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CHARACTERISTIC CLASSES OF FLAT BUNDLES

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INTRODUCTION

IN THIS article we construct by a *modified splitting principle* characteristic classes of bundles with supplementary structures.

On an analytic manifold X a bundle E is said to be flat if it has an holomorphic integrable connection ∇. We construct classes

$$c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \to \mathbb{C}) \simeq H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)).$$

More generally, if $\tau_0: \Omega_X^{\cdot} \to A^{\cdot}$ is a quotient of the holomorphic De Rham complex $\Omega_X^{\cdot}$ such that $A^0 = \mathcal{O}_X$ and A^i is a quotient bundle of Ω_X^i, we say that E is τ_0-flat if it has an integrable connection ∇ with values in A^1 (2.1). We construct *classes*

$$c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \to A^{\cdot})$$

which are *functorial* and *additive* for exact sequences compatible with ∇. They are *uniquely* determined by those two properties and the definition of $c_1(E, \nabla)$ (due to P. Deligne [2], (1.3), [6]). The group $H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to A^1)$ is identified with the group of isomorphism classes (E, ∇) of rank one bundles E with an A^1-connection ∇. Therefore ∇ is integrable if and only if (E, ∇) lies in the subgroup $H^2(X, \mathbb{Z}(1) \to A^{\cdot})$. This defines $c_1(E, \nabla)$.

M. Karoubi ([11] and [12]) constructed with K-theory and cyclic homology classes $\check{c}_p(E) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p))$ when X is a simplicial set and E is a flat bundle. He told the author that his classes are functorial and additive (and that he will write it down in the planned "Homologie cyclique et K-théorie III"). This would imply $c_p(E, \nabla) = \check{c}_p(E)$ for flat bundles (2.25.2).

The cohomology $H^{2p}(X, \mathbb{Z}(p) \to \Omega_X^{\cdot})$ maps to the Deligne cohomology $H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$, where $\mathbb{Z}(p)_{\mathscr{D}} = \mathbb{Z}(p) \to \mathcal{O}_X \to \ldots \to \Omega_X^{p-1}$. *Our classes* $c_p(E, \nabla)$ *lift the Chern classes* $c_p^{\mathscr{D}}(E)$ *in the Deligne cohomology* (2.25.1) (see [2] for a definition of $c_p^{\mathscr{D}}(E)$). If X has an Hodge structure, for example if X is algebraic proper over $\mathbb{C}$, then the *projection* of $c_p(E, \nabla)$ in $H^{2p}(X, \mathbb{Z}(p) \to \mathbb{R}(p))$ is identified with $c_p^{\mathscr{D}}(E)$ (2.25.1).

J. Cheeger and J. Simons ([4]) constructed in a differential geometric framework classes $\hat{c}_p(E) \in H^{2p-1}(X, \mathbb{R}/\mathbb{Z})$ when X is a C^∞ manifold and E is a flat bundle. In general the relationship between $\hat{c}_p(E)$ and $c_p^{\mathscr{D}}(E)$ is not known. When E is unitary and X has an Hodge structure, S. Bloch [3] and C. Soulé [13] proved that $\hat{c}_p(E)$ lifts $c_p^{\mathscr{D}}(E)$. Therefore *in this case our classes lift the Cheeger–Simons classes* (2.25.3) via the map $\mathbb{C}/\mathbb{Z}(p) \to \mathbb{R}(p)/\mathbb{Z}(p) \xrightarrow{\sim} \mathbb{R}/\mathbb{Z}$.

Our method consists of two parts: the definitions of the "τ-*construction*" and of the "τ-*product*". If f: $P \to X$ is the flag bundle of E(2.7), the connection ∇ defines a morphism τ: $\Omega_P^1 \to f^* A^1$. The integrability condition implies that $(\tau d)^2 = 0$ and that τ extends to the

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De Rham complex $\tau\colon \Omega_P^{\cdot} \to A_\tau^{\cdot}$, where $A_\tau^i = f^* A^i$, with $Rf_* A_\tau^{\cdot} = A^{\cdot}$ (2.7,i). This defines an integrable τ-connection (i.e. with values in A_τ^1; see (2.1) for definition) ∇_τ on $f^* E$, which is compatible with the canonical filtration of $f^* E$. Therefore it defines integrable τ-connections $\nabla_{\tau,k}$ on the splitting rank one bundles L_k of $f^* E$, and classes $(L_k, \nabla_{\tau,k})$ in $H^2(P, \mathbb{Z}(1) \to A_\tau^{\cdot})$ (2.7, ii). Next, one has to define a "τ-product" on the "τ-cohomology" $H^{2p}(P, \mathbb{Z}(p) \to A_\tau^{\cdot})$. We do it just by multiplying on the left with $H^{2p}(P, \mathbb{Z}(p))$ (2.9). This defines classes

$$c_p(f^* E, f^* \nabla) \in H^{2p}(P, \mathbb{Z}(p) \to A_\tau^{\cdot})$$

as p-symmetric sums of $(L_k, \nabla_{\tau,k})$ (2.10).

The τ-cohomology is *not* a free module over the τ_0-cohomology. So one can not apply Hirzebruch–Grothendieck's splitting principle. However the construction implies immediately that our classes are defined on X and lift the Deligne classes (2.15). It is technical but straightforward to prove functoriality (2.16) and additivity (2.17).

We hope that our quite elementary method may cast a new light on the role played by the flat structure ∇ on the multiplicative behaviour of $c_p^{\mathcal{D}}(E)$. (Recall in this connection that S. Bloch's conjecture [3], asking whether $c_p^{\mathcal{D}}(E)$ is torsion if $p \geq 2$ when E is flat and X is projective, is still unsolved.)

Flat or τ_0-flat bundles are quite rigid. It is a larger class to consider holomorphic integrable connections ∇ with logarithmic poles along a divisor with normal crossing D. More generally if $\tau_0\colon \Omega_X^{\cdot}\langle D\rangle \to A_D^{\cdot}$ is a quotient of the holomorphic De Rham complex with logarithmic poles along D, such that $A_D^0 = \mathcal{O}_X$ and A_D^i is a quotient bundle of $\Omega_X^i\langle D\rangle$, we consider bundles E with integrable connections ∇ with values in A_D^1 (3.1). We construct classes

$$c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \to \mathcal{O}_X \to \dots \to \Omega_Z^{p-1} \xrightarrow[\tau_0 d]{} A_D^p \to \dots \xrightarrow[\tau_0 d]{} A_D^{\dim X})$$

which are functorial and additive for exact sequences compatible with ∇. They are uniquely determined by those two properties and the definition of $c_1(E, \nabla)$. Moreover they lift the classes $c_p^{\mathcal{D}}(E)$ (3.6). The method is the same τ-construction as in the τ_0-flat case, but the τ-product has to be refined a little bit to obtain classes lifting the Deligne classes $c_p^{\mathcal{D}}(E)$ (3.6).

P. Deligne [6] in the rank one case, H. Gillet and C. Soulé [8] in the higher rank case defined characteristic classes of hermitian bundles in Chow groups in the Arakelov theory, i.e. groups of cycles with supplementary structures. A good cohomological theory of characteristic classes of hermitian bundles should factor through their classes, lift $c_p^{\mathcal{D}}(E)$ and the Chern forms. A very small step in this direction is described in §4.

PRELIMINARIES

(0.1) Throughout this article, X is an analytic manifold over $\mathbb{C}$ of complex dimension n, D is a normal crossing divisor on X, $j\colon X - D \to X$ is the open embedding. As usual one denotes by $\Omega_X^{\cdot}$ ($\Omega_X^{\cdot}\langle D\rangle$) the holomorphic (holomorphic with logarithmic poles along D) De Rham complex of X with Kähler differential d. The complex $\Omega_X^{\cdot}\langle D\rangle$ is quasi-isomorphic to $Rj_*\mathbb{C}$ ([5]). One defines $\mathcal{O}_X = \Omega_X^0$, the sheaf of holomorphic functions. A vector bundle E is a locally free $\mathcal{O}_X$-sheaf of finite rank. We call r its rank. The vector bundle $\mathcal{E}nd\, E$ of endomorphisms of E splits into $\mathcal{O}_X \oplus \mathcal{E}nd^0 E$, where $\mathcal{E}nd^0 E$ is the vector bundle of endomorphisms of trace 0, via $\varphi = \frac{1}{r}$ trace $\varphi \cdot$ identity $+ \varphi^0$. For a $\mathbb{Z}$-module A we set $A(p) = (2i\pi)^p A$.

(0.2) An holomorphic *connection*

∇: $E \to \Omega^1_X \otimes E$ is a $\mathbb{C}$-linear morphism verifying the Leibniz-rule
$\nabla(\lambda \cdot x) = \lambda \cdot \nabla(x) + d\lambda \cdot x$, for $\lambda \in \mathcal{O}_X$ and $x \in E$. One defines
∇: $\Omega^p_X \otimes E \to \Omega^{p+1}_X \otimes E$ by
$\nabla(\omega \otimes x) = d\omega \wedge x + (-1)^p\, \omega \wedge \nabla x$, for $\omega \in \Omega^p_X$ and $x \in E$.

One says that ∇ is *integrable* if $(\Omega^{\cdot}_X \otimes E, \nabla)$ is a complex, or equivalently, if the curvature $\nabla^2 \in \mathrm{Hom}_{\mathcal{O}_X}(E, \Omega^2_X \otimes E)$ vanishes. The bundle E is said to be *flat* if E admits an integrable connection. Flat bundles are in one-to-one correspondence with local constant systems by the Riemann–Hilbert correspondence

$$\{(E, \nabla)\} \to \{L = \mathrm{Ker}\nabla\}, \quad \{L\} \to \{L \otimes_{\mathbb{C}} \mathcal{O}_X, 1 \otimes d\}.$$

(0.3) On a trivializing open cover U_i of E on X define ∇_i by declaring some basis to be flat. Since two holomorphic connections differ by a holomorphic one form with values in $\mathcal{E}nd\ E$, the class in $H^1(X, \Omega^1_X \otimes \mathcal{E}nd\ E)$ of the cocycle $(\nabla_i - \nabla_j) \in \Gamma(U_i \cap U_j, \Omega^1_X \otimes \mathcal{E}nd\ E)$ does not depend on the trivialization chosen. This is the *Atiyah class* $\mathrm{at}E$ of E ([1]). Its vanishing is the obstruction for E to have an holomorphic connection. One has $c^{DR}_p(E) = (-1)^p$ trace $\bigwedge^p$ at $E \in H^p(X, \Omega^p_X)$, where $c^{DR}_p(E)$ is the De Rham Chern class. One has

$$\mathrm{at}E = -\frac{1}{r}\, c^{DR}_1(E) \cdot \mathrm{identity} \oplus \mathrm{at}^0 E,$$

according to the splitting $\mathcal{E}nd\ E = \mathcal{O}_X \oplus \mathcal{E}nd^0 E$ (0.1). If ξ_{ij} is a cocycle representing the class of E in $H^1(X, \mathcal{G}\ell_r(\mathcal{O}_X))$, then $-\xi_{ij}^{-1} d\xi_{ij}$ represents $\mathrm{at}E$.

(0.4) One defines $\mathrm{at}_D E$ to be the image of $\mathrm{at}E$ in $H^1(X, \Omega^1_X\langle D\rangle \otimes \mathcal{E}nd\ E)$. Its vanishing is the obstruction for E to have an holomorphic connection with logarithmic poles along D (same definition as in (0.2) where one replaces Ω^1_X by $\Omega^1_X\langle D\rangle$). Integrable logarithmic connections were studied by P. Deligne [5].

(0.5) Define $P = P(E) = \mathrm{Proj}_X(\bigoplus_{n \geq 0} S^{\cdot}(E))$ the *projective bundle* of E, where $S^{\cdot}(E)$ are the symmetric powers of E, f: $P \to X$, $\mathcal{O}(1)$ as the relatively ample sheaf uniquely determined by the surjection

$$f^* E \twoheadrightarrow \mathcal{O}(1).$$

One has two fundamental sequences

$$(0)\quad 0 \to f^*\Omega^1_X \xrightarrow{i} \Omega^1_P \xrightarrow{p} \Omega^1_{P/X} \to 0$$

$$(1)\quad 0 \to \Omega^1_{P/X}(1) \to f^* E \xrightarrow{q} \mathcal{O}(1) \to 0$$

where $\Omega^1_{P/X}$ is the sheaf of relative holomorphic one forms. Denote by $T^1_{P/X}$ the relative tangent sheaf.

(0.6) Let V_i be an open cover of P such that p admits a section σ_i on V_i. Two sections of p differ by an element in $f^*\Omega^1_X \otimes T^1_{P/X}$. Therefore the class in

$$H^1(P, f^*\Omega^1_X \otimes T^1_{P/X}) = H^1(X, \Omega^1_X \otimes Rf_* T^1_{P/X}) = H^1(X, \Omega^1_X \otimes \mathcal{E}nd^0 E)$$

of the cocycle

$$(\sigma_i - \sigma_j) \in \Gamma(V_i \cap V_j, f^*\Omega^1_X \otimes T^1_{P/X})$$

does not depend on the trivialization chosen. This is the extension class (0.5.0).

LEMMA. *This class is* $-\mathrm{at}^0 E$.

Proof. By the descriptions (0.3) and (0.6) it is enough to see that on any trivializing open set U for E on X, any connection ∇ defines a section of p, and that $\Omega^1_U \otimes \mathcal{E}nd^0 E$ acts on the connections on U as $f^*\Omega^1_U \otimes T^1_{P/X}$ does on the sections of p on $f^{-1}U$.

Given ∇, define

$$\sigma \otimes 1_{\mathcal{O}(1)} = (1_{\Omega^1_P} \otimes q) f^*\nabla|_{\Omega^1_{P/X}(1)} \quad \text{from} \quad \Omega^1_{P/X}(1) \text{ to } \Omega^1_P(1),$$

where $f^*\nabla$ *is defined to be* $f^{-1}\nabla$ *on* $f^{-1}E$, d *on* $\mathcal{O}_P$ *via the Leibniz rule.* Then $\sigma \otimes 1_{\mathcal{O}(1)}$ is $\mathcal{O}_P$-linear.

(0.6.1) *Claim.* $-\sigma$ *is a section of* p.

Proof. Let e^k be a basis of E on U. Define $t^k = q(e^k)$. Then t^k are the homogeneous coordinates of $\mathbb{P}^{r-1}$ in $f^{-1}U \simeq U \times \mathbb{P}^{r-1}$. On $f^{-1}U \cap (t^0 \neq 0)$ the holomorphic coordinates of $\mathbb{P}^{r-1} \cap (t^0 \neq 0)$ are $\frac{t^k}{t^0}$, t^0 is a basis of $\mathcal{O}(1)$, $x^k = e^k - \frac{t^k}{t^0} e^0$ $(k = 1, \ldots, r-1)$ is a basis of $\Omega^1_{P/X}$ with $pd\left(\frac{t^k}{t^0}\right) = \frac{x^k}{t^0}$. One has

$$\sigma \otimes 1(x^k) = (1 \otimes q)\left(f^*\nabla e^k - \frac{t^k}{t^0} f^*\nabla e^0\right) - d\left(\frac{t^k}{t^0}\right) \cdot t^0$$

and therefore

$$\sigma\left(\frac{x^k}{t^0}\right) = \frac{1}{t^0}(1 \otimes q)\left(f^*\nabla e^k - \frac{t^k}{t^0} f^*\nabla e^0\right) - d\left(\frac{t^k}{t^0}\right).$$

Since $f^*\nabla e^k \in f^*\Omega^1_X \otimes E$, one has

$$p\left(\frac{1}{t^0}(1 \otimes q)\left(f^*\nabla e^k - \frac{t^k}{t^0} f^*\nabla e^0\right)\right) = 0.$$

Therefore one has

$$p\sigma\left(\frac{x^k}{t^0}\right) = -pd\left(\frac{t^k}{t^0}\right) = -\frac{x^k}{t^0}.$$

If $\nabla' = \nabla + \alpha$, with $\alpha = \alpha^{kl} \in \Gamma(U, \Omega^1_X \otimes \mathcal{E}nd E)$, one has

$$\begin{aligned}(\sigma' - \sigma) \otimes 1(x^k) &= (1 \otimes q)\left(\alpha e^k - \frac{t^k}{t^0} \alpha e^0\right) \\ &= \sum_l \alpha^{kl} t^l - \frac{t^k}{t^0} \sum_l \alpha^{0l} t^l.\end{aligned}$$

One sees that $\frac{1}{r}$ *trace* $\alpha \cdot$identity acts trivially and that $(\sigma' - \sigma)$ defines a $\mathcal{O}_P$-linear map from $\Omega^1_{P/X}$ to $f^*\Omega^1_X$. Therefore $f^*\Omega^1_U \otimes \mathcal{E}nd^0 E$ acts as $f^*\Omega^1_U \otimes T^1_{P/X}$ does.

(0.7) Assume that E has an holomorphic connection ∇. This defines σ as in (0.6) and $\tau = 1 + \sigma p$ is a section of i. With the notations of (0.6) one has

$$\tau d\left(\frac{t^k}{t^0}\right) = \frac{1}{t^0}(1 \otimes q)\left(f^*\nabla e^k - \frac{t^k}{t^0} f^*\nabla e^0\right).$$

Define a *τ-connection* ∇_τ on a sheaf F on P to be a $\mathbb{C}$-linear morphism $\nabla_\tau: F \to f^*\Omega^1_X \otimes F$ verifying the *τ-Leibniz rule* $\nabla_\tau(\lambda . x) = \lambda . \nabla_\tau(x) + \tau(d\lambda) . x$, for $\lambda \in \mathcal{O}_P$ and $x \in F$.

LEMMA. *$\tau f^*\nabla$ is a τ-connection on f^*E such that $\tau f^*\nabla|_{\Omega^1_{P/X}(1)}$ is a τ-connection ∇'_τ on $\Omega^1_{P/X}(1)$ and $(1 \otimes q)\tau f^*\nabla$ is a well defined τ-connection ∇_τ on $\mathcal{O}(1)$.*

Proof. As $-\sigma$ is a section of p, $(1 \otimes q)(\tau f^*\nabla)|_{\Omega^1_{P/X}(1)} = 0$. Therefore $\Omega^1_{P/X}(1)$ is stable under $\tau f^*\nabla$, and the quotient $(1 \otimes q)\tau f^*\nabla$ is defined.

(0.8) *Remark.* In an effort to understand conditions for a bundle to be flat, we computed some time ago (0.6) and (0.7) with B. Angéniol. The point (0.6) is well known whereas the point (0.7) will play an important role in this article.

§1. SOME CONDITIONS FOR A BUNDLE TO BE FLAT.

(1.1) Let E be a rank one bundle. Its isomorphism class is a class in

$$H^1(X, \mathcal{O}^*) \xleftarrow[\exp]{\sim} H^2(X, \mathbb{Z}(1) \to \mathcal{O}),$$

say with cocycle $\xi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ in a Čech cover U_i. Consider the exact sequence of complexes

$$\begin{array}{l} 0 \to (0 \quad \to \ 0 \to 0 \quad \to \Omega^2_X \ \to \ldots \to \Omega^n_X) \\ \quad\ \downarrow \\ \quad\ (\mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \ \to \ldots \to \Omega^n_X) \\ \quad\ \downarrow \\ \quad\ (\mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X) \to 0 \end{array}$$

As $H^2(X, 0 \to 0 \to 0 \to \Omega^2_X \to \ldots \to \Omega^n_X) = 0$, the morphism

$$\begin{array}{c} H^2(X, \mathbb{Z}(1) \to \Omega^\cdot_X) \xrightarrow{\sim} H^1(X, \mathbb{C}^*) \\ \downarrow \\ H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X) \end{array}$$

is injective. One considers also the morphism

$$\begin{array}{c} H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X) \\ \downarrow \\ H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X) \xrightarrow{\sim} H^1(X, \mathcal{O}^*). \end{array}$$

LEMMA. (i) *The isomorphism classes of rank one bundles E with holomorphic connections ∇ build a group identified with $H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X)$. Denote by (E, ∇) a class in $H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega^1_X)$. Its image (E) in $H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X)$ is the isomorphism class of E.*

(ii) *∇ is integrable if and only if $(E, \nabla) \in H^2(X, \mathbb{Z}(1) \to \Omega^\cdot_X)$.*

Proof. (i) This is *Deligne's observation* ([2], (1.3), [6]) In some Čech cover ∇ is given by one forms $\omega_i \in \Gamma(U_i, \Omega_X^1)$ verifying $\xi_{ij}^{-1} . d\xi_{ij} = \omega_i - \omega_j$. (ξ_{ij}, ω_i) is the class wanted. It is isomorphic to the class of $(\mathcal{O}, d)$ if and only if they are functions $f_i \in \Gamma(U_i, \mathcal{O})$ verifying $\xi_{ij} = f_i . f_j^{-1}$ and $\omega_i = f_i^{-1} . df_i$.

(ii) The curvature $d\omega_i \in H^0(X, \Omega_X^2 \to \ldots \to \Omega_X^n)$, where ω_i is as in (i), vanishes if and only if

$$(E, \nabla) \in \operatorname{Ker}(H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega_X^1) \underset{d}{\to} H^0(X, \Omega_X^2 \to .. \to \Omega_X^n))$$
$$= H^2(X, \mathbb{Z}(1) \to \Omega_X^{\cdot}).$$

(1.2) In this language it is easy to see the well known

CLAIM. *If X has an Hodge structure, then $H^1(X, \mathbb{C}^*) = H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to \Omega_X^1)$. Therefore if E is a rank one bundle with vanishing Atiyah class, all the holomorphic connections on E are integrable.*

Proof. By (1.1) the second statement is a consequence of the first one. One has the commutative square of complexes

$$\begin{array}{ccccc}
\mathbb{Z}(1) \to \mathcal{O} & \to \Omega^1 & & & \\
\downarrow & \downarrow d_1 & & & \\
\mathbb{Z}(1) \to \mathcal{O} & \Omega^2 & \to \ldots & \to \Omega^n & \\
\downarrow d_0 & \downarrow i & & & \\
\Omega^1 & \to \Omega^2 & \to \ldots & \to \Omega^n &
\end{array}$$

This gives a commutative diagram

$$\begin{array}{ccc}
H^2(\mathbb{Z}(1) \to \mathcal{O} \to \Omega^1) & \xrightarrow{H(d_1)} & H^0(\Omega^2 \to \ldots \to \Omega^n) \\
\downarrow & & \downarrow H(i) \\
H^2(\mathbb{Z}(1) \to \mathcal{O}) & \xrightarrow{H(d_0)} & H^1(\Omega^1 \to \Omega^2 \to \ldots \to \Omega^n)
\end{array}$$

The first statement is equivalent to $H(d_1) = 0$. The image of $H(d_0)$ is contained in $H^1(\Omega^1)$ and therefore meets in 0 the image of $H(i)$. Since $H(i)$ is injective, this implies $H(d_1) = 0$.

(1.3) Let E be a bundle of rank r with an holomorphic connection ∇. Introduce τ, $(\mathcal{O}(1), \nabla_\tau)$ and $(\Omega_{P/X}^1(1), \nabla_\tau')$ as in (0.7). If $(\tau d)^2 = 0$ denote by $\Omega_\tau^{\cdot}$ the complex

$$\mathcal{O}_P \underset{\tau d}{\longrightarrow} f^*\Omega_X^1 \underset{\tau d}{\longrightarrow} \ldots \underset{\tau d}{\longrightarrow} f^*\Omega_X^n,$$

and call it the *τ-De Rham complex*. We say that $\nabla_\tau (\nabla_\tau')$ is *integrable* if $\nabla_\tau^2 = 0$ $(\nabla_\tau'^2 = 0)$. Define the *τ-flat* sections to be those which are annihilated by a τ-connection.

LEMMA.

(i) *One has* $Rf_* (f^*\Omega_X^k \xrightarrow{\tau d} f^*\Omega_X^{k+1}) = \Omega_X^k \xrightarrow{d} \Omega_X^{k+1}$

(ii) *One has* $Rf_* \nabla_\tau = \nabla$

(iii) $(\tau d)^2 = 0$ *if and only if* ∇_τ^2 *is* $\mathcal{O}_P$*-linear. In this case,* $\tau: \Omega_P^1 \to f^*\Omega_X^1$ *extends to a morphism of complexes* $\tau: \Omega_P^{\cdot} \to \Omega_\tau^{\cdot}$. *One has* $Rf_* \Omega_\tau^{\cdot} = \Omega_X^{\cdot}$. *This defines a morphism* $Rf_* \mathbb{C}_P \to \mathbb{C}_X$ *in the derived category.*

(iv) ∇ *is integrable if and only if* ∇_τ *is. In this case,* ∇_τ' *is integrable. Moreover* $\mathcal{O}(1)$ *and* $\Omega_{P/X}^1(1)$ *are generated by* τ*-flat sections.*

Proof. (i) As $Rf_*f^*\Omega_X^k=\Omega_X^k$, one just has to see that $f_*\tau d=d$. This is a local condition on X. On an open set U on X, one has $\Gamma(f^{-1}U,f^*\Omega_X^k)=\Gamma(f^{-1}U,f^{-1}\Omega_X^k)$ on which $\tau d=d$.

(ii) As in (i), one just has to see $f_*\nabla_\tau=\nabla$. As f^*E is by construction the sheaf generated by relative global sections of $\mathcal{O}(1)$, and as $\nabla_\tau=(1\otimes q)\tau f^*\nabla$, this is equivalent to see $f_*(\tau f^*\nabla)=\nabla$. This is the same as in (i).

(iii) One has $\nabla_\tau^2(\lambda\cdot x)=\lambda\cdot\nabla_\tau^2(x)+(\tau d)^2(\lambda)\cdot x$, for $\lambda\in\mathcal{O}_p$ and $x\in\mathcal{O}(1)$. Ω_P^k is additively generated by elements $y=\lambda\cdot d\omega$, for $\omega\in\Omega_P^{k-1}$, $\lambda\in\mathcal{O}_P$. Then $\tau dy=\tau d\lambda\wedge\tau d\omega$, whereas $\tau d(\lambda\cdot\tau d\omega)=\tau d\lambda\wedge\tau d\omega+\lambda(\tau d)^2\omega$. If $(\tau d)^2=0$, then $\tau dy=\tau d(\lambda\cdot\tau d\omega)$. In other words, one has a morphism of complexes $\tau:\Omega_P^{\cdot}\to\Omega_\tau^{\cdot}$.

Since $\Omega_\tau^{\cdot}$ is f-acyclic and $f_*\tau d=d$ (i), one has $Rf_*\Omega_\tau^{\cdot}=\Omega_X^{\cdot}$. Since $\Omega_P^{\cdot}$ and $\Omega_X^{\cdot}$ are quasi-isomorphic to $\mathbb{C}_P$ and $\mathbb{C}_X$, τ defines a morphism $Rf_*\mathbb{C}_P\to\mathbb{C}_X$ in the derived category.

(iv) If $\nabla^2=0$ then $E=L\underset{\mathbb{C}}{\otimes}\mathcal{O}_X$ where L is a local constant system, and $\nabla=1\otimes d$. Then $\tau f^*\nabla=1\otimes\tau d$. If e^k is a basis of L on U, one has (with the notations of (0.7)) $\tau d\left(\frac{t^k}{t^0}\right)=0$. Therefore $(\tau d)^2=0$. This implies $(\tau f^*\nabla)^2=0$, as well as $\nabla_\tau^2=\nabla_\tau'^2=0$.

Conversely if $\nabla_\tau^2=0$, then $f_*\nabla_\tau^2=\nabla^2=0$. One may generate $\mathcal{O}(1)$ by t^k and $\Omega_{P/X}^1(1)$ by x^k, which are τ-flat sections.

(1.4) *Remark.* To see that the integrability of ∇ implies $(\tau d)^2=0$ (which means that one has a τ-complex), one does not need in (iv) the description of E by its flat sections. If e^k is any basis of E on U, one has (0.7):

$$\tau du^k=\sum_s\omega^{ks}\cdot u^s-u^k\cdot\sum_s\omega^{0s}\cdot u^s$$

where one sets for simplicity $u^k=\frac{t^k}{t^0}$, where ω^{kl} is the connection matrix of ∇ on U. Therefore one has:

$$(\tau d)^2u^k=A+B+C+D+E$$

where

$$A=\sum_s d\omega^{ks}\cdot u^s$$

$$B=-\sum_s\omega^{ks}\sum_{s'}\omega^{ss'}\cdot u^{s'}+\sum_s\omega^{ks}\cdot u^s\sum_{s'}\omega^{0s'}\cdot u^{s'}=b+b'$$

$$C=-\sum_s\omega^{ks}\cdot u^s\sum_s\omega^{0s}\cdot u^s+u^k\sum_s\omega^{0s}\cdot u^s\sum_s\omega^{0s}\cdot u^s=c+c'$$

$$D=-u^k\sum_s d\omega^{0s}\cdot u^s$$

$$E=u^k\sum_s\omega^{0s}\sum_{s'}\omega^{ss'}\cdot u^{s'}-u^k\sum_s\omega^{0s}\cdot u^s\sum_{s'}\omega^{0s'}\cdot u^{s'}=e+e'.$$

The integrability condition

$$d\omega^{kl}-\sum_s\omega^{ks}\cdot\omega^{sl}=0$$

implies $A+b=0$, $D+e=0$. On the other hand one has $b'+c=0$, $c'=e'=0$. This proves that $(\tau d)^2u^k=0$.

This remark is important for §3 and §4, where the τ-flat sections for the connections considered there do not generate the bundle E.

(1.5) If $(\tau d)^2=0$, one has as in (1.1) an injection

$$\begin{array}{c} H^2(P, \mathbb{Z}(1) \to \Omega^{\cdot}_{\tau}) \\ \downarrow \\ H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P \xrightarrow[\tau d]{} f^*\Omega^1_X). \end{array}$$

$(\mathcal{O}_P, \tau d)$ is the trivial (integrable) τ-connection. One considers the morphism

$$\begin{array}{c} H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P \xrightarrow[\tau d]{} f^*\Omega^1_X) \\ \downarrow \\ H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P). \end{array}$$

For (F, ∇_τ) and (F', ∇'_τ) two rank one bundles with (integrable) τ-connections, define the (integrable) τ-connection on $F\otimes F'$: $\nabla_\tau\otimes\nabla'_\tau(e\otimes e')=\nabla_\tau e\otimes e'+e\otimes\nabla'_\tau e'$. If $\varphi: F'\to F$ is a $\mathcal{O}_P$-morphism, define on F' the (integrable) τ-connection: $\varphi^*\nabla_\tau(e')=\nabla_\tau(\varphi(e))$. Then φ is an isomorphism from (F', ∇'_τ) to (F, ∇_τ) if it is an isomorphism from F' to F verifying $\varphi^*\nabla_\tau=\nabla'_\tau$.

LEMMA. (i) *The isomorphism classes of rank one bundles F with τ-connections ∇_τ build a group identified with $H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P \xrightarrow[\tau d]{} f^*\Omega^1_X)$. Denote by $(\mathcal{O}(1), \nabla_\tau)$ the class defined in* (0.7). *Its image in $H^2(P, \mathbb{Z}(1)\to\mathcal{O}_P)$ is the isomorphism class of $\mathcal{O}(1)$.*

(ii) *Assume that $(\tau d)^2=0$. Then ∇_τ is integrable if and only if $(\mathcal{O}(1), \nabla_\tau)\in H^2(P, \mathbb{Z}(1)\to\Omega^{\cdot}_\tau)$.*

Proof. (i) We mimic (1.1). If $u_{\alpha\beta}$ is a cocycle representing F on some Čech cover, then ∇_τ is given by $\omega_\alpha\in\Gamma(U_\alpha, f^*\Omega^1_X)$ such that $u_{\alpha\beta}^{-1}\cdot\tau du_{\alpha\beta}=\omega_\alpha-\omega_\beta$. Then $(u_{\alpha\beta}, \omega_\alpha)$ is the class wanted.

This class is isomorphic to $(\mathcal{O}, \tau d)$ if and only if they are $f_\alpha\in\Gamma(U_\alpha, \mathcal{O}^*_P)$ verifying $u_{\alpha\beta}=f_\alpha\cdot f_\beta^{-1}$, and $\omega_\alpha=f_\alpha^{-1}\cdot\tau df_\alpha$.

(ii) By (1.3)(iv), $\nabla^2=0$ if and only if $\nabla^2_\tau=0$. This is equivalent to

$$0=\tau d\omega_\alpha\in H^0(P, f^*\Omega^2_X\to\ldots\to f^*\Omega^n_X)$$

or

$$\begin{array}{l} (\mathcal{O}(1), \nabla_\tau)\in \operatorname{Ker}(H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P\to f^*\Omega^1_X) \\ \qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad H^0(P, f^*\Omega^2_X\to\ldots\to f^*\Omega^n_X) \\ \qquad\qquad\qquad = H^2(P, \mathbb{Z}(1)\to\Omega^{\cdot}_\tau). \end{array}$$

(1.6) CLAIM. *If X has an Hodge structure, and E is a bundle on X with an holomorphic connection ∇ such that $(\tau d)^2=0$, then one has*

$$H^2(P, \mathbb{Z}(1)\to\Omega^{\cdot}_\tau)=H^2(P, \mathbb{Z}(1)\to\mathcal{O}_P\xrightarrow[\tau d]{} f^*\Omega^1_X).$$

In particular $\nabla^2=0$ if and only if $(\tau d)^2=0$.

Proof. By (1.5) the second statement is a consequence of the first one. From the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}(1) \to \mathcal{O}_P & \longrightarrow & f^*\Omega^1_X & & & & \\
\downarrow & & \downarrow d_1 & & & & \\
\mathbb{Z}(1) \to \mathcal{O}_P & & & & & & \\
\downarrow d_0 & & f^*\Omega^2_X & \to \dots \to & f^*\Omega^n_X & & \\
& & | & & & & \\
\Omega^1_P & \longrightarrow & \Omega^2_P & \to \dots & \longrightarrow & & \Omega^{n+r-1}_P \\
\downarrow & & \downarrow i & & & & \\
f^*\Omega^1_X & \to & f^*\Omega^2_X & \to \dots \to & f^*\Omega^n_X & &
\end{array}$$

one has the commutative diagram

$$\begin{array}{ccccc}
H^2(\mathbb{Z}(1) \to \mathcal{O}_P \to f^*\Omega^1_X) & & \xrightarrow{H(d_1)} & & H^0(f^*\Omega^2_X \to \dots \to f^*\Omega^n_X) \\
\downarrow & & & & \downarrow H(i) \\
H^2(\mathbb{Z}(1) \to \mathcal{O}_P) & \xrightarrow[H(d_0)]{} & H^1(\Omega^1_P \to \dots \to \Omega^{n+r-1}_P) & \xrightarrow[H(\tau)]{} & H^1(f^*\Omega^1_X \to \dots \to f^*\Omega^n_X).
\end{array}$$

The first statement is equivalent to $H(d_1)=0$. The image of $H(d_0)$ is contained in $H^1(\Omega^1_P)$, therefore the image of $H(\tau)H(d_0)$ is contained in $H^1(f^*\Omega^1_X.)$.

It meets in 0 the image of $H(i)$. Since $H(i)$ is injective, one has $H(d_1)=0$.

Remark. Compare (1.3)(iv) and (1.6). In general one has $\nabla^2=0$ if and only if $\nabla^2_\tau=0$. With an Hodge structure, one has $\nabla^2=0$ if and only if $(\tau d)^2=0$. This is slightly weaker. This corresponds to (1.2).

§2. CHARACTERISTIC CLASSES OF A BUNDLE E WITH AN INTEGRABLE CONNECTION

(2.1) Let Y be a smooth analytic variety. Let $(A^k, k \geq 0)$ be a complex such that there is a morphism of complexes $\tau\colon \Omega^{\cdot}_Y \to A^{\cdot}$ where $A^0=\mathcal{O}_Y$, $\Lambda^k A^1 = A^k$ is a quotient bundle of Ω^k_Y. Define $B^1=\operatorname{Ker}\tau\colon \Omega^1_Y \to A^1$. As the differential of $A^{\cdot}$ is the factorization through $A^{\cdot}$ of τd, write simply τd for it.

A bundle F is said to have a *τ-connection* if there is a $\mathbb{C}$-linear morphism $\nabla\colon F \to A^1 \otimes F$ verifying the *τ-Leibniz rule* $\nabla_\tau(\lambda \cdot x) = \lambda \cdot \nabla_\tau(x) + \tau d(\lambda) \otimes x$.

∇_τ is said to be *integrable* if $\nabla^2_\tau=0$.

F is said to be *generated* by *τ-flat sections* if it is locally generated by sections x verifying $\nabla_\tau x=0$. In this case one may find a cocycle $u_{\alpha\beta}$ representing F with $u^{-1}_{\alpha\beta} \cdot \tau d u_{\alpha\beta}=0$.

$(\mathcal{O}, \tau d)$ is the trivial (integrable) τ-connection. As in (1.5) the isomorphism class of (F, ∇_τ) is in $H^2(Y, \mathbb{Z}(1) \to \mathcal{O}_Y \to A^1)$, and $\nabla^2_\tau=0$ if and only if (F, ∇_τ) is in $H^2(Y, \mathbb{Z}(1) \to A^{\cdot})$.

(2.2) One has the standard operations for bundles with τ-connections.

Let F and F' be bundles with (integrable) τ-connections ∇_τ and ∇'_τ. One defines (integrable) τ-connections on

$\overset{l}{\bigwedge} F$ by $(\overset{l}{\bigwedge}\nabla)(f_1 \wedge \dots \wedge f_l) = \sum f_1 \wedge \dots \wedge f_{i-1} \wedge \nabla f_i \wedge f_{i+1} \wedge \dots \wedge f_l$

$F \otimes F'$ by $\nabla_\tau \otimes \nabla'_\tau(f \otimes f') = \nabla_\tau(f) \otimes f' + f \otimes \nabla'_\tau(f')$

$\mathcal{H}om_{\mathcal{O}_Y}(F, F')$ by $(\nabla\varphi)(f) = \nabla'_\tau \varphi(f) - \varphi(\nabla_\tau f)$.

Denote by $\nabla^\vee_\tau$ the connection on $\mathcal{H}om_{\mathcal{O}_Y}(F, \mathcal{O}_Y) = F^\vee$.

If (F, ∇_τ) and (F', ∇'_τ) are of rank one, of cocycles $(u_{\alpha\beta}, \omega_\alpha)$ and $(u'_{\alpha\beta}, \omega'_\alpha)$, then $(F \otimes F', \nabla_\tau \otimes \nabla'_\tau)$ is of cocycle $(u_{\alpha\beta} \cdot u'_{\alpha\beta},\ \omega_\alpha + \omega'_\alpha)$. Therefore $(F \otimes F', \nabla_\tau \otimes \nabla'_\tau) = (F, \nabla_\tau) + (F', \nabla'_\tau)$ in $H^2(Y, \mathbb{Z}(1) \to \mathcal{O}_Y \to A^1)$ (resp. in $H^2(Y, \mathbb{Z}(1) \to A^\cdot)$). Similarly $(F^\vee, \nabla_\tau^\vee) = -(F, \nabla_\tau)$ in $H^2(Y, \mathbb{Z}(1) \to \mathcal{O}_Y \to A^1)$ (resp. $H^2(Y, \mathbb{Z}(1) \to A^\cdot)$).

A *filtration* $F_{k-1} \subset F_k$ of a higher rank bundle F by subbundles F_k such that $\nabla_\tau F_k \subset A^1 \otimes F_k$ is said to be τ-compatible (τ-*flat* if $\nabla_\tau^2 = 0$). This defines (integrable) τ-connections $\nabla_{\tau,k}$ on F_k/F_{k-1}.

An *exact sequence* $0 \to F' \to F \to F'' \to 0$ is said to be τ-compatible (τ-*flat*) if the filtration $F' \subset F$ is.

(2.3) Let $g\colon Z \to Y$ be a morphism between two manifolds, and F and τ be as in (2.1). Define $\Omega^1_{Z,\tau}$ by the exact sequence

$$g^* B^1 \to \Omega^1_Z \xrightarrow{r} \Omega^1_{Z,\tau} \to 0.$$

One has the exact sequence

$$g^* A^1 \to \Omega^1_{Z,\tau} \xrightarrow{p'} \Omega^1_{Z/Y} \to 0$$

Define $\bigwedge^k \Omega^1_{Z,\tau} = \Omega^k_{Z,\tau}$.

CLAIM. *r extends to a surjective morphism of complexes $r\colon \Omega^\cdot_Z \to \Omega^\cdot_{Z,\tau}$.*

Proof. In order to extend r as a morphism of complexes, one has to see that the kernel of $\Omega^k_Z \to \Omega^k_{Z,\tau}$ is generated by the image in Ω^k_Z of $g^* B^1 \wedge \Omega^{k-1}_Z$. By the Leibniz rule it is enough to see that $d g^* B^1 \subset g^* B^1 \wedge \Omega^1_Z$. Since $A^\cdot$ is a quotient complex of $\Omega^\cdot_Y$ one has $dB^1 \subset B^1 \wedge \Omega^1_Y$. Write $g^* B^1 = \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_Y} g^{-1} B^1$. By the Leibniz rule again one has

$$d g^* B^1 \subset g^* B^1 \wedge \Omega^1_Z + \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}(B^1 \wedge \Omega^1_Y) \subset g^* B^1 \wedge \Omega^1_Z.$$

Now since $\Omega^1_Z \to \Omega^1_{Z,\tau}$ is surjective by definition, $\Omega^k_Z \to \Omega^k_{Z,\tau}$ is surjective, and r is a surjective morphism of complexes.

Denote by rd the differential on $\Omega^\cdot_{Z,\tau}$. One has $(rd)^2 = 0$. *One defines the r-connection*

$$g^*\nabla_\tau\colon g^* F \to \Omega^1_{Z,\tau} \underset{\mathcal{O}_Z}{\otimes} g^* F$$

by writing

$$g^* F = \mathcal{O}_Z \underset{g^{-1}\mathcal{O}_Z}{\otimes} g^{-1} F$$

and

$$g^*\nabla_\tau(\lambda \otimes \varphi) = rd\lambda \underset{\mathcal{O}_Z}{\otimes} \varphi + \lambda \underset{g^{-1}\mathcal{O}_Y}{\otimes} \nabla_\tau \varphi$$

for $\varphi \in g^{-1} F$ and $\lambda \in \mathcal{O}_Z$.

The corresponding B'^1 is the image of $g^* B^1$ in Ω^1_Z. As $(rd)^2 = 0$, $g^*\nabla_\tau$ *is integrable if* ∇_τ *is,* and $g^* F$ is generated by r-flat sections if F is generated by τ-flat sections.

(2.4) Set $Z = P(F)$ the projective bundle of F. One has the other exact sequence

$$0 \to \Omega^1_{Z/Y}(1) \to g^* F \xrightarrow{q} \mathcal{O}(1) \to 0$$

Define as in (0.6) $\sigma\colon \Omega^1_{Z/Y} \to \Omega^1_{Z,\tau}$ by $\sigma \otimes 1 = (1 \otimes q)\, g^*\nabla_\tau|_{\Omega^1_{Z/Y}(1)}$. By the same computation as in (0.6.1) one has $-\sigma$ *is a section of* p'.

In this case, $g^* A^1$ is embedded in $\Omega^1_{Z,\tau}$.

One obtains a section

$$\tau'=(1+p'\sigma)\colon \Omega^1_{Z,\tau}\to g^*A^1$$

which may be written with the notations (0.7) as $\tau' rd\left(\frac{t^k}{t^0}\right)=\frac{1}{t^0}(1\otimes q)\left(g^*\nabla_\tau e^k-\frac{t^k}{t^0}g^*\nabla_\tau e^0\right)$.

(2.5) Assume now that ∇_τ is integrable. By (1.4) one has $(\tau' rd)^2=0$. This defines (1.3, iii) a *morphism of complexes*

$$\tau' r\colon \Omega^{\cdot}_Z\to g^*A^{\cdot}$$

where the differential on $g^*A^{\cdot}$ is defined by $\tau' rd$. As in (1.3) one has $Rg_*g^*A^{\cdot}=A^{\cdot}$. The morphism $\tau' r$ defines a morphism in the derived category $\tau' r\colon Rg_*\mathbb{C}_Z\to A^{\cdot}$.

(2.6) Further one may define: *$\tau' r$-connections* $\nabla_{\tau' r}$ and $\nabla'_{\tau' r}$ on $\mathcal{O}(1)$ and $\Omega^1_{Z/Y}(1)$ by

$$\nabla'_{\tau' r}=\tau' g^*\nabla_\tau|_{\Omega^1_{Z/Y}(1)}$$

$$\nabla_{\tau' r}=(1\otimes q)\tau' g^*\nabla_\tau.$$

They are *integrable* if ∇_τ is.

(2.7) *Through the rest of* §2, *one considers on a manifold X a morphism of complexes $\tau_0\colon \Omega^{\cdot}_X\to A^{\cdot}$ as in* (2.1) *and a bundle E with an integrable τ_0-connection* ∇.

On the projective bundle $P(E)$ one has defined $r\tau_0$ and integrable $r\tau_0$-connections on $\mathcal{O}(1)$ and $\Omega^1_{P(E)/X(1)}$. One may repeat this construction (rank E-1) times. One has the following data on *the flag bundle* of E which we call $f\colon P\to X$, with *f the splitting morphism.*

(i) There is a morphism $\tau\colon \Omega^1_P\to f^*A^1$ with $(\tau d)^2=0$. The complex

$$A^{\cdot}_\tau=\mathcal{O}_P\xrightarrow[\tau d]{} f^*A^1\to\dots\xrightarrow[\tau d]{} f^*A^n$$

verifies $Rf_*A^{\cdot}_\tau=A^{\cdot}$. If $\tau_0=$identity (which means E flat), write $\Omega^{\cdot}_\tau$ for $A^{\cdot}_\tau$. One has $Rf_*\Omega^{\cdot}_\tau=\Omega^{\cdot}_X$.

τ extends to a morphism of complexes $\tau\colon \Omega^{\cdot}_P\to A^{\cdot}_\tau$.

(ii) The integrable τ_0-connection ∇ defines an integrable τ-connection $(f^*\nabla)_\tau$ *on* f^*E. The *canonical filtration* $0=E_0\subset\dots\subset E_r=f^*E$ of f^*E is τ-flat (see 2.2). This defines an integrable τ-connection $\nabla_{\tau,k}$ on the *splitting rank one bundle* $L_k=E_k/E_{k-1}$, and therefore a class $(L_k,\nabla_{\tau,k})\in H^2(P,\mathbb{Z}(1)\to A^{\cdot}_\tau)$ whose image in $H^2(P,\mathbb{Z}(1)\to\mathcal{O}_P)$ is the isomorphism class of L_k (and whose image in $H^2(P,\mathbb{Z}(1))$ is $c_1^{\text{top}}(L_k)$, the topological Chern class (2.1)). This class is represented on some Čech cover by $(u^k_{\alpha\beta},\omega^k_\alpha)\in\Gamma(U_{\alpha\beta},\mathcal{O}^*)\times\Gamma(U_\alpha,A^1_\tau)$ such that $\delta u=0$, $u^{-1}\cdot\tau du=\delta\omega$, $\tau d\omega=0$, where δ is the Čech differential.

(2.8) The Deligne complexes (see [2]) on a manifold Z are

$$\mathbb{Z}(p)_{\mathcal{D}}=\mathbb{Z}(p)\to\mathcal{O}_Z\to\dots\to\Omega^{p-1}_Z$$

$$=\text{cone}(\mathbb{Z}(p)\oplus F^p\xrightarrow[\alpha-i]{}\Omega^{\cdot}_Z)[-1]$$

where $\alpha\colon \mathbb{Z}(p)\to\mathbb{C}$ is the natural embedding and $i\colon F^p\to\Omega^{\cdot}_Z$ is the Hodge–Deligne F-filtration. There is a product

$$\mathbb{Z}(p)_{\mathcal{D}}\times\mathbb{Z}(q)_{\mathcal{D}}\to\mathbb{Z}(p+q)_{\mathcal{D}}$$

which is uniquely defined by

$$x\cdot x'=\begin{cases}\alpha(x)\cdot x' & \text{if } \deg x=0\\ x\wedge dx' & \text{if } \deg x>0 \text{ and } \deg x'=q\\ 0 & \text{otherwise,}\end{cases}$$

for x homogeneous in $\mathbb{Z}(p)_{\mathscr{D}}$ and x' homogeneous in $\mathbb{Z}(q)_{\mathscr{D}}$.

In the cone language this corresponds to $(n\oplus f\oplus\omega)\cdot(n'\oplus f'\oplus\omega')=(n\cdot n'+f\cdot f', \alpha(n)\cdot\omega'+\omega\wedge i(f))$, for $(n\oplus f)\in\mathbb{Z}(p)\oplus F^p$, $(n'\oplus f')\in\mathbb{Z}(q)\oplus F^q$, ω and $\omega'\in\Omega^{\cdot}_Z$. One has $x\cdot x' =(-1)^{\deg x\cdot\deg x'}x'\cdot x$ up to homotopy, for x homogeneous in $\mathbb{Z}(p)_{\mathscr{D}}$ and x' homogeneous in $\mathbb{Z}(q)_{\mathscr{D}}$ (see [2], and [7] for precise computations).

This defines a product in the cohomology

$$H^{p'}(\mathbb{Z}(p)_{\mathscr{D}})\times H^{q'}(\mathbb{Z}(q)_{\mathscr{D}})\to H^{p'+q'}(\mathbb{Z}(p+q)_{\mathscr{D}})$$

which is anticommutative, that is

$$x\cdot x'=(-1)^{p'\cdot q'}x'\cdot x \quad\text{for}\quad x\in H^{p'}(\mathbb{Z}(p)_{\mathscr{D}}) \text{ and } x'\in H^{q'}(\mathbb{Z}(q)_{\mathscr{D}}).$$

Therefore the p-symmetric functions of the classes of L_k in $H^2(P,\mathbb{Z}(1)_{\mathscr{D}})=H^1(P,\mathcal{O}_P^*)$ on the flag bundle P of E define classes $c_p^{\mathscr{D}}(f^*E)\in H^{2p}(P,\mathbb{Z}(p)_{\mathscr{D}})$.

Define

$$\mathbb{Z}(p)_{\mathscr{D},\tau_0}=\mathbb{Z}(p)\to A^0\to A^1\to\dots\to A^{p-1}.$$

One has the morphism $\tau_0\colon\mathbb{Z}(p)_{\mathscr{D}}\to\mathbb{Z}(p)_{\mathscr{D},\tau_0}$.

(2.9) On a manifold Y with a morphism $\tau\colon\Omega^{\cdot}_Y\to A^{\cdot}$ as in (2.1), define $\mathbb{Z}(p)_\tau=\mathbb{Z}(p)\xrightarrow{\tau}A^{\cdot}$. One defines a map

$$\mathbb{Z}(p)_\tau\times\mathbb{Z}(q)_\tau\to\mathbb{Z}(p+q)_\tau$$

by

$$x\cdot x'=\begin{cases}\tau(x)\cdot x' & \text{if } \deg x=0\\ 0 & \text{otherwise}\end{cases}$$

for x and x' homogeneous in $\mathbb{Z}(p)_\tau$ and $\mathbb{Z}(q)_\tau$. This defines a *product*, that is it factorizes through

$$\mathbb{Z}(p)_\tau\otimes_{\mathbb{Z}}\mathbb{Z}(q)_\tau\to\mathbb{Z}(p+q)_\tau.$$

One has to verify that

$$d(x\cdot x')=dx\cdot x'+(-1)^{\deg x}x\cdot dx'$$

for x and x' homogeneous in $\mathbb{Z}(p)_\tau$ and $\mathbb{Z}(q)_\tau$, where d is the differential in the corresponding complex. The left hand side is

$$\begin{cases}\tau(x)\cdot dx' & \text{if } \deg x=0\\ 0 & \text{otherwise}\end{cases}$$

whereas the right hand side is

$$\begin{cases}xd(x')=\tau(x)\cdot x' & \text{if } \deg x=0 \quad (\text{since } \deg dx=1)\\ 0 & \text{otherwise.}\end{cases}$$

Define a map

$$h\colon(\mathbb{Z}(p)_\tau\otimes_{\mathbb{Z}}\mathbb{Z}(q)_\tau)^l\to(\mathbb{Z}(p+q)_\tau)^{l-1}$$

where l is the homogeneous degree of the complex considered, by

$$h(x \otimes x') = (-1)^{\deg x} x \wedge x' \quad \text{if } \deg x > 0 \text{ and } \deg x' > 0$$
$$0 \quad \text{otherwise.}$$

One verifies immediately that

$$(hd + dh)(x \otimes x') = h(dx \otimes x' + (-1)^{\deg x} x \otimes dx') + dh(x \otimes x')$$

is

$$\begin{array}{ll} -xx' & \text{if } \deg x = 0, \ \deg x' > 0 \\ (-1)^{2\deg x} xx' & \text{if } \deg x > 0, \ \deg x' = 0 \\ 0 & \text{otherwise.} \end{array}$$

This means that

$$x' \cdot x - (-1)^{\deg x \,.\, \deg x'} x' \cdot x = (hd + dh)(x \otimes x').$$

In other words, this product is *anticommutative.*

This defines a *τ-product* in the *τ-cohomology*

$$H^{p'}(\mathbb{Z}(p)_\tau) \times H^{q'}(\mathbb{Z}(q)_\tau) \to H^{p'+q'}(\mathbb{Z}(p+q)_\tau).$$

The anticommutativity implies

$$x \cdot x' = (-1)^{p'q'} x' \cdot x \quad \text{for} \quad x \in H^{p'}(\mathbb{Z}(p)_\tau) \text{ and } x' \in H^{q'}(\mathbb{Z}(q)_\tau),$$

and in particular $x \cdot x' = x' \cdot x$ for $x \in H^{2p}(\mathbb{Z}(p)_\tau)$.

The *τ-product factorizes* through the product

$$\mathbb{Z}(p) \times \mathbb{Z}(q)_\tau \to \mathbb{Z}(p+q)_\tau$$

defined by

$$(x, x') \to \tau(x) \cdot x'.$$

Therefore the τ-product in the τ-cohomology factorizes through the product

$$H^{p'}(\mathbb{Z}(p)) \times H^{q'}(\mathbb{Z}(q)_\tau) \to H^{p'+q'}(\mathbb{Z}(p+q)_\tau)$$

which is defined by $\tau(x) \cdot x'$.

Finally the product on $\mathbb{Z}(p)_\tau$ maps to the cup-product on $\mathbb{Z}(p)$. Therefore the *τ-product in the τ-cohomology maps to the cup-product in cohomology*: the following diagram

$$\begin{array}{ccc} H^{p'}(\mathbb{Z}(p)_\tau) \times H^{q'}(\mathbb{Z}(q)_\tau) & \to & H^{p'+q'}(\mathbb{Z}(p+q)_\tau) \\ \downarrow & & \downarrow \\ H^{p'}(\mathbb{Z}(p)) \times H^{q'}(\mathbb{Z}(q)) & \to & H^{p'+q'}(\mathbb{Z}(p+q)) \end{array}$$

is commutative.

(2.10) Define the *characteristic classes of* $(f^* E, f^* \nabla)$ *by* $c_p(f^* E, f^* \nabla) =$ *p-th symmetric function of*

$$(L_k, \nabla_{\tau, k}) \in H^{2p}(P, \mathbb{Z}(p)_\tau)$$

for E, L_k, $\nabla_{\tau,k}$ as in (2.7), $\mathbb{Z}(p)_\tau = \mathbb{Z}(p) \to A^{\cdot}_\tau$ and the product as in (2.9).

(2.11) Denote by a_p the morphism

$$A^{\cdot}_\tau \to (\mathcal{O}_P \underset{\tau d}{\to} \cdots \to f^* A^{p-1}),$$

by τ the morphism

$$(\mathcal{O}_P \to \ldots \to \Omega_P^{p-1}) \to (\mathcal{O}_P \underset{\tau d}{\to} \ldots \to f^* A^{p-1}),$$

by $\mathbb{Z}(p)_{\mathcal{D},\tau}$ the complex

$$\mathbb{Z}(p) \to \mathcal{O}_P \underset{\tau d}{\to} \ldots . \to f^* A^{p-1}$$

and similarly for τ_0.

PROPOSITION. *One has*

$$\tau c_p^{\mathcal{D}}(f^* E) = a_p c_p(f^* E, f^* \nabla) \text{ in } H^{2p}(P, \mathbb{Z}(p)_{\mathcal{D},\tau}).$$

Proof. The product on $\mathbb{Z}(p)_{\mathcal{D}}$ (2.8) defines a product $\times$ on $\mathbb{Z}(p)_{\mathcal{D},\tau}$ by

$$\begin{array}{ll} x \times x' = \tau(x)x' & \text{if } \deg x = 0 \\ x \wedge \tau dx' & \text{if } \deg x > 0, \deg x' = q \\ 0 & \text{otherwise,} \end{array}$$

for x and x' homogeneous elements in $\mathbb{Z}(p)_{\mathcal{D},\tau}$ and $\mathbb{Z}(q)_{\mathcal{D},\tau}$. Define a map

$$h: (\mathbb{Z}(p)_\tau \otimes_{\mathbb{Z}} \mathbb{Z}(q)_\tau)^l \to (\mathbb{Z}(p+q)_{\mathcal{D},\tau})^{l-1},$$

where l is the degree in the complex considered, by

$$\begin{array}{ll} h(x \otimes x') = (-1)^{\deg x} x \wedge x' & \text{if } \deg x > 0, \deg x' \geqq q+1 \\ 0 & \text{otherwise.} \end{array}$$

We prove in (3.3)—actually in a greater generality—that

$$a_p x \times a_q x - a_{p+q} x \cdot x' = (hd + dh)(x \otimes x').$$

Therefore the diagram

$$\begin{array}{ccc} \mathbb{Z}(p)_\tau \otimes_{\mathbb{Z}} \mathbb{Z}(q)_\tau & \xrightarrow{\cdot} & \mathbb{Z}(p+q)_\tau \\ \downarrow a_p \otimes a_q & & \downarrow a_{p+q} \\ \mathbb{Z}(p)_{\mathcal{D},\tau} \otimes_{\mathbb{Z}} \mathbb{Z}(q)_{\mathcal{D},\tau} & \xrightarrow{\times} & \mathbb{Z}(p+q)_{\mathcal{D},\tau} \end{array}$$

commutes up to homotopy. Since $\tau c_p^{\mathcal{D}}(f^* E)$ and $a_p c_p(f^* E, f^* \nabla)$ are defined as symmetric products ((2.8) and (2.10)), it is enough to verify

$$\begin{aligned} \tau c_1^{\mathcal{D}}(L_k)(&= c_1^{\mathcal{D}}(L_k) \text{ in } H^2(P, \mathbb{Z}(1)_{\mathcal{D},\tau}) = H^2(P, \mathbb{Z}(1)_{\mathcal{D}})) \\ &= a_1(L_k, \nabla_{\tau,k}). \end{aligned}$$

This is (2.7, ii).

(2.12) If g: $M \to X$ is the projective bundle of E, then one has ([2], 1.7.2)

$$H^q(M, \mathbb{Z}(p)_{\mathcal{D}}) = \bigoplus_{\substack{0 \leqq j \leqq r-1 \\ 0 \leqq q-2j \\ 0 \leqq p-j}} g^{-1} H^{q-2j}(X, \mathbb{Z}(p-j)_{\mathcal{D}}) \cdot \mathcal{O}(1)^j$$

The Deligne cohomology of M is a free module over the Deligne cohomology of X, with bases $\mathcal{O}(1)^j$, $0 \leqq j \leqq r-1$. By taking the coefficients of the expansion of $\mathcal{O}(1)^r$, one defines the Chern classes $c_p^{\mathcal{D}}(E) \in H^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}})$. With the formalism of Hirzebruch–Grothendieck ([9]), one proves they are functorial and additive, and thereby verify $f^{-1} c_p^{\mathcal{D}}(E) = c_p^{\mathcal{D}}(f^* E)$, where $c_p^{\mathcal{D}}(f^* E)$ was defined in (2.8) (see [2], 1.7.2 and 1.7.3).

The image of $c_p^{\mathcal{D}}(f^* E)$ in $H^{2p}(P, \mathbb{Z}(p))$ is the topological Chern class $c_p^{\mathrm{top}}(f^* E) = f^{-1} c_p^{\mathrm{top}}(E)$, where $c_p^{\mathrm{top}}(E)$ is the image of $c_p^{\mathcal{D}}(E)$ in $H^{2p}(X, \mathbb{Z}(p))$.

(2.13) The formula (2.12) is no longer true for the τ-cohomology: $H^{\cdot}(M, \mathbb{Z}(\cdot)_\tau)$ is *not* a free module over $H^{\cdot}(X, \mathbb{Z}(\cdot)_{\tau_0})$. Therefore one can not use Hirzebruch–Grothendieck's formalism to prove that our classes $c_p(f^* E, f^* \nabla)$ verify the standard properties of Chern classes.

The rest of this chapter is essentially devoted to the definition of classes $c_p(E, \nabla)$ on X (2.15), to the proof of the functoriality (2.16) and the additivity (2.17), and to some simple comments.

(2.14) Lemma. *With the notations of* (2.7) *and* (2.9), *one has the following commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
0 \to & f^{-1} H^q(X, \mathbb{Z}(p)_{\tau_0}) & \to & H^q(P, \mathbb{Z}(p)_\tau) & \to & H^q(P, \mathbb{Z}(p))/f^{-1} H^q(X, \mathbb{Z}(p)) & \to 0 \\
& \downarrow a_p & & \downarrow a_p & & \| & \\
0 \to & f^{-1} H^q(X, \mathbb{Z}(p)_{\mathcal{D}, \tau_0}) & \to & H^q(P, \mathbb{Z}(p)_{\mathcal{D}, \tau}) & \to & H^q(P, \mathbb{Z}(p))/f^{-1} H^q(X, \mathbb{Z}(p)) & \to 0.
\end{array}$$

Proof. Just write

$$\begin{array}{ll}
\mathbb{Z}(p)_\tau & = \mathrm{cone}(\mathbb{Z}(p) \to A^{\cdot})[-1] \\
\downarrow a_p & \\
\mathbb{Z}(p)_{\mathcal{D}, \tau} & = \mathrm{cone}(\mathbb{Z}(p) \to (\mathcal{O}_P \xrightarrow[\tau d]{} \cdots \to (A_\tau^{p-1}))[-1]
\end{array}$$

and remember that

$$Rf_*(A_\tau^k \xrightarrow[\tau d]{} A_\tau^{k+1}) = A^k \xrightarrow[\tau_0 d]{} A^{k+1}$$

(2.15) Theorem. *Let E be a bundle on a manifold X with an integrable τ_0-connection ∇. They are classes $c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p)_{\tau_0})$ whose images in $H^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}, \tau_0})$ are the images by τ_0 of the Chern classes $c_p^{\mathcal{D}}(E) \in H^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}})$ in the Deligne cohomology, and whose images in $H^{2p}(X, \mathbb{Z}(p))$ are the topological Chern classes $c_p^{\mathrm{top}}(E)$.*

Proof. The τ-product is compatible with the cup-product (2.9). Therefore the image of $c_p(f^* E, f^* \nabla)$ in $H^{2p}(P, \mathbb{Z}(p))$ is precisely $f^{-1} c_p^{\mathrm{top}}(E)$. This shows via (2.14) that

$$c_p(f^* E, f^* \nabla) = f^{-1} c_p(E, \nabla)$$

for a class

$$c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p)_{\tau_0})$$

which is uniquely determined. Its image c' in $H^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}, \tau_0})$ verifies

$$\begin{aligned}
f^{-1} c' &= a_p c_p(f^* E, f^* \nabla) \\
&= \tau c_p^{\mathcal{D}}(f^* E) && (2.11) \\
&= \tau f^{-1} c_p^{\mathcal{D}}(E). && (2.12)
\end{aligned}$$

One has the commutative diagram

$$\begin{array}{ccc} f^{-1}H^q(X,\mathbb{Z}(p)_{\mathscr{D}}) & \subsetneq & H^q(P,\mathbb{Z}(p)_{\mathscr{D}}) \\ {\scriptstyle\tau_0}\downarrow & & \downarrow{\scriptstyle\tau} \\ f^{-1}H^q(X,\mathbb{Z}(p)_{\mathscr{D},\tau_0}) & \subsetneq & H^q(P,\mathbb{Z}(p)_{\mathscr{D},\tau}) \end{array}$$

Therefore $c'=\tau_0 c_p^{\mathscr{D}}(E)$.

This proves also that the image of $c_p(E,\nabla)$ is the topological class $c_p^{\mathrm{top}}(E)$.

(2.16) At this point we want to prove the *functoriality*. Let $g\colon Y\to X$ be a morphism between two manifolds, and E be a bundle with an integrable τ_0-connection on X (2.1). As in (2.3) τ_0 defines a morphism $\tau_0'\colon \Omega_Y^{\cdot}\to\Omega_{Y,\tau_0'}^{\cdot}$. Write for simplicity $A'^{\cdot}=\Omega_{Y,\tau_0'}^{\cdot}$ and set $B'^1=\operatorname{im} g^*B^1$ in Ω_Y^1. Let $r\colon A'^{\cdot}\to A''^{\cdot}$ be a morphism of complexes with $A''^0=\mathcal{O}_Y$, A''^k is a quotient bundle of A'^k. Set $B'^1\subset B''^1=\mathscr{K}er(\Omega_Y^1\to A''^1)$. Then $rg^*\nabla=r\nabla'$ is a well defined integrable τ_0''-connection on $g^*E=E'$ for $\tau_0''\colon\Omega_Y^{\cdot}\to A''^{\cdot}$. Define $g^{-1}A^{\cdot}$ by $g^{-1}A^0\to g^{-1}A^1\to\ldots$ as a complex of $\mathbb{C}$-modules. One has a natural map of $\mathbb{C}$-complexes $\rho\colon g^{-1}(\mathbb{Z}(p)_{\tau_0})\to\mathbb{Z}(p)_{\tau_0''}$. This defines

$$\rho g^{-1}\colon H^{2p}(X,\mathbb{Z}(p)_{\tau_0})\to H^{2p}(Y,\mathbb{Z}(p)_{\tau_0''}).$$

PROPOSITION. (i) *One has* $\rho g^{-1}c_p(E,\nabla)=c_p(E',\nabla')$.

(ii) *One has* $c_1(E,\nabla)=(\overset{r}{\bigwedge}E,\ \overset{r}{\bigwedge}\nabla)$ *as defined in* (2.15) *and* (2.2).

Proof. The second statement is a consequence of the first.

If (i) is true, then one has

$$\rho' f^{-1}(\overset{r}{\bigwedge}E,\ \overset{r}{\bigwedge}\nabla)=(\overset{r}{\bigwedge}f^*E,\ \overset{r}{\bigwedge}(f^*\nabla)_\tau),$$

for

$$\rho'\colon f^{-1}\mathbb{Z}(p)_{\tau_0}\to\mathbb{Z}(p)_\tau,$$

and τ and f as in (2.7). One has

$$\begin{aligned}(\overset{r}{\bigwedge}f^*E,\ \overset{r}{\bigwedge}(f^*\nabla)_\tau)&=(\bigotimes_j L_j,\ \bigotimes_j\nabla_{\tau,j}) && \text{by construction}\\ &=\sum_j(L_j,\nabla_{\tau,j}) && (2.2)\\ &=c_1(f^*E,f^*\nabla) && (2.10)\\ &=f^{-1}c_1(E,\nabla) && (2.15)\end{aligned}$$

Therefore one has $(\overset{r}{\bigwedge}E,\ \overset{r}{\bigwedge}\nabla)=c_1(E,\nabla)$.

Let us prove the first statement. Consider the Cartesian product

$$\begin{array}{ccc} P' & \xrightarrow{h} & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

where P is the flag bundle of E and P' is the flag bundle of E'(2.7). The canonical filtration E_k'

(resp. splitting L'_k) of f'^*E' is the pull-back by h of the canonical filtration E_k (resp. splitting L_k) of f^*E.

On P and P' one has τ: $\Omega^{\cdot}_P \to A^{\cdot}_\tau$ and τ'': $\Omega^{\cdot}_{P'} \to A''^{\cdot}_{\tau''}$ as defined in (2.7).

One wants to see that there is a natural map

$$\rho' : h^{-1}\mathbb{Z}(p)_\tau \to \mathbb{Z}(p)_{\tau''}$$

such that the image by ρ' of

$$(L_j, \nabla_{\tau, j}) \in H^2(P, \mathbb{Z}(1)_\tau) \text{ is } (L'_j, \nabla_{\tau'', j}) \text{ in } H^2(P', \mathbb{Z}(1)_{\tau''}).$$

Assume that $P = P(E)$ and $P' = P'(E')$ (this means that rank $E \leqq 2$).

One has the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 \to & h^*f^*A^1 & \to & h^*\Omega^1_P/f^*B^1 & \to & h^*\Omega^1_{P/X} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow \iota & \\ 0 \to & f'^*A''^1 & \to & \Omega^1_{P'}/f'^*B''^1 & \longrightarrow & \Omega^1_{P'/Y} & \longrightarrow 0 \end{array}$$

Recall that σ is defined by

$$\begin{array}{ccc} \Omega^1_{P/X}(1) & \longrightarrow & f^*E \\ & \searrow^{\sigma \otimes 1} & \downarrow f^*\nabla \\ & & \Omega^1_P/f^*B^1 \otimes f^*E \\ & & \downarrow 1 \otimes q \\ & & \Omega^1_P/f^*B^1 \otimes \mathcal{O}(1) \end{array}$$

This gives a commutative diagram

$$\begin{array}{ccccc} h^*A^1_\tau = h^*f^*A^1 & \xleftarrow{h^*\tau} & h^*\Omega^1_P/f^*B^1 & & \\ \downarrow & & & \searrow & (0) \\ A''^1_{\tau''} = f'^*A''^1 & \xleftarrow{\tau'} & \Omega^1_P/f'^*B''^1 & \xleftarrow{\alpha} & \Omega^1_{P'}/f'^*B'^1 \end{array}$$

One has $\tau'\alpha h^*f^*\nabla = \tau' f'^* r g^*\nabla$.

Define $C^1 = \mathcal{K}er\, \Omega^1_P \to A^1_\tau$. Then $h^*(f^*\nabla)_\tau$ is a connection with values in $\Omega^1_{P'}/h^*C^1$. Define the morphisms r' and r''

$$\Omega^1_{P'}/f'^*B'^1 \xrightarrow{r'} \Omega^1_{P'}/h^*C^1 \xrightarrow{r''} A''^1_{\tau''}.$$

One has $r''r' = \tau'\alpha$. Therefore one has

$$(1) \qquad r''h^*(f^*\nabla)_\tau = \tau'\alpha h^*f^*\nabla.$$

Call ∇_τ and ∇'_τ the integrable τ-connections on $\mathcal{O}_P(1)$ and $\Omega^1_{P/X}(1)$, $\nabla_{\tau''}$ and $\nabla'_{\tau''}$ the integrable τ''-connections on $\mathcal{O}_{P'}(1)$ and $\Omega^1_{P'/Y}(1)$.

(1) implies

$$r''h^*\nabla_\tau = \nabla_{\tau''}$$
$$r''h^*\nabla'_\tau = \nabla'_{\tau''}.$$

Now (0) implies that the map $h^{-1}A^k_\tau \to A''^k_{\tau''}$ extends to well defined maps of complexes $h^{-1}A^{\cdot}_\tau \to A''^{\cdot}_{\tau''}$ and $\rho' : h^{-1}\mathbb{Z}(p)_\tau \to \mathbb{Z}(p)_{\tau''}$ such that

$$\rho'(\mathcal{O}_P(1), \nabla_\tau) = (\mathcal{O}_{P'}, (1), \nabla_{\tau'}) \quad \text{and}$$
$$\rho'(\Omega^1_{P/X}(1), \nabla'_\tau) = (\Omega^1_{P'/Y}(1), \nabla'_{\tau''}).$$

One repeats the construction inductively for $(\Omega^1_{P/X}(1), \nabla'_\tau)$ and $(\Omega^1_{P'/Y}(1), \nabla'_{\tau''})$.

(2.17) The next points (2.18) and (2.19) are devoted to the following *additivity* property. Let $0 \to (G,\nabla) \to (E,\nabla) \xrightarrow{\pi} (F,\nabla) \to 0$ be a τ_0-flat sequence ((2.2)), with $r = \operatorname{rank} E$ and $s = \operatorname{rank} G$.

PROPOSITION. *One has* $c_p(E,\nabla) = \sum_{k+l=p} c_k(G,\nabla)\cdot c_l(F,\nabla)$.

To prove it we need a standard geometrical compatibility of the flag bundles of F, G, E and further we need that this compatibility respects the complexes $\mathbb{Z}(p)_\tau$.

(2.18) We consider the flat exact sequence and

$$\begin{array}{ccc} P(F) & & P(E) \\ & {\scriptstyle\varepsilon}\searrow \quad \swarrow{\scriptstyle\varepsilon'} & \\ & X & \end{array}$$

The surjective morphism $\varepsilon^*E \to \mathcal{O}_{P(F)}(1)$ defines an injection $j: P(F) \to P(E)$ such that $j^*\mathcal{O}_{P(E)}(1) = \mathcal{O}_{P(F)}(1)$ [10]. One obtains the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \varepsilon^*G & \simeq & \varepsilon^*G & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & \Omega^1_{P(E)/X}(1)\,|_{P(F)} & \to & \varepsilon^*E & \xrightarrow{j^*q_E} & \mathcal{O}_{P(F)}(1) & \to 0 \\
 & \downarrow & & \downarrow{\scriptstyle \varepsilon^*\pi} & & \| & \\
0 \to & \Omega^1_{P(F)/X}(1) & \longrightarrow & \varepsilon^*F & \xrightarrow{q_F} & \mathcal{O}_{P(F)}(1) & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array} \qquad (*)$$

Call σ_E and σ_F the sections defined in (2.4). $\varepsilon'^*\nabla$ is a connection with values in $\Omega^1_{P(E)}/\varepsilon'^*B^1$, and we have $j^*\varepsilon'^*\nabla = \varepsilon^*\nabla$ by construction. Call $j^*\varepsilon'^*\nabla$ simply the *restriction of* $\varepsilon'^*\nabla$ to $P(F)$.

One has

$$\begin{aligned}
\varepsilon^*\pi j^*\sigma_E \otimes 1 &= \varepsilon^*\pi j^*(1 \otimes q_E)\,(\varepsilon'^*\nabla) \\
&= (1 \otimes q_F)\,\varepsilon^*\nabla \\
&= \sigma_F \otimes 1
\end{aligned}$$

Therefore the diagram

$$\begin{array}{ccc}
\Omega^1_{P(E)\,|\,P(F)} & \xrightarrow{\tau_{E|P(F)}} & \varepsilon^*A^1 \\
\downarrow & \nearrow{\scriptstyle \tau_F} & \\
\Omega^1_{P(F)} & &
\end{array}$$

is commutative and extends to the commutative diagram

$$\begin{array}{ccc}
\Omega^{\cdot}_{P(E)\,|\,P(F)} & \xrightarrow{\tau_{E|P(F)}} & A^{\cdot}_{\tau_F} \\
\downarrow & \nearrow{\scriptstyle \tau_F} & \\
\Omega^{\cdot}_{P(F)} & &
\end{array}$$

Especially the restriction of the τ_E-connection of $\mathcal{O}_{P(E)}(1)$ to $P(F)$ is the τ_F-connection of $\mathcal{O}_{P(F)}(1)$, and the vertical left-hand side sequence of $(*)$ is an exact sequence of integrable τ_F-connections. This shows that our situation is inductive. We repeat the previous first step to reach the following state at the $(r-(s+1))$-st step.

One has the commutative diagram

$$\begin{array}{ccc} D(F) & \xrightarrow{i'} & Z' \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h'} & \\ X & & \end{array}$$

where $D(F)$ is the flag bundle of F, i' is injective. On Z' one has the canonical "half-splitting" of

$$E: E_s' \subset E_{s+1}' \subset \ldots \subset E_r' = h^*E$$

such that

$$i'^*E_s' = f^*G$$

$$i'^*E_k'/E_{k-1}' = F_k/F_{k-1} \quad \text{for} \quad s+1 \leqq k \leqq r$$

where F_k is the canonical splitting of F:

$$0 = F_s \subset F_{s+1} \subset \ldots \subset F_r = f^*F .$$

Call $\tau_F: \Omega^{\cdot}_{D(F)} \to A^{\cdot}_{\tau_F}$ the morphism defined in (2.7), with $A^k_{\tau(F)} = f^*A^k$, $Rf_*A^{\cdot}_{\tau(F)} = A^{\cdot}$, and $\tau_1: \Omega^{\cdot}_{Z'} \to A^{\cdot}_{\tau_1}$ the morphism in Z' defined in (2.5) and (2.7). The filtration E_k' (F_k) is $\tau_1 - (\tau_F -)$ flat. The restriction of the integrable τ_1-connection $\nabla_{\tau_1,k}$ on E_k'/F_{k-1}' to $D(F)$ is the integrable τ_F-connection $\nabla_{\tau_F,k}$ on F_k/F_{k-1}.

One has the commutative diagram of complexes

$$\begin{array}{ccc} \Omega^{\cdot}_{Z'|D(F)} & \xrightarrow{\tau_{1|D(F)}} & A^{\cdot}_{\tau_F} \\ \downarrow & \nearrow{\scriptstyle \tau_F} & \\ \Omega^{\cdot}_{D(F)} & & \end{array}$$

This defines a morphism $\mathbb{Z}(p)_{\tau_{1|D(F)}} \to \mathbb{Z}(p)_{\tau_F}$. The classes $(E_k'/E_{k-1}', \nabla_{\tau_1,k})$ in $H^2(Z', \mathbb{Z}(1)_{\tau_1})$ are mapped to the classes $(F_k/F_{k-1}, \nabla_{\tau_F,k})$ in $H^2(D(F), \mathbb{Z}(1)_{\tau_F})$.

(2.19) Consider now the Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{i} & Z \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle h''} \\ D(F) & \xrightarrow{i'} & Z' \end{array}$$

where Y is the flag bundle of f^*G, Z is the flag bundle of E_s'. Of course $Y = D(F) \times_X D(G)$ and Z is also the flag bundle of E. Write E_k the canonical filtration of $h^*E = h''^*h'^*E$. Then $0 = E_0 \subset E_1 \subset \ldots \subset E_s = h''^*E_s'$ is the canonical filtration of h''^*E_s' and $E_k = h''^*E_k'$ for $s+1 \leqq k \leqq r$. Call $0 = G_0 \subset G_1 \subset \ldots \subset G_s = \gamma^*G = \beta^*f^*G = i^*E_s$ the canonical filtration of γ^*G, and set $F_k' = \beta^*F_k$. One has $i^*E_k = G_k$ for $0 \leqq k \leqq s$ and $i^*E_k/E_{k-1} = F_k'/F_{k-1}'$ for $s+1 \leqq k \leqq r$.

Call $\tau':\Omega_Y^{\cdot} \to A_{\tau'}^{\cdot}$, the morphism defined in (2.7) (with respect to f^*G and $\tau_F:\Omega_{D(F)}^{\cdot} \to A_{\tau_F}^{\cdot}$ on $D(F)$).

One has $A_{\tau'}^k = \gamma^* A^k$ and $R\gamma_* A_{\tau'}^{\cdot} = A^{\cdot}$. Call $\tau:\Omega_Z^{\cdot} \to A_\tau^{\cdot}$ the morphism defined in (2.7) (with respect to h''^*E_s' and $\tau_1:\Omega_{Z'}^{\cdot} \to A_{\tau_1}^{\cdot}$, or if one prefers, with respect to h^*E and $\tau_0:\Omega_X^{\cdot} \to A^{\cdot}$).

We apply now the functoriality (2.16). There is a morphism $\rho':\mathbb{Z}(p)_\tau \to \mathbb{Z}(p)_{\tau'}$ which sends the class of $(E_k/E_{k-1}, \nabla_{\tau,k})$ in $H^2(Z, \mathbb{Z}(1)_\tau)$ to the class of $(F_k'/F_{k-1}', \nabla_{\tau',k})$ for $s+1 \leqq k \leqq r$ or to the class of $(G_k/G_{k-1}, \nabla_{\tau',k})$ for $0 \leqq k \leqq s$ in $H^2(Y, \mathbb{Z}(1)_{\tau'})$.

By the functoriality again, one knows that

$$\gamma^{-1}c_p(F,\nabla) = c_p(\gamma^*F, (\gamma^*\nabla)_{\tau'}) \quad \text{and}$$
$$\gamma^{-1}c_p(G,\nabla) = c_p(\gamma^*G, (\gamma^*\nabla)_{\tau'}).$$

Therefore one obtains

$$\rho' i^{-1} c_p(h^*E, (h^*\nabla)_\tau) = \sum_{k+l=p} \gamma^{-1} c_k(G,\nabla)\cdot\gamma^{-1}c_l(F,\nabla).$$

The latter is $\gamma^{-1} \sum_{k+l=p} c_k(G,\nabla)\cdot c_l(F,\nabla)$.

This finishes the proof.

(2.20) Corollary. *Let $g: Y \to X$ be as in (2.16). Assume that $(g^*E, rg^*\nabla)$ has a τ_0''-flat filtration $(E_k, rg^*\nabla = \nabla_k')$. For $c(E_k/E_{k-1}, \nabla_k') = \sum_i c_i(E_k/E_{k-1}, \nabla_k')$ one has $\rho g^{-1}c(E,\nabla) = \prod_k c(E_k/E_{k-1}, \nabla_k')$.*

Proof. Apply the functoriality and the additivity.

(2.21) Corollary. *Let X be a smooth projective variety. Let $0 \to (G,\nabla) \to (E,\nabla) \to (F,\nabla) \to 0$ be a flat exact sequence with rank $E = r$ and rank $G = s$. Then $c_p(E,\nabla)$ is torsion for $p \geqq \sup(s, r-s)+1$.*

Proof. One has ((2.17) and (2.9)), assuming $r-s<p$ and $s<p$:

$$c_p(E,\nabla) = \sum_{k+l=p} c_k^{\text{top}}(G)\cdot c_l(F,\nabla)$$

As $c_k^{\text{top}}(G)$ is torsion for $k \geqq 1$ and as $l<p$, one obtains (2.21).

Remark. This implies (2.15) that the image $c_p^{\mathscr{D}}(E)$ is torsion also.

(2.22) Multiplicativity.

Let E and F be two bundles on X with integrable τ_0-connections ∇ and ∇'. Consider a morphism $f: P \to X$ realizing a splitting L_i of E, M_j of F with integrable τ-connections ∇_i and ∇_j'. One has the splitting of $f^*(E\otimes F)$ by $L_i \otimes M_j$, of $f^*(\nabla\otimes\nabla')_\tau$ by $\nabla_i \otimes \nabla_j'$. Then one has

$$\sum_{p\geqq 0} f^{-1}c_p(E\otimes F, \nabla\otimes\nabla')\cdot t^p = \prod_{i,j}(1+[(L_i,\nabla_i)+(M_j,\nabla_j')]\cdot t)$$

$$\sum_{p\geq 0} f^{-1}c_p\left(\overset{k}{\bigwedge} E, \overset{k}{\bigwedge}\nabla\right)\cdot t^p = \prod(1+[(L_{i_1}, \nabla_{i_1}) + \ldots + (L_{i_k}, \nabla_{i_k})]\cdot t).$$

$$1 \leqq i_1 < \ldots < i_k \leqq \text{rank } E$$

$$c_p(E, \nabla) = (-1)^p\, c_p(E^\vee, \nabla^\vee).$$

(2.23) *One summarizes the previous statements for standard flat bundles.*

THEOREM. *Let E be a flat bundle on X with an integrable connection* ∇. *There are classes* $c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \to \mathbb{C})$ *whose images in* $H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$ *are the classes* $c_p^{\mathscr{D}}(E)$, *whose images in* $H^2(X, \mathbb{Z}(p))$ *are the Chern classes* $c_p^{\text{top}}(E)$. *They are functorial and additive. The class* $c_1(E, \nabla)$ *is the isomorphism class of* $\left(\overset{r}{\bigwedge} E, \overset{r}{\bigwedge}\nabla\right)$ *in* $H^2(X, \mathbb{Z}(1) \to \mathbb{C})$. *Moreover* $c_p(E, \nabla)$ *is torsion for* $p \geqq 2$ *as soon as E has a flat splitting by rank one bundles (and X is projective).*

(2.24) *Axiomatic description of the classes* $c_p(E, \nabla)$. Let X be a manifold, $\tau_0 : \Omega^{\cdot}_X \to A^{\cdot}$ be as in (2.1).

THEOREM. *For any bundle E with an integrable* τ_0*-connection* ∇, *there are classes* $c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p) \to A^{\cdot})$ *which are uniquely determined by the following conditions:*
(i) $c_1(E, \nabla)$ *is the class* (det E, det ∇) *defined in* (2.1)
(ii) $c_p(E, \nabla)$ *is functorial in the sense of* (2.16)
(iii) $c_p(E, \nabla)$ *is additive in the sense of* (2.17).

Proof. The existence has been shown in (2.15), (2.16) and (2.17) whereas the unicity follows from the existence of the τ-flat canonical filtration $(L_k, \nabla_{\tau,k})$ on the flag bundle $f: P \to X$ of E: by (2.16), one has $f^{-1}\, c_p(E, \nabla) = c_p(f^*E, f^*\nabla)$ and by (2.17) this class is the p-symmetric sum of the classes $(L_k, \nabla_{\tau,k})$.

(2.25) *Comparison with other classes.*

(2.25.0) Consider the morphisms

$$a_p: H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \to H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$$
$$\beta: H^{2p-1}(X, \mathbb{R}(p)/\mathbb{Z}(p)) \to H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$$
$$\gamma: H^{2p-1}(X, \mathbb{R}/\mathbb{Z}) \to H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$$

with
$$\gamma = \frac{1}{(2i\pi)^p}\beta$$

$$\pi_p: \mathbb{C} \to \mathbb{R}(p).$$

If X has a Hodge structure, for example if X is an algebraic proper manifold, then one knows that γ is an isomorphism onto its image. Call γ^{-1} its inverse.

(2.25.1) *Comparison with the Deligne classes* $c_p^{\mathscr{D}}(E)$. We have seen in (2.15) that

$$a_p c_p(E, \nabla) = c_p^{\mathscr{D}}(E)$$

for flat bundles. This implies the following

LEMMA. *Let X be a manifold with a Hodge structure. Then* $\frac{1}{(2i\pi)^p}\pi_p c_p(E,\nabla)$ *is uniquely determined by $c_p^{\mathscr{D}}(E)$ for E flat; one has*

$$\frac{1}{(2i\pi)^p}\pi_p c_p(E,\nabla) = \gamma^{-1} c_p^{\mathscr{D}}(E).$$

(2.25.2) M. Karoubi [11, 12] defined for a simplicial set X and a flat bundle E classes $\check{c}_p(E) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p))$, using K-theory, cyclic homology and the cohomology of the classifying space BG. In a personal communication, he hold the author that his classes verify the conditions (i), (ii) and (iii) of (2.24) (work in preparation). Therefore one would have (2.24): $c_p(E,\nabla) = \check{c}_p(E)$ for flat bundles.

(2.25.3) J. Cheeger and J. Simons [4] defined for a differentiable manifold X and a flat bundle E classes $\hat{c}_p(E) \in H^{2p-1}(X, \mathbb{R}/\mathbb{Z})$ using the cohomology of the classifying space BG and the existence of universal unitary connections after M. S. Narasimhan and S. Ramanan.

If X has a Hodge structure and E is *unitary* (that is coming from a unitary representation of the fundamental group), S. Bloch ([2], (3.1)) and C. Soulé ([8], (5)) proved that $\gamma\hat{c}_p(E) = c_p^{\mathscr{D}}(E)$. Therefore one has in this case (2.25.1)

$$\frac{1}{(2i\pi)^p}\pi_p c_p(E,\nabla) = \hat{c}_p(E).$$

(2.25.4) It would be nice to know that the Cheeger–Simons classes lift the Deligne classes if E is a flat bundle which is not necessarily unitary. This would imply that our classes lift the Cheeger–Simons classes as in (2.25.3). However it might be difficult to show, since the construction of Cheeger–Simons is quite complicated in the non unitary case (at least for me!).

§3. LOGARITHMIC THEORY

(3.1) Let D be a normal crossing divisor on X and $j: X - D \to X$ be the open embedding. One considers a morphism of complexes

$$\tau_0 : \Omega_X^{\cdot} < D > \to A_D^{\cdot},$$

where $\mathcal{O}_X = A_D^0$, A_D^k is a quotient bundle of $\Omega_X^k \langle D \rangle$. One defines

$$\begin{aligned} \mathbb{Z}(p)_{D,\tau_0} &= \mathbb{Z}(p) \to A_D^{\cdot}, \\ \mathbb{Z}(p)_{\mathscr{D},D,\tau_0} &= \mathbb{Z}(p) \to A_D^0 \ldots \to A_D^{p-1} \\ a_p : \mathbb{Z}(p)_{D,\tau_0} &\to \mathbb{Z}(p)_{\mathscr{D},D,\tau_0} \\ \tau_0 : \mathbb{Z}(p)_{\mathscr{D}} &\to \mathbb{Z}(p)_{\mathscr{D},D,\tau_0}. \end{aligned}$$

One may perform the whole construction of §2. One finds classes $c_{p,D}(E,\nabla) \in H^{2p}(X, \mathbb{Z}(p)_{D,\tau_0})$. For $\tau_0 =$ identity, i.e. for standard logarithmic connections, this gives classes in $H^{2p}(X, \mathbb{Z}(p) \to Rj_*\mathbb{C})$. One has $a_p c_{p,D}(E,\nabla) = \tau_0 c_p^{\mathscr{D}}(E)$. As one sees, those classes do not lift the Deligne classes. We have to refine the construction to obtain this property.

(3.2) On X define the complex

$$\mathbb{Z}(p)_{\tau_0} = \mathbb{Z}(p) \to \mathcal{O}_X \to \ldots \to \Omega_X^{p-1} \underset{\tau_0 d}{\to} A_D^p \to \ldots \underset{\tau_0 d}{\to} A_D^n$$

and the morphism

$$a_p : \mathbb{Z}(p)_{\tau_0} \to \mathbb{Z}(p)_{\mathscr{D}}.$$

On the flag bundle $f: P \to X$ of E, the integrable τ_0-connection defines a morphism of complexes $\tau: \Omega_P^{\cdot}\langle D'\rangle \to A_{D,\tau'}^{\cdot}$ where $D' = f^{-1}D$, with $Rf_* A_{D,\tau}^{\cdot} = A_D^{\cdot}$, and integrable τ-connections $\nabla_{\tau,k}$ on L_k. This defines of course

$$(L_k, \nabla_{\tau,k}) \in H^2(P, \mathbb{Z}(1)_{D,\tau}).$$

Define

$$\mathbb{Z}(p)_\tau: \mathbb{Z}(p) \to \mathcal{O}_P \to \ldots \to \Omega_P^{p-1} \underset{\tau d}{\to} f^* A_D^p \to \ldots \to f^* A_D^n.$$

One has $\mathbb{Z}(1)_\tau = \mathbb{Z}(1)_{D,\tau}$ and morphisms:

$$\tau: \mathbb{Z}(p)_\tau \to \mathbb{Z}(p)_{D,\tau}, \text{ with } \mathbb{Z}(p)_{D,\tau} = \mathbb{Z}(p) \to \mathcal{O}_P \underset{\tau d}{\to} f^* A_D^1 \to \ldots \to f^* A_D^n$$

as in (3.1), and $a_p: \mathbb{Z}(p)_\tau \to \mathbb{Z}(p)_{\mathcal{D}}$. We define a product

$$\cup: \mathbb{Z}(p)_\tau \times \mathbb{Z}(q)_\tau \to \mathbb{Z}(p+q)_\tau$$

by the following data:

$$x \cup x' = \begin{cases} \tau(x)x' & \text{if } \deg x = 0, \quad \deg x' \leqq q \\ x \wedge dx' & \text{if } 0 < \deg x < p, \quad \deg x' = q \\ x \wedge \tau dx' & \text{if } p \leqq \deg x, \quad \deg x' = q \\ 0 & \text{otherwise} \end{cases}$$

for x and x' homogeneous in $\mathbb{Z}(p)_\tau$ and $\mathbb{Z}(q)_\tau$.

LEMMA. *$\cup$ is a well defined product, that is it defines a morphism of complexes*

$$\mathbb{Z}(p)_\tau \otimes_{\mathbb{Z}} \mathbb{Z}(q)_\tau \to \mathbb{Z}(p+q)_\tau.$$

Proof. One has to verify

$$d(x \cup x') = dx \cup x' + (-1)^{\deg x} x \cup dx',$$

where d is the differential in the corresponding complex. The left hand side is

$$\begin{array}{lll} x dx' & \text{if } \deg x = 0, & \deg x' \leqq q \\ dx \wedge dx' & \text{if } 0 < \deg x \leqq p-2, & \deg x' = q \\ \tau(dx \wedge dx') = \tau dx \wedge \tau dx' & \text{if } \deg x = p-1, & \deg x' = q \\ \tau dx \wedge \tau dx' & \text{if } \deg x \geqq p, & \deg x' = q \\ 0 & \text{otherwise,} & \end{array}$$

whereas the right hand side is

$$\begin{array}{lll} x \cup dx' = x dx' & \text{if } \deg x = 0, & \deg x' < q \\ dx \cup x' = x dx' & \text{if } \deg x = 0, & \deg x' = q \\ dx \cup x' = dx \wedge dx' & \text{if } 0 < \deg x \leqq p-2, & \deg x' = q \\ \qquad = \tau dx \wedge \tau dx' & \text{if } \deg x \geqq p-1, & \deg x' = q \\ 0 & \text{if } \deg x = 0, & \deg x' \geqq q+1 \\ & \text{or if } \quad 0 < \deg x, & \deg x' \leqq q-2 \\ (-1)^{\deg x} x \cup dx' = 0 & \text{if } \quad 0 < \deg x, & \deg x' = q-1. \end{array}$$

(3.3) On $\mathbb{Z}(p)_{D,\tau}$ we define the same product as in (2.9):

$$x\cdot x' = \tau(x)\cdot x' \text{ if } \deg x = 0$$
$$0 \qquad \text{otherwise.}$$

The product $\cup$ on $\mathbb{Z}(p)_\tau$ defines via τ a product on $\mathbb{Z}(p)_{D,\tau}$, still denoted by $\cup$, by

$$\begin{aligned} x\cup x' &= \tau(x)\cdot x' && \text{if } \deg x = 0, \quad \deg x' \leqq q\\ &\ x\wedge \tau dx' && \text{if } \deg x > 0, \quad \deg x' = q\\ &\ 0 && \text{otherwise,}\end{aligned}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{Z}(p)_\tau \otimes_{\mathbb{Z}} \mathbb{Z}(q)_\tau & \xrightarrow{\ \cup\ } & \mathbb{Z}(p+q)_\tau \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \mathbb{Z}(p)_{D,\tau} \otimes_{\mathbb{Z}} \mathbb{Z}(q)_{D,\tau} & \xrightarrow{\ \cup\ } & \mathbb{Z}(p+q)_{D,\tau} \end{array}$$

Recall that the Deligne product (2.8) on $\mathbb{Z}(p)_{\mathcal{D}}$ is defined by:

$$\begin{aligned} x\cdot_d x' &= x\cdot x' && \text{if } \deg x = 0\\ &\ x\wedge dx' && \text{if } \deg x > 0, \quad \deg x' = q\\ &\ 0 && \text{otherwise.}\end{aligned}$$

LEMMA. (i) *The following diagram is commutative*

$$\begin{array}{ccc} \mathbb{Z}(p)_\tau \otimes_{\mathbb{Z}} \mathbb{Z}(q)_\tau & \xrightarrow{\ \cup\ } & \mathbb{Z}(p+q)_\tau \\ \downarrow{\scriptstyle a_p\otimes a_q} & & \downarrow{\scriptstyle a_{p+q}} \\ \mathbb{Z}(p)_{\mathcal{D}} \otimes_{\mathbb{Z}} \mathbb{Z}(q)_{\mathcal{D}} & \xrightarrow{\ \cdot d\ } & \mathbb{Z}(p+q)_{\mathcal{D}} \end{array}$$

(ii) *The products $\cup$ and $\cdot$ on $\mathbb{Z}(p)_{D,\tau}$ are homotopic.*
(iii) *The product $\cup$ (on $\mathbb{Z}(p)_\tau$ and on $\mathbb{Z}(p)_{D,\tau}$) is anticommutative.*

Proof. (i) is obvious
(ii) Define a map

$$h\colon (\mathbb{Z}(p)_{D,\tau} \otimes_{\mathbb{Z}} \mathbb{Z}(q)_{D,\tau})^l \to \mathbb{Z}(p+q)^{l-1}_{D,\tau},$$

where l denotes the degree of the corresponding complex, by

$$h(x\otimes x') = (-1)^{\deg x}\, x\wedge x' \text{ if } \deg x > 0,\ \deg x' \geqq q+1$$
$$0 \qquad \text{otherwise.}$$

One has to verify that h is the homotopy wanted, that is

$$x\cup x' - x\cdot x' = (hd + dh)(x\otimes x'),$$

where d is the differential in the corresponding complex.

The left hand side is

$$\begin{aligned} &-x\cdot x' && \text{if } \deg x = 0, \quad \deg x' \geqq q+1\\ &\ x\wedge\tau dx' && \text{if } \deg x > 0, \quad \deg x' = q\\ &\ 0 && \text{otherwise.}\end{aligned}$$

The right hand side is

$$h(\mathrm{d}x\otimes x'+(-1)^{\deg x}x\otimes \mathrm{d}x')+\mathrm{d}h(x\otimes x').$$

Recall that if x is in Ω_P^s or in $f^*A_D^s$, its degree is $(s+1)$. One has

$$\begin{aligned} h(\mathrm{d}x\otimes x') &= (-1)^{\deg x+1}\tau \mathrm{d}x\wedge x' && \text{if } \deg x'\geqq q+1\\ & \quad 0 && \text{otherwise}\\ (-1)^{\deg x}h(x\otimes \mathrm{d}x') &= (-1)^{2\deg x}x\wedge \tau \mathrm{d}x' && \text{if } \deg x>0,\ \deg x'\geqq q\\ & \quad 0 && \text{otherwise}\\ \mathrm{d}h(x\otimes x') &= (-1)^{\deg x}(\tau \mathrm{d}x\wedge x'+(-1)^{\deg x-1}x\wedge \tau \mathrm{d}x')\\ & \quad \text{if } \deg x>0,\ \deg x'\geqq q+1\\ & \quad 0 && \text{otherwise.} \end{aligned}$$

Altogether this gives the equality wanted.

(iii) Define the homotopy

$$h:(\mathbb{Z}(p)_\tau\otimes_{\mathbb{Z}}\mathbb{Z}(q)_\tau)^l\to \mathbb{Z}(p+q)_\tau^{l-1}$$

by

$$\begin{aligned} h(x\otimes x') &= (-1)^{\deg x}x\wedge x' && \text{if } 0<\deg x\leqq p\\ & && \quad 0<\deg x'\leqq q\\ & \quad 0 && \text{otherwise.} \end{aligned}$$

One verifies in the same way as in (ii) that

$$(-1)^{\deg x \deg x'}x'\cup x-x\cup x'=(hd+\mathrm{d}h)(x\otimes x').$$

This proves the anticommutativity of $\cup$ on $\mathbb{Z}(p)_\tau$. On $\mathbb{Z}(p)_{D,\tau}$ either one takes the same homotopy, or one remembers that $\cup$ and $\cdot$ are homotopic (ii), and that $\cdot$ is anticommutative (2.9).

(3.4) The product $\cup$ on $\mathbb{Z}(p)_\tau$ defines a product $\cup$ in the cohomology $H^{p'}(\mathbb{Z}(p)_\tau)$, which verifies $x\cup x'=(-1)^{p'q'}x'\cup x$ for $x\in H^{p'}(\mathbb{Z}(p)_\tau)$ and $x'\in H^{q'}(\mathbb{Z}(q)_\tau)$, and especially $x\cup x'=x'\cup x$ if $p'=2p$. Define the *characteristic classes* $c_p(f^*E, f^*\nabla)$ as the p-symmetric sum of

$$(L_k,\nabla_{\tau,k})\in H^2(P,\mathbb{Z}(1)_\tau)\quad (3.2)$$

By (3.3, i), one has

$$a_p c_p(f^*E,f^*\nabla)=c_p^{\mathscr{D}}(f^*E)(=f^{-1}c_p^{\mathscr{D}}(E)\quad \text{by (2.8)}).$$

Proposition. *There is a class $c_p(E,\nabla)\in H^{2p}(X,\mathbb{Z}(p)_{\tau_0})$, which is uniquely determined such that*

$$f^{-1}c_p(E,\nabla)=c_p(f^*E,f^*\nabla).$$

It verifies $a_p c_p(E,\nabla)=c_p^{\mathscr{D}}(E)$.

Proof. One considers the exact sequence

$$0\to f^{-1}H^{2p}(X,\mathbb{Z}(p)_{\tau_0})\to H^{2p}(P,\mathbb{Z}(\mathrm{p})_\tau)\to \frac{H^{2p}(P,\mathbb{Z}(p)_{\mathscr{D}})}{f^{-1}H^{2p}(X,\mathbb{Z}(p)_{\mathscr{D}})}\to 0$$

As $a_p c_p(f^*E, f^*\nabla) \in f^{-1}H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}})$, one has $c_p(f^*E, f^*\nabla) = f^{-1}c_p(E, \nabla)$ for a well defined class $c_p(E, \nabla)$. The second assertion follows from the commutative diagram

$$\begin{array}{ccc} f^{-1}H^{2p}(X, \mathbb{Z}(p)_{\tau_0}) & \hookrightarrow & H^{2p}(P, \mathbb{Z}(p)_\tau) \\ \downarrow a_p & & \downarrow a_p \\ f^{-1}H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}}) & \hookrightarrow & H^{2p}(P, \mathbb{Z}(p)_{\mathscr{D}}). \end{array}$$

(3.5) Consider the morphism $\tau_0 : \mathbb{Z}(p)_{\tau_0} \to \mathbb{Z}(p)_{D,\tau_0}$.

LEMMA. *One has*

$$\tau_0 c_p(E, \nabla) = c_{p,D}(E, \nabla) \qquad (\textit{defined in } (3.1)).$$

In particular, if $D = \phi$ *and* $\tau_0 = $ *identity (that is if* (E, ∇) *is a flat bundle), then the two definitions of* $c_p(E, \nabla)$ *(in* (2.15) *and* (3.4)) *coincide. If* $D = \phi$ *and* $\tau_0 \neq$ *identity, one has* $\tau_0 c_p(E, \nabla)$ *(as in* (3.4)) $= c_p(E, \nabla)$ *(as in* (2.15)).

Proof. By (3.3,ii), the products $\cup$ and $\cdot$ are homotopic. Therefore one has $\tau c_p(f^*E, f^*\nabla) = c_{p,D}(f^*E, f^*\nabla)$. From the commutative diagram

$$\begin{array}{ccc} f^{-1}H^{2p}(X, \mathbb{Z}(p)_{\tau_0}) & \hookrightarrow & H^{2p}(P, \mathbb{Z}(p)_\tau) \\ \downarrow \tau_0 & & \downarrow \tau \\ f^{-1}H^{2p}(X, \mathbb{Z}(p)_{D,\tau_0}) & \hookrightarrow & H^{2p}(P, \mathbb{Z}(p)_{D,\tau}) \end{array}$$

one obtains the result.

(3.6) One proves now in exactly the same way as in (2.16) and (2.17), replacing each time it is necessary $\mathbb{Z}(p)_\tau$ defined in §2 by $\mathbb{Z}(p)_\tau$ defined in §3, that the classes defined in (3.4) are functorial and additive.

Summarizing everything, one has the following

THEOREM. *Let* $\tau_0 : \Omega^{\cdot}_X\langle D\rangle \to A^{\cdot}_D$, (E, ∇) *be as in* (3.1), $\mathbb{Z}(p)_{\tau_0}$ *and* $a_p : \mathbb{Z}(p)_{\tau_0} \to \mathbb{Z}(p)_{\mathscr{D}}$ *be as in* (3.2). *There are classes*

$$c_p(E, \nabla) \in H^{2p}(X, \mathbb{Z}(p)_{\tau_0})$$

which are uniquely determined by the following conditions:

(i) $c_1(E, \nabla)$ *is the class* (det E, det ∇) *defined in* (2.1)

(ii) $c_p(E, \nabla)$ *is functorial in the sense of* (2.16).

(iii) $c_p(E, \nabla)$ *is additive in the sense of* (2.17).

Moreover one has $a_p c_p(E, \nabla) = c_p^{\mathscr{D}}(E)$. The *proof* is of course the same as in (2.24).

(3.7) *Remark.* The theorem (3.6) is interesting essentially in the case $\tau_0 = $ identity, that is if (E, ∇) is an integrable logarithmic connection. However, even in the case $D = \phi$ and $\tau_0 \neq$ identity, one obtains slightly more than in (2.15), because the classes (3.4) lift the Deligne classes, whereas the classes (2.15) lift "only" the "τ_0-Deligne classes".

§4. MISCELLANEOUS ABOUT HERMITIAN BUNDLES

(4.0) *In* [6] *P. Deligne introduces the group* $\widehat{\text{Pic}}\, X$ *of isomorphism classes of rank one bundles E with an hermitian metric h* on an analytic manifold X. He identifies it with the *cohomology group.*

$$H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \to S_X^0)$$

where S_X^0 is the sheaf of $\mathbb{R}$-valued C^∞ functions, and the map $\mathcal{O}_X \to S_X^0$ is described by $f \to f + \bar{f}$. This may be seen with Čech cohomology as in (1.1). If U_α is a trivializing cover of E, with $E|_{U_\alpha} \simeq \mathcal{O}_X \cdot e_\alpha$, and $h_\alpha = \log\ h(e_\alpha, e_\alpha)$, the Čech cocycle of (E, h) in $\mathscr{C}^2(\mathbb{Z}(1)) \times \mathscr{C}^1(\mathcal{O}_X) \times \mathscr{C}^0(S_X^0)$ is $(2i\pi m_{ijk}, f_{ij}, \log h_i)$, with $\delta f = 2i\pi m, f + \bar{f} = \delta \log h$, where δ is the Čech differential, coming from $e_i = \exp(f_{ij}) \cdot e_j$, $h_i = \exp(f_{ij}) \cdot \overline{\exp(f_{ij})} \cdot h_j$.

(4.1) Let $A_X^{\cdot}$ be the C^∞ $\mathbb{C}$-valued De Rham complex of X. Its differential d decomposes in $\partial + \bar{\partial}$, where $\bar{\partial}: A_X^{ij} \to A_X^{i,j+1}$ and $\partial: A_X^{ij} \to A_X^{i+1,j}$, and A_X^{ij} is the sheaf of C^∞ differential forms of type (i,j). On the other hand, the cohomology group

$$H^2(X, \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{d} A_X^{10} \xrightarrow{\partial} A_X^{20} \to \ldots \xrightarrow{\partial} A_X^{n0})$$

is the group of isomorphism classes of rank one bundles E with A_X^1-valued connection ∇ which is compatible with the complex structure and whose curvature in the $(2,0)$ direction vanishes: $(\nabla^2)^{20} = 0$. This may be seen in the Čech cohomology in the same way as in (4.1). A cocycle for (E, ∇) in $\mathscr{C}^2(\mathbb{Z}(1) \times \mathscr{C}^1(\mathcal{O}_X) \times \mathscr{C}^0(A_X^{10})$ is $(2i\pi m_{ijk} f_{ij}, \omega_i)$, with $\delta f = 2i\pi m$, $\delta\omega = \mathrm{d}f$ and $\partial\omega = 0$.

(4.2) Define the complexes

$$G^p = 0 \to \ldots \to 0 \to A_X^{pp} \xrightarrow{d} A_X^{p+1,p} \oplus A_X^{p,p+1} \xrightarrow{d} \ldots$$

where A_X^{pp} is in degree $2p$,

$$\mathbb{Z}(p)_G = \text{Cone}\ (\mathbb{Z}(p) \oplus G^p \to A_X^{\cdot})[-1],$$

and the cohomology

$$H_G^q(X, p) = H^q(X, \mathbb{Z}(p)_G).$$

Since G^p has a natural product, it defines as in (2.8) a product on $\mathbb{Z}(p)_G$ and therefore a product on $H_G^q(X, p)$. From the natural map $G^p \to F^p$, where F^p is the Hodge–Deligne filtration, one obtains a map

$$\alpha: \mathbb{Z}(p)_G \to \mathbb{Z}(p)_{\mathscr{D}}$$

which is compatible with the products, and therefore a ring map

$$\alpha: \bigoplus_{p,q} H_G^q(X, p) \to \bigoplus_{p,q} H^q(X, \mathbb{Z}(p)_{\mathscr{D}}).$$

This defines a commutative square of rings

$$\begin{array}{ccc} \bigoplus_p H_G^{2p}(X, p) & \xrightarrow{\alpha} & \bigoplus_p H^{2p}(X, \mathbb{Z}(p)_{\mathscr{D}}) \\ \big\downarrow \beta & & \big\downarrow \\ \bigoplus_p H^0(A_X^{pp})_{\mathrm{d\ closed}} & \to & \bigoplus_p H^{2p}(X, F^p) \end{array}$$

(see [2] for the compatibility between the Deligne and the F products, and [7] for precise computations).

Let $S_X^{\cdot}$ be the C^∞ $\mathbb{R}$-valued De Rham complex of X, and $\hat{H}^k(X, \mathbb{R}/\mathbb{Z})$ be the group of differential characters defined by Cheeger–Simons ([4]).

LEMMA. *There is a natural map*

$$H_G^{2p}(X,p) \underset{\gamma}{\rightarrow} \hat{H}^{2p-1}(X, \mathbb{R}/\mathbb{Z}).$$

Proof. Define

$$K^{k+1} = 0 \rightarrow 0 \rightarrow S_X^{k+1} \rightarrow \dots \rightarrow S_X^{2n}$$

where S_X^{k+1} is in degree $(k+1)$, and

$${}^{k+1}K = \mathrm{cone}(C^{\cdot}(\mathbb{Z}) \oplus K^{k+1} \rightarrow C^{\cdot}(\mathbb{R}))\,[-1],$$

where $C^{\cdot}(\Lambda)$ is the complex of smooth singular Λ-valued cochains on X. The definition given in [4] implies immediately:

$$\hat{H}^k(X, \mathbb{R}/\mathbb{Z}) = H^{k+1}(X, {}^{k+1}K).$$

On the other hand one has maps $\dot{\mathbb{Z}}(p) \rightarrow C^{\cdot}(\mathbb{Z}(p))$,

$$G^p \rightarrow K^{2p}(p), \quad A_X^{\cdot} \rightarrow S_X^{\cdot}(p) \rightarrow C^{\cdot}(\mathbb{R}(p)),$$

which define a map

$$\mathbb{Z}(p)_G \rightarrow {}^{2p}K(p)$$

and therefore a map

$$H_G^{2p}(X,p) \underset{\gamma'}{\rightarrow} H^{2p}(X, {}^{2p}K(p)) = (2i\pi)^p \hat{H}^{2p-1}(X, \mathbb{R}/\mathbb{Z}).$$

Then $\gamma = \dfrac{1}{(2i\pi)^p}\gamma'$.

(4.3) By definition one has

$$H_G^2(X,1) = H^2(X, \mathbb{Z}(1) \rightarrow \mathcal{O}_X \xrightarrow{d} A_X^{10} \rightarrow \dots \xrightarrow{\bar\partial} A_X^{n0}).$$

Consider the morphism of complexes

$$\begin{array}{ccccccc} \partial: \mathbb{Z}(1) & \rightarrow & \mathcal{O}_X & \rightarrow & S_X^0 & & \\ \downarrow \int & & \downarrow \int & & \downarrow \partial & & \\ \mathbb{Z}(1) & \rightarrow & \mathcal{O}_X & \rightarrow & A_X^{10} & \rightarrow \dots \xrightarrow{\bar\partial} & A_X^{n0} \end{array}$$

which sends $\widehat{\mathrm{Pic}}(X)$ to $H_G^2(X,1)$.

LEMMA. *The morphism*

$$\partial\colon \widehat{\mathrm{Pic}}(X) \rightarrow H_G^2(X,1)$$

is injective. One has $\partial(E,h) = (E, \nabla_h)$ where ∇_h is the unique connection on E compatible with the complex structure and the hermitian metric h. One has $\alpha(E,\nabla_h) = c_1^{\mathcal{D}}(E)$, $\beta(E,\nabla_h) =$ Chern form defined by ∇_h.

Proof. In the Čech representation, one has

$$\partial(2i\pi m, f, \log h) = (2i\pi m, f, \partial h/h).$$

Since $\omega=\partial h/h\in\mathscr{C}^0(A_X^{10})$ defines the unique connection compatible with the complex structure and h, one has $\partial(E,h)=(E,\nabla_h)$.

One considers the following commutative diagram with exact columns:

$$\begin{array}{ccc}
H^1(\mathbb{Z}(1)\to\mathscr{O}_X) & \xrightarrow{\sim} & H^1(\mathbb{Z}(1)\to\mathscr{O}_X)\\
\downarrow & & \downarrow\\
H^0(S_X^0) & \xrightarrow{\partial} & H^0(A_X^{10}\xrightarrow{\partial}\ldots\to A_X^{n0})\\
\downarrow & & \downarrow\\
H^2(\mathbb{Z}(1)\to\mathscr{O}_X\to S_X^0) & \xrightarrow{\partial} & H^2(\mathbb{Z}(1)\to\mathscr{O}_X\to A_X^{10}\to\to A_X^{n0})\\
\downarrow & & \downarrow\\
H^2(\mathbb{Z}(1)\to\mathscr{O}_X) & \xrightarrow{\sim} & H^2(\check{\mathbb{Z}}(1)\to\mathscr{O}_X)\\
\downarrow & & \\
H^1(S_X^0)=0 & &
\end{array}$$

If $x\in H^2(\mathbb{Z}(1)\to\mathscr{O}_X\to S_X^0)$ with $\partial x=0$, then $x=\operatorname{Im} y$, with $y\in H^0(S_X^0)$, and $\partial y=\operatorname{Im} z$, with $z\in\operatorname{Im} H^1(\mathbb{Z}(1)\to\mathscr{O}_X)$. Therefore $(y-\operatorname{Im} z)$ maps to x and verifies $\partial(y-\operatorname{Im} z)=0$. Since $(y-\operatorname{Im} z)\in H^0(S_X^0)$ this implies $y-\operatorname{Im} z=0$. Therefore $x=0$. Finally, since the Chern form is $\bar\partial\omega=\bar\partial\partial h|h$, it is $\beta(E,\nabla_h)$.

We would like to define classes $c_p(E,h)\in H_G^{2p}(X,p)$ such that $\alpha c_p(E,h)=c_p^{\mathscr{D}}(E)$, $\beta c_p(E,h)=$ Chern form defined by h, (eventually $\gamma c_p(E,h)=\hat{c}_p(E)$ as defined in [4]), and such that $c_1(E,h)$ is the class

$$(\det E,\ \det h)=(\det E,\ \det\nabla_h)$$

described in (4.1), (4.2) and (4.3). We present a small step in this direction based on the τ-construction. Since it does not lead to the goal, we just sketch the proof.

(4.4) Consider an hermitian bundle (E,h). One has $(\nabla_h^2)^{20}=0$, where ∇_h is the connection compatible with the complex structure and h. Consider the flag bundle $f\colon P\to X$ of E as in (2.7). Similarly to there, ∇_h defines a splitting $\tau\colon A_P^{10}\to f^*A_X^{10}$, where

$$f^*A_X^{ij}=f^{-1}A_X^{ij}\otimes_{f^{-1}A_X^0}A_P^0.$$

The condition $(\nabla_h^2)^{20}=0$ implies $(\tau\partial)^2=0$. Consider the sheaf $\mathscr{O}_{P_m}$ of functions which are $\mathbb{C}$-valued, C^∞ in the X direction and holomorphic in the fibre direction. Then $\mathscr{O}_{P_m}$ is quasi-isomorphic to the complex

$$A_P^0\xrightarrow{\bar\partial_r}A_{P/X}^{01}\xrightarrow{\bar\partial_r}A_{P/X}^{02}\to\ldots$$

where $\bar\partial_r$ is defined by

$$\bar\partial_r\colon A_P^0\xrightarrow{\bar\partial}A_P^{01}\to A_{P/X}^{01}$$

and $$A_{P/X}^{01}=A_P^{01}/f^*A_X^{01},\quad\text{and}\quad A_{P/X}^{0k}=\bigwedge^k A_{P/X}^{01}.$$

Define $A_\tau^{i0}=f^{-1}A_X^{i0}\otimes_{f^{-1}A_X^0}\mathscr{O}_{P_m}$. One verifies $\tau\partial A_\tau^{i0}\subset A_\tau^{i+1,0}$. One shows by standard arguments of the sheaf theory that $Rf_*\mathscr{O}_{P_m}=A_X^0$ and thereby that $Rf_*A_\tau^{\cdot 0}=A_X^{\cdot 0}$ where $A_\tau^{\cdot 0}$ is the complex $(A_\tau^{i0},\tau\partial)$. Now the connection ∇_h defines τ-connections

$$\nabla_{\tau,h}\colon L_k\to A_\tau^{10}\otimes_{\mathscr{O}_P}L_k$$

on the splitting rank one bundles L_k, and thereby classes

$$(L_k, \nabla_{\tau,k}) \in H^2(P, \mathbb{Z}(1) \to \mathcal{O}_P \to A_\tau^{10} \underset{\tau\partial}{\to} A_\tau^{20} \to \dots)$$

Using the product $\cup$ of (3.2), one finds classes

$$c_p(f^*E, f^*\nabla_h) \in H^{2p}(P, \mathbb{Z}(p) \to \mathcal{O}_P \to \dots \to \Omega_P^{p-1} \to A_\tau^{p0} \to \dots \to A_\tau^{n0})$$

which are coming from X as in (3.4). Altogether this defines classes

$$c_p(E, \nabla_h) \in H^{2p}(X, \mathbb{Z}(p) \to \mathcal{O}_X \to \dots \to \Omega_X^{p-1} \xrightarrow{d} A_X^{p0} \xrightarrow{\partial} \dots \xrightarrow{\partial} A_X^{n0})$$

with the standard properties, and which lift the Deligne classes. However they don't lift the Chern form: the natural map

$$H_G^{2p}(X, p) \to H^{2p}(X, \mathbb{Z}(p) \to \mathcal{O}_X \to \dots \to \Omega_X^{p-1} \xrightarrow{d} A_X^{p0} \to \dots \xrightarrow{\partial} A_X^{n0})$$

is not injective.

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