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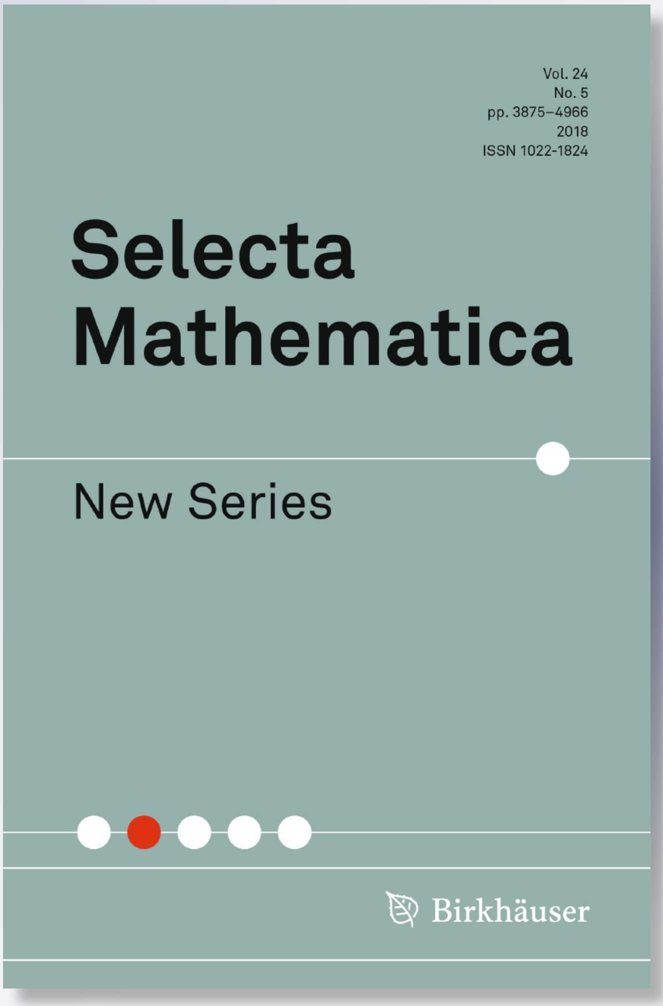

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Cohomologically rigid local systems and integrality

Hélène Esnault[1] · Michael Groechenig[1]

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Abstract We prove that the monodromy of an irreducible cohomologically complex rigid local system with finite determinant and quasi-unipotent local monodromies at infinity on a smooth quasiprojective complex variety X is integral. This answers positively a special case of a conjecture by Carlos Simpson. On a smooth projective variety, the argument relies on Drinfeld's theorem on the existence of ℓ-adic companions over a finite field. When the variety is quasiprojective, one has in addition to control the weights and the monodromy at infinity.

Mathematics Subject Classification 14D07 · 14G15 · 14F35

1 Introduction

Let X be a smooth connected quasiprojective complex variety, $j : X \hookrightarrow \bar{X}$ be a *good compactification*, that is a smooth compactification such that $D = \bar{X} \setminus X$ is a normal crossings divisor. An irreducible complex local system $\mathcal{V}$ is said to be *cohomologically rigid* if $\mathbb{H}^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V})) = 0$. The finite dimensional complex vector space $\mathbb{H}^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V}))$ is the Zariski tangent space at the moduli point of $\mathcal{V}$ of the Betti

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moduli stack of complex local systems of rank r with prescribed determinant and prescribed local monodromies along the components of D (see Sect. 2). So a cohomologically rigid complex local system is *rigid*, that is its moduli point is isolated, and in addition it is smooth.

Simpson conjectures that *rigid irreducible complex local systems with torsion determinant and quasi-unipotent monodromies around the components of D are of geometric origin*. In particular, they should be *integral*, where a complex local system is said to be integral if it is coming by extension of scalars from a local system of projective $\mathcal{O}_L$-modules of finite type, where $\mathcal{O}_L$ is the ring of integers of a number field $L \subset \mathbb{C}$. See [15, Conj. 0.1, 0.2] for the formulation of the conjectures in the projective case. We prove:

Theorem 1.1 *Let X be a smooth connected quasiprojective complex variety. Then irreducible cohomologically rigid complex local systems with finite determinant and quasi-unipotent local monodromies around the components at infinity of a good compactification are integral.*

When X is projective, a first proof of Theorem 1.1 using F-isocrystals and p to ℓ-companions [1] has been given in [10]. The short proof presented in this note only uses the ℓ to ℓ' companions, the existence of which has been proved by Drinfeld [9, Thm. 1.1]. We outline it in the projective case.

We use the following criterion for integrality. Let $\mathcal{V}$ be rigid complex local system which comes by extension of scalars from a local system $\mathcal{V}_{\mathcal{O}_{K,\Sigma}}$ of projective $\mathcal{O}_{K,\Sigma}$-modules where $K \subset \mathbb{C}$ is a number field, Σ is a finite set of places of K and $\mathcal{O}_{K,\Sigma}$ is the ring of Σ-integers of K. For any place λ of K, let K_λ be the completion of K at λ, $K_\lambda \subset \bar{K}_\lambda$ be an algebraic closure, $\mathcal{O}_{K_\lambda} \subset \bar{\mathcal{O}}_{K_\lambda}$ be the underlying extension of rings of integers. Then $\mathcal{V}$ is integral if and only if for any λ in Σ, $\mathcal{V}_{\bar{K}_\lambda} = \mathcal{V}_K \otimes_K \bar{K}_\lambda$ comes by extension of scalars from a local system $\mathcal{V}_{\bar{\mathcal{O}}_{K_\lambda}}$ of free $\bar{\mathcal{O}}_{K_\lambda}$-modules. The local system of projective $\mathcal{O}_K$-modules is defined as the inverse image of $\prod_{\lambda\in\Sigma} \mathcal{V}_{\bar{\mathcal{O}}_{K_\lambda}} \subset \prod_{\lambda\in\Sigma} \mathcal{V}_{\bar{K}_\lambda}$ via the localization map $\mathcal{V}_{\mathcal{O}_{K,\Sigma}} \to \prod_{\lambda\in\Sigma} \mathcal{V}_{\bar{K}_\lambda}$. (See [3, Cor. 2.3, Cor. 2.5] for the same criterion expressed in term of traces of representations).

We fix natural numbers r and d. The moduli stack of irreducible complex local systems of rank r and determinant of order d is of finite type. Thus the set of isomorphism classes of such local systems which are cohomologically rigid has finite cardinality $N(r, d)$. For any algebraic closed field C of characteristic 0, $N(r, d)$ is also the number of isomorphism classes of irreducible C-local systems of rank r and determinant of order d.

The $N(r, d)$ complex local systems come by extension of scalars of local systems $\mathcal{V}_K$ defined over a number field $K \subset \mathbb{C}$. As the topological fundamental group is finitely generated, the $N(r, d)$ local systems $\mathcal{V}_K$ come by extension of scalars from local systems $\mathcal{V}_{\mathcal{O}_{K,\Sigma}}$ of free $\mathcal{O}_{K,\Sigma}$-modules, where Σ is a finite set of places of K and $\mathcal{O}_{K,\Sigma}$ is the ring of Σ-integers of K. We want to show that $\mathcal{V}_{\bar{K}_\lambda} = \mathcal{V}_{K_\lambda} \otimes_{K_\lambda} \bar{K}_\lambda$ comes from a local system defined over $\mathcal{O}_{\bar{K}_\lambda}$ for all λ in Σ.

One chooses a place λ of K which is not in Σ. It divides a prime number ℓ. We complete the finitely many $\mathcal{V}_{\mathcal{O}_{K,\Sigma}}$ considered at λ so as to obtain λ-adic lisse sheaves on X. We take a model X_S of X over a scheme S of finite type over $\mathbb{Z}$ such that X_S/S

is smooth, and take a closed point s of S of characteristic p prime to the order of the residual representations, to ℓ, to d, and to the residual characteristics of the places in Σ. Then the λ-adic sheaves descend to lisse sheaves on $X_{\bar{s}}$, where $k(s) \hookrightarrow k(\bar{s}) \simeq \bar{\mathbb{F}}_p$ is an algebraic closure of the residue field of s.

By a variant of an argument of Simpson (see Proposition 3.1), after finite base change on s, the lisse sheaves descend to arithmetic lisse sheaves. They are irreducible on $X_{\bar{s}}$ and can be taken to have finite determinant. By Drinfeld's existence theorem, for all places λ' of K not dividing p, in particular for all λ' in Σ, they have λ'-companions, which are $\bar{K}_{\lambda'}$-lisse sheaves, thus by definition, come from $\bar{\mathcal{O}}_{K_{\lambda'}}$-lisse sheaves. Using purity, the L-function of the trace 0 endomorphisms of the companions, base change, local acyclicity, and Betti to étale comparison, one shows that viewed back as representations of the topological fundamental group with values in $GL(r, \bar{\mathcal{O}}_{K_{\lambda'}})$, the λ'-companions define on X the required number $N(r, d)$ of cohomologically rigid non-isomorphic $\bar{K}_{\lambda'}$-local systems of the type we want, all coming from $\bar{\mathcal{O}}_{K_{\lambda'}}$-local systems.

To generalize the argument to the quasiprojective case and quasi-unipotent monodromies around the components at infinity, one needs that specializiation, purity, base change, local acyclicity are applicable in the non-proper case. This being acquired, we bound the problem by bounding the rank r, and fixing two natural numbers d and h such that the order of the determinant divides d and the order of the eigenvalues of the monodromies around the components at infinity divides h. There are finitely many isomorphism classes of cohomologically rigid local systems with those invariants, say $N(r, d, h)$. Deligne's theorem on compatible systems on curves [6, Thm. 9.8] implies that those data are preserved by passing to companions, which enables us to make the counting argument with $N(r, d)$ replaced by $N(r, d, h)$.

2 Cohomologically rigid local systems with prescribed monodromy around the components of the divisor at infinity

Let X be a smooth connected quasiprojective complex variety, $j : X \hookrightarrow \bar{X}$ be a good compactification, $D = \bar{X} \setminus X$ be the normal crossings divisor at infinity. We write $D = \cup_{i=1}^N D_i$ where the D_i are the irreducible components. In this section we clarify the notion of *rigid irreducible complex local systems with torsion determinant and quasi-unipotent monodromies around the components of* D.

Let $U = \bar{X} \setminus D_{\text{sing}}$, where D_{sing} is the singular locus of D, and $j : X \xrightarrow{a} U \xrightarrow{b} \bar{X}$ be the open embeddings. We choose a complex point $x \in X$ and $r \in \mathbb{N}_{\geq 1}$. For each $i = 1, \ldots, N$, we fix a conjugacy class $\mathcal{K}_i \subset GL(r, \mathbb{C})$. It is the set of complex points of a subvariety of $GL(r)$ which we denote by the same letter $\mathcal{K}_i$. If $\Delta_i \subset U$ is a ball around $y_i \in D_i \cap U$ and $x_i \in \Delta_i^\times$ with $\Delta_i^\times = \Delta_i \setminus D_i \cap \Delta_i$, the fundamental group $\pi_1(\Delta_i^\times, x_i)$ is freely generated by the local monodromy T_i, determined uniquely up to sign, around $D_i \cap \Delta_i$, so $\pi_1(\Delta_i^\times, x_i) = \mathbb{Z} \cdot T_i$. Any complex linear local system $\mathcal{V}$ defines by restriction a complex linear local system $\mathcal{V}|_{\Delta_i^\times}$ on $\Delta_i^\times$. We say that $\mathcal{V}|_{\Delta_i^\times}$ *is defined by* $\mathcal{K}_i$ if the image of T_i lies in $\mathcal{K}_i$ in its monodromy representation. We say that $\mathcal{V}$ *is defined by* $\mathcal{K}_i$ *along* D_i if $\mathcal{V}|_{\Delta_i^\times}$ is *for all points* $y_i \in D_i \cap U$. As D_i is irreducible, $D_i \cap U$ is smooth and connected, so the condition that $\mathcal{V}$ be defined by $\mathcal{K}_i$ along D_i is equivalent to the condition that $\mathcal{V}|_{\Delta_i^\times}$ is defined by $\mathcal{K}_i$ for the choice of one single $y_i \in D_i \cap U$. For each $i = 1, \ldots, N$, we fix one such $y_i \in D_i \cap U$, Δ_i and $x_i \in \Delta_i^\times$ as above. We also fix a rank 1 local system $\mathcal{L}$ of order d defined by a character $\chi_{\mathcal{L}} : \pi_1^{\mathrm{top}}(X, x) \to \mu_d(\mathbb{C}) \subset \mathbb{C}^\times$.

Let $T = \mathrm{Spec}(B)$ be a complex connected affine variety of finite type. A *T-local system* $\mathcal{V}_T$ over X is a local system of locally free T-modules, or equivalently a locally free T-sheaf W together with a representation $\rho : \pi_1^{\mathrm{top}}(X, x) \to \mathrm{Aut}(W)$. The *rank* of $\mathcal{V}_T$ is the rank of W. A *geometrically irreducible T-local system* is a T-local system $\mathcal{V}_T$ such that $\mathcal{V} \times_T \bar{\eta}$ is an irreducible local system over $\bar{\eta}$ for all geometric generic points $\bar{\eta}$ of T. We define a stack from the category of affine varieties of finite type over K to the category of groupoids, sending T to the groupoid of isomorphism classes of geometrically irreducible T-local systems $\mathcal{V}_T$ of rank r together with an isomorphism $\wedge^r \mathcal{V}_T \cong \mathcal{L} \otimes_{\mathbb{C}} T$. We denote it by $\mathsf{IrrLoc}(X, r, \mathcal{L})$. For each $i = 1, \ldots, N$, we choose a basis of $(\mathcal{V}_T)_{x_i}$. We then define the substack $\underline{M}$ of $\mathsf{IrrLoc}(X, r, \mathcal{L})$ by the condition that the image of T_i by the monodromy representation of $\mathcal{V}_T|_{\Delta_i^\times}$ is a section of $\mathcal{K}_i \times_K T \subset GL(r) \times_K T$. As $\mathcal{K}_i \subset GL(r)$ is locally closed, $\underline{M} \subset \mathsf{IrrLoc}(X, r, \mathcal{L})$ is a locally closed substack.

If for any $i = 1, \ldots, N$, the $\mathcal{K}_i$ is the conjugacy class of a quasi-unipotent matrix, we say that a point $[\mathcal{V}] \in \underline{M}(K)$ has *quasi-unipotent monodromies along the components of D*. This implies in particular that the varieties $\mathcal{K}_i$ are defined over a number field K, and consequently $\underline{M}$ is defined over the same number field. We shall need two properties of $\underline{M}$.

Proposition 2.1 *Assume that for any $i = 1, \ldots, N$, the $\mathcal{K}_i$ is the conjugacy class of a quasi-unipotent matrix. Then $\underline{M}$ is an algebraic stack of finite type defined over the number field K. In particular, it has finitely many* 0*-dimensional irreducible components.*

Proof As $\underline{M}$ is a locally closed substack of $\mathsf{IrrLoc}(X, r, \mathcal{L})$, it suffices to show that the stack $\mathsf{IrrLoc}(X, r, \mathcal{L})$ is an algebraic stack of finite type.

We recall the classical argument for this fact. First the automorphism group of any $\mathcal{V}_T$ is $\mu_r(T)$, which defines the finite constant group scheme μ_r. Since the topological fundamental group of X is a finitely presented group, we equivalently consider a finitely presented group Γ and the stack $\mathsf{IrrRep}(\Gamma, \mathcal{L})$ of geometrically irreducible families of Γ-representations of rank r given with an isomorphism of its determinant to a fixed $\mathcal{L}$. At first we remark that this stack admits a fully faithful morphism to the stack of rank r Γ-representations $\mathsf{Rep}(\Gamma, \mathcal{L})$ with an isomorphism of its r-th exterior power with $\mathcal{L} := \mathcal{O}_T$ on which Γ acts by $\chi_{\mathcal{L}}$. We claim that this morphism is an open immersion. To see this we have to prove that for a T-family of Γ-representations, that is, a rank r-representation $\rho \colon \Gamma \to \mathrm{Aut}(W)$, where W is a locally free B-module of rank r, together with the determinant condition, there exists an open subset $T_0 \subset T$, such that the Γ-representation over a geometric point x of T is irreducible, if and only if x is a geometric point of T_0.

The Γ-representation ρ induces an action of Γ on the fibre bundle $\pi \colon \bigsqcup_{k=0}^r \mathsf{Gr}(W, k) \to T$, where $\mathsf{Gr}(W, k)$ is the Grassmann bundle of k-planes in W. We define T_0 as the complement of $\pi(\bigsqcup_{k=0}^{r}\mathsf{Gr}(W,k)^{\Gamma})$. Since π is proper, and the fixed point set is closed, we see that T_0 is an open subscheme of T. It is clear that for a geometric point x of T, the induced representation ρ_x is irreducible if and only if there does not exist an integer k, and a k-dimensional subspace fixed by Γ.

Since an open substack of a stack of finite type is also of finite type, we are now reduced to proving that $\mathsf{Rep}(\Gamma,\mathcal{L})$ is an algebraic stack of finite type. To see this we choose a presentation $\Gamma \simeq \langle r_1,\ldots,r_e|s_1,\ldots,s_f\rangle$ and $L_1,\ldots,L_e\in\mu_r(K)$, where K is a number field, such that $r_i\mapsto L_i$ defines the character $\chi_{\mathcal{L}}$. Let $\mathcal{R}(\Gamma,\mathcal{L})$ be the affine K-variety of finite type, which represents the functor

$$T\mapsto (A_1,\ldots,A_e)\in GL(r)_K^e$$

such that the relations $\det(A_j)=L_i$ for $j=1,\ldots,e$ and $s_i(A_1,\ldots,A_e)=1$ for $i=1,\ldots,f$ hold. This is by definition an affine variety of finite type over K. To such a tuple $(A_1,\ldots,A_e)$ one attaches the representation of Γ on $\mathcal{O}_T^{\oplus r}$ given by the A_i, and the isomorphism $e_1\wedge\ldots\wedge e_r\to 1$ of the determinant of this representation with $\mathcal{L}$. This construction induces an equivalence of the quotient stack $[\mathcal{R}(\Gamma,\mathcal{L})/SL(r)]$ with $\mathsf{Rep}(\Gamma,\mathcal{L})$. Hence $\mathsf{Rep}(\Gamma,\mathcal{L})$ is an algebraic stack of finite type over K. □

Remark 2.2 It follows from general theory that the stack $\underline{M}$ has a coarse moduli space. Indeed, according to [2, Thm. 5.1.5], there exists an algebraic stack $\underline{M}^{\mu_r}$, such that for any $z\in\underline{M}^{\mu_r}(T)$ we have $\mathsf{Aut}(z)=\{1\}$, and a morphism $\underline{M}\to\underline{M}^{\mu_r}$ which is the universal morphism to an algebraic stack with this property. This implies that for a K-scheme T, the groupoid $\underline{M}^{\mu_r}(T)$ is equivalent to a set. Hence the algebraic stack $\underline{M}^{\mu_r}$ is actually (equivalent to) an algebraic space. The universal property of the morphism $\underline{M}\to\underline{M}^{\mu_r}$ shows that $\underline{M}^{\mu_r}$ is a coarse moduli space.

The $\mathbb{C}$-points corresponding to 0-dimensional components are isolated points of the moduli space (so-called *rigid local systems*). Thus they are all defined over a finite extension of $\mathbb{Q}$.

Proposition 2.3 *The Zariski tangent space $T_{[\mathcal{V}]}$ at a point $[\mathcal{V}]\in\underline{M}$ associated to $\mathcal{V}$ defined over K is the finite dimensional K-vector space $H^1(U,a_*\mathcal{E}nd^0(\mathcal{V}))$. In particular, if $H^1(U,a_*\mathcal{E}nd^0(\mathcal{V}))=0$, the geometrically irreducible K-local system is rigid, and there are finitely many such.*

Proof We follow Deligne's line of proof. Set $K[e]=K[e]/(e^2)$. By definition $T_{[\mathcal{V}]}=\underline{M}(\mathrm{Spec}(K[e])$ where $[\mathcal{V}]=\underline{M}(\mathrm{Spec}(K[e]\otimes_{K[e]}K))$. So we want to identify $H^1(U,a_*\mathcal{E}nd^0(\mathcal{V}))$ with the set of isomorphism classes of $K[e]$-local systems $\mathcal{V}_{K[e]}$ of rank r, with an isomorphism $\wedge^r\mathcal{V}_T\cong\wedge^r\mathcal{V}\otimes_K T$ and with the following conditions:

0) $\mathcal{V}_{K[e]}\otimes_{K[e]}K\cong\mathcal{V}$;
1) for any complex point $y_i\in D_i\cap U$, and Δ_i a ball around y_i, $\mathcal{V}_{K[e]}|_{\Delta_i^\times}\cong\mathcal{V}|_{\Delta_i^\times}\otimes_K K[e]$.

We have to show that the $K[e]$-local systems with a fixed isomorphism $\wedge^r\mathcal{V}_T\cong\wedge^r\mathcal{V}\otimes_K T$ and fulfilling the conditions 0), 1) form a non-empty torsor under $a_*\mathcal{E}nd^0(\mathcal{V})$. So we choose a cover $X=\cup_\alpha\mathcal{X}_\alpha$ by balls $\mathcal{X}_\alpha$. On each $\mathcal{X}_\alpha$, $\mathcal{V}_{K[e]}|_{\mathcal{X}_\alpha}$

and $\mathcal{V}|_{\mathcal{X}_\alpha}$ are trivialized, thus one has an isomorphism $\mathcal{V}_{K[e]}|_{\mathcal{X}_\alpha} \cong \mathcal{V}|_{\mathcal{X}_\alpha} \otimes_K K[e]$. The constraint 0) implies that on $\mathcal{V}|_{\mathcal{X}_\alpha \cap \mathcal{X}_\beta}$ the composition of one such isomorphism by the inverse of the other one yields an isomorphism in $(\mathcal{E}nd^0(\mathcal{X}_\alpha \cap \mathcal{X}_\beta, \mathcal{V}), +) = (1 + \mathcal{E}nd^0(\mathcal{X}_\alpha \cap \mathcal{X}_\beta, \mathcal{V}), \times) \subset \mathrm{Aut}_{K[e]}(\mathcal{X}_\alpha \cap \mathcal{X}_\beta, \mathcal{V} \otimes_K K[e])$. This defines a cocycle, thus a cohomology class in $H^1(X, \mathcal{E}nd^0(\mathcal{V}))$. The constraint 1) implies that on $\mathcal{X}_\alpha \cap \mathcal{X}_\beta \cap \Delta_i^\times$ the cohomology class defined by the cocycle is trivial. This shows that the cohomology class has values in

$$H^1(U, a_*\mathcal{E}nd^0(\mathcal{V})) = \mathrm{Ker}\big(H^1(X, \mathcal{E}nd^0(\mathcal{V})) \to \oplus_{i=1}^N \oplus_{j=1}^{j_i} H^1(\Delta_{ij}^\times, \mathcal{E}nd^0(\mathcal{V}))\big),$$

where for each $i = 1, \ldots, N$, we covered D_i by $\cup_{j=1}^{j_i} \Delta_{ij} \supset D_i$ with $\Delta_{ij} \subset U$. One trivially checks that the change of isomorphisms $\mathcal{V}_{K[e]}|_{\mathcal{X}_\alpha} \cong \mathcal{V}|_{\mathcal{X}_\alpha} \otimes_K K[e]$ changes the cocycle with values in $a_*\mathcal{E}nd^0(\mathcal{V})$ by a coboundary. This finishes the proof of the cohomological part. The rest follows directly from Proposition 2.1.

Remark 2.4 Let $j_{!*}\mathcal{V}$ be the *intermediate* extension on $\bar{X}$ of a K-local system $\mathcal{V}$ on X. From [4, Prop. 2.1.11] one derives that there is an exact triangle $j_{!*}\mathcal{V} \to Rb_*a_*\mathcal{V} \to \mathcal{C}$ in the bounded derived category of $\bar{X}$ such that $\mathcal{C}$ is supported on D_{sing} and is concentrated in degrees ≥ 2. Thus it induces an isomorphism on $\mathbb{H}^1$. So Proposition 2.3 says $T_{[\mathcal{V}]} = \mathbb{H}^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V}))$. We also remark that if the local monodromies of $\mathcal{V}$ along the components of D are finite, then $j_{!*}\mathcal{V} = j_*\mathcal{V},\ j_{!*}\mathcal{E}nd^0\mathcal{V} = j_*\mathcal{E}nd^0\mathcal{V}$. □

3 Proof of Theorem 1.1

Let X be a smooth connected quasiprojective complex variety, $x \in X$ be a geometric point, $j : X \hookrightarrow \bar{X}$ be a good compactification, thus $D = \bar{X} \backslash X$ is a strict normal crossings divisor. We write $j : X \xrightarrow{a} U = \bar{X} \backslash D_{\mathrm{sing}} \xrightarrow{b} \bar{X}$. Let $\rho : \pi_1^{\mathrm{top}}(X, x) \to GL(r, \mathbb{C})$ be a complex linear representation, defining the local system $\mathcal{V}$.

We fix a natural number h and define the set $\mathcal{S}(r, d, h)$ consisting of isomorphism classes of rank r irreducible cohomologically rigid complex local systems $\mathcal{V}$ on X with determinant of order dividing d, such that the local monodromies at infinity, which are quasi-unipotent, have eigenvalues of order dividing h. There are finitely many local systems of rank 1 of order dividing d and once the Jordan type of the unipotent part of the monodromy representation along D_i is fixed, there are finitely many possibilities for $\mathcal{K}_i \subset GL(r, \mathbb{C})$ (see notations of Sect. 2). As there are finitely many such Jordan types, Proposition 2.3 implies that $\mathcal{S}(r, d, h)$ is finite, of cardinality $\mathfrak{N} = N(r, d, h)$. In addition, one has finitely many possibilities for $\mathcal{K}_i$ and the determinant $\mathcal{L}$, defining finitely many stacks $\underline{M}_m$. Let K_0 be a number field over which they are all defined are defined. The disjoint union $\underline{N} = \sqcup_m \underline{M}_m$ is a stack of finite type defined over K_0.

There is a connected regular scheme S of finite type over $\mathbb{Z}$ with a complex generic point $\mathrm{Spec}(\mathbb{C}) \to S$ such that $(j : X \hookrightarrow \bar{X}, x, D, D_J)$ is the base change from S to $\mathrm{Spec}(\mathbb{C})$ of $(j_S : X_S \hookrightarrow \bar{X}_S, x_S, D_S = \bar{X}_S \backslash X_S, D_{J,S} = \cap_{j \in J} D_{j,S})$ with the following properties. The scheme $\bar{X}_S$ is smooth projective over S, D_S is a relative normal crossings divisor with strata $D_{J,S}$, $X_S = \bar{X}_S \backslash D_S$, x_S is a S-point of X_S. This also defines $X_S \xrightarrow{a_S} U_S \xrightarrow{b_S} \bar{X}_S$. We say for short that the objects with lower index S are *models* over S of the objects without lower index S (which means that the latter ones are defined over $\mathbb{C}$).

As the $\mathcal{V}_i \in \mathcal{S}(r, d, h)$ are cohomologically rigid, they are in particular rigid. As in addition the local monodromies at infinity are quasi-unipotent, there is a number field $K \subset \mathbb{C}$ containing K_0 such that up to conjugacy the underlying complex linear representations factor as $\rho_i : \pi_1^{\mathrm{top}}(X, x) \to GL(r, K) \to GL(r, \mathbb{C})$, and such that the rank 1 local systems $\det(\rho_i)$ factor through $\pi_1^{\mathrm{top}}(X, x) \to \mu_d(K)$. As $\pi_1^{\mathrm{top}}(X, x)$ is finitely generated, there is a finite set Σ of places of K such that one has a factorization

$$\rho_i : \pi_1^{\mathrm{top}}(X, x) \xrightarrow{\rho_i^0} GL(r, \mathcal{O}_{K,\Sigma}) \to GL(r, K) \to GL(r, \mathbb{C}),$$

where $\mathcal{O}_{K,\Sigma} \subset K$ is the ring of Σ-integers of K.

We fix a finite place $\lambda \in \mathrm{Spec}(\mathcal{O}_{K,\Sigma})$, dividing $\ell \in \mathrm{Spec}(\mathbb{Z})$. We denote by K_λ the completion of K at λ and by $\mathcal{O}_{K_\lambda}$ its ring of integers. This defines the representations

$$\rho_{i,\lambda}^{\mathrm{top}} : \pi_1^{\mathrm{top}}(X, x) \xrightarrow{\rho_i^{0,\mathrm{top}}} GL(r, \mathcal{O}_{K,\Sigma}) \to GL(r, \mathcal{O}_{K_\lambda})$$

with factorization

$$\rho_{i,\lambda} : \pi_1^{\text{ét}}(X, x) \xrightarrow{\rho_{i,\lambda}^0} GL(r, \mathcal{O}_{K_\lambda}) \to GL(r, K_\lambda).$$

By extension of the ring of coefficients for Betti cohomology, one has

$$0 = H^1(U, a_*\mathcal{E}nd^0(\rho_i)) = H^1(U, a_*\mathcal{E}nd^0(\rho_i^{0,\mathrm{top}})) \otimes_{\mathcal{O}_{K,\Sigma}} \mathbb{C},$$

thus $H^1(U, a_*\mathcal{E}nd^0(\rho_i^{0,\mathrm{top}}))$ is torsion, while by comparison between Betti and étale cohomology one has

$$0 = H^1(U, a_*\mathcal{E}nd^0(\rho_i^{0,\mathrm{top}})) \otimes_{\mathcal{O}_{K,\Sigma}} K_\lambda = H^1(U, a_*\mathcal{E}nd^0(\rho_{i,\lambda})).$$

Let $\mathfrak{m}_{K_\lambda} \subset \mathcal{O}_{K_\lambda}$ be the maximal ideal, and

$$\overline{\rho_{i,\lambda}^0} : \pi_1^{\text{ét}}(X, x) \to GL(r, \mathcal{O}_{K_\lambda}/\mathfrak{m}_{K_\lambda})$$

be the residual representations.

We choose a closed point $s \in S$ such that its characteristic p is prime to

the cardinality of the residual monodromy groups $\overline{\rho_{i,\lambda}^0}(\pi_1^{\text{ét}}(X, x))$,
to d,
to ℓ,
to the residual characteristics of the places in Σ,
and to h.

There is a *specialization homomorphism*

$$sp : \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét,t}}(X_{\bar{s}}, x_{\bar{s}})$$

defined by Grothendieck, with target the tame quotient $\pi_1^{\text{ét,t}}(X_{\bar{s}}, x_{\bar{s}})$ of $\pi_1^{\text{ét}}(X_{\bar{s}}, x_{\bar{s}})$, which is surjective, and induces an isomorphism $\pi_1^{\text{ét,p}'}(X, x) \xrightarrow{\cong} \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}})$ on the prime to p quotients [11, X, Cor. 2.4], [11, XIII,4.7]), [11, XIII, 2.10]). The complex point $\mathrm{Spec}(\mathbb{C}) \to S$ factors through the choice of a complex point $\mathrm{Spec}(\mathbb{C}) \to T$ where $T = \mathrm{Spec}(\bar{\mathcal{O}}_{S,s})$ and $\bar{\mathcal{O}}_{S,s}$ is the strict henselization of S at s. The tame étale coverings of $X_{\bar{s}}$ lift to X_T, defining a faithful functor from the category of tame lisse sheaves on $X_{\bar{s}}$ to lisse sheaves on $X(\mathbb{C})$. This functor is an equivalence when restricted to "monodromy prime to p" part of the fundamental group. That is, omitting the base points, sp comes from the induced map $X \to X_T$ and base change $\pi_1^{\text{ét,p}'}(X_{\bar{s}}) \xrightarrow{\cong} \pi_1^{\text{ét,p}'}(X_T)$.

The push-out by sp of the homotopy exact sequence [11, IX, Thm. 6.1]

$$1 \to \pi_1^{\text{ét}}(X_{\bar{s}}, x_{\bar{s}}) \to \pi_1^{\text{ét}}(X_s, x_s) \to \pi_1^{\text{ét}}(s, \bar{s}) \to 1$$

yields the "prime to p homotopy exact sequence"

$$1 \to \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}}) \to \pi_1^{\text{ét,p}'}(X_s, x_s) \to \pi_1^{\text{ét}}(s, \bar{s}) \to 1$$

defining $\pi_1^{\text{ét,p}'}(X_s, x_s)$. As $\mathrm{Ker}(\rho^0_{i,\lambda}(\pi_1^{\text{ét}}(X, x)) \to \overline{\rho^0_{i,\lambda}}(\pi_1^{\text{ét}}(X, x)))$ is a pro-ℓ-group, by the choice of s, one has a factorization

$$\rho^0_{i,\lambda} : \pi_1^{\text{ét}}(X, x) \xrightarrow{sp} \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}}) \xrightarrow{\rho^0_{i,\lambda,\bar{s}}} GL(r, \mathcal{O}_{K_\lambda}),$$

where and $\bar{s}$ is an $\bar{\mathbb{F}}_p$-point of X_s above x_s.

This defines the diagram

$$X_s \xrightarrow{a_s} U_s \xrightarrow{b_s} \bar{X}_s, \qquad j_s = b_s \circ a_s : X_s \to \bar{X}_s$$

and above it the diagram

$$X_{\bar{s}} \xrightarrow{a_{\bar{s}}} U_{\bar{s}} \xrightarrow{b_{\bar{s}}} \bar{X}_{\bar{s}}, \qquad j_{\bar{s}} = b_{\bar{s}} \circ a_{\bar{s}} : X_{\bar{s}} \to \bar{X}_{\bar{s}}$$

Finally, still by the choice of s, one has a factorization

$$\det(\rho_i) : \pi_1^{\text{top}}(X, x) \to \pi_1^{\text{ét}}(X, x) \xrightarrow{sp} \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}}) \to \mu_d(K).$$

We define the representation $\rho_{i,\lambda,\bar{s}} : \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}}) \xrightarrow{\rho^0_{i,\lambda,\bar{s}}} GL(r, \mathcal{O}_{K_\lambda}) \to GL(r, K_\lambda)$. The next proposition is a variant Simpson's Theorem [17, Thm. 4].

Proposition 3.1 *After replacing $s \in S$ by any point $s' \in S(k')$, with $k(s) \subset k' \subset k(\bar{s})$ with degree s'/s sufficiently divisible, one has a factorization*

$$\begin{array}{ccc} \pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}}) & \longrightarrow & \pi_1^{\text{ét,p}'}(X_s, x_s) \\ {\scriptstyle \rho_{i,\lambda,\bar{s}}}\downarrow & \swarrow {\scriptstyle \rho_{i,\lambda,s}} & \\ GL(r, K_\lambda) & & \end{array}$$

such that $\det(\rho_{i,\lambda,s})$ *is finite.*

Proof The representation $\rho_{i,\lambda}$, or equivalently the representation $\rho_{i,\lambda,\bar{s}}$, defines a K_λ-point $[\rho_{i,\lambda}] \in \underline{M}(K_\lambda)$. The point $x_s \in X_s$ is rational, thus splits the prime to p homotopy exact sequence. We still denote by g the lift to $\pi_1^{\text{ét}}(X_s, x_s)$ of an element in $\pi_1^{\text{ét}}(s, \bar{s})$. For such a g, we define the representation

$$\rho^g_{i,\lambda} : \pi_1^{\text{ét}}(X, x) \to GL(r, K_\lambda)$$
$$\gamma \mapsto \rho_{i,\lambda,\bar{s}}(g \cdot sp(\gamma) \cdot g^{-1}).$$

Lemma 3.2 $\rho^g_{i,\lambda} \in \underline{N}(K_\lambda)$.

Proof We have to prove that the determinant of $\rho^g_{i,\lambda}$ has order dividing d, and that the monodromies along the components of D at infinity are quasi-unipotent with order of the eigenvalues dividing h.

We first show we may assume that X is a curve. We take in $\bar{X}$ a smooth projective curve $\bar{C}$ which is a complete intersection of ample divisors, all of which containing x and all the y_ι, in good position with respect to D. In particular, the curve C contains x and the points y_ι. One may assume that $\bar{C}$ is defined over S, and that $y_{\iota,S}$ is a S-point of $D_{\iota,S}$. Let $C = \bar{C} \cap X$. Via the surjections $\pi_1^{\text{top}}(C, x) \to \pi_1^{\text{top}}(X, x)$, $\pi_1^{\text{ét}}(C, x) \to \pi_1^{\text{ét}}(X, x)$ inducing via the specialization the surjection $\pi_1^{\text{ét}}(C_{\bar{s}}, x_{\bar{s}}) \to \pi_1^{\text{ét}}(X_{\bar{s}}, x_{\bar{s}})$, one reduces the statement to the case where X has dimension 1.

Let $k_{y_{\iota,s}}$ be the local field which is the field of fractions of the complete ring $\mathcal{O}_{\bar{X}_s, y_{\iota,s}}$ of $\bar{X}_s$ at $y_{\iota,s}$. A uniformizer t of $\mathcal{O}_{\bar{X}_s, y_{\iota,s}}$ yields an identification of the category of finite étale prime to p extensions of $k_{y_{\iota,s}}$ with the category of finite étale prime to p covers of $\mathbb{G}_m$ (see [12, Thm. 1.41] and [8, Section 15] for the theory of tangential base points at infinity). The rational point 1 of $\mathbb{G}_m$ defines a fiber functor 1_t, thus yields the prime to p homotopy exact exact sequence

$$1 \to \pi_1^{\text{ét,p}'}(k_{y_{\iota,\bar{s}}}, 1_t) \to \pi_1^{\text{ét,p}'}(k_{y_{\iota,s}}, 1_t) \to \pi_1^{\text{ét}}(s, \bar{s}) \to 1$$

for $k_{y_{\iota,s}}$, together with a splitting of $\pi_1^{\text{ét}}(s,\bar{s})$ in $\pi_1^{\text{ét},p'}(k_{y_{\iota,s}},1_t)$. It maps to the prime to p homotopy exact exact sequence

$$1 \to \pi_1^{\text{ét},p'}(X_{\bar{s}},1_t) \to \pi_1^{\text{ét},p'}(X_s,1_t) \to \pi_1^{\text{ét}}(s,\bar{s}) \to 1$$

for X_s based at 1_t, where 1_t is viewed as a tangential base point on X_s. An equivalence θ_s of fiber functors between 1_t and $x_{\bar{s}}$ for the category of finite étale prime to p covers of X_s yields a isomorphism of exact sequences from the prime to p homotopy exact sequence for X_s based at 1_t with the one based at $x_{\bar{s}}$, compatibly with the section. Thus conjugacy by g stabilizes the image of $\pi_1^{\text{ét},p'}(k_{y_{\iota,\bar{s}}},1_t) \to \pi_1^{\text{ét},p'}(X_{\bar{s}},1_t) \xrightarrow{\theta_s} \pi_1^{\text{ét},p'}(X_{\bar{s}},x_{\bar{s}})$. Let $\mathcal{O}_{\bar{X},y_\iota}$ be the complete ring $\mathcal{O}_{\bar{X},y_\iota}$ of $\bar{X}$ at y_ι, and k_{y_ι} be its field of fractions. The specialization homomorphism for the fundamental groups based at 1_t sends $\pi_1^{\text{ét}}(k_{y_\iota},1_t)$ to $\pi_1^{\text{ét},p'}(k_{y_{\iota,\bar{s}}},1_t)$, and the profinite completion of $\pi_1(\Delta_\iota^\times,1_t)$ is $\pi_1^{\text{ét}}(k_{y_\iota},1_t)$. Here we abused notations: $\mathbb{G}_m$ over s lifts to $\mathbb{G}_m$ over T then over $\mathbb{C}$, as well as the uniformizer t. We used the same notation 1_t for the base point on those lifts, and used that the category of finite étale extensions (resp. topological covers) of k_{y_ι} (resp. $\Delta_\iota^\times$) is equivalent to the corresponding one on $\mathbb{G}_m$. Finally, identifying $\mathbb{G}_m(\mathbb{C})$ with $\Delta_\iota^\times$ identifies 1_t and x_ι with two points in $\Delta_\iota^\times$. We choose a path θ_ι between the two in $\Delta_\iota^\times$, and a path θ between 1_t and x on X. This defines isomorphisms $\pi_1(\Delta_\iota^\times,1_t) \xrightarrow{\theta_\iota} \pi_1(\Delta_\iota^\times,x_\iota)$ and $\pi_1(X,1_t) \xrightarrow{\theta} \pi_1(X,x)$. Set $U = \theta_\iota^{-1}(T_\iota)$. Then by assumption $\rho_i \circ \theta(U) \in \mathcal{K}_\iota$. Let $\hat{U}$ be its image via the profinite completion $\pi_1(\Delta_\iota^\times,1_t) \to \pi_1^{\text{ét}}(k_{y_\iota},1_t)$. Summarizing the information we have

$$\rho^g_{i,\lambda} \circ \theta(\hat{U}) = \rho_{i,\lambda,\bar{s}}(g \cdot sp(\theta(\hat{U})) \cdot g^{-1}) = \rho_{i,\lambda,\bar{s}} \circ \theta_s(g \cdot sp(\hat{U}) \cdot g^{-1})$$

and

$$g \cdot sp(\hat{U}) \cdot g^{-1} \in \pi_1^{\text{ét},p'}(k_{y_{\iota,\bar{s}}},1_t).$$

As $sp(\hat{U})$ is a topological generator of $\pi_1^{\text{ét},p'}(k_{y_{\iota,\bar{s}}},1_t) \cong \hat{\mathbb{Z}}^{(p')}$, there is an element $(c_n) \in \hat{\mathbb{Z}}^{(p')}$, with $c_n \in \mathbb{Z}/n$, such that $g \cdot sp(\hat{U}) \cdot g^{-1} = sp(\hat{U})^{c_n} \in \pi_1^{\text{ét},p'}(k_{y_{\iota,\bar{s}}},1_t) \in \mathbb{Z}/n$. As the subset A of matrices in $GL(r,K_\lambda)$ of quasi-unipotent matrices with order of the eigenvalues dividing h and the order of the determinant dividing d is closed, and $\rho_{i,\lambda,\bar{s}} \circ \theta_s(sp(\hat{U})^{c_n}) \in A$, one has $\rho^g_{i,\lambda} \circ \theta(\hat{U}) \in A$. Thus the representation $\rho^g_{i,\lambda}$ defines a point $[\rho^g_{i,\lambda}] \in \underline{N}(K_\lambda)$. □

The map

$$\pi_1^{\text{ét}}(s,\bar{s}) \to \underline{N}(K_\lambda),\ g \mapsto [\rho^g_{i,\lambda}]$$

is continuous for the profinite topology on $\pi_1^{\text{ét}}(s,\bar{s})$ and the λ-adic topology on $\underline{N}(K_\lambda)$. As $[\rho_{i,\lambda}] \in \underline{N}(K_\lambda)$ is isolated, there is an open subgroup of $\pi_1^{\text{ét}}(s,\bar{s})$ on which the map is constant with image $[\rho_{i,\lambda}]$. This defines a point s' with $\bar{s} \to s' \to s$ with $\pi_1^{\text{ét}}(s',\bar{s})$ being this open subgroup. We abuse notations and set $s = s'$.

Let $K_\lambda \subset \bar{K}_\lambda$ be an algebraic closure. The representation $\rho_{i,\lambda} \otimes \bar{K}_\lambda$ is irreducible as ρ_i is irreducible over $\mathbb{C}$. Thus the equation $[\rho^g_{i,\lambda}] = [\rho_{i,\lambda}] \in \underline{N}(K_\lambda)$ implies that there is a $T(g) \in GL(r, K_\lambda)$ such that $\rho_{i,\lambda,\bar{s}}(g\gamma g^{-1}) = T(g)\rho_{i,\lambda,\bar{s}}(\gamma)T(g)^{-1} \in GL(r, K_\lambda)$ for all $\gamma \in \pi_1^{\text{ét}}(X_{\bar{s}}, x_{\bar{s}})$, and moreover $T(g)$ is uniquely defined up to multiplication by a scalar in $K_\lambda^\times$. The so defined map $\pi_1^{\text{ét}}(s, \bar{s}) \to PGL(r, K_\lambda),\ g \mapsto \bar{T}(g)$, where $\bar{T}(g)$ is the image of $T(g)$, is continuous for the profinite topology on $\pi_1^{\text{ét}}(s, \bar{s})$ and the λ-adic topology on $PGL(r, K_\lambda)$. Writing $\pi_1^{\text{ét,p}'}(X_s, x_s)$ as a semi-direct product of $\pi_1^{\text{ét}}(s, \bar{s})$ by $\pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}})$, we define $p\rho_{i,\lambda,s} : \pi_1(X_s, x_s) \to PGL(r, K_\lambda)$ by sending $(\gamma \cdot g)$ to $\rho_{i,\lambda,\bar{s}}(\gamma) \cdot \bar{T}(g)$. It remains to lift $p\rho_{i,\lambda,s}$ to $\rho_{i,\lambda,s}$ as in the proposition. Our initial argument consisted in saying that the Brauer obstruction to the lift dies as $k(s)$ is finite, and in then using class field theory [7, Prop. 1.3.4] to ensure that the so constructed representation has finite determinant after twist by a character of $k(s)$. We present here Deligne's argument which has the advantage to work on all base fields, but necessitates a new change of s. The representation $\rho_{i,\lambda,\bar{s}}$ has values in the subgroup $G \subset GL(r, K_\lambda)$ consisting of the elements with order d' determinant, where d' is any divisor of d. The composite homomorphism $G \to GL(r, K_\lambda) \to PGL(r, K_\lambda)$ is finite étale onto its image. Thus again base changing s to $s' \to s$ finite with $\bar{s} \to s'$, the pull-back Π of $\pi_1^{\text{ét}}(s', \bar{s})$ in $\pi_1^{\text{ét}}(X_s, x_s)$ is open and the restriction of $p\rho_{i,\lambda,s}$ to Π lifts to G in a unique way so that on $\pi_1^{\text{ét}}(X_{\bar{s}}, x_{\bar{s}})$, it is precisely $\rho_{i,\lambda,\bar{s}}$. □

We denote the lisse sheaves associated to $\rho_{i,\lambda}$ resp. $\rho_{i,\lambda,\bar{s}}$ resp. $\rho_{i,\lambda,s}$ by $\mathcal{V}_{i,\lambda}$ resp. $\mathcal{V}_{i,\lambda,\bar{s}}$ resp. $\mathcal{V}_{i,\lambda,s}$.

Lemma 3.3 *The lisse sheaves $\mathcal{V}_{i,\lambda,\bar{s}}$, resp. $\mathcal{V}_{i,\lambda,s}$ are tame, have quasi-unipotent monodromies at infinity, and the order of the eigenvalues of the local monodromies at infinity divides h.*

Proof Since $\mathcal{V}_{i,\lambda,\bar{s}}$ is defined by a representation of $\pi_1^{\text{ét,p}'}(X_{\bar{s}}, x_{\bar{s}})$ it is tame, so so is $\mathcal{V}_{i,\lambda,s}$. We denote by $\Gamma_{i,\lambda} \subset GL(r, \mathcal{O}_{K_\lambda})$ the monodromy group of $\mathcal{V}_{i,\lambda}$, which is the one of $\mathcal{V}_{i,\lambda,\bar{s}}$. The rest of the statement is proven in the proof of Proposition 3.1. □

Fix any prime number $\ell' \neq p$ and denote by n the dimension of X.

Lemma 3.4 *Let $\mathcal{A}$ be a pure tame $\bar{\mathbb{Q}}_{\ell'}$-lisse sheaf of weight* 0 *on X_s. Then*

$$H^1(\bar{X}_{\bar{s}}, j_{\bar{s}!*}\mathcal{A}) = H^1(U_{\bar{s}}, a_{\bar{s}*}\mathcal{A})$$

and is the $\bar{\mathbb{Q}}_{\ell'}$-sub vector space of $\oplus_{j\in\mathbb{N}} H^j(X_{\bar{s}}, \mathcal{A})$ consisting of all the elements of weight precisely 1.

Proof The proof of Remark 2.4 yields over X_s the relation $H^1(\bar{X}_{\bar{s}}, j_{\bar{s}!*}\mathcal{A}) = H^1(U_{\bar{s}}, a_{\bar{s}*}\mathcal{A})$. The weight of $H^0(X_{\bar{s}}, \mathcal{A})$ is 0, and for any j, the weight of $H^j(X_{\bar{s}}, \mathcal{A})$ is $\geq j$ (see [7, Thm. 3.3.1]). Thus the weight 1 part of $\oplus_{j\in\mathbb{N}} H^j(X_{\bar{s}}, \mathcal{A})$ lies in $H^1(X_{\bar{s}}, \mathcal{A})$. One has a short $\pi_1^{\text{ét}}(s, \bar{s})$ equivariant exact sequence

$$0 \to H^1(\bar{X}_{\bar{s}}, j_{\bar{s}!*}\mathcal{A}) \to H^1(X_{\bar{s}}, \mathcal{A}) \to H^0(U_{\bar{s}}, R^1 a_{\bar{s}*}\mathcal{A}).$$

The group $H^1(\bar{X}_{\bar{s}}, j_{\bar{s}!*}\mathcal{A})$ is pure of weight 1 while $R^1 a_{\bar{s}*}\mathcal{A}$ has weights ≥ 2 at closed points, which is seen on curves, and on them $\mathcal{A}$ is tame [13, Thm. 1.1], thus the local inertia $\mathbb{Z}_\ell(1)$ acts at the punctures at infinity. Thus a fortiori the weight of $H^0(U_{\bar{s}}, R^1 a_{\bar{s}*}\mathcal{A})$ is ≥ 2. This finishes the proof. □

Proof of Theorem 1.1 We fix an embedding $K_\lambda \subset \bar{\mathbb{Q}}_\ell$. Let $\sigma : \bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_{\ell'}$ be a field isomorphism, where $\ell' \neq p$. By Drinfeld's theorem [9, Thm. 1.1], there is a σ-companion $\mathcal{V}^\sigma_{i,\lambda,s}$ to $\mathcal{V}_{i,\lambda,s}$. By definition of the indexing, for $i \in \{1, \ldots, \mathfrak{N}\}$, the complex local systems $\mathcal{V}_i$ are irreducible and pairwise non-isomorphic. Thus by definition, the $\bar{\mathbb{Q}}_\ell$ lisse sheaves $\mathcal{V}_{i,\lambda,s}$ are irreducible and pairwise non-isomorphic. If $\mathcal{V}^\sigma_{i,\lambda,\bar{s}}$ was not irreducible, it would split after a finite base change $s' \to s$ with $\bar{s} \to s' \to s$, thus on $X_{s'}$, thus the σ^{-1}-companion of $\mathcal{V}^\sigma_{i,\lambda,s'}$, which is $\mathcal{V}_{i,\lambda,s'}$, would split as well, a contradiction. Likewise, if $\mathcal{V}^\sigma_{i,\lambda,\bar{s}}$ is isomorphic to $\mathcal{V}^\sigma_{j,\lambda,\bar{s}}$, since $H^0(X_{\bar{s}}, (\mathcal{V}^\sigma_{i,\lambda,\bar{s}})^\vee \otimes \mathcal{V}^\sigma_{j,\lambda,\bar{s}})$ has weight 0 [14, Prop. VII.7], the isomorphism is defined over X_s, thus $i = j$. Thus the $\bar{\mathbb{Q}}_\ell$ lisse sheaves $\mathcal{V}_{i,\lambda,\bar{s}}$ are irreducible and pairwise non-isomorphic. In addition, since $\det(\mathcal{V}_{i,\lambda,\bar{s}})^\sigma = \det(\mathcal{V}^\sigma_{i,\lambda,\bar{s}})$ is constructed by post-composing the $K_\lambda^\times$ character by σ, it has order precisely d.

As $\mathcal{V}_{i,\lambda,s}$ is tame by Lemma 3.3, it is tame in restriction to all curves [13, Thm. 1.1]. Taking a smooth curve $\bar{C} \subset \bar{X}$ which is a complete intersection of smooth ample divisors in good position with respect to $\bar{X} \setminus X$, and denoting by C its intersection with X, the restriction $\mathcal{V}_{i,\lambda,\bar{s}}|_{C_{\bar{s}}}$ has the same monodromy group as the one of $\mathcal{V}_{i,\lambda,\bar{s}}$, thus has quasi-unipotent monodromies at the points $C_{\bar{s}} \cap D_{\bar{s}}$, and their eigenvalues have order dividing h. By [6, Thm. 9.8], $\mathcal{V}^\sigma_{i,\lambda,\bar{s}}$ has the same property.

As $\mathcal{V}_{i,\lambda,s}$ and $\mathcal{V}^\sigma_{i,\lambda,s}$ are pure of weight 0, so are $\mathcal{A}_{is} = \mathcal{E}nd^0(\mathcal{V}_{i,\lambda,s})$ and $\mathcal{A}^\sigma_{is} = \mathcal{E}nd^0(\mathcal{V}^\sigma_{i,\lambda,s})$. By local acyclicity [16, Lem. 3.14] applied to $X_T \to T$ used to define the specialization,

$$H^1(U_{\bar{s}}, a_{\bar{s}*}\mathcal{A}_{i,\bar{s}}) \to H^1(U, a_*\mathcal{A}_i)$$

is an isomorphism. Thus $H^1(\bar{X}_{\bar{s}}, j_{!*}\mathcal{A}_{i\bar{s}}) = 0$. The L- functions $L(X_{\bar{s}}, \mathcal{A}_{i,s})$ and $L(X_{\bar{s}}, \mathcal{A}^\sigma_{i,s})$ defined by a product formula are equal [5, 5.2.3]. In particular, for any natural number w, $d(j, \ell) = d(j, \ell')$ where $d(i, \ell)$ (resp. $d(i, \ell')$) denotes the dimension over $\bar{\mathbb{Q}}_\ell$ (resp. $\bar{\mathbb{Q}}_{\ell'}$) of the pure weight w summand of $H^j_c(X_{\bar{s}}, \mathcal{A}_{i,\bar{s}})$ (resp. $H^j_c(X_{\bar{s}}, \mathcal{A}^\sigma_{i,\bar{s}})$.) By duality, the same it true replacing the cohomologies $H^j_c(X_{\bar{s}}, \mathcal{A}_{i,\bar{s}})$ and $H^j_c(X_{\bar{s}}, \mathcal{A}^\sigma_{i,\bar{s}})$ by the cohomologies $H^j(X_{\bar{s}}, \mathcal{A}_{i,\bar{s}})$ and $H^j(X_{\bar{s}}, \mathcal{A}^\sigma_{i,\bar{s}})$. Applying this for $w = 1$, from Lemma 3.4, we conclude $H^1(\bar{X}_{\bar{s}}, j_{\bar{s}!*}\mathcal{A}^\sigma_{i\bar{s}}) = H^1(U_{\bar{s}}, a_{\bar{s}*}\mathcal{A}^\sigma_{i,\bar{s}}) = 0$.

Pulling back along the specialization homomorphism $sp : \pi_1^{\text{ét}}(X, x) \to \pi_1^{\text{ét,t}}(X_{\bar{s}}, x_{\bar{s}})$ defines the $\bar{\mathbb{Q}}_{\ell'}$-lisse sheaves $\mathcal{V}^\sigma_{i\lambda}$ and $\mathcal{A}^\sigma_i$ on X, together with the specialization homomorphism

$$H^1(U_{\bar{s}}, a_{\bar{s}*}\mathcal{A}^\sigma_{i,\bar{s}}) \to H^1(U, a_*\mathcal{A}^\sigma_i).$$

By local acyclicity again, it is an isomorphism thus $H^1(U, a_*\mathcal{A}^\sigma_i) = 0$.

We now define $\mathcal{V}_i^{\sigma\ \mathrm{top}}$ to be the $\bar{\mathbb{Q}}_{\ell'}$-local system on X which is defined by composing the representation with the homomorphism $\pi_1^{\mathrm{top}}(X,x) \to \pi_1^{\mathrm{\acute{e}t}}(X,x)$. By the comparison between Betti and étale cohomology one has $H^1(U, a_*\mathcal{E}nd(\mathcal{V}_i^{\sigma\ \mathrm{top}})) = 0$. Furthermore, the $\mathcal{V}_i^{\sigma\ \mathrm{top}}$ on X are irreducible and pairwise non-isomorphic, and $\det(\mathcal{V}_i^{\sigma\ \mathrm{top}})$ has order d. Since $\mathcal{V}_{i,\lambda,\bar{s}}^{\sigma}$ has quasi-unipotent monodromies with eigenvalues of order dividing h, $\mathcal{V}_i^{\sigma\ \mathrm{top}}$ has quasi-unipotent monodromies along the components of $\bar{X}\setminus X$, with eigenvalues of order dividing h. (There is a slight abuse of notations here, the curve $\bar{C}$ chosen is perhaps not defined over the same S, we just take a s in the construction on which C is defined).

As $\pi_1^{\mathrm{top}}(X,x)$ is a finitely generated group, there is a subring $A \subset \bar{\mathbb{Q}}_{\ell'}$ of finite type such that the monodromy representations of the $\mathcal{V}^{\sigma_i\ \mathrm{top}}$ factor through $\pi_1^{\mathrm{top}}(X,x) \to GL(r,A)$. We fix a complex embedding $A \hookrightarrow \mathbb{C}$. This defines the complex local systems $\mathcal{V}_i^{\sigma\ \mathbb{C}}$. By definition, there are irreducible, pairwise different, have determinant of order precisely d and the eigenvalues of the monodromies at infinity have order at most h. By comparison between Betti and étale cohomology, they are cohomologically rigid. Thus the set of isomorphism classes of the $\mathcal{V}_i^{\sigma\ \mathbb{C}}$ is precisely $\mathcal{S}(r,d,h)$. On the other hand, they are integral at all places of number rings in $\bar{\mathbb{Z}}_{\ell'}$ which divide ℓ'. We conclude the proof by doing the construction for all ℓ' divided by places in Σ. □

Remark 3.5 This remark is due to Pierre Deligne. If in Theorem 1.1, the irreducible complex local system $\mathcal{V}$ with quasi-unipotent monodromies along the components of D is assumed to be orthogonal, then it is integral under the assumption $\mathbb{H}^1(\bar{X}, j_{!*}\wedge^2\mathcal{V}) = 0$. This assumption is weaker than the assumption $\mathbb{H}^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V})) = 0$ of Theorem 1.1, as, as $\mathcal{V}$ is self-dual, $\wedge^2\mathcal{V}$ is a summand of $\mathcal{E}nd^0(\mathcal{V})$, and thus $\mathbb{H}^1(\bar{X}, j_{!*}\wedge^2\mathcal{V})$ is a summand of $\mathbb{H}^1(\bar{X}, j_{!*}\mathcal{E}nd^0(\mathcal{V}))$. Likewise, if $\mathcal{V}$ is assumed to be symplectic, then it is integral under the assumption $\mathbb{H}^1(\bar{X}, j_{!*}Sym^2\mathcal{V}) = 0$. Again $Sym^2\mathcal{V}$ is a summand $\mathcal{E}nd^0(\mathcal{V})$. We do not detail the proof. One has to perform the stack construction in Sect. 2 for the corresponding category of orthogonal, resp. symplectic local systems, and eventually see that the companion construction in Sect. 3 preserves this category.

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