi^t = \pi_1^t(X)$ when there is no confusion. We write an exact sequence of profinite groups

(3.1)
$$1 \to K \to \pi \to \pi^t \to 1$$

defining K.

Lemma 3.3. *For a prime finite field $\mathbb{F}$ of characteristic different from p we have $H^1(K, \mathbb{F}) = 0$.*

Proof. The proof is inspired by [GM71], Proposition 5.2.4 a). Let $\widetilde{X} \to X$ be the universal cover with base point x and Galois group π and $\widetilde{X} \to X^t \to X$ be the intermediate 'universal tame cover' with Galois group π^t. Then $K = \pi_1(X^t, x)$. By definition $H^1(\widetilde{X}, \mathbb{F}) = 0$ thus the Lyndon-Hochshild-Serre spectral sequence applied to $\widetilde{X} \to X^t$ implies $H^1(K, \mathbb{F}) = H^1(X^t, \mathbb{F})$.

On the other hand, $H^1(X^t, \mathbb{F}) = \varinjlim_\alpha H^1(X_\alpha, \mathbb{F})$ for all the intermediate covers $X^t \to X_\alpha \to X$ with $X_\alpha \to X$ finite. Let $a \in H^1(K, \mathbb{F})$, so there is $X_\alpha \to X$ as above so that a is the image of a class $a_\alpha \in H^1(X_\alpha, \mathbb{F}) = H^1(\pi^\alpha, \mathbb{F}) = H^1(\pi^{\alpha, t}, \mathbb{F})$ where $\pi^\alpha = \pi_1(X_\alpha) \subset \pi$, and $\pi^{\alpha, t}$ is its tame quotient. The latter factorization follows from the fact that the pro-ℓ-completion of π^α is a quotient of $\pi^{t, \alpha}$. This defines an $\mathbb{F}$-torsor based at x

$$X_{a(\alpha)} = \widetilde{X} \times_{\pi^\alpha} \mathbb{F} = X^t \times_{\pi^{\alpha, t}} \mathbb{F} \to X_\alpha$$

which is tame, thus the composite $X_{a(\alpha)} \to X_a \to X$ is dominated by X^t. Finally by construction the image of a_α in $H^1(X_{a(\alpha)}, \mathbb{F})$ vanishes. This finishes the proof.

$\square$

Corollary 3.4. *Let M be a representation of π^t on a finite dimensional $\mathbb{F}$-vector space, where the characteristic of $\mathbb{F}$ is not p. Then the maps*

$$H^2(\pi^t, M) \to H^2(\pi, M) \to H^2(X, M)$$

are injective.

Proof. Let $\widetilde{X} \to X$ be the universal cover with base point x. Then by definition $H^1(\widetilde{X}, \mathbb{F}) = 0$. By the Lyndon-Hochshild-Serre spectral sequence we conclude

$$(3.2) \qquad H^i(\pi, M) = H^i(X, M) \ 0 \le i \le 1, \ H^2(\pi, M) \hookrightarrow H^2(X, M)$$

regardless of the characteristic of $\mathbb{F}$. We have the exact sequence

$$(H^1(K, \mathbb{F}) \otimes M)^{\pi^t} \to H^2(\pi^t, M) \to H^2(\pi, M).$$

We apply Lemma 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 3.5. *Let X be a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. There is a constant $\gamma \in \mathbb{R}_{>0}$ such that for any r-dimensional linear representation M of π^t over a prime field $\mathbb{F}$ of characteristic prime to p one has*

$$\dim_{\mathbb{F}} H^2(\pi^t, M) \le \gamma \cdot r.$$

Proof. By the prime to ℓ-alteration theorem [Tem17, Thm. 1.2.5] by de Jong-Gabber-Temkin, there is an alteration $h : Y \to X$ of degree δ a power of p, with a smooth projective compactification $Y \hookrightarrow \bar{Y}$ such that $\bar{Y} \setminus Y$ is a strict normal crossings divisor.

Let d be the dimension of X. For $x \in H^2(X, M), y \in H_c^{2d-2}(X, M^\vee)$, one has $\mathrm{Tr}(h^*x \cup h^*y) = \delta(x \cup y) \in H_c^{2d}(X, \mathbb{F})$ where the trace map $\mathrm{Tr} : H_c^{2d}(Y, \mathbb{F}) \to H_c^{2d}(X, \mathbb{F})$ is an isomorphism. (We do not write Tate twists as k is algebraically closed). If $h^*x = 0$, then $h^*x \cup h^*y = 0$; as δ is prime to the characteristic of $\mathbb{F}$, this implies $x \cup y = 0$. Thus if $h^*x = 0$ then $x \cup H_c^{2d-2}(X, M^\vee) = 0$, which by Poincaré duality [SGA4, Exposé XVIII, Théorème 3.2.5] implies that $x = 0$. We conclude that $h^* : H^2(X, M) \to H^2(Y, h^*M)$ is injective, thus by Corollary 3.4 that $h^* : H^2(\pi^t, M) \to H^2(\pi_1^t(Y), h^*M)$ is injective as well. In other words, we may assume that X has a smooth projective compactification $X \hookrightarrow \bar{X}$ such that $\bar{X} \setminus X$ is a strict normal crossings divisor.

We apply [EK16, Theorem 1.1 b)] (using the definitions of the Lefschetz condition introduced in [SGA2, X, pp. 89-90]) to reduce to the case where $d = 2$. Let $\bar{C}$ be smooth ample curve on $\bar{X}$ in good position with respect to $\bar{X} \setminus X$. Set $C = X \cap \bar{C}$. By purity [SGA4, XVI, Théorème. 3.7], the Gysin morphism $H^0(C, M) \to H_C^2(X, M)$ is an isomorphism, thus the sequence $H^0(C, M) \to H^2(X, M) \to H^2(X \setminus C, M)$ is exact. As $\dim_{\mathbb{F}} H^0(C, M) \le r$ we may assume that X is affine and we compute $H^2(X, M)$. By Poincaré duality, Deligne's theorem [Ill81, Corollaire 2.9], and the tameness of M, one has

$$\chi(X, M) = \chi_c(X, M^\vee) = r \cdot \chi(X, \mathbb{F}).$$

By Artin's vanishing theorem [SGA4, Corollaire 3.5] we have $H^3(X, M) = H^4(X, M) = 0$. Thus

$$\dim_{\mathbb{F}} H^2(X, M) \leq \dim_{\mathbb{F}} H^1(X, M) + r \cdot \chi(X, \mathbb{F}).$$

We apply Claim 2.6 and (3.2) to $H^1(X, M)$.

It remains to bound $|\chi(X, \mathbb{F})|$ independently of the choice of $\mathbb{F}$. This is a general argument for a normal crossings divisor compactification $(\bar{X}, D)$. Write $C = \cup_i C_i = \bar{X} \setminus X$, where the C_i are the irreducible components. For $I = \{i_1, \ldots, i_s\}$, set $C_I = \cap_{i \in I} C_i$. Then

$$\chi(X, \mathbb{F}) = \chi(\bar{X}, \mathbb{F}) + \sum_{|I|=a} (-1)^a \chi(C_I, \mathbb{F}).$$

On the other hand, the integers $|\chi(\bar{X}, \mathbb{F})|, |\chi(C_I, \mathbb{F})|$ are seen to be independent of the choice of $\mathbb{F}$, as follows. The cohomology groups $H^i(-, \mathbb{Z}_\ell)$ are modules of finite type over $\mathbb{Z}_\ell$. If V is a finite type $\mathbb{Z}_\ell$-module, so $V = L \oplus T$ where L is free and $T = \oplus_1^t \mathbb{Z}_\ell/\ell^{n_i}, n_i \geq 1$ is torsion, one has $\dim_{\mathbb{F}_\ell}(V \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell) = \dim_{\mathbb{Q}_\ell}(V \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) + t$ while $\dim_{\mathbb{F}_\ell} \mathrm{Ker}(\cdot \ell : V \to V) = t$. This together with the (universal coefficient) exact sequences (for $i \geq 0$)

$$0 \to H^i(-, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell \to H^i(-, \mathbb{F}_\ell) \to$$
$$\mathrm{Ker}(\cdot \ell : (H^{i+1}(-, \mathbb{Z}_\ell) \to H^{i+1}(-, \mathbb{Z}_\ell))) \to 0.$$

shows $\chi(-, \mathbb{F}_\ell) = \chi(-, \mathbb{Q}_\ell)$ and reduces the problem to the independence of ℓ of $|\chi(\bar{X}, \mathbb{Q}_\ell)|, |\chi(C_I, \mathbb{Q}_\ell)|$. This last point can be seen in at least two ways. We can argue using Deligne's purity of cohomology [Del80, Théorème 3.3.1] out of which we conclude that even the single dimensions $\dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell)$ and $\dim_{\mathbb{F}} H^i(C_I, \mathbb{Q}_\ell)$ do not depend on $\ell \neq p$. Or if $k = \bar{\mathbb{F}}_p$ we argue that χ is the difference of the degree of the denominator and the degree of the numerator of the zeta function. If $k \neq \bar{\mathbb{F}}_p$ one reduces to this case by specialization. Either way this finishes the proof.

$\square$

We now deduce the finite presentation of $\pi_1(X)^{(p')}$ (see the introduction).

Corollary 3.6. *Let X be a smooth connected quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. Then $\pi_1(X)^{(p')}$ is finitely presented.*

Proof. We set $\pi^{(p')} = \pi_1^{(p')}(X, x)$ to simplify the notation, which by Proposition 3.2 is finitely generated. We apply Theorem 1.3 to an r-dimensional linear representation M of $\pi^{(p')}$. As the order $|\pi^{(p')}|$ is

prime to p as a supernatural number, one has $H^2(\pi^{(p')}, M) = 0$ from [Ser97, §3. Proposition 11, Corollary 2] when $\mathbb{F} = \mathbb{F}_p$. We now assume that the characteristic of $\mathbb{F}$ is prime to p. We define the exact sequence

$$1 \to K^{(p')} \to \pi^t \to \pi^{(p')} \to 1$$

yielding the exact sequence of cohomology

$$(3.3) \qquad (H^1(K^{(p')}, \mathbb{F}) \otimes_{\mathbb{F}} M)^{\pi^{(p')}} \to H^2(\pi^{(p')}, M) \to H^2(\pi^t, M).$$

By [RZ10, Lemma 3.4.1(d)], applied with $K = K^{(p')}$, and the formation $\mathcal{C}$ of finite groups of order prime to p, one has

$$(3.4) \qquad\qquad\qquad H^1(K^{(p')}, \mathbb{F}) = 0.$$

Thus (3.4) implies that the term on the left in (3.3) vanishes. We now apply Proposition 3.5 to the term on the right to finish the proof. $\qquad\square$

4. THE TAME FUNDAMENTAL GROUP OF A SMOOTH QUASI-PROJECTIVE VARIETY ADMITTING A GOOD COMPACTIFICATION

The aim of this section is to prove Theorem 1.1. We set $\pi^t = \pi_1^t(X)$ to simplify the notation. Given Proposition 3.5 which is true even if one does not have a good compactification, it remains to bound $H^2(\pi^t, M)$ linearly in r when M is defined over $\mathbb{F} = \mathbb{F}_p$. By $\underline{M}$ we denote the local system associated to M.

Proposition 4.1. *Let X be as in Theorem 1.1, M be a linear representation of $\pi_1^t(X)$ defined over $\mathbb{F}_p$. Then there is an injective $\mathbb{F}_p$-linear map $H^2(\pi_1^t(X), M) \to H^2(\bar{X}, j_*\underline{M})$.*

Proof. For each open normal subgroup $U \lhd_o \pi^t$ we define the fibre square

$$
\begin{array}{ccc}
X_U & \xrightarrow{j_U} & \bar{X}_U \\
h_U \downarrow & & \downarrow \bar{h}_U \\
X & \xrightarrow{j} & \bar{X}
\end{array}
$$

where h_U is the tame Galois cover defined by $\pi^t \to G_U = \pi^t/U$ and $\bar{X}_U$ is the normalization of $\bar{X}$ in the field of functions $k(X_U)$ of X_U. Then $j_{U*}\underline{M}^U$ is a G_U-equivariant constructible sheaf on $\bar{X}_U$. By [Suw79, Theorem], Grothendieck's spectral sequence [Gro57, Théorème 5.2.1]

$$E_2^{ab} = H^a(G_U, H^b(\bar{X}_U, j_{U*}\underline{M}^U)) \Rightarrow H^{a+b}(\bar{X}_U, G_U, j_{U*}\underline{M}^U).$$

exists where on the right stands equivariant cohomology. By definition for $U \subset \mathrm{Ker}(\pi^t \to GL(M))$, $\underline{M}^U = h_U^* \underline{M}$ thus

(4.1) $$\bar{h}_{U*}^{G_U} j_{U*} \underline{M}^U = j_* \underline{M},$$

where $\bar{h}_{U*}^{G_U}$ is the G_U-invariant direct image functor.

As j is a normal crossings compactification, $\bar{h}_U$ is numerically tame in the sense of [KS10, Section 5], see [KS10, Theorem 5.4 (a)]. Thus by [Gro57, Corollaire p.204] one has

(4.2) $$R^{>0} \bar{h}_{U*}^{G_U} j_{U*} \underline{M}^U = 0,$$

$$H^n(\bar{X}_U, G_U, j_{U*} \underline{M}^U) = H^n(\bar{X}, j_* \underline{M}).$$

We conclude that the spectral sequence E_2^{ab} yields for each U a short exact sequence

$$H^0(G_U, H^1(\bar{X}_U, j_{U*}\underline{M}^U)) \to H^2(G_U, H^0(\bar{X}_U, j_{U*}\underline{M}^U)) \to H^2(\bar{X}, j_*\underline{M})$$

which is compatible with the restriction maps as we make U smaller. For U contained in $\mathrm{Ker}\big(\pi_1^t(X) \to GL(M)\big)$, we have $H^1(\bar{X}_U, j_{U*}\underline{M}^U) = H^1(\bar{X}_U, \mathbb{F}_p) \otimes_{\mathbb{F}_p} M$ where on the right $\otimes_\mathbb{F} M$ means the $\mathbb{F}_p$-vector space as a G_U-representation. There is an open (normal if we wish) subgroup $\bar{V} \triangleleft_o \pi_1(\bar{X}_U)$ such that the induced map $H^1(\bar{X}_U, \mathbb{F}_p) \to H^1((\bar{X}_U)_{\bar{V}}, \mathbb{F}_p)$ is equal to 0. We take $V \triangleleft_o \pi^t$ open normal and contained in the inverse image of $\bar{V}$ under the surjection $\pi_1^t(X_U) \to \pi_1(\bar{X}_U)$. Then the map $H^1(\bar{X}_U, j_{U*}\underline{M}^U) \to H^1(\bar{X}_V, j_{V*}\underline{M}^V)$ is equal to zero. We conclude that

$$\varinjlim_U H^2(G_U, M^U) \to H^2(\bar{X}, j_*\underline{M})$$

is injective. By [NSW13, Proposition 1.2.5] we have

$$H^2(\pi^t, M) = \varinjlim_U H^2(G_U, M^U)$$

where U ranges over all open normal subgroups of π^t and $G_U = \pi^t/U$. This finishes the proof.

$\square$

Proof of Theorem 1.1. As j is a normal crossings compactification, we apply [EK16, Theorem 1.1 b)] (using the definitions of the Lefschetz condition introduced in [SGA2, X, pp. 89-90] and [SGA2, X, Exemple 2.2]) to reduce to the case of dimension equal to 2. We fix a linear representation M of rank r of π^t over $\mathbb{F}_p$. For $U = \mathrm{Ker}(\pi^t \to GL(M))$ we define $h_U, \bar{h}_U$ as in the proof of Proposition 4.1. Then $h_U^* M$ is a trivial representation of U. We set $L = H^0(X_U, h_U^* \underline{M})$ for its r-dimensional vector space of global sections. We define $\mathcal{L} = L \otimes_{\mathbb{F}_p} \mathcal{O}_{\bar{X}_U}$, a locally

free sheaf of rank r which admits an action of the Frobenius F_U on $\bar{X}_U$ with invariant sections L. The Artin-Schreier sequence on $(\bar{X}_U)_{\text{ét}}$

$$0 \to \underline{L} \to \mathcal{L} \xrightarrow{1 \otimes F_U - 1} \mathcal{L} \to 0$$

is an exact sequence of G_U-equivariant sheaves. By definition, we have $\underline{L} = j_{U*}h_U^*\underline{M}$. Applying (4.1) and (4.2) we obtain the induced exact sequence

$$0 \to j_*\underline{M} \to \mathcal{M} \xrightarrow{\psi} \mathcal{M} \to 0$$

on $\bar{X}_{\text{ét}}$ where $\mathcal{M} = \bar{h}_{U*}^{G_U}\mathcal{L}$, $\psi = \bar{h}_{U*}^{G_U}(1 \otimes F_U - 1)$. On the other hand, the map ψ is of the form $\psi_F - 1$, where $\psi_F = \bar{h}_{U*}^{G_U}(1 \otimes F_U)$ is a p-linear endomorphism of $\mathcal{M}$. Since $H^1(\bar{X}, \mathcal{M})$ is finite dimensional over k, we have that

$$\psi_F - 1 : H^1(\bar{X}, \mathcal{M}) \to H^1(\bar{X}, \mathcal{M})$$

is surjective, by [SGA7.2, XXII, Proposition 1.2]. Similarly the finite dimensionality of $H^2(\bar{X}, \mathcal{M})$ and [SGA7.2, XXII, (1.0.10) and Proposition 1.1] imply that

$$H^2(\bar{X}, j_*\underline{M}) \otimes_{\mathbb{F}_p} k \subset H^2(\bar{X}, \mathcal{M})$$

and it suffices to bound the dimension over k of the right hand side.

Since $\bar{X}_U$ is normal of dimension 2, hence Cohen-Macaulay, $\bar{X}$ is smooth, and $\bar{h}_U$ is finite and surjective, we have that $\bar{h}_U$ is flat, and hence $\bar{h}_{U*}\mathcal{L}$ is locally free on $\bar{X}$. As a consequence, the invariant direct image subsheaf $\mathcal{M} \subset \bar{h}_{U*}\mathcal{L}$, which is locally a direct summand (by tameness), is also locally free.

By definition

$$(4.3) \qquad\qquad\qquad \bar{h}_U^*\mathcal{M} \subset \mathcal{L}$$

and by construction this inclusion restricts to an isomorphism on X_U. On the other hand, we claim that the natural inclusion

$$(4.4) \qquad\qquad \mathcal{L} \otimes_{\mathcal{O}_{\bar{X}_U}} \bar{h}_U^*\mathcal{O}_{\bar{X}}(-D) \subset \mathcal{L}$$

(determined by the ideal sheaf $\bar{h}_U^*\mathcal{O}_{\bar{X}}(-D) \subset \mathcal{O}_{\bar{X}_U}$) factors through an inclusion compatible with (4.3

$$(4.5) \qquad\qquad \mathcal{L} \otimes_{\mathcal{O}_{\bar{X}_U}} \bar{h}_U^*\mathcal{O}_{\bar{X}}(-D) \subset \bar{h}_U^*\mathcal{M}$$

where D is the reduced normal crossings divisor $\bar{X} \setminus X$. Indeed, as both sheaves in 4.5 are locally free, both are subsheaves of $\mathcal{L}$, $\bar{X}_U$ is normal, and $\bar{h}_U^*\mathcal{M} = \mathcal{L}$ on X_U, it is enough to check the inclusion in restriction to $\Delta \to \bar{X}_U$ where $\Delta = \text{Spec}(R)$, R is a complete discrete valuation ring with residue field k, so that Δ is in good position with respect to D (that is, Δ is the complete local ring on $\bar{C}$ of a point

$x \in D \cap \bar{C}$, with $\bar{C}$ chosen as in Remark 3.1). Then $\bar{X}_U \times_{\bar{X}} \Delta \to \Delta$ is a disjoint union of Kummer covers $T \to \Delta$, where $T = \mathrm{Spec}(S)$, $S = R[t]/(t^n - x \cdot u)$, $u \in R^\times$, x is the uniformizer of R, and $(n,p) = 1$, which are permuted by the quotient of G_U by the stabilizer of T. Thus $\mathcal{M}|_\Delta$, as an R-module, is of the shape $\oplus_1^r t^{a_i} R$, where $0 \le a_i < n$, so $\mathcal{M}|_\Delta \otimes_R S = \oplus_1^r t^{a_i} S \supset \oplus_1^t t^n S = \mathcal{L} \otimes_{\mathcal{O}_{\bar{X}_U}} \bar{h}_U^* \mathcal{O}_{\bar{X}}(-D) \otimes_{\mathcal{O}_{\bar{X}_U}} S$. So we conclude.

By Serre duality $H^2(\bar{X}, \mathcal{M}) = H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}})^\vee$ where $\omega_{\bar{X}}$ is the dualizing sheaf of $\bar{X}$. Let $\mathcal{A}$ be a very ample line bundle on $\bar{X}$ such that $\mathcal{A} \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(D) = \mathcal{B}$ is very ample as well. Let C and C' be smooth curves on $\bar{X}$ such that $\mathcal{B} = \mathcal{O}_{\bar{X}}(C) = \mathcal{O}_{\bar{X}}(C')$, and C, C' intersect transversally and are both in good position with respect to D in the sense of Remark 3.1 (since $\mathcal{B}$ is very ample on the smooth projective surface $\bar{X}$, such curves C, C' may be obtained by Bertini's theorem). Then

$$\bar{h}_U^*(\mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C)) \subset$$
$$\mathcal{L}^\vee \otimes_{\mathcal{O}_{\bar{X}_U}} \bar{h}_U^*(\omega_{\bar{X}}(D - C)) = \mathcal{L}^\vee \otimes_{\mathcal{O}_{\bar{X}_U}} \bar{h}_U^* \mathcal{A}^{-1}.$$

Since $\mathcal{L}$ is a trivial vector bundle, and by choice, $\omega_{\bar{X}}^\vee(C - D) = \mathcal{A}$ is very ample on $\bar{X}$, we deduce that

$$H^0(\bar{X}_U, \bar{h}_U^*(\mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}_U}} \omega_{\bar{X}}(-C))) =$$
$$H^0(\bar{X}_U \times_{\bar{X}} C, \bar{h}_U^*(\mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}_U}} \omega_{\bar{X}}(-C'))) = 0$$

which implies

$$H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C)) = H^0(C, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C')|_C) = 0.$$

(We used here the fact that the global sections of a vector bundle on an integral scheme inject in the one of its pullback to a an integral scheme, dominant and proper over the original one.) We have exact sequences of sheaves

$$0 \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C) \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}} \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_C \to 0,$$

$$0 \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C')|_C \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_C \to \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_{C \cap C'} \to 0$$

coming from the ideal sheaves $\mathcal{O}_{\bar{X}}(-C) \subset \mathcal{O}_{\bar{X}}$ and $\mathcal{O}_{\bar{X}}(-C')|_C \subset \mathcal{O}_C$, determined by closed subschemes $C \subset \bar{X}$ and $C \cap C' \subset C$. These lead to left exact sequences on global sections

$$(4.6) \qquad 0 \to H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C)) \to H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}})$$
$$\to H^0(C, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_C),$$

and

$$(4.7) \quad 0 \to H^0(C, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}(-C')|_C) \to H^0(C, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_C)$$
$$\to H^0(C \cap C', \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_{C \cap C'})$$

In both the sequences, the left extreme terms vanish, as seen above. So the restriction maps

$$H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}) \to H^0(C, (\mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}})|_C)$$
$$\to H^0(C \cap C', \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_{C \cap C'})$$

are injective, out of which we conclude

$$(4.8)$$
$$\dim_k H^0(\bar{X}, \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}) \le \dim_k H^0(C \cap C', \mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_{C \cap C'})$$
$$\le r \cdot c_1(\mathcal{B})^2.$$

Here, for the last inequality, we note that since C, C' are nonsingular projective curves meeting transversally, both representing the class of the very ample divisor $c_1(\mathcal{B})$ on the smooth projective surface $\bar{X}$, the intersection scheme $C \cap C'$ consists of $c_1(\mathcal{B})^2$ distinct points; we then observe that $\mathcal{M}^\vee \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}}|_{C \cap C'}$ is a vector bundle of rank r on this 0-dimensional reduced k-scheme. This finishes the proof.

$\square$

5. REMARKS

5.1. **Perfect fields.** Theorem 1.1 remains true if we relax the assumption on k being algebraically closed under the following assumptions:

(i) k is perfect;
(ii) the Galois group G_k of k is finitely presented.

For example, k could be a finite field.

Proof. We have the homotopy exact sequence

$$1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X) \to G_k \to 1$$

[SGA1, Théorème 9.6.1]. On the other hand, tameness is preserved by base change thus the composite $\pi_1(X_{\bar{k}}) \to \pi_1(X) \to \pi_1^t(X)$ factors through $\pi_1^t(X_{\bar{k}}) \to \pi_1^t(X)$.

Claim 5.1. $\pi_1^t(X_{\bar{k}}) \to \pi_1^t(X)$ is injective.

To check this we have a criterion.

Claim 5.2. $\pi_1^t(X_{\bar{k}}) \to \pi_1^t(X)$ is injective if and only if for any $Y \to X_{\bar{k}}$ finite étale tame connected, there is $Z \to X$ finite étale tame connected such that one has a factorization

$$
\begin{array}{ccc}
Y & \overset{\exists}{\dashrightarrow} & Z \\
\downarrow & & \downarrow \\
X_{\bar{k}} & \longrightarrow & X
\end{array}
$$

Proof. We have the base points $x \in X_{\bar{k}} \mapsto x \in X$, so

$$\pi_1^t(X_{\bar{k}}) = \mathrm{Aut}(\omega_x), \omega_x : \mathrm{Rev}^t(X_{\bar{k}}) \to \mathrm{Sets}, (\sigma : Y \to X_{\bar{k}}) \mapsto \sigma^{-1}(x)$$
$$\pi_1^t(X_t) = \mathrm{Aut}(\omega_x), \omega_x : \mathrm{Rev}^t(X) \to \mathrm{Sets}, (\pi : Z \to X) \mapsto \pi^{-1}(x).$$

Here $\mathrm{Rev}^t(X_{\bar{k}})$ resp. $\mathrm{Rev}^t(X)$ denotes the category of finite étale tame covers, and Sets the one of sets. So $\pi_1^t(X_{\bar{k}}) \to \pi_1^t(X)$ injective is equivalent to the $\mathrm{Aut}(\omega_x)$-representation $\sigma^{-1}(x)$ for each σ to be a sub-$\mathrm{Aut}(\omega_x)$-representation of a $\mathrm{Aut}(\omega_x)$ representation $\pi^{-1}(x)$. □

We have fixed a geometric point x which defines the algebraic closure $k \hookrightarrow \bar{k} \hookrightarrow k(x)$. As k is perfect, $\bar{k}$ is the separable closure thus the surjection $\pi_1(X) \to G_k$ factors through $\pi_1^t(X) \to G_k$. As $X_{\bar{k}}$ is of finite type, any $\sigma : Y \to X_{\bar{k}}$ finite étale tame is defined over a finite extension $k' \supset k$, so say $Y_{k'} \to X_{k'}$, where $k' \supset k$ is étale. Thus $Y_{k'} \to X_{k'} \to X$ is finite étale tame. This proves Claim 5.1. On the other hand, as the image of $\pi_1(X_{\bar{k}}) \to \pi_1(X)$ is normal, and $\pi_1(X_{\bar{k}}) \to \pi_1^t(X_{\bar{k}})$ is surjective, the image of $\pi_1^t(X_{\bar{k}}) \to \pi_1^t(X)$ is normal as well. Thus the homotopy exact sequence descends to an exact sequence

$$1 \to \pi_1^t(X_{\bar{k}}) \to \pi_1^t(X) \to G_k \to 1.$$

We now apply Claim 2.7 (i) to finish the proof. □

5.2. **Analogy with Deligne's canonical extension.** The complex

$$\mathcal{M} \overset{\psi}{\to} \mathcal{M}$$

in the proof of Proposition 4.1 is analog on $X_{\mathbb{C}}$ smooth over $\mathbb{C}$ with a normal crossings divisor compactification $j : X_{\mathbb{C}} \hookrightarrow \bar{X}_{\mathbb{C}}$ to the logarithmic de Rham complex $\Omega^{\bullet}_{X_{\mathbb{C}}}(\log(\bar{X}_{\mathbb{C}} \setminus X_{\mathbb{C}})) \otimes \mathcal{M}_{\mathbb{C}}$ where $\mathcal{M}_{\mathbb{C}}$ is Deligne's canonical extension associated to a complex local system $\underline{M}_{\mathbb{C}}$ [Del70, Proposition .5.2, p. 91]. The property that under the numerical tameness assumption this complex on $\bar{X}_{\text{ét}}$ computes $j_*\underline{M}$ is analog to the analytic property on the local monodromies along $\bar{X}_{\mathbb{C}} \setminus X_{\mathbb{C}}$ which force the condition $j_*\underline{M}_{\mathbb{C}} = Rj_*\underline{M}_{\mathbb{C}}$.

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