TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 371, Number 2, 15 January 2019, Pages 1333–1358
https://doi.org/10.1090/tran/7342
Article electronically published on September 10, 2018

CHERN CLASSES OF CRYSTALS

HÉLÈNE ESNAULT AND ATSUSHI SHIHO

ABSTRACT. The crystalline Chern classes of the value of a locally free crystal vanish on a smooth variety defined over a perfect field. Out of this we conclude new cases of de Jong's conjecture relating the geometric étale fundamental group of a smooth projective variety defined over an algebraically closed field and the constancy of its category of isocrystals. We also discuss the case of the Gauß–Manin convergent isocrystal.

1. INTRODUCTION

On a smooth algebraic variety X defined over the field $\mathbb{C}$ of complex numbers, a vector bundle E endowed with an integrable connection $\nabla : E \to \Omega^1_X \otimes_{\mathcal{O}_X} E$ has vanishing Chern classes $c_i^{dR}(E)$ in de Rham cohomology $H^{2i}_{dR}(X/\mathbb{C})$ for $i \geq 1$.

The standard way to see this is applying Chern–Weil theory: the classes in $H^{2i}_{\mathrm{dR}}(X)$ of the successive traces of the iterates of the curvature $\nabla^2 \in \mathrm{Hom}(E, \Omega^2_X \otimes E)$ are identified with the Newton classes $N_i(E)$, and the $\mathbb{Q}$-vector spaces spanned by $c_i^{dR}(E), 1 \leq i \leq n$, and the $N_i(E)$, $1 \leq i \leq n$, are the same in $H^{2i}_{\mathrm{dR}}(X)$ [10]. In particular, the method loses torsion information, and, for example, a torsion class in integral ℓ-adic cohomology $H^2(X_{\bar{k}}, \mathbb{Z}_\ell(1))$ for some ℓ is the first ℓ-adic Chern class of some line bundle which carries an integrable connection.

If (X, E) is defined over a field k of characteristic 0 and E admits an integrable connection after base changing to $\mathbb{C}$ for a complex embedding $k \hookrightarrow \mathbb{C}$, one still has vanishing $0 = c_i^{dR}(E) \in H^{2i}_{dR}(X/k)$ for $i \geq 1$ because Chern classes in de Rham cohomology are functorial and de Rham cohomology satisfies the base change property.

The first purpose of this article is to present a similar vanishing statement where k is now a perfect field of characteristic $p > 0$, X is projective, E is replaced by the value E_X on X of a p-torsion free crystal E, and de Rham cohomology is replaced by crystalline cohomology. Let $W = W(k)$ be the ring of Witt vectors on k. We denote by $H^i_{\mathrm{crys}}(X/W)$ the integral crystalline cohomology of X and by $c_i^{\mathrm{crys}}(E_X)$ the crystalline Chern classes of E_X in $H^{2i}_{\mathrm{crys}}(X/W)$.

Recall (see [19, Sec. 1] for an overview of the concepts) that a crystal is a sheaf of $\mathcal{O}_{X/W}$-modules of finite presentation on the crystalline site of X/W such that the transition maps are isomorphisms. Crystals build a W-linear category $\mathrm{Crys}(X/W)$, of which the $\mathbb{Q}$-linearization $\mathrm{Crys}(X/W) \xrightarrow{\mathbb{Q}\otimes} \mathrm{Crys}(X/W)_{\mathbb{Q}}$ is the category of isocrystals. Then $\mathrm{Crys}(X/W)_{\mathbb{Q}}$ is a Tannakian category over K, the

Received by the editors November 29, 2015, and, in revised form, May 25, 2017.
2010 *Mathematics Subject Classification.* Primary 11S99, 14G99.
The first author was supported by the Einstein program.
The second author was partly supported by JSPS Grants-in-Aid for Scientific Research (C)25400008, (A)15H02048, and (C)17K05162.

fraction field of W. Any $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$ is of the shape $\mathbb{Q} \otimes E$ where E is a lattice, that is, a p-torsion free crystal. However it is an open question whether one can choose E to be locally free.

Theorem 1.1. *Let X be a smooth variety defined over a perfect field k of characteristic $p > 0$. If $E \in \mathrm{Crys}(X/W)$ is a locally free crystal, then $c_i^{\mathrm{crys}}(E_X) = 0$ for $i \geq 1$.*

It is proved in [19, Prop. 3.1] that given an isocrystal $\mathbb{Q} \otimes E \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$, $c_i^{\mathrm{crys}}(E_X)$ does not depend on the choice of the lattice E. In particular, if it were true that any isocrystal carries a locally free lattice, then Theorem 1.1 would imply that $c_i^{\mathrm{crys}}(E_X) = 0$ for $i \geq 1$ for any p-torsion free crystal $E \in \mathrm{Crys}(X/W)$.

The proofs of Theorem 1.1 imply also the following variant for Chern classes in torsion crystalline cohomology: Let $W_n := W/p^nW$. Then, if X is as in Theorem 1.1 and if E is a locally free crystal on X/W_n, then $c_i^{\mathrm{crys}}(E_X)$ is zero in the torsion crystalline cohomology group $H^{2i}_{\mathrm{crys}}(X/W_n)$ for $i \geq 1$. See Remarks 2.1 and 3.4.

Recall that the Frobenius acts on $\mathrm{Crys}(X/W)$ and $\mathrm{Crys}(X/W)_{\mathbb{Q}}$. Locally on X, $\mathrm{Crys}(X/W)$ is equivalent to the category of quasi-nilpotent integrable connections on a formal lift of X over W, and the action is just given by the Frobenius pull-back of a connection. The category $\mathrm{Conv}(X/K)$ of convergent isocrystals is the largest full subcategory of $\mathrm{Crys}(X/W)_{\mathbb{Q}}$ which is stabilized by the Frobenius action. It is proved in [19, Prop. 3.1] that $c_i^{\mathrm{crys}}(E_X) = 0$ for any lattice E of a convergent isocrystal, regardless of the existence of a locally free lattice.

For a natural number r, set $N(r)$ to be the maximum of the lower common multiples of a and b for all choices $a, b \geq 1, a + b \leq r$. As in [19], Theorem 1.1 enables one to prove the following case of de Jong's conjecture ([19, Conj. 2.1]).

Theorem 1.2. *Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$. If the étale fundamental group of $X \otimes_k \overline{k}$ is trivial and the maximal Mumford slope of the sheaf of 1-forms is bounded above by $N(r)^{-1}$, then any isocrystal $\mathcal{E}$ which is an iterated extension of irreducible isocrystals of rank $\leq r$ having locally free lattices is isomorphic to $\mathcal{O}_{X/K}^{\oplus \mathrm{rank}(\mathcal{E})}$, where $\mathcal{O}_{X/K} := \mathbb{Q} \otimes \mathcal{O}_{X/W}$.*

Given Theorem 1.1, the proof is nearly the same as the one of the main Theorem [19, Thm. 1.1], with some differences which we explain in Section 4.

The second main theorem is de Jong's conjecture for convergent isocrystals coming from geometry. Recall that for a smooth proper morphism $f : Y \to X$ of varieties over a perfect field k of characteristic $p > 0$, the Gauß–Manin convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ is defined by Ogus [32].

Theorem 1.3. *Let $f : Y \to X$ be a smooth proper morphism between smooth proper varieties over a perfect field k of characteristic $p > 0$. If the étale fundamental group of $X \otimes_k \overline{k}$ is trivial, then the Gauß–Manin convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ is isomorphic to $\mathcal{O}_{X/K}^{\oplus r}$, where r is its rank.*

Remark 1.4. [1] For a smooth projective morphism $f : Y \to X$ between varieties defined over a perfect field k of characteristic $p > 0$, Lazda [29, Cor. 5.4] proved recently that the Gauß–Manin convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ canonically lifts to

[1] Added in proof: The authors are informed that there is an error in the proof of [29, Lem. 5.1] and so the proof of [29, Cor. 5.4] is incomplete. In this article, we *assume* the existence of the Gauß-Manin overconvergent isocrystal $R^i f_* \mathcal{O}^{\dagger}_{Y/K}$ when X is not proper.

an overconvergent isocrystal on X, which we denote by $R^if_*\mathcal{O}^\dagger_{Y/K}$ and call the Gauß–Manin overconvergent isocrystal. When X is smooth, geometrically simply connected, k is a finite field, and $p \geq 3$, we prove that $R^if_*\mathcal{O}^\dagger_{Y/K}$ is constant as an overconvergent isocrystal on X. See Section 5.

We now explain the methods used in order to prove Theorem 1.1.

There are two ways to prove the vanishing of $c_i^{dR}(E) \in H^{2i}_{dR}(X/k)\,(i>0)$ for a locally free sheaf E equipped with an integrable connection which does not use Chern–Weil theory.

One method uses a *modified splitting principle* as developed in [16] and [18]. On the projective bundle $\mathbb{P}(E) \xrightarrow{\pi} X$, the integrable connection induces a differential graded algebra $\Omega^\bullet_\tau$, which is a quotient $\Omega^\bullet_{\mathbb{P}(E)} \to \Omega^\bullet_\tau$ of the de Rham complex and which cohomologically splits $H^i_{dR}(X/k)$ in $H^i(\mathbb{P}(E), \Omega^\bullet_\tau)$. Then one shows that the Ω^1_τ-connection on π^*E induced by ∇ stabilizes $\mathcal{O}_{\mathbb{P}(E)}(1)$. Hence, when the rank of E is two, π^*E is an extension of Ω^1_τ-connections of rank 1, and so one can prove the vanishing of the classes in $H^{2i}(\mathbb{P}(E), \Omega^\bullet_\tau)$, thus in $H^{2i}_{dR}(X/k)$. In the case of rank r, one repeats the above argument $(r-1)$-times to obtain a filtration by Ω^1_τ-connections of rank 1.

We adapt this construction to the crystalline case as follows. (In the introduction, we assume the existence of a closed embedding $X \hookrightarrow P$ of X into a smooth p-adic formal scheme P over W to ease the explanation.) One considers the projective bundle $\pi : \mathbb{P}(E_D) \to D$ of the value E_D of a locally free crystal E on the PD-hull $X \hookrightarrow D \to \mathbb{P}_W$ of the embedding $X \hookrightarrow P$. One shows that the connection on E_D induces a quotient differential graded algebra $\bar{\Omega}^\bullet_{\mathbb{P}(E_D)} \to \bar{\Omega}^\bullet_\tau$ of the PD-de Rham complex $\bar{\Omega}^\bullet_{\mathbb{P}(E_D)}$. Then one shows that the $\bar{\Omega}^1_\tau$-connection on $\pi^*(E_D)$ respects $\mathcal{O}_{\mathbb{P}(E_D)}(1)$. Thus we can argue as in the de Rham case and obtain the required vanishing. See Section 2.

Another way on the de Rham side is to say that local trivializations of E yield a simplicial scheme $X_\bullet$ augmenting to X together with a morphism $e : X_\bullet \to BGL(r)$, defined by the transition functions, to the simplicial classifying scheme $BGL(r)$, where r is the rank of E. This induces the maps $H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}_{dR}(X_\bullet/k) \cong H^{2i}_{dR}(X/k)$. If $k = \mathbb{C}$ and E carries an integrable connection, we have a similar map $H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}_{dR}(X_{\mathrm{an},\bullet}/k) \xleftarrow{\cong} H^{2i}_{dR}(X_{\mathrm{an}}/k)$ which is identified with the previous one and factors through the cohomology $H^{2i}(BGL(r)_{\mathrm{disc}}, \Omega^\bullet) = H^{2i}(BGL(r)_{\mathrm{disc}}, \mathcal{O})$ of the discrete classifying simplicial space $BGL(r)_{\mathrm{disc}}$. Thus $c_i^{dR}(E)$ is in the image of the composite map $H^{2i}(BGL(r), \Omega^{\geq i}) \to H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(BGL(r), \mathcal{O}) \to H^{2i}(BGL(r)_{\mathrm{disc}}, \mathcal{O})$, which is zero for $i \geq 1$.

We adapt this construction to the crystalline case as follows. Given a closed emdedding $X \subset P$ into a smooth p-adic formal scheme over W, one defines $D_\bullet$ to be the simplicial scheme such that D_n is the PD-hull of the diagonal in $P^{\times n+1}$. Then the crystalline Poincaré lemma and the Čech–Alexander resolution equate $H^i_{\mathrm{crys}}(X/W)$ both with $H^i(D_\bullet, \bar{\Omega}^\bullet)$ and with $H^i(D_\bullet, \mathcal{O})$ (see Proposition 3.3). Thus, defining a certain simplicial version $\mathcal{D}_\bullet$ of $D_\bullet$, to which $X_\bullet$ maps, E has the free value $E_{\mathcal{D}_\bullet}$, and this induces the map $H^{2i}_{\mathrm{crys}}(BGL(r)/W) = H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(\mathcal{D}_\bullet, \bar{\Omega}_{\mathcal{D}_\bullet}) \xrightarrow{\cong} H^{2n}(\mathcal{D}_\bullet, \mathcal{O}) \cong H^{2i}_{\mathrm{crys}}(X/W)$. Thus $c_i^{\mathrm{crys}}(E_X)$ is in the image of the composite map $H^{2i}(BGL(r), \Omega^{\geq i}) \to H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(BGL(r), \mathcal{O}) \to H^{2i}(\mathcal{D}_\bullet, \mathcal{O}) \cong H^{2i}_{\mathrm{crys}}(X/W)$, which is zero for $i \geq 1$. See Section 3.

We now explain the methods used in order to prove Theorem 1.3, assuming k is a finite field.

Let us assume first that $f : Y \to X$ is an abelian scheme and present then an ℓ-adic argument due to G. Faltings. The arithmetic fundamental group of X acts on $R^i f_* \mathbb{Q}_\ell$ via $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Thus by Tate's theorem [38], all fibers of f over $\bar{\mathbb{F}}_p$-points of X are isogeneous; thus the Gauß–Manin convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ is constant.

In general, one has to replace Tate's motivic theorem by a result of Chiarellotto-Le Stum [11] (generalizing the Katz–Messing theorem [25]) and Abe's Čebotarev density theorem [1, Prop. A.3.1], which yields the constancy of the semi-simplification of $R^i f_* \mathcal{O}^\dagger_{Y/K}$ (see Remark 1.4 for notation) in the category $\mathrm{Conv}^\dagger(X/K)$ of overconvergent isocrystals on X. Finally one has to go from the semi-simplifcation to the original Gauß–Manin isocrystal by showing that there are no extensions of the constant overconvergent isocrystal by itself on X when X is proper or $p \geq 3$ (Theorem 5.1).

Over a non-finite field (when X is proper), one reduces the proof to the case of a finite ground field by a specialization argument. See Section 5.

Finally, in Section 6 we prove a very weak form of a Lefschetz theorem for isocrystals.

2. Crystalline modified splitting principle

The aim of this section is to prove Theorem 1.1 using a crystalline modified splitting principle.

Let X be a smooth variety over a perfect field k of characteristic $p > 0$ and let $X_{(\bullet)} \to X$ be a simplicial scheme augmented to X defined as the Čech hypercovering associated to an open covering $X = \bigcup_{i \in I} X_i$ which admits a closed embedding $\iota_{(\bullet)} : X_{(\bullet)} \to D_{(\bullet)}$ into a simplicial p-adic formal scheme $D_{(\bullet)}$ over W such that, for any $n \in \mathbb{N}$ and Zariski locally on $X_{(n)}$, $\iota_{(n)} : X_{(n)} \to D_{(n)}$ is the PD-envelope of a closed immersion $X_{(n)} \to Y$ of $X_{(n)}$ into some smooth p-adic formal scheme Y over W which may depend on n.

Note that, for any smooth variety X over k, there exists such a system: Indeed, if we take $X_{(\bullet)} \to X$ to be the Čech hypercovering associated to an affine open covering $X = \bigcup_{i \in I} X_i$, take a closed immersion $X_i \to Y_i$ into a p-adic formal scheme Y_i over W, and define $Y_{(n)}$ to be the fiber product of $(n+1)$ copies of $\coprod_{i \in I} Y_i$ over W, we naturally obtain a simplicial formal scheme $Y_{(\bullet)}$ and an immersion $X_{(\bullet)} \to Y_{(\bullet)}$. If we define the morphism $\iota_{(\bullet)} : X_{(\bullet)} \to D_{(\bullet)}$ to be the PD-envelope of $X_{(\bullet)}$ in $Y_{(\bullet)}$, it satisfies the above assumption.

2.1. Generalities on the first crystalline Chern class and variants of it.

One denotes by $(X/W)_{\mathrm{crys}}, (X_{(\bullet)}/W)_{\mathrm{crys}}$ the crystalline topos of X/W, $X_{(\bullet)}/W$ respectively. (See [9, p. 5.3] for the former one. The latter one is the topos associated to the diagram of topoi $\{(X_{(n)}/W)_{\mathrm{crys}}\}_{n \in \mathbb{N}}$, whose definition is given in [35, 1.2.8]. See also [35, 1.2.12]. It also appears, for example, in [4, p. 344].) One denotes by $(X_{(\bullet)}/W)_{\mathrm{crys}}|_{D_{(\bullet)}}$ the localization of $(X_{(\bullet)}/W)_{\mathrm{crys}}$ at $D_{(\bullet)}$ ([9, p. 5.23]) and by $(-)_{\mathrm{Zar}}$ the Zariski topos. In particular the canonical morphism of topoï $X_{(\bullet)\mathrm{Zar}} \to D_{(\bullet)\mathrm{Zar}}$ is an equivalence. On $D_{(\bullet)\mathrm{Zar}}$ one defines the PD-de Rham complex $\bar{\Omega}^\bullet_{D_{(\bullet)}}$, which is a quotient differential graded algebra of the de Rham complex

$\Omega^\bullet_{D_{(\bullet)}}$ of $D_{(\bullet)}$. The submodule $\mathcal{K} \subset \Omega^1_{D_{(\bullet)}}$, topologically spanned by

$$da^{[m]} - a^{[m-1]}da \, (a \in I = \mathrm{Ker}(\mathcal{O}_{D_{(n)}} \to \iota_{(n)*}\mathcal{O}_{X_{(n)}}), n, m \in \mathbb{N}),$$

spans all relations, that is,

$$\bar{\Omega}^\bullet_{D_{(\bullet)}} = \Omega^\bullet_{D_{(\bullet)}}/\mathcal{K} \wedge \Omega^{\bullet-1}_{D_{(\bullet)}}, \tag{2.1}$$

and in addition, each $\bar{\Omega}^i_{D_{(n)}}$ is locally free over $\mathcal{O}_{D_{(n)}}$ with the relation

$$\bar{\Omega}^i_{D_{(\bullet)}} = \bigwedge^i_{\mathcal{O}_{D_{(\bullet)}}} \bar{\Omega}^1_{D_{(\bullet)}} \tag{2.2}$$

([24, Prop. 3.1.6]).

One has the following commutative diagram of topoï [9, p. 6.12]:

$$\begin{array}{ccc} (X_{(\bullet)}/W)_{\mathrm{crys}}|_{D_{(\bullet)}} & \xrightarrow{\varphi} & D_{(\bullet)\mathrm{Zar}} = X_{(\bullet)\mathrm{Zar}} \\ \downarrow j & \nearrow u & \\ (X_{(\bullet)}/W)_{\mathrm{crys}} & & \end{array} \tag{2.3}$$

The complex

$$L(\bar{\Omega}^\bullet_{D_{(\bullet)}}) := j_*\varphi^*(\bar{\Omega}^\bullet_{D_{(\bullet)}}) \tag{2.4}$$

in $(X_{(\bullet)}/W)_{\mathrm{crys}}$ is defined in [9, p. 6.13], and it is proved in [9, Thm. 6.12] that the natural map

$$\mathcal{O}_{X_{(\bullet)}/W} \to L(\bar{\Omega}^\bullet_{D_{(\bullet)}}) \tag{2.5}$$

is a quasi-isomorphism in $(X_{(\bullet)}/W)_{\mathrm{crys}}$. For each n and an object $(U \to T, \delta)$ in the crystalline site on $X_{(n)}/W$, this quasi-isomorphism is locally written as

$$\begin{aligned} \mathcal{O}_T \xrightarrow{\cong} (\mathcal{O}_T\langle x_1, \ldots, x_d\rangle \to \bigoplus_{1\le i\le d} \mathcal{O}_T\langle x_1, \ldots, x_d\rangle dx_i \\ \to \bigoplus_{1\le i<j\le d} \mathcal{O}_T\langle x_1, \ldots, x_d\rangle dx_i \wedge dx_j \to \cdots) \end{aligned} \tag{2.6}$$

for some d, where $\mathcal{O}_T\langle x_1, \ldots, x_d\rangle$ denotes the p-adically completed PD-polynomial algebra. The map (2.6) induces the quasi-isomorphism

$$\begin{aligned} \mathcal{O}^\times_T \xrightarrow{\cong} (\mathcal{O}_T\langle x_1, \ldots, x_d\rangle^\times \xrightarrow{d\log} \bigoplus_{1\le i\le d} \mathcal{O}_T\langle x_1, \ldots, x_d\rangle dx_i \\ \to \bigoplus_{1\le i<j\le d} \mathcal{O}_T\langle x_1, \ldots, x_d\rangle dx_i \wedge dx_j \to \cdots), \end{aligned} \tag{2.7}$$

and if we denote the sheaf $j_*(\varphi^*(\mathcal{O}_{D_{(\bullet)}}))^\times$ by $L(\mathcal{O}_{D_{(\bullet)}})^\times$, the quasi-isomorphisms (2.7) for $(U \to T, \delta)$'s induce the quasi-isomorphism

$$\mathcal{O}^\times_{X_{(\bullet)}/W} \xrightarrow{\cong} (L(\mathcal{O}_{D_{(\bullet)}})^\times \xrightarrow{d\log} L(\bar{\Omega}^1_{D_{(\bullet)}}) \to L(\bar{\Omega}^1_{D_{(\bullet)}}) \to \cdots).$$

By applying Ru_* and using [9, 5.27.2], we obtain a quasi-isomorphism in $X_{(\bullet)\mathrm{Zar}}$,

$$Ru_*\mathcal{O}^\times_{X_{(\bullet)}/W} \xrightarrow{\cong} (\mathcal{O}^\times_{D_{(\bullet)}} \xrightarrow{d\log} \bar{\Omega}^1_{D_{(\bullet)}} \xrightarrow{d} \bar{\Omega}^2_{D_{(\bullet)}} \to \ldots), \tag{2.8}$$

which is stated in [21, I, (5.1.12)]. The exact sequence

$$0 \to \mathcal{I}_{X_{(\bullet)}/W} \to \mathcal{O}_{X_{(\bullet)}/W} \to \iota_*\mathcal{O}_{X_{(\bullet)}} \to 0 \tag{2.9}$$

in $(X_{(\bullet)}/W)_{\mathrm{crys}}$, defining $\mathcal{I}_{X_{(\bullet)}/W}$, yields an exact sequence

$$1 \to (1+\mathcal{I}_{X_{(\bullet)}/W}) \to \mathcal{O}^\times_{X_{(\bullet)}/W} \to \iota_*\mathcal{O}^\times_{X_{(\bullet)}} \to 1 \tag{2.10}$$

in $(X_{(\bullet)}/W)_{\mathrm{crys}}$. By [8, 2.1] and the functoriality of the construction there, the connecting homomorphism

$$H^1(X,\mathcal{O}_X^\times) \to H^1(X_{(\bullet)},\mathcal{O}^\times_{X_{(\bullet)}}) \to H^2((X_{(\bullet)}/W)_{\mathrm{crys}}, 1+\mathcal{I}_{X_{(\bullet)}/W}),$$

followed by the logarithm

$$H^2((X_{(\bullet)}/W)_{\mathrm{crys}}, 1+\mathcal{I}_{X_{(\bullet)}/W}) \to H^2((X_{(\bullet)}/W)_{\mathrm{crys}}, \mathcal{I}_{X_{(\bullet)}/W})$$

and the natural map

$$H^2((X_{(\bullet)}/W)_{\mathrm{crys}}, \mathcal{I}_{X_{(\bullet)}/W}) \to H^2_{\mathrm{crys}}(X_{(\bullet)}/W) \cong H^2_{\mathrm{crys}}(X/W),$$

precisely computes c_1^{crys}. Thus applying Ru_* to (2.10) and using (2.8), we conclude that the connecting homomorphism

$$H^1(X,\mathcal{O}_X^\times) \to H^1(X_{(\bullet)},\mathcal{O}^\times_{X_{(\bullet)}}) \to H^2(D_{(\bullet)}, (1+I_{(\bullet)}) \xrightarrow{d\log} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}})$$

of the exact sequence in $D_{(\bullet)\mathrm{Zar}}$,

$$1 \to ((1+I_{(\bullet)}) \xrightarrow{d\log} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}}) \to (\mathcal{O}^\times_{D_{(\bullet)}} \xrightarrow{d\log} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}}) \to \mathcal{O}_X^\times \to 1, \tag{2.11}$$

followed by the logarithm $H^2(D_{(\bullet)}, (1+I_{(\bullet)}) \xrightarrow{d\log} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}}) \to H^2(D_{(\bullet)}, I_{(\bullet)} \xrightarrow{d} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}})$ and the natural map $H^2(D_{(\bullet)}, I_{(\bullet)} \xrightarrow{d} \bar{\Omega}^{\geq 1}_{D_{(\bullet)}}) \to H^2(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}}) = H^2_{\mathrm{crys}}(X_{(\bullet)}/W) = H^2_{\mathrm{crys}}(X/W)$, precisely computes c_1^{crys}. In particular, if E is a locally free sheaf of rank 1 on X, then by (2.8), $c_1^{\mathrm{crys}}(E_X) = 0$ where E_X is the value of E at X.

More generally, let $\tau : \bar{\Omega}^\bullet_{D_{(\bullet)}} \twoheadrightarrow A^\bullet_{(\bullet)}$ be a surjection of sheaves of differential graded algebras on $D_{(\bullet)\mathrm{Zar}}$ such that, for any $n \in \mathbb{N}$, $A^1_{(n)}$ is locally free over $\mathcal{O}_{D_{(n)}}$, $A^0_{(n)} = \mathcal{O}_{D_{(n)}}$, $A^m_{(n)} = \bigwedge^m_{\mathcal{O}_{D_{(n)}}} A^1_{(n)}$. A *$\tau$-connection* (see [16, (2.1)]) on a locally free sheaf $E_{D_{(\bullet)}}$ on $D_{(\bullet)}$ is an additive map $\nabla_\tau : E_{D_{(\bullet)}} \to A^1_{(\bullet)} \otimes_{\mathcal{O}_{D_{(\bullet)}}} E_{D_{(\bullet)}}$ which fulfills the *τ-Leibniz* rule $\nabla_\tau(\lambda e) = \tau d(\lambda) \otimes e + \lambda \otimes \nabla_\tau(e)$. Then $\nabla_\tau \circ \nabla_\tau : E_{D_{(\bullet)}} \to A^2_{(\bullet)} \otimes E_{D_{(\bullet)}}$, where $\nabla_\tau(\omega \otimes e) = (-1)^i \tau d(\omega) + \omega \otimes \nabla_\tau(e)$ for $\omega \in A^i_{(n)}$, is $\mathcal{O}_{D_{(\bullet)}}$-linear. The τ-connection is *integrable* if $\nabla_\tau \circ \nabla_\tau = 0$. One pushes down (2.11) along τ and obtains the exact sequence in $D_{(\bullet)\mathrm{Zar}}$:

$$1 \to ((1+I_{(\bullet)}) \xrightarrow{d\log} A^{\geq 1}_{(\bullet)}) \to (\mathcal{O}^\times_{D_{(\bullet)}} \xrightarrow{d\log} A^{\geq 1}_{(\bullet)}) \to \mathcal{O}^\times_{X_{(\bullet)}} \to 1. \tag{2.12}$$

Then, for a locally free sheaf E_X of rank 1 on X, $0 = \tau(c_1^{\mathrm{crys}}(E_X)) \in H^2(D_{(\bullet)}, A^\bullet_{(\bullet)})$ if $E_X|_{X_{(\bullet)}}$ is the restriction to $X_{(\bullet)}$ of a line bundle $E_{D_{(\bullet)}}$ on $D_{(\bullet)}$ which is endowed with an integrable τ-connection.

2.2. **Modified splitting principle.** Let $\tau : \bar{\Omega}^\bullet_{D_{(\bullet)}} \twoheadrightarrow A^\bullet_{(\bullet)}$ be a surjection of sheaves of differential graded algebras on $D_{(\bullet)\mathrm{Zar}}$ such that, for any $n \in \mathbb{N}$, $A^1_{(n)}$ is locally free over $\mathcal{O}_{D_{(n)}}$, $A^0_{(n)} = \mathcal{O}_{D_{(n)}}$, and $A^m_{(n)} = \bigwedge^m_{\mathcal{O}_{D_{(n)}}} A^1_{(n)}$. Also, let E_X be a locally free sheaf on X such that its restriction $E_X|_{X_{(\bullet)}}$ to $X_{(\bullet)}$ extends to a locally free sheaf $E_{D_{(\bullet)}}$ on $D_{(\bullet)}$ endowed with an integrable τ-connection. One defines $X' = \mathbb{P}(E_X), X'_{(\bullet)} = \mathbb{P}(E_X|_{X_{(\bullet)}}),\ D'_{(\bullet)} = \mathbb{P}(E_{D_{(\bullet)}})$, together with the augmentation $X'_{(\bullet)} \to X'$ and the closed embedding $\iota' : X'_{(\bullet)} \to D'_{(\bullet)}$. One has cartesian squares of (simplicial formal) schemes (over W)

(2.13)
$$\begin{array}{ccccc} X' & \longleftarrow & X'_{(\bullet)} & \xrightarrow{\iota'} & D'_{(\bullet)} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ X & \longleftarrow & X_{(\bullet)} & \xrightarrow[\iota]{} & D_{(\bullet)}. \end{array}$$

The PD-structure on $I_{(\bullet)}$ extends uniquely to a PD-structure on

$$I'_{(\bullet)} = \mathrm{Ker}(\mathcal{O}_{D'_{(\bullet)}} \to \mathcal{O}_{X'_{(\bullet)}})$$

as π is flat ([9, Prop. 3.21]), and ι' is again the PD-envelope of a closed immersion from $X'_{(n)}$ to a smooth p-adic formal scheme over W Zariski locally on $X'_{(n)}$ for each n. Thus one can define $\bar{\Omega}^\bullet_{D'_{(\bullet)}}$ and one has an exact sequence

$$0 \to \pi^*\bar{\Omega}^1_{D_{(\bullet)}} \to \bar{\Omega}^1_{D'_{(\bullet)}} \to \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}} \to 0 \tag{2.14}$$

on $X'_{(\bullet)\mathrm{Zar}}$. Setting $\mathcal{K}_\tau = \mathrm{Ker}(\bar{\Omega}^1_{D_{(\bullet)}} \to A^1_{(\bullet)})$, one defines $\bar{\Omega}^1_{D'_{(\bullet)},\tau} = \bar{\Omega}^1_{D'_{(\bullet)}}/\pi^*\mathcal{K}_\tau$. By definition, (2.14) pushes down to an exact sequence

$$0 \to \pi^* A^1_{(\bullet)} \xrightarrow{i} \bar{\Omega}^1_{D'_{(\bullet)},\tau} \xrightarrow{p} \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}} \to 0 \tag{2.15}$$

on $X'_{(\bullet)\mathrm{Zar}}$. One defines $\bar{\Omega}^n_{D'_{(\bullet)},\tau} = \bigwedge^n_{\mathcal{O}_{D'_{(\bullet)}}} \bar{\Omega}^1_{D'_{(\bullet)},\tau}$. By [16, Claim, p. 332], the quotient homomorphism $r : \bar{\Omega}^1_{D'_{(\bullet)}} \to \bar{\Omega}^1_{D'_{(\bullet)},\tau}$ extends to a quotient

$$r : \bar{\Omega}^\bullet_{D'_{(\bullet)}} \to \bar{\Omega}^\bullet_{D'_{(\bullet)},\tau} \tag{2.16}$$

of differential graded algebras on $X'_{(\bullet)\mathrm{Zar}}$, where the differential on $\bar{\Omega}^\bullet_{D'_{(\bullet)},\tau}$ is denoted by rd.

The τ-connection ∇_τ on $E_{D_{(\bullet)}}$ induces a pull-back connection $\pi^*\nabla_\tau : \pi^* E_{D_{(\bullet)}} \to \bar{\Omega}^1_{D'_{(\bullet)},\tau} \otimes_{\mathcal{O}_{D'_{(\bullet)}}} \pi^* E_{D_{(\bullet)}}$. Its restriction to $\Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1)$ via the exact sequence

$$0 \to \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1) \to \pi^* E_{D_{(\bullet)}} \to \mathcal{O}_{D'_{(\bullet)}}(1) \to 0, \tag{2.17}$$

followed by the projection $\bar{\Omega}^1_{D'_{(\bullet)},\tau} \otimes_{\mathcal{O}_{D'_{(\bullet)}}} \pi^* E_{D_{(\bullet)}} \to \bar{\Omega}^1_{D'_{(\bullet)},\tau} \otimes_{\mathcal{O}_{D'_{(\bullet)}}} \otimes \mathcal{O}_{D'_{(\bullet)}}(1)$ defines a section

$$\sigma : \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}} \to \bar{\Omega}^1_{D'_{(\bullet)},\tau}$$

of $-p$ ([16, (2.4)]). Thus $\tau' = 1 + p \circ \sigma : \bar{\Omega}^1_{D'_{(\bullet)},\tau} \to \pi^* A^1_{(\bullet)}$ is a section of i. By [16, (2.5)], τ' induces a surjective homomorphism

$$\tau' : \bar{\Omega}^\bullet_{D'_{(\bullet)},\tau} \twoheadrightarrow \pi^* A^\bullet_{(\bullet)} \tag{2.18}$$

of differential graded algebras, where the differential on the right is induced by $\tau' \circ rd$.

The main point is then that the $\pi^* A^1_{(\bullet)}$-valued connection $\nabla' : \pi^* E_{D_{(\bullet)}} \to \pi^* A^1_{(\bullet)} \otimes_{\mathcal{O}_{D'_{(\bullet)}}} \pi^* E_{D_{(\bullet)}}$, which is $\pi^*\nabla_\tau$ followed by τ', is integrable and respects the flag (2.17), thus induces τ'-integrable connections

$$\nabla' : \mathcal{O}_{D'_{(\bullet)}}(1) \to \pi^* A^1_{(\bullet)} \otimes \mathcal{O}_{D'_{(\bullet)}}(1), \qquad (2.19)$$
$$\nabla' : \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1) \to \pi^* A^1_{(\bullet)} \otimes \Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1).$$

Consequently we can iterate the construction, replacing X by X', $X_{(\bullet)}$ by $X'_{(\bullet)}$, $D_{(\bullet)}$ by $D'_{(\bullet)}$, E_X by the descent of $\Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1)|_{X'_{(\bullet)}}$ to X', $E_{D_{(\bullet)}}$ by $\Omega^1_{D'_{(\bullet)}/D_{(\bullet)}}(1)$, and τ by $\tau' \circ r$.

2.3. **Proof of Theorem 1.1.** Further iterating, after $(r-1)$-steps, one obtains a diagram as (2.13), where now X' is the complete flag bundle over X, with (2.18) becoming a surjective homomorphism

$$\tau : \bar{\Omega}^\bullet_{D'_{(\bullet)}} \to \pi^* \bar{\Omega}^\bullet_{D_{(\bullet)}} \qquad (2.20)$$

of differential graded algebras and with a filtration on $\pi^* E_X$ with graded sheaf being a sum of locally free sheaves L_j of rank 1 such that the restriction $L_j|_{X'_{(\bullet)}}$ of L_j to $X'_{(\bullet)}$ extends to a locally free sheaf on $D'_{(\bullet)}$ endowed with an integrable τ-connection (thus with values in $\pi^*\bar{\Omega}^1_{D_{(\bullet)}}$). In addition, from [16, Lemma in (1.3), (2.7)], the composite

$$\bar{\Omega}^\bullet_{D_{(\bullet)}} \to R\pi_* \bar{\Omega}^\bullet_{D'_{(\bullet)}} \xrightarrow{R\pi_*\tau} R\pi_*\pi^*\bar{\Omega}^\bullet_{D_{(\bullet)}} \overset{\cong}{\leftarrow} \bar{\Omega}^\bullet_{D_{(\bullet)}} \qquad (2.21)$$

is the identity on $X_{(\bullet)\mathrm{Zar}}$. By the standard Whitney product formula for crystalline Chern classes [21, III, Thm. 1.1.1] for $i \geq 1$, $c_i^{\mathrm{crys}}(\pi^* E_X)$ is a sum of products of $c_1^{\mathrm{crys}}(L_j)$, which by Subsection 2.1 maps to 0 in $H^2(D'_{(\bullet)}, \pi^*\bar{\Omega}^\bullet_{D_{(\bullet)}})$. Thus $c_i^{\mathrm{crys}}(E_X) \in H^{2i}(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}})$ maps to 0 in $H^{2i}(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}})$ via the composite map in (2.21), which is the identity. This shows Theorem 1.1.

Remark 2.1. Let $W_n := W/p^n W$. Then, by replacing W by W_n and $D_{(\bullet)}$ by its mod p^n reduction, we see that the proof above gives the following variant of Theorem 1.1: If X is as in Theorem 1.1 and E is a locally free crystal on X/W_n, then $c_i^{\mathrm{crys}}(E_X)$ is zero in the torsion crystalline cohomology group $H^{2i}_{\mathrm{crys}}(X/W_n)$ for $i \geq 1$. Because $H^{2i}_{\mathrm{crys}}(X/W) \otimes_W W_n \to H^{2i}_{\mathrm{crys}}(X/W_n)$ is injective, it implies that the Chern classes $c_i^{\mathrm{crys}}(E_X)$ in $H^{2i}_{\mathrm{crys}}(X/W)$ $(i \geq 1)$ are divisible by p^n in this case.

2.4. **Remark on Chern–Simons theory.** In [17] and [18], a version of the modified splitting principle which is slightly more elaborate than the one used in Subsection 2.2 was performed in order to construct classes $c_i(E, \nabla) \in H^i(X, \mathcal{K}_i^M \xrightarrow{d\log} \Omega^i_X \xrightarrow{d} \ldots)$ of a bundle with an integrable connection (E, ∇), depending on ∇, where $\mathcal{K}_i^M$ is the Zariski sheaf of Milnor K-theory. Those classes lift both the Chow classes in $CH^i(X) = H^i(X, \mathcal{K}_i^M)$ via the obvious forgetful map and the Chern–Simons classes in $H^{2i-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(i))$, if $k = \mathbb{C}$. We hope to be able to define a crystalline version of Chern–Simons theory, yielding classes lifting both Chow classes and classes in syntomic cohomology.

3. The crystalline version of the discrete classifying space $BGL(r)$

The aim of this section is to prove Theorem 1.1 using a crystalline version of the discrete classifying space $BGL(r)$.

Let X be a smooth variety defined over a perfect field k of characteristic $p > 0$.

3.1. **Čech–Alexander resolutions of $\mathcal{O}_{X/W}$.** Fix a closed embedding $X \hookrightarrow Y$ into a p-adic smooth formal scheme Y over W, and define $D(n)$ as the PD-envelope of X in the diagonal embedding Y^{n+1}, where ${}^{n+1}$ means the product over W. Then one has the canonical morphism of topoï

$$j_n : (X/W)_{\mathrm{crys}}|_{D(n)} \to (X/W)_{\mathrm{crys}} \tag{3.1}$$

and the Čech–Alexander resolution [9, 5.29] of the abelian sheaf $\mathcal{O}_{X/W}$,

$$\mathcal{O}_{X/W} \xrightarrow{\cong} (j_{0*}j_0^*\mathcal{O}_{X/W} \to j_{1*}j_1^*\mathcal{O}_{X/W} \to \dots). \tag{3.2}$$

We also use the following variant of (3.2). Fix two closed embeddings $X \hookrightarrow Y$ and $X \hookrightarrow Z$, where both Y/W and Z/W are p-adic formal schemes and Y/W is smooth. One defines $D(n)$ as the PD-envelope of X in the diagonal embedding $Y^{n+1} \times_W Z$. Then one has the canonical morphisms of topoï

$$j_n : (X/W)_{\mathrm{crys}}|_{D(n)} \xrightarrow{\ell_n} (X/W)_{\mathrm{crys}}|_{D(-1)} \xrightarrow{j_{-1}} (X/W)_{\mathrm{crys}}. \tag{3.3}$$

On $(X/W)_{\mathrm{crys}}|_{D(-1)}$, one has the Čech–Alexander resolution [9, 5.29] of the abelian sheaf $j_{-1}^*\mathcal{O}_{X/W}$,

$$j_{-1}^*\mathcal{O}_{X/W} \xrightarrow{\cong} (\ell_{0*}j_0^*\mathcal{O}_{X/W} \to \ell_{1*}j_1^*\mathcal{O}_{X/W} \to \dots). \tag{3.4}$$

Hence, applying the exact functor j_{-1*} ([9, Cor. 5.27.1]), one obtains the resolution of the abelian sheaf $j_{-1*}j_{-1}^*\mathcal{O}_{X/W}$:

$$j_{-1*}j_{-1}^*\mathcal{O}_{X/W} \xrightarrow{\cong} (j_{0*}j_0^*\mathcal{O}_{X/W} \to j_{1*}j_1^*\mathcal{O}_{X/W} \to \dots). \tag{3.5}$$

3.2. **Various simplicial constructions to compute crystalline cohomology.** Let $X = \bigcup_{i \in I} X_i$ be a finite covering of X by affine open subvarieties. We assign to it the standard Mayer–Vietoris simplicial scheme, the definition of which we recall now.

We choose a total order on I, define the set of tuples $I^n_\leq := \{(i_0, \dots, i_n) \ ; \ i_0 \leq i_1 \leq \dots \leq i_n)\}$, and set $I_\leq$ to be the disjoint union of the $I^n_\leq$. For $J = (i_0, \dots, i_n) \in I_\leq$, one sets $X_J = \bigcap_{i_j \in J} X_{i_j}$. One upgrades $I_\leq$ to a category. The Hom-set $\mathrm{Hom}_{I_\leq}(J, J')$, for $J = (i_0, \dots, i_n)$ and $J' = (i'_0, \dots, i'_{n'})$, consists of those non-decreasing maps $\varphi : [n] \to [n']$, where $[n] = \{0, 1, \dots, n\}$, with the property that $i_a = i'_{\varphi(a)}$ for all $a \in \{0, \dots, n\}$. Thus to $\varphi \in \mathrm{Hom}_{I_\leq}(J, J')$, one assigns the open embedding $X_{J'} \hookrightarrow X_J$, which one denotes by φ^*.

If $\varphi \in \mathrm{Hom}_{I_\leq}(J, J')$ with $J' = (i'_0, \dots, i'_{n'})$, J is necessarily equal to the tuple $(i'_{\varphi(0)}, \dots, i'_{\varphi(n)})$. Thus, given any non-decreasing map $\varphi : [n] \to [n']$ and $J' \in I^{n'}_\leq$, there is one and only one $J \in I^n_\leq$ such that $\varphi \in \mathrm{Hom}_{I_\leq}(J, J')$, in particular the open embedding $\varphi^* : X_{J'} \to X_J$ is determined as well. Denoting by $X_{(n)} = \coprod_{J \in I^n_\leq} X_J$ the disjoint union of the X_J over all the $J \in I^n_\leq$, one defines the map

$$\varphi^* : X_{(n')} \to X_{(n)} \tag{3.6}$$

for a non-decreasing map $\varphi : [n] \to [n']$ as the disjoint union of the maps $X_{J'} \xrightarrow{\varphi^*} X_J \hookrightarrow X^{(n)}$ for $J' \in I_{\le}^{n'}$. Using the definition in [15, Section 5], (3.6) defines the simplicial scheme $X_{(\bullet)}$ which augments to X:

$$X_{(\bullet)} \xrightarrow{\epsilon} X. \tag{3.7}$$

Remark 3.1. The simplicial scheme $X_{(\bullet)}$ here differs from the simplicial scheme $X_{(\bullet)}$ which appeared in Section 2.

The aim of this subsection is to prove the following.

Proposition 3.2. *The augmentation map induces a quasi-isomorphism*

$$\epsilon^* : R\Gamma((X/W)_{\mathrm{crys}}, \mathcal{O}_{X/W}) \to R\Gamma((X_{(\bullet)}/W)_{\mathrm{crys}}, \mathcal{O}_{X_{(\bullet)}/W}) \tag{3.8}$$

in the derived category $D(\mathrm{Ab})$ of abelian groups.

Proof. The proof goes by induction on the cardinality $|I|$ of I. If $|I| = 1$, one sets $G = R\Gamma((X/W)_{\mathrm{crys}}, \mathcal{O}_{X/W}) \in D(\mathrm{Ab})$. Then the right hand side of (3.8) reads

$$G \xrightarrow{\alpha_0} G \to \cdots \xrightarrow{\alpha_n} G \to \cdots \tag{3.9}$$

where $\alpha_n = \mathrm{id}$ for n even, and $\alpha_n = 0$ for n odd.

For $|I| > 1$, we subdivide the simplicial construction $X_{(\bullet)}$ as follows. Let 0 be the minimal element of I. One sets $= I' \sqcup I''$, with $I' = \{0\}$ and $I'' = I \setminus I'$. Then one has

$$I_{\le} = I'_{\le} \sqcup (\bigsqcup_{(n,m)\in\mathbb{N}} I'^{n}_{\le} \times I''^{m}_{\le}) \sqcup I''_{\le}. \tag{3.10}$$

Setting $X_{J',J''} = X_{J'} \cap X_{J''}$ for $J' \in I'_{\le}$ and $J'' \in I''_{\le}$, one sets $X_{(n,m)} = \bigsqcup_{J'\in I'^{n}_{\le}, J''\in I''^{m}_{\le}} X_{J',J''}$. Then $X_{(\bullet,\bullet)}$ forms a bisimplicial scheme ([20, p. 17]), and one has the commutative diagram

$$\begin{array}{ccccc} X'_{(\bullet)} & \longleftarrow & X_{(\bullet,\bullet)} & \longrightarrow & X''_{(\bullet)} \\ \downarrow{\scriptstyle \epsilon'} & & & & \downarrow{\scriptstyle \epsilon''} \\ X' & & & & X'' \end{array} \tag{3.11}$$

with $X' = X_0$ and $X'' = \bigcup_{i\in I\setminus\{0\}} X_i$. By induction, Proposition 3.2 applies to ϵ' and ϵ''. On the other hand, $X_{(\bullet,m)}$ is the constant simplicial scheme on $X' \cap X''_{(m)}$. So by the case $|I| = 1$, one has

$$R\Gamma((X' \cap X''_{(m)}/W)_{\mathrm{crys}}, \mathcal{O}_{X'\cap X''_{(m)}/W}) \xrightarrow{\cong} R\Gamma((X_{(\bullet,m)}/W)_{\mathrm{crys}}, \mathcal{O}_{X_{(\bullet,m)}/W}). \tag{3.12}$$

From this and the induction hypothesis one deduces the isomorphism

(3.13)
$$\begin{aligned} \epsilon_1^* : R\Gamma((X' \cap X''/W)_{\mathrm{crys}}, \mathcal{O}_{X'\cap X''/W}) &\xrightarrow{\cong} R\Gamma((X' \cap X''_{(\bullet)}/W)_{\mathrm{crys}}, \mathcal{O}_{X'\cap X''_{(\bullet)}/W}) \\ &\xrightarrow{\cong} R\Gamma((X_{(\bullet,\bullet)}/W)_{\mathrm{crys}}, \mathcal{O}_{X_{(\bullet,\bullet)}/W}). \end{aligned}$$

One now extends (3.8) to a diagram

(3.14)
$$\begin{array}{ccc} R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) & \xrightarrow{\epsilon^*} & R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X''_{(\bullet)}/W}) \\ \downarrow & & \downarrow \\ R\Gamma((X'/W)_{\text{crys}}, \mathcal{O}_{X'/W}) \oplus R\Gamma((X''/W)_{\text{crys}}, \mathcal{O}_{X''/W}) & \xrightarrow{\epsilon'^* \oplus \epsilon''^*} & R\Gamma((X'_{(\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X'_{(\bullet)}/W}) \oplus R\Gamma((X''_{(\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X''_{(\bullet)}/W}) \\ \downarrow & & \downarrow \\ R\Gamma((X' \cap X''/W)_{\text{crys}}, \mathcal{O}_{X' \cap X''/W}) & \xrightarrow{\epsilon_1^*} & R\Gamma((X_{(\bullet,\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X_{(\bullet,\bullet)}/W}) \end{array}$$

where the left vertical triangle is induced by the quasi-isomorphism in [4, V, (3.5.4)], and the right one exists by construction. As $\epsilon'^*, \epsilon''^*, \epsilon_1^*$ are isomorphisms, so is ϵ^*. □

3.3. **Lifting the simplicial construction to PD-envelopes.** Keeping the same notation, we choose for each affine X_i a closed embedding $X_i \hookrightarrow Y_i$ into a smooth p-adic formal scheme Y_i over W. One defines the PD-envelope $\alpha_J : X_J \hookrightarrow D_J$ of $X_J \hookrightarrow Y_{i_0} \times_W \cdots \times_W Y_{i_n}$ for $J = (i_0, \ldots, i_n) \in I^n_{\leq}$. As in (3.7) and (3.11) one has the simplicial formal scheme $D_{(\bullet)}$ and the diagram of simplicial formal schemes

(3.15)
$$D'_{(\bullet)} \longleftarrow D_{(\bullet,\bullet)} \longrightarrow D''_{(\bullet)},$$

this time without augmentation. On the other hand, for each $J \in I_{\leq}$, one has as in (2.3) the diagram of topoï

(3.16)
$$\begin{array}{ccc} (X_J/W)_{\text{crys}}|_{D_J} & \xrightarrow{\varphi_J} & D_{J,\text{Zar}} = X_{J,\text{Zar}} \\ {\scriptstyle j_J}\downarrow & \nearrow{\scriptstyle u_J} & \\ (X_J/W)_{\text{crys}} & & \end{array}$$

to which one applies (2.4) and the quasi-isomorphism (2.5), which in addition is functorial. Thus, combined with (3.8), this yields quasi-isomorphisms

(3.17)
$$\begin{aligned} R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) &\xrightarrow{\cong} R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X_{(\bullet)}/W}) \\ &\xrightarrow{\cong} R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\bar{\Omega}^\bullet_{D_{(\bullet)}})) \xleftarrow{\cong} R\Gamma(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}}). \end{aligned}$$

(The last isomorphism follows from [9, Cor. 5.27.2].)

Proposition 3.3. *The forgetful morphism*

(3.18)
$$R\Gamma(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}}) \to R\Gamma(D_{(\bullet)}, \mathcal{O}_{D_{(\bullet)}})$$

is a quasi-isomorphism.

Proof. It is enough to show that the map

(3.19)
$$R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) \to R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet)}}))$$

is a quasi-isomorphism. As in the proof of Proposition 3.2, we argue by induction in $|I|$. For $|I| = 1$, the right hand side is $R\Gamma((X/W)_{\text{crys}}, (-))$ of the right hand side of (3.2), thus computes $R\Gamma((X/W)_{\mathrm{crys}}, \mathcal{O}_{X/W})$. For general I, we argue as in the proof of Proposition 3.2. We have the isomorphism

$$R\Gamma((X' \cap X''_{(m)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D''_{(m)}})) \xrightarrow{\cong} R\Gamma((X_{(\bullet,m)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D_{(\bullet,m)}})), \tag{3.20}$$

which is $R\Gamma((X' \cap X''_{(m)}/W)_{\mathrm{crys}}, (-))$ of the resolution (3.5) (with X there replaced by $X' \cap X''_{(m)}$.) From this and the induction hypothesis one deduces the isomorphism

(3.21)

$$\begin{aligned}\epsilon_1^* : R\Gamma((X' \cap X''/W)_{\mathrm{crys}}, \mathcal{O}_{X' \cap X''/W}) &\xrightarrow{\cong} R\Gamma((X' \cap X''_{(\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D''_{(\bullet)}})) \\ &\xrightarrow{\cong} R\Gamma((X_{(\bullet,\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D_{(\bullet,\bullet)}})).\end{aligned}$$

Then one has the diagram as in (3.14), with the triangle in the right column replaced by the triangle

$$\begin{aligned}&R\Gamma((X_{(\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D_{(\bullet)}})) \\ &\to R\Gamma((X'_{(\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D'_{(\bullet)}})) \oplus R\Gamma((X''_{(\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D''_{(\bullet)}})) \\ &\to R\Gamma((X_{(\bullet,\bullet)}/W)_{\mathrm{crys}}, L(\mathcal{O}_{D_{(\bullet,\bullet)}})),\end{aligned} \tag{3.22}$$

which exists by construction. Thus one concludes by induction.

3.4. **Proof of Theorem 1.1.** One defines G to be the group scheme $\mathrm{GL}(r)$ over $\mathbb{Z}$, and for any scheme S, one writes G_S for the induced group scheme over S or, by abuse of notation, G_R for $S = \mathrm{Spec}(R)$. So one has G_k, G_W, and its p-adic completion $\hat{G}_W$. One has the classifying (formal) simplicial schemes $BG_k, BG_W, B\hat{G}_W$ [15, 6.1.2]: The degree n part $BG_{k,(n)}$ of BG_k is defined by $BG_{k,(n)} := G_k^{[n]}/G_k$ with the action of G_k on $G_k^{[n]}$ given by $g(g_0, \ldots, g_n) := (g_0 g^{-1}, \ldots, g_n g^{-1})$ (where $[n] := \{0, \ldots, n\}$), and the transition morphism $BG_{k,(n')} \to BG_{k,(n)}$ associated to a non-decreasing map $\varphi : [n] \to [n']$ is defined as the one induced by $\varphi^* : G^{[n']} \to G^{[n]}$. If we use the identification

$$G_k^n \cong G_k^{[n]}/G_k = BG_{k,(n)}; \quad (g_1, \ldots, g_n) \mapsto (g_1 g_2 \cdots g_n, g_2 \cdots g_n, \ldots, g_n, 1),$$

the face maps σ_i^* and degeneracy maps δ_i^* are described respectively as

$$\sigma_i^* : (g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, \ldots, g_n) & (i = 0), \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & (1 \le i \le n-1), \\ (g_1, \ldots, g_{n-1}) & (i = n), \end{cases}$$

$$\delta_i^* : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n) \quad (0 \le i \le n).$$

BG_W and $B\hat{G}_W$ are defined in a similar way. The datum of a locally free crystal E of rank r, together with a trivialization of $E_X|_{X_i} = E_{X_i}$, where $X = \bigcup_{i \in I} X_i$ is a finite affine covering, and a lift of the trivialization to a trivialization of $E_D|_{D_i}$, yields, via the transition functions, a commutative diagram

$$\begin{array}{ccccc} X_{(\bullet)} & \xrightarrow{\alpha} & D_{(\bullet)} & & \\ \downarrow f & & \downarrow g & \overset{h}{\searrow} & \\ BG_k & \xrightarrow{\hat{\beta}} & B\hat{G}_W & \xrightarrow{\iota} & BG_W \end{array} \tag{3.23}$$

where α, $\hat{\beta}$, and ι are the canonical morphisms.

Let

$$C_i \in \mathrm{Fil}^i H^{2i}(BG_W, \Omega^\bullet_{BG_W}) := \mathrm{Im}(H^{2i}(BG_W, \Omega^{\geq i}_{BG_W}) \to H^{2i}(BG_W, \Omega^\bullet_{BG_W}))$$

be the universal de Rham Chern class and let $C_i^{\mathrm{crys}} \in H^{2i}((BG_k/W)_{\mathrm{crys}}, \mathcal{O}_{BG_k/W})$ be its image by the map

$$H^{2i}(BG_W, \Omega^\bullet_{BG_W}) \xrightarrow{\iota^*} H^{2i}(B\hat{G}_W, \Omega^\bullet_{B\hat{G}_W}) \cong H^{2i}((BG_k/W)_{\mathrm{crys}}, \mathcal{O}_{BG_k/W}).$$

Then we can define the crystalline Chern classes of E also as the pull-back of C_i^{crys} by

$$f^* : H^{2i}((BG_k/W)_{\mathrm{crys}}, \mathcal{O}_{BG_k/W}) \to H^{2i}((X_{(\bullet)}/W)_{\mathrm{crys}}, \mathcal{O}_{X_{(\bullet)}/W}). \tag{3.24}$$

Indeed, the crystalline Chern class is characterized by functoriality, normalization, and the Whitney sum formula (additivity) [8, Théorème 2.4], and the definition as the pull-back of C_i^{crys} also satisfies these properties by [3, Thm. 4.2]. So, by Proposition 3.3, the crystalline Chern class of E is computed as the image of C_i by

$$\begin{aligned} h^* : H^{2i}(BG_W, \Omega^\bullet_{BG_W}) &\xrightarrow{\iota^*} H^{2i}(B\hat{G}_W, \Omega^\bullet_{B\hat{G}_W}) \\ &\xrightarrow{g^*} H^{2i}(D_{(\bullet)}, \bar{\Omega}^\bullet_{D_{(\bullet)}/W}) \xrightarrow{\cong} H^{2i}(D_{(\bullet)}, \mathcal{O}^\bullet_{D_{(\bullet)}/W}). \end{aligned} \tag{3.25}$$

Because the map h^* factors through the forgetful map

$$H^{2i}(BG_W, \Omega^\bullet_{BG_W}) \to H^{2i}(BG_W, \mathcal{O}_{BG_W})$$

and C_i belongs to $\mathrm{Fil}^i H^{2i}(BG_W, \Omega^\bullet_{BG_W})$, it is enough to prove that the composition

$$H^{2i}(BG_W, \Omega^{\geq i}_{BG_W}) \to H^{2i}(BG_W, \Omega^\bullet_{BG_W}) \to H^{2i}(BG_W, \mathcal{O}_{BG_W})$$

is zero for $i > 0$, which is obvious. So the proof of Theorem 1.1 is finished. □

Remark 3.4. By replacing W with $W_n = W/p^n W$, we see that the proof above also gives the following variant of Theorem 1.1: If X is as in Theorem 1.1 and E is a locally free crystal on X/W_n, then $c_i^{\mathrm{crys}}(E_X)$ is zero in the torsion crystalline cohomology group $H^{2i}_{\mathrm{crys}}(X/W_n)$ for $i \geq 1$. In the final step, we need the surjectivity of the map $H^{2i}(BG_{W_n}, \Omega^{\geq i}_{BG_{W_n}}) \to H^{2i}(BG_{W_n}, \Omega^\bullet_{BG_{W_n}})$, but it follows from the isomorphism

$$H^{2i}(BG_W, \Omega^\bullet_{BG_W}) \otimes_W W_n = H^{2i}(BG_{W_n}, \Omega^\bullet_{BG_{W_n}}),$$

which is true because $H^n(BG_W, \Omega^\bullet_{BG_W})$ $(n \in \mathbb{N})$ are free over W ([22, II, Thm. 1.1]).

4. Generalization of the main theorem [19, Thm. 1.1] from convergent isocrystals to isocrystals possessing a locally free lattice

The aim of this section is to prove Theorem 1.2. The proof is the same as the one of [19, Thm. 1.1] except for some points which we explain now.

First assume that k is algebraically closed. We have $H^1_{\mathrm{crys}}(X/W) = 0$ by [19, Prop. 2.9(2)]. (See also Theorem 5.1.) So we may assume that $\mathcal{E}$ is irreducible (of rank $s \leq r$) to prove the theorem. Also, there exists $N \in \mathbb{N}$ such that the restriction $H^1_{\mathrm{crys}}(X/W_n) \to H^1_{\mathrm{crys}}(X/k)$ is zero for any $n \geq N$. By using Theorem 1.1 (and the remark after it) in place of [19, Prop. 3.1], we see that $\mathcal{E}$ admits a lattice E with E_X strongly μ-stable as an $\mathcal{O}_X$-module by [19, Prop. 4.2]. Because E_X has vanishing Chern classes, we can argue as in Section 3 in [19], and by [19, Cor. 3.8] we see that there exists $a \in \mathbb{N}$ such that $(F^a)^* E_N \in \mathrm{Crys}(X/W_N)$ is trivial.

Let $\mathcal{D}_{n,m}$ be the category of pairs (G, φ), where $G \in \mathrm{Crys}(X/W_{n+m})$ and φ is an isomorphism between the restriction of G to $\mathrm{Crys}(X/W_n)$ and $\mathcal{O}^s_{X/W_n}$. Then the same computation as [19, Prop. 3.6] implies the isomorphism as pointed sets

$$\mathcal{D}_{n,m} \cong M(s \times s, H^1_{\mathrm{crys}}(X/W_m)) \tag{4.1}$$

for $1 \le m \le n$, which is compatible with respect to m.

Applying (4.1) to the pairs $(n, m) = (N, N), (N, 1)$, we conclude that $(F^a)^* E_{N+1} \in \mathrm{Crys}(X/W_{N+1})$, which is the image of $(F^a)^* E_{2N} \in \mathrm{Crys}(X/W_{2N})$ via the restriction

$$\mathcal{D}_{N,N} \cong M(s \times s, H^1_{\mathrm{crys}}(X/W_N)) \to M(s \times s, H^1_{\mathrm{crys}}(X/W_1)) \cong \mathcal{D}_{N,1}$$

is constant. We continue similarly to show that $(F^a)^* E_n \in \mathrm{Crys}(X/W_n)$ is constant for all $n \ge N$. Hence $(F^a)^* E$ is constant. Because the endofunctor $F^* : \mathrm{Crys}(X/W) \to \mathrm{Crys}(X/W)$ is fully faithful at least when restricted to locally free crystals ([33, Ex. 7.3.4]), we see that E itself is constant. Hence $\mathcal{E}$ is also constant.

Now we prove the theorem for general k. If we take a locally free lattice E of $\mathcal{E}$ and denote the pull-back of E to $(X \otimes_k \overline{k}/W(\overline{k}))_{\mathrm{crys}}$ by $\overline{E}$, we see by the base change theorem that

$$\mathbb{Q} \otimes_{\mathbb{Z}} (W(\overline{k}) \otimes_W H^0((X/W)_{\mathrm{crys}}, E)) \cong \mathbb{Q} \otimes_{\mathbb{Z}} H^0((X \otimes_k \overline{k}/W(\overline{k}))_{\mathrm{crys}}, E),$$

and the latter is s-dimensional over the fraction field of $W(\overline{k})$. So

$$\mathbb{Q} \otimes_{\mathbb{Z}} H^0((X/W)_{\mathrm{crys}}, E)$$

is s-dimensional over K, and hence $\mathcal{E}$ is constant.

5. Gauss–Manin convergent isocrystal

The aim of this section is to prove Theorem 1.3. The proof is inspired by the discussion with G. Faltings related in the introduction.

5.1. First rigid cohomology of a smooth simply connected variety is trivial.

We shall prove the following theorem, which is used in order to pass from the constancy of the semi-simplification of the Gauß–Manin overconvergent isocrystal to the constancy of the Gauß–Manin overconvergent isocrystal itself.

As usual, if k is a perfect field, then one denotes by $W = W(k)$ its ring of Witt vectors and by K the field of fractions of W.

Theorem 5.1. *Let X be a smooth connected variety defined over a perfect field k of characteristic $p > 0$, and assume that X is proper or $p \ge 3$. If $\pi_1^{\text{ét,ab}}(X \otimes_k \overline{k}) = 0$, then $H^1_{\mathrm{rig}}(X/K) = 0$.*

Proof. We can reduce to the case where k is algebraically closed, because $H^1_{\mathrm{rig}}(X/K)$ is compatible with base extension of K [6, Remarque, p. 498]. Let ℓ be a prime not equal to p. When X is proper, the assumption $\pi_1^{\text{ét,ab}}(X) = 0$ implies that $0 = H^1_{\text{ét}}(X, \mathbb{Q}_\ell) = H^1_{\text{ét}}(\mathrm{Pic}^0_{\mathrm{red}}(X), \mathbb{Q}_\ell)$, and so $\mathrm{Pic}^0_{\mathrm{red}}(X) = 0$. Hence $H^1_{\mathrm{crys}}(X/W) = 0$ because it is the Dieudonné crystal associated to $\mathrm{Pic}^0_{\mathrm{red}}(X)$, and so $H^1_{\mathrm{rig}}(X/K)$ is also equal to 0.

In case X is not proper, we use the Picard 1-motive $M := \mathrm{Pic}^+(X)$ defined in [2]. Put $M = [L \to G]$, where L is a $\mathbb{Z}$-module and G is a semi-abelian variety.

Because the ℓ-adic realization $V_\ell M$ of M is equal to $H^1_{\text{ét}}(X,\mathbb{Q}_\ell)$ by [30] and fits into the exact sequence

$$0 \longrightarrow V_\ell G \longrightarrow V_\ell M \longrightarrow V_\ell L \longrightarrow 0,$$

we have $G = 0, L = 0$, hence $M = 0$. So the crystalline realization of M, which is equal to $H^1_{\text{rig}}(X/K)$ by [2] when $p \geq 3$, vanishes. □

Remark 5.2. We need the assumption $p \geq 3$ in the theorem (when X is not proper) because it is imposed in [2] (see [2, 7.4] for details). We expect that the theorem should be true without this assumption, but we don't know how to prove it.

5.2. **Gauß–Manin convergent isocrystal.** In this subsection, we give some preliminaries on convergent F-isocrystals and then recall Ogus's definition of Gauß–Manin convergent F-isocrystal $R^if_*\mathcal{O}_{Y/K}$ for a smooth proper morphism $f : Y \to X$ with X smooth over a perfect field k (see [32, Sec. 3]).

Recall that objects in the convergent site on X over K are enlargements, which are the diagrams of the form $(X \leftarrow (T \otimes_W k)_{\text{red}} \hookrightarrow T)$ over W, where T is a p-adic formal scheme of finite type and flat over W. One defines a convergent isocrystal on X/K as a crystal of $\mathbb{Q}\otimes_{\mathbb{Z}}\mathcal{O}_T$-modules on enlargements. Crystal means a sheaf of coherent $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_T$-modules with transition maps being isomorphisms. We denote the category of convergent isocrystals on X/K by $\text{Conv}(X/K)$. The convergent site is functorial for X/W, and so we can define the pull-back functor $F^* : \text{Conv}(X/K) \to \text{Conv}(X/K)$ induced by the Frobenius (F_X, F_W) on X and W. Then we define the category F-$\text{Conv}(X/K)$ of convergent F-isocrystals on X/K as the category of pairs (E, Φ), where $E \in \text{Conv}(X/K)$ and Φ is an isomorphism $F^*E \to E$. Φ is called a Frobenius structure on E.

We define the p-adic convergent site on X over K as a variant of the convergent site: The objects in it are the p-adic enlargements, which are the diagrams of the form $(X \leftarrow T \otimes_W k \hookrightarrow T)$ over W, where T is as before. As in the case of the convergent site, we can define the category of p-adic convergent isocrystals and that of p-adic convergent F-isocrystals, which we denote by $p\text{Conv}(X/K)$, F-$p\text{Conv}(X/K)$ respectively.

Then we have the sequence of functors

$$F\text{-Conv}(X/K) \to F\text{-Crys}(X/W)_{\mathbb{Q}} \to F\text{-}p\text{Conv}(X/K), \tag{5.1}$$

in which the first one is the inverse of the functor $M \mapsto M^{\text{an}}$ in [5, Thm. 2.4.2]. For any p-adic enlargement $T := (X \overset{h}{\leftarrow} T \otimes_W k \hookrightarrow T)$, $T_n := (T \otimes_W k \hookrightarrow T \otimes W/p^nW)$ $(n \in \mathbb{N})$ are objects in the crystalline site $(T \otimes k/W)_{\text{crys}}$. Then, for any $E \in \text{Crys}(X/W)$, $T \mapsto \varprojlim_n (h^*E)_{T_n}$ defines an object in $p\text{Conv}(X/K)$, and this induces the second functor in (5.1). The functors in (5.1) are known to be equivalences by [5, Thm. 2.4.2] and [32, Prop. 2.18].

We recall the definition of the Gauß–Manin convergent F-isocrystal

$$R^if_*\mathcal{O}_{Y/K} \in F\text{-Conv}(X/K)$$

for a smooth proper morphism $f : Y \to X$ (see [32, Sec. 3]). It is defined as the unique object such that, for any p-adic enlargement $T := (X \leftarrow T \otimes_W k \hookrightarrow T)$, the value $(R^if_*\mathcal{O}_{Y/K})_T$ at T of $R^if_*\mathcal{O}_{Y/K}$ as an object in F-$p\text{Conv}(X/K)$ is given by $\mathbb{Q} \otimes R^n(f_T)_{\text{crys}*}\mathcal{O}_{Y_T/T}$, where $(f_T)_{\text{crys}} : (Y_T/T)_{\text{crys}} \to T_{\text{Zar}}$ is the morphism of topoï induced by the pull-back $f_T : Y_T \to T \otimes k$ of f by $T \otimes k \to X$.

Assume now that X admits a closed embedding $\iota : X \hookrightarrow P$ into a p-adic smooth formal scheme P over $\operatorname{Spf} W$ and give another description of convergent F-isocrystals and the Gauß–Manin convergent isocrystals. For $n = 0, 1, 2$, let $X \leftarrow X(n) \hookrightarrow \mathcal{Z}(n)$ be the universal p-adic enlargement [32, Prop. 2.3] of $X \hookrightarrow \underbrace{P \times_W \cdots \times_W P}_{n+1}$, and let $F\text{-Str}(X \hookrightarrow P/W)$ be the category of triples (E, ϵ, Φ) consisting of a coherent $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{Z}(0)}$-module E, an isomorphism $\epsilon : p_2^* E \xrightarrow{\cong} p_1^* E$ ($p_i : \mathcal{Z}(1) \to \mathcal{Z}(0)$ are the projections) satisfying the cocycle condition, and an isomorphism $\Phi : F^*(E, \epsilon) \xrightarrow{\cong} (E, \epsilon)$ (where F is the pull-back by Frobenius on $X(n) \hookrightarrow \mathcal{Z}(n)$). Then we have an equivalence of categories [32, (2.11)]

$$F\text{-}p\mathrm{Conv}(X/K) \cong F\text{-Str}(X \hookrightarrow P/W). \tag{5.2}$$

Via the equivalences (5.1) and (5.2), the Gauß–Manin convergent isocrystal $R^i f_* \mathcal{O}_{Y/K}$ is described as the triple $(\mathbb{Q} \otimes R^n f(0)_{\mathrm{crys}*} \mathcal{O}_{Y(0)/\mathcal{Z}(0)}, \epsilon, \Phi) \in F\text{-Str}(X \hookrightarrow P/W)$, where $f(0)_{\mathrm{crys}} : (Y(0)/\mathcal{Z}(0))_{\mathrm{crys}} \to \mathcal{Z}(0)_{\mathrm{Zar}}$ is the morphism of topoï induced by the pull-back $f(0) : Y(0) \to X(0)$ of f by $X(0) \to X$ and ϵ, Φ are defined by the functoriality of crystalline cohomology sheaves.

Let us denote the category of overconvergent F-isocrystals on X/K [5, (2.3.7)] by $F\text{-Conv}^\dagger(X/K)$, and the structure of overconvergent isocrystals by $\mathcal{O}^\dagger_{X/K}$ to distinguish it from $\mathcal{O}_{X/K}$. There exists a natural restriction functor $F\text{-Conv}^\dagger(X/K) \to F\text{-Conv}(X/K)$ which is fully faithful ([26]). When $f : Y \to X$ is smooth projective, Lazda [29, Cor. 5.4][2] proved that the convergent F-isocrystal $R^i f_* \mathcal{O}_{Y/K}$ lifts to an overconvergent F-isocrystal on X, which we denote by $R^i f_* \mathcal{O}^\dagger_{Y/K}$. When X is proper, the categories $F\text{-Conv}(X/K)$ and $F\text{-Conv}^\dagger(X/K)$ are the same, and so one can equate $R^i f_* \mathcal{O}^\dagger_{Y/K}$ and $R^i f_* \mathcal{O}_{Y/K}$.

Remark 5.3. In this subsection, the Frobenius structure on a(n) (over)convergent isocrystal is defined with respect to the pull-back functor induced by the Frobenius (F_X, F_W) on X and W. For any $d \geq 1$, we can also define the Frobenius structure with respect to the pull-back functor induced by (F_X^d, F_W^d), and the Frobenius structure in the former case induces the one in the latter case. Such a Frobenius structure will appear in the next subsection.

5.3. **Case where the ground field k is finite.** In this subsection, we prove Theorem 1.3 in the case where k is a finite field and the statement in Remark 1.4. In order to prove them, we may replace k by a finite extension k' and K by a finite possibly ramified extension of the field of fractions of the ring of Witt vectors over k', such that the following hold:

(1) X has a k-rational point x.
(2) The eigenvalues of the action of Frobenius F_x on $L_\ell := H^i_{\text{ét}}(Y \times_X \bar{x}, \mathbb{Q}_\ell)$ belong to a number field K_0 contained in K.
(3) There exists a K_0-vector space L with linear action F and an inclusion of fields $K_0 \hookrightarrow \mathbb{Q}_\ell$ such that $L \otimes_{K_0} \mathbb{Q}_\ell$ is isomorphic to L_ℓ as vector spaces with an action.

(Note that overconvergent isocrystals and rigid cohomologies are defined even when the base complete discrete valuation ring O_K is ramified over $W(k)$ ([5], [37]) and

[2] Added in proof: See the footnote for Remark 1.4.

the rigid cohomology satisfies the base change property for finite extension of the base ([12, Cor. 11.8.2]).)

Until the end of this subsection, we consider the Frobenius structure with respect to the pull-back induced by (F_X^d, id), where $d = \log_p |k|$. (See Remark 5.3.) For any closed point y in X, one has the base change isomorphism

$$y^* R^i f_* \mathcal{O}^\dagger_{Y/K} = y^* R^i f_* \mathcal{O}_{Y/K} = H^i_{\mathrm{rig}}(Y \times_X y/K) \tag{5.3}$$

by [32, Rmk. 3.7.1] and [7, Props. 1.8 and 1.9], and the set of Frobenius eigenvalues on it is the same as that on ℓ-adic cohomology $H^i(Y \times_X \bar{y}, \mathbb{Q}_\ell)$ by [25, Thm. 1], [11, Cor. 1.3]. The assumption $\pi_1^{\text{ét}}(X \otimes \overline{k}) = \{1\}$ implies that the action of Frobenius F_y on $H^i(Y \times_X \bar{y}, \mathbb{Q}_\ell)$ is identified with the action of $F_x^{d_y}$ (where $d_y = \deg(y/k)$) on L_ℓ, hence the action of $F \otimes \mathrm{id}$ on $L \otimes_{K_0} \mathbb{Q}_\ell$.

Let $\mathcal{E}_0$ be the overconvergent F-isocrystal on X defined by $((L \otimes_{K_0} K) \otimes_K \mathcal{O}^\dagger_{X/K}, F \otimes \mathrm{id})$. Then, by construction, $R^i f_* \mathcal{O}^\dagger_{Y/K}$ and $\mathcal{E}_0$ have the same eigenvalues of Frobenius action on any closed point of X. Then Abe's Čebotarev's density theorem [1, A.3] implies that the semi-simplification of $R^i f_* \mathcal{O}^\dagger_{Y/K}$ is the same as that of $\mathcal{E}_0$. Hence the semi-simplification of $R^i f_* \mathcal{O}^\dagger_{Y/K}$ is constant as an overconvergent isocrystal on X. By Theorem 5.1, extensions of $\mathbb{Q} \otimes \mathcal{O}^\dagger_{X/W}$ by itself are constant when X is proper or $p \geq 3$. This finishes the proof.

5.4. **General case.** In this subsection, we prove Theorem 1.3 by a spreading out argument allowing k to be a perfect field, but assuming X to be proper. Let $f : Y \to X$ be as in the statement of Theorem 1.3 and let $g : X \to \operatorname{Spec} k$ be the structure morphism. Also, let $X = \bigcup_{i \in I} X_i$ be an affine open covering of X and take a closed embedding $X_i \to P_i$ into a smooth p-adic formal scheme P_i over W for each $i \in I$. We prove that the Gauß–Manin convergent isocrystal $\mathcal{E}_{\mathrm{conv}} := R^i f_* \mathcal{O}_{Y/K} \in F\text{-}\mathrm{Conv}(X/K)$ is constant as an object in $\mathrm{Conv}(X/K)$. Denote by $\mathcal{E}_{\mathrm{crys}}, \mathcal{E}_{p\mathrm{conv}}$ the image of $\mathcal{E}_{\mathrm{conv}}$ in $F\text{-}\mathrm{Crys}(X/W)_{\mathbb{Q}}, F\text{-}p\mathrm{Conv}(X/K)$ via (5.1).

We can find a connected affine scheme $T = \operatorname{Spec} A_1$ smooth of finite type over $\mathbb{F}_p$, a p-adic formal lift $\mathcal{T} := \operatorname{Spf} A$ of T which is smooth over $\operatorname{Spf} \mathbb{Z}_p$ and endowed with a lift of Frobenius, and proper smooth morphisms $Y_T \overset{f_T}{\to} X_T \overset{g_T}{\to} T$ which fit into the commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & Y_T & = & Y_T \\
{\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f_T} \\
X & \xrightarrow[\alpha]{} & X_T & = & X_T \\
{\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle g_T} & & \downarrow \\
\operatorname{Spec} k & \longrightarrow & T & \longrightarrow & \operatorname{Spec} \mathbb{F}_p \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\operatorname{Spf} W & \longrightarrow & \mathcal{T} & \longrightarrow & \operatorname{Spf} \mathbb{Z}_p,
\end{array} \tag{5.4}$$

where the squares with symbol $\square$ are cartesian squares. Also, we may assume the existence of an open covering $X_T = \bigcup_{i \in I} X_{i,T}$ which induces $X = \bigcup_{i \in I} X_i$ and closed immersions $X_{i,T} \hookrightarrow P_{i,T}$ $(i \in I)$ into a p-adic formal scheme $P_{i,T}$ smooth over $\mathcal{T}$ with $P_{i,T} \times_{\mathcal{T}} \operatorname{Spf} W = P_i$.

Let $\widetilde{\mathcal{E}}_{T,\mathrm{conv}} := R^n f_{T,\mathrm{conv},*}\mathcal{O}_{Y_T/\mathbb{Q}_p} \in F\text{-Conv}(X_T/\mathbb{Q}_p)$ be the Gauß–Manin convergent F-isocrystal defined by the right column of (5.4), and let $\widetilde{\mathcal{E}}_{T,\mathrm{crys}}$, $\widetilde{\mathcal{E}}_{T,p\mathrm{conv}}$ be its images in the categories $F\text{-Crys}(X_T/\mathbb{Z}_p)_{\mathbb{Q}}$, $F\text{-}p\mathrm{Conv}(X_T/\mathbb{Q}_p)$. Then the morphism α in (5.4) induces the pull-back functor

$$F\text{-Crys}(X_T/\mathbb{Z}_p)_{\mathbb{Q}} \to F\text{-Crys}(X_T/\mathcal{T})_{\mathbb{Q}} \xrightarrow{\alpha^*} F\text{-Crys}(X/\mathbb{Z}_p)_{\mathbb{Q}}. \tag{5.5}$$

We denote this functor by $\widetilde{\alpha}^*$. Then we have the map

$$\widetilde{\alpha}^*\widetilde{\mathcal{E}}_{T,\mathrm{crys}} \to \mathcal{E}_{\mathrm{crys}} \tag{5.6}$$

defined by functoriality.

To prove that the map (5.6) is an isomorphism, we may work locally on each X_i. So we may assume that X_T admits a closed embedding $X_T \to P_T$ into a p-adic formal scheme P_T smooth over $\mathcal{T}$. Put $P := P_T \times_{\mathcal{T}} W$. Then the functor (5.5) is identified with the composite

$$\begin{aligned} F\text{-}p\mathrm{Conv}(X_T/\mathbb{Q}_p) &\cong F\text{-Str}(X_T \hookrightarrow P_T/\mathbb{Z}_p) \\ &\xrightarrow{\alpha^*} F\text{-Str}(X \hookrightarrow P/W) \cong F\text{-}p\mathrm{Conv}(X/\mathbb{Q}_p) \end{aligned} \tag{5.7}$$

via the second equivalence in (5.1). It suffices to prove that the map

$$\widetilde{\alpha}^*\widetilde{\mathcal{E}}_{T,p\mathrm{conv}} \to \mathcal{E}_{p\mathrm{conv}} \tag{5.8}$$

induced by (5.6) via the second equivalence in (5.1) is an isomorphism, and to see this, it suffices to prove that the base change map

$$\alpha^*_{\mathcal{Z}(n)}(\mathbb{Q} \otimes R^i g(n)_{T,\mathrm{crys},*}\mathcal{O}_{Y_T(n)/\mathcal{Z}_T(n)}) \to \mathbb{Q} \otimes R^i g(n)_{\mathrm{crys},*}\mathcal{O}_{Y(n)/\mathcal{Z}(n)}$$

induced by diagrams

$$\begin{array}{ccc} Y(n) & \longrightarrow & Y_T(n) \\ {\scriptstyle g(n)}\downarrow & \square & \downarrow{\scriptstyle g(n)_T} \\ X(n) & \longrightarrow & X_T(n) \\ \downarrow & & \downarrow \\ \mathcal{Z}(n) & \xrightarrow[\alpha_{\mathcal{Z}(n)}]{} & \mathcal{Z}_T(n) \end{array} \tag{5.9}$$

is an isomorphism, where $X_T \leftarrow X_T(n) \hookrightarrow \mathcal{Z}_T(n)$ is the universal p-adic enlargement of $X_T \hookrightarrow \underbrace{P_T \times_{\mathbb{Z}_p} \cdots \times_{\mathbb{Z}_p} P_T}_{n+1}$ and $X \leftarrow X(n) \hookrightarrow \mathcal{Z}(n)$ is the universal p-adic enlargement of $X \hookrightarrow \underbrace{P \times_W \cdots \times_W P}_{n+1}$. This follows from the base change theorem of crystalline cohomology

$$L\alpha^*_{\mathcal{Z}(n)} Rg(n)_{T,\mathrm{crys},*}\mathcal{O}_{Y_T(n)/\mathcal{Z}_T(n)} \xrightarrow{\cong} Rg(n)_{\mathrm{crys},*}\mathcal{O}_{Y(n)/\mathcal{Z}(n)}$$

([9, Thm. 7.8], [32, Cor. 3.2]) and the flatness of $\mathbb{Q} \otimes R^j g(n)_{T,\mathrm{crys},*}\mathcal{O}_{Y_T(n)/\mathcal{Z}_T(n)}$ for $j \geq 0$ [32, Cor. 2.9]. Hence the map (5.6) is an isomorphism.

Let $\mathcal{E}_{T,\mathrm{crys}}$ be the image of $\widetilde{\mathcal{E}}_{T,\mathrm{crys}}$ by $F\text{-Crys}(X_T/\mathbb{Z}_p)_{\mathbb{Q}} \to F\text{-Crys}(X_T/\mathcal{T})_{\mathbb{Q}}$. Then the map (5.6) can be rewritten as

$$\alpha^* \mathcal{E}_{T,\mathrm{crys}} \to \mathcal{E}_{\mathrm{crys}}. \tag{5.10}$$

Hence the map (5.10) is also an isomorphism.

Next, let $s = \operatorname{Spec} \mathbb{F}_q$ be a closed point of T and put $\mathcal{S} = \operatorname{Spf} W(\mathbb{F}_q) =: \operatorname{Spf} \mathbb{Z}_q, \mathbb{Q}_q := \mathbb{Q} \otimes \mathbb{Z}_q$. Then, by taking the fiber at s, we obtain the following diagram:

$$\begin{array}{ccccc}
Y & \longrightarrow & Y_T & \longleftarrow & Y_s \\
{\scriptstyle f}\downarrow & \square & {\scriptstyle f_T}\downarrow & \square & {\scriptstyle f_s}\downarrow \\
X & \xrightarrow{\ \alpha\ } & X_T & \xleftarrow{\ \beta\ } & X_s \\
{\scriptstyle g}\downarrow & \square & {\scriptstyle g_T}\downarrow & \square & {\scriptstyle g_s}\downarrow \\
\operatorname{Spec} k & \longrightarrow & T & \longleftarrow & s \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\operatorname{Spf} W & \longrightarrow & \mathcal{T} & \longleftarrow & \mathcal{S}.
\end{array} \tag{5.11}$$

Let $\mathcal{E}_{s,\mathrm{conv}} := R^i f_{s,\mathrm{conv},*} \mathcal{O}_{Y_s/\mathbb{Q}_q} \in F\text{-Conv}(X_s/\mathbb{Q}_q)$ be the Gauß–Manin convergent F-isocrystal defined by the right column of (5.11), and denote its image in $F\text{-Crys}(X_s/\mathbb{Z}_q)_{\mathbb{Q}}$ by $\mathcal{E}_{s,\mathrm{crys}}$. By the same method as above, one can prove the isomorphism

$$\beta^* \mathcal{E}_{T,\mathrm{crys}} \to \mathcal{E}_{s,\mathrm{crys}}, \tag{5.12}$$

where β^* is the pull-back $F\text{-Crys}(X_T/\mathcal{T})_{\mathbb{Q}} \to F\text{-Crys}(X_s/\mathbb{Z}_q)_{\mathbb{Q}}$ by β.

We consider the crystalline cohomology $H^j((X_T/A)_{\mathrm{crys}}, \mathcal{E}_{T,\mathrm{crys}})$ $(j \geq 0)$. This is a finitely generated $\mathbb{Q} \otimes A$-module as we will prove below. Since A_{pA} is a discrete valuation ring, we may assume that $H^j((X_T/A)_{\mathrm{crys}}, \mathcal{E}_{T,\mathrm{crys}})$'s are free $\mathbb{Q} \otimes A$-modules, by shrinking $\mathcal{T} = \operatorname{Spf} A$. Then we have the base change isomorphisms

$$\mathbb{Q}_q \otimes_{\mathbb{Q} \otimes A} H^j((X_T/A)_{\mathrm{crys}}, \mathcal{E}_{T,\mathrm{crys}}) \cong H^j((X_s/\mathbb{Z}_q)_{\mathrm{crys}}, \mathcal{E}_{s,\mathrm{crys}}), \tag{5.13}$$

$$K \otimes_{\mathbb{Q} \otimes A} H^j((X_T/A)_{\mathrm{crys}}, \mathcal{E}_{T,\mathrm{crys}}) \cong H^j((X/W)_{\mathrm{crys}}, \mathcal{E}_{\mathrm{crys}}), \tag{5.14}$$

which we will prove below.

Let r be the rank of $\mathcal{E}_{T,\mathrm{crys}}$. (As we will explain below, $\mathcal{E}_{T,\mathrm{crys}}$ is locally free as an isocrystal and so its rank is well-defined. Note that r is equal to the rank of $\mathcal{E}_{s,\mathrm{crys}}, \mathcal{E}_{\mathrm{crys}}$.) By the constancy of the Gauß–Manin convergent isocrystal in the finite field case, $\mathcal{E}_{s,\mathrm{crys}}$ is constant as a convergent isocrystal. (Note that $\pi_1^{\text{ét}}(X_{\overline{s}})$ is constant by [23, X, Théorème 3.8].) Hence, by (5.13), $H^0((X_T/A)_{\mathrm{crys}}, \mathcal{E}_{T,\mathrm{crys}}) \cong (\mathbb{Q} \otimes A)^r$, and by (5.14), $H^0((X/W)_{\mathrm{crys}}, \mathcal{E}_{\mathrm{crys}}) \cong K^r$. Hence $\mathcal{E}_{\mathrm{crys}}$ is constant (and so $\mathcal{E}_{\mathrm{conv}}$ is also constant), as required. So the proof of the theorem is finished modulo the finiteness and the base change property we used above.

Finally, we prove the finiteness and the base change property. In the following, for an affine p-adic formal scheme A flat over $\operatorname{Spf} \mathbb{Z}_p$ and a smooth scheme X over $\operatorname{Spec} A_1$ $(A_1 := A/pA)$, we say that an isocrystal $\mathcal{E} = \mathbb{Q} \otimes E$ on $(X/A)_{\mathrm{crys}}$ is locally free if, for any affine open subscheme $U = \operatorname{Spec} B_1$ of X and any lift $\mathcal{U} = \operatorname{Spf} B$ of it to an affine p-adic formal scheme which is smooth over $\operatorname{Spf} A$, the value $\mathcal{E}_{\mathcal{U}} := \mathbb{Q} \otimes E_{\mathcal{U}} = \mathbb{Q} \otimes (\varprojlim_n E_{\mathcal{U} \otimes \mathbb{Z}/p^n\mathbb{Z}})$ of $\mathcal{E}$ at $\mathcal{U}$ is a finitely generated projective $\mathbb{Q} \otimes B$-module.

For example, when $A = \mathbb{Z}_p$, any isocrystal on $(X/A)_{\mathrm{crys}}$ is locally free because of the existence of integrable connection associated to the structure of the isocrystal. In particular, the isocrystal $\widetilde{\mathcal{E}}_{T,\mathrm{crys}}$ above is locally free, and this implies that the isocrystal $\mathcal{E}_{T,\mathrm{crys}}$ above is also locally free.

Proposition 5.4. *Let* $\mathrm{Spec}\, A_1$ *be an affine regular scheme of characteristic* $p > 0$, *and let* $\mathrm{Spf}\, A$ *be a* p*-adic formal scheme flat over* $\mathrm{Spf}\, \mathbb{Z}_p$ *such that* $A/pA = A_1$. *Let* $X \to \mathrm{Spec}\, A_1$ *be a smooth proper morphism, and let* $\mathcal{E}$ *be an isocrystal on* $(X/A)_{\mathrm{crys}}$ *which is locally free.*

Moreover, assume that we are given the following cartesian diagram such that $\mathrm{Spec}\, A_1'$ *is also regular*:

$$\begin{array}{ccc} X & \xleftarrow{\ \alpha\ } & X' \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}\, A_1 & \longleftarrow & \mathrm{Spec}\, A_1' \\ \downarrow & \square & \downarrow \\ \mathrm{Spf}\, A & \longleftarrow & \mathrm{Spf}\, A'. \end{array}$$

Then we have the following:

(1) $R\Gamma((X/A)_{\mathrm{crys}}, \mathcal{E})$ *is a perfect complex of* $(\mathbb{Q} \otimes A)$*-modules.*

(2) $A' \otimes^L_A R\Gamma((X/A)_{\mathrm{crys}}, \mathcal{E}) \to R\Gamma((X'/A')_{\mathrm{crys}}, \alpha^*\mathcal{E})$ *is a quasi-isomorphism.*

Because $\mathcal{E}_{T,\mathrm{crys}}$ is locally free as an isocrystal, we can apply Proposition 5.4 to it and we obtain the isomorphisms (5.13) and (5.14) as required.

Remark 5.5. Although $\mathcal{E}$ is locally free as an isocrystal, it is not clear at all that there exists a lattice E of $\mathcal{E}$ with E locally free as a crystal. This is the reason that the base change theorem in [9] is not enough for us.

Proof. Let $X_{(\bullet)} \to X$ be a simplicial scheme augmented to X defined as the Čech hypercovering associated to an open covering $X = \bigcup_{i \in I} X_i$ which admits a closed embedding $X_{(\bullet)} \to Y_{(\bullet)}$ into a simplicial p-adic formal scheme $Y_{(\bullet)}$ smooth over A, and let $D_{(\bullet)}$ be the PD-envelope of it. For $n \in \mathbb{N}$, put $A_n := A/p^nA, D_{(\bullet)n} := D_{(\bullet)} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$. Also, let E be a p-torsion free crystal on X/A with $\mathcal{E} = \mathbb{Q} \otimes E$, let E_n be the restriction of E to $(X/A_n)_{\mathrm{crys}}$, and let $E_{(\bullet)n}$ be the value of E at $D_{(\bullet)n}$. Then $R\Gamma((X/A_n)_{\mathrm{crys}}, E_n) = R\Gamma(D_{(\bullet)n}, E_{(\bullet)n} \otimes \bar{\Omega}^{\bullet}_{D_{(\bullet)n}})$, and this is quasi-isomorphic to a bounded complex of A_n-modules flat over $\mathbb{Z}/p^n\mathbb{Z}$ which is compatible with respect to n. Hence

$$\begin{aligned} R\Gamma((X/A_n)_{\mathrm{crys}}, E_n) \otimes^L_{A_n} A_{n-1} &= R\Gamma((X/A_n)_{\mathrm{crys}}, E_n) \otimes^L_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^{n-1}\mathbb{Z} \\ &= R\Gamma((X/A_{n-1})_{\mathrm{crys}}, E_{n-1}). \end{aligned}$$

Also, $R\Gamma((X/A_1)_{\mathrm{crys}}, E_1) = R\Gamma(X, E_X \otimes \Omega^{\bullet}_X)$ is a perfect complex of A_1-modules, because A_1 is regular. Hence, by [9, B.10],

$$R\Gamma((X/A)_{\mathrm{crys}}, E) = R \varprojlim_n R\Gamma((X/A_n)_{\mathrm{crys}}, E_n)$$

is a perfect complex of A-modules, and so $R\Gamma((X/A)_{\mathrm{crys}}, \mathcal{E}) := \mathbb{Q} \otimes R\Gamma((X/A)_{\mathrm{crys}}, E)$ is a perfect complex of $(\mathbb{Q} \otimes A)$-modules. This proves (1).

Next we prove (2). The left hand side is equal to

$$\mathbb{Q}\otimes(A'\otimes^L_A R\Gamma((X/A)_{\mathrm{crys}},E)) = \mathbb{Q}\otimes(R\varprojlim_n(A'_n\otimes^L_{A_n} R\Gamma((X/A_n)_{\mathrm{crys}},E_n)))$$

because $R\Gamma((X/A)_{\mathrm{crys}},E)$ is perfect, and the right hand side is equal to

$$\mathbb{Q}\otimes R\varprojlim_n R\Gamma((X/A'_n)_{\mathrm{crys}},\alpha^*E_n).$$

Let us define C_n by

$$C_n := \mathrm{Cone}(A'_n\otimes^L_{A_n} R\Gamma((X/A_n)_{\mathrm{crys}},E_n)\to R\Gamma((X/A'_n)_{\mathrm{crys}},\alpha^*E_n)).$$

Then it suffices to prove that $\mathbb{Q}\otimes R\varprojlim_n C_n = 0$. To prove this, we may work Zariski locally on X (because of cohomological descent). So we can reduce to the case that $X\to\operatorname{Spec} A_1$ is liftable to an affine smooth morphism $\mathcal{X}=\operatorname{Spf} B\to\operatorname{Spf} A$. (But we lose the properness of X.)

Let $\mathcal{X}_n := \mathcal{X}\otimes_{\mathbb{Z}_p}\mathbb{Z}/p^n\mathbb{Z}$, $\mathcal{X}'_n := \mathcal{X}\otimes_{A_n} A'_n$ and let $E_n^{\mathcal{X}}$ be the value of E_n at $\mathcal{X}_n$. Also, let $E^{\mathcal{X}} := \varprojlim_n E_n^{\mathcal{X}}$ be the value of E at $\mathcal{X}$.

Then each term of $\Gamma(\mathcal{X},E^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}/A})$ is a finitely generated p-torsion free B-module (because E is p-torsion free), and it will become a projective $\mathbb{Q}\otimes B$-module if we apply $\mathbb{Q}\otimes -$, due to the local freeness of $\mathcal{E}$. Also, we have

$$\Gamma(\mathcal{X},E^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}/A})\otimes_A A_n = \Gamma(\mathcal{X}_n,E_n^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}_n/A_n}),$$
$$\Gamma(\mathcal{X}_n,E_n^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}_n/A_n})\otimes_{A_n} A'_n = \Gamma(\mathcal{X}'_n,\alpha^*E_n^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}'_n/A'_n}),$$

and C_n is written in the following way:

$$C_n = \mathrm{Cone}(A'_n\otimes^L_{A_n}\Gamma(\mathcal{X}_n,E_n^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}_n/A_n})\to\Gamma(\mathcal{X}'_n,\alpha^*E_n^{\mathcal{X}}\otimes\Omega^\bullet_{\mathcal{X}'_n/A'_n})).$$

For a finitely generated p-torsion free B-module M, put $M_n := M\otimes_A A_n$ and

$$C_n(M) := \mathrm{Cone}(A'_n\otimes^L_{A_n} M_n\to A'_n\otimes_{A_n} M_n).$$

Then, to prove that $\mathbb{Q}\otimes R\varprojlim_n C_n = 0$, it suffices to prove that $\mathbb{Q}\otimes R\varprojlim_n C_n(M) = 0$ when $\mathbb{Q}\otimes M$ is a projective $\mathbb{Q}\otimes B$-module. If we take a resolution $N^\bullet\to M$ of M by finitely generated free B-modules, $N_n^\bullet\to M_n$ is also a resolution because M is p-torsion free. Then

$$\begin{aligned}(5.15)\qquad &\mathbb{Q}\otimes R\varprojlim_n C_n(M)\\ &=\mathbb{Q}\otimes R\varprojlim_n \mathrm{Cone}(A'_n\otimes_{A_n} N_n^\bullet\to A'_n\otimes_{A_n} M_n)\\ &=\mathrm{Cone}((\mathbb{Q}\otimes A')\otimes_{\mathbb{Q}\otimes A}(\mathbb{Q}\otimes N^\bullet)\to(\mathbb{Q}\otimes A')\otimes_{\mathbb{Q}\otimes A}(\mathbb{Q}\otimes M)) = 0.\end{aligned}$$

(The last equality follows from the projectivity of $\mathbb{Q}\otimes N^\bullet$, $\mathbb{Q}\otimes M$.) So the proof of (2) is finished. □

6. A very weak form of a Lefschetz theorem for isocrystals

Let X be a smooth projective variety defined over a perfect field k of characteristic $p>0$. One expects that given an irreducible isocrystal $\mathcal{E}\in\mathrm{Crys}(X/W)_{\mathbb{Q}}$, there is an ample divisor $Y\subset X$ defined over the same field such that $\mathcal{E}|_Y$ is irreducible in $\mathrm{Crys}(Y/W)_{\mathbb{Q}}$. One could also expect the existence of such Y which is independent of $\mathcal{E}$. One could also weaken all those variants by requesting the irreducibility only for convergent isocrystals $\mathcal{E}\in\mathrm{Conv}(X/K)$. Note that any subisocrystal of a convergent isocrystal is a convergent isocrystal; thus this version for convergent isocrystals is just the restriction to $\mathrm{Conv}(X/K)$ of the version for isocrystals.

We prove here a very weak form of this expectation.

Theorem 6.1. *Let X be a smooth projective variety defined over a perfect field k of characteristic $p>0$ which is liftable to $W_2(k)$ and let $\mathcal{H}$ be an ample line bundle on X. Then there exists an integer n_0 such that, for any $n \geq n_0$, the following property is satisfied: If $\mathcal{E}$ is an isocrystal on X/W of rank $\leq p$ which admits a lattice E with E_X μ-stable in* $\mathrm{Crys}(X/k)$ (*with respect to $\mathcal{H}$*), *and if Y is a smooth divisor in $|\mathcal{H}^n|$ liftable to $W_2(k)$ such that the restriction E_Y of E_X to Y is torsion free, then $E_Y \in \mathrm{Crys}(Y/k)$ is μ-stable as well. Moreover, if $\mathcal{E}$ as above admits a locally free lattice* (*which is not necessarily equal to E*), *E_X is locally free, and so we can take Y above independently of $\mathcal{E}$.*

To link Theorem 6.1 to the expectation, one needs the following theorem.

Theorem 6.2. *Let X be a smooth projective variety defined over a perfect field k of characteristic $p>0$. Let $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$ which admits a lattice E such that E_X is μ-stable in $\mathrm{Crys}(X/k)$. Then $\mathcal{E}$ is irreducible.*

Proof. Let $\mathcal{E}' \subset \mathcal{E}$ be an irreducible subisocrystal and let E' be a lattice of $\mathcal{E}'$ which is μ-semistable in $\mathrm{Crys}(X/k)$. (See [19, Prop. 4.1] for the existence of such a lattice.) Then there is an integer n such that $E'(nX) \subset E$ and such that this inclusion is maximal for this property. By replacing E' by $E'(nX)$, we can assume that the inclusion $\iota : E' \to E$ in $\mathrm{Crys}(X/W)$ induces a non-zero map $\iota_X : E'_X \to E_X$ in $\mathrm{Crys}(X/k)$. The locus $\Sigma \subset X$ on which E'_X, E_X are possibly not locally free has codimension ≥ 2. Let $\mathcal{H}$ be the ample line bundle defining the μ-(semi-)stability, and let C be a smooth complete intersection curve in a high power of the linear system $|\mathcal{H}|$, which is disjoint of Σ. Then the restrictions E'_C, E_C of E'_X, E_X to C, which is the same as the values of E', E at C, are locally free. By Theorem 1.1, the degree of E'_C, E_C are 0. Thus the slopes $\mu(E'_X), \mu(E_X)$ are 0. Then $0 = \mu(E'_X) \leq \mu(\mathrm{Im}\, \iota_X) < \mu(E_X) = 0$ unless ι_X is surjective. Hence ι_X is surjective and so is ι. Hence $\mathcal{E}' = \mathcal{E}$. □

Remark 6.3. The condition in Theorem 6.2 implies irreducibility but is not equivalent to it. We give two examples.

(1) Let X be a smooth projective curve of genus ≥ 2, with lift X_W to $W = W(k)$, and let $a, b \in H^0(X_W, \Omega^1_{X_W/W})$ be linearly independent differential forms. Define the connection on $E = \mathcal{O}_{X_W} e_1 \oplus \mathcal{O}_{X_W} e_2$ by the matrix $A = \begin{pmatrix} 0 & pa \\ pb & 0 \end{pmatrix}$, which is the value of ∇ on the two basis vectors e_i. Then E_X is a constant connection. As a consequence, E is locally nilpotent, thus is a crystal. It is irreducible as an isocrystal, as any subline bundle of degree 0 of $\mathcal{O}_{X_K} e_1 \oplus \mathcal{O}_{X_K} e_2$ is isomorphic to $\mathcal{O}_{X_K}$, spanned by $\lambda_1 e_1 + \lambda_2 e_2$, $(\lambda_1, \lambda_2) \neq (0,0)$, where $\lambda_i \in K$ and there is no form $\omega \in H^0(X_K, \omega_{X_K})$ such that $\lambda_2 pa = \lambda_1 \omega$, $\lambda_1 pb = \lambda_2 \omega$.

We prove now that there is no lattice E' of $\mathcal{E} := \mathbb{Q} \otimes E$ such that E'_X is μ-stable in $\mathrm{Crys}(X/k)$. The argument is essentially the same as in [27, Thm. 5.2]. Assuming there is such a lattice E', after replacing E' by $E'(nX)$ for a suitable integer n, there is an inclusion $\iota : E' \to E$ inducing the identity on $\mathcal{E}$ such that the induced map $\iota_X : E'_X \to E_X$ is non-zero. By Theorem 1.1, $\mu(E_X) = \mu(E'_X) = 0$. If ι_X is not surjective, $\mu(E'_X) < \mu(\mathrm{Im}\iota_X) \leq \mu(E_X)$, a contradiction. So ι_X, and thus ι are surjective. Then $E' = E$ and thus $E_X = E'_X$, which is impossible as E_X is not stable.

In this example, it is not likely that $\mathcal{E}$ is a convergent isocrystal.

(2) We give an example in which $\mathcal{E}$ is a convergent isocrystal. Let k be an algebraically closed field of characteristic $p > 0$ and let X be a smooth projective curve such that the p-rank of its Jacobian is $r \geq 2$. Let K, K_0 be the fraction field of $W(k)$, $W(\mathbb{F}_{p^2})$ respectively. Let $\rho : \pi_1^{\text{ét}}(X) \to \mathrm{GL}_2(\mathbb{Z}_p)$ be a continuous representation such that $\rho_{K_0} : \pi_1^{\text{ét}}(X) \to \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(K_0)$ is irreducible and that ρ modulo p is trivial. Because the pro-p completion $\pi_1^{\text{ét}}(X)^p$ of the étale fundamental group $\pi_1^{\text{ét}}(X)$ is isomorphic to a free pro-p group $\langle \gamma_1, \ldots, \gamma_r \rangle$ of rank r ([36], [13, Thm. 1.9]), the composition

$$\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(X)^p \cong \langle \gamma_1, \ldots, \gamma_r \rangle \to \mathrm{GL}_2(\mathbb{Z}_p)$$

defined by

$$\gamma_1 \mapsto \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \qquad \gamma_2 \mapsto \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad \gamma_i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (i \geq 3)$$

satisfies the conditions required for ρ. Let $(\mathcal{E}, \Phi)$, $\Phi : F^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$, be the unit-root convergent F-isocrystal associated to $\rho_{\mathbb{Q}_p} : \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Q}_p)$ by [14, Thm. 2.1], and let E be the lattice of $\mathcal{E}$ associated to ρ by the construction given in [14, Prop. 2.3, Rmk. 2.3.2]. Namely, if we denote the finite Galois covering associated to the kernel of ρ modulo p^n by $f_n : Y_n \to X$ and denote its Galois group by G_n, E is defined to be the inverse limit of $(f_{n\,\mathrm{crys}*}\mathcal{O}^2_{Y_n/W_n})^{G_n}$ $(n \in \mathbb{N})$, where the action of G_n is induced by its action on Y_n and ρ modulo p^n. Then $E_X = (f_{1\,\mathrm{crys}*}\mathcal{O}^2_{Y_1/k})^{G_1}$ is constant because ρ is trivial modulo p.

We prove that $\mathcal{E}$ is irreducible as a convergent isocrystal. Assume on the contrary that $\mathcal{E}$ has a convergent subisocrystal $\mathcal{E}'$ of rank 1. Then $0 \subsetneq \mathcal{E}' \subsetneq \mathcal{E}$ is a Jordan-Hölder sequence of $\mathcal{E}$, and so is $0 \subsetneq F^*\mathcal{E}' \subsetneq F^*\mathcal{E} \cong \mathcal{E}$. Hence $\{\mathcal{E}', \mathcal{E}/\mathcal{E}'\} = \{F^*\mathcal{E}', F^*\mathcal{E}/F^*\mathcal{E}'\}$, and so we have an isomorphism $(F^*)^2\mathcal{E}' \xrightarrow{\cong} \mathcal{E}'$, which we denote by α. By replacing α by $p^n\alpha$ for a suitable integer n, we may assume that the slope of $(\mathcal{E}', \alpha)$ is 0. We denote the composite $\Phi \circ F^*\Phi : (F^*)^2\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ by Φ_2. Let $\iota_1 : \mathcal{E}' \to \mathcal{E}$ be the inclusion map and let $\iota_2 : \mathcal{E}' \to \mathcal{E}$ be the composite $\mathcal{E}' \xrightarrow{\alpha^{-1}} (F^*)^2\mathcal{E}' \to (F^*)^2\mathcal{E} \xrightarrow{\Phi_2} \mathcal{E}$. Both ι_1, ι_2 are maps of convergent isocrystals. If ι_2 is equal to ι_1 up to scalar $a \in K \setminus \{0\}$, $(\mathcal{E}', a\alpha)$ is a convergent F-subisocrystal of $(\mathcal{E}, \Phi_2)$. Because $(\mathcal{E}, \Phi_2)$, corresponding to $\rho_{K_0} : \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(K_0)$, has pure slope 0, $(\mathcal{E}', a\alpha)$ is necessarily of slope 0 and so it induces a non-trivial subrepresentation of ρ_{K_0}, which contradicts the irreducibility of ρ_{K_0}. Hence ι_2 is not a scalar multiple of ι_1, and so $\iota := \iota_1 \oplus \iota_2$ defines an isomorphism $\mathcal{E}' \oplus \mathcal{E}' \xrightarrow{\cong} \mathcal{E}$ of convergent isocrystals. If we identify $\mathcal{E}' \oplus \mathcal{E}'$ and $\mathcal{E}$ via ι, the isomorphism $\Phi_2 : (F^*)^2\mathcal{E}' \oplus (F^*)^2\mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}'$ is written in the form $f \circ (\alpha \oplus \alpha)$, where $f \in \mathrm{End}(\mathcal{E}' \oplus \mathcal{E}') \cong M(2 \times 2, K)$. Because $(\mathcal{E}, \Phi_2)$, $(\mathcal{E}', \alpha)$ have pure slope 0, the F-isocrystal (K^2, f) on K has pure slope 0. Hence there exists a non-zero $(x_1, x_2) \in K^2$ with $f(F^2(x_1), F^2(x_2)) = (x_1, x_2)$. Then we see that the image of $x_1 \oplus x_2 : \mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}' \cong \mathcal{E}$ is stable under Φ_2 and thus defines a convergent F-subisocrystal of $(\mathcal{E}, \Phi_2)$, which again leads to a contradiction. Hence $\mathcal{E}$ is irreducible as a convergent isocrystal, as required.

Finally, the same argument as in (1) shows that there is no lattice E' of $\mathcal{E}$ such that E'_X is μ-stable in $\mathrm{Crys}(X/k)$.

Proof of Theorem 6.1. As E_X is torsion free, we find a smooth complete intersection curve C in $|\mathcal{H}^n|$ such that E_C as a coherent sheaf is locally free. Thus by Theorem 1.1, E_C has degree 0. Thus $\mu(E_X) = 0$, and for all smooth divisors $Y \in |\mathcal{H}^n|$ with E_Y torsion free, $\mu(E_Y) = 0$. By [27, Cor. 5.10] the Higgs sheaf (V, θ) associated to E_X by the Ogus–Vologodsky correspondence [34, Thm. 2.8] is μ-stable of degree 0 as well. By [28, Thm. 10], $(V, \theta)|_Y$ is μ-stable of degree 0; thus again by [27, Cor. 5.10], E_Y is μ-stable of degree 0. When $\mathcal{E}$ admits a locally free lattice, Theorem 1.1, and the independence of crystalline Chern classes with respect to lattices [19, Proof of Prop. 3.1], the normalized Hilbert polynomial of E_X is the same as that of $\mathcal{O}_X$. Thus E_X is locally free by [28, Thm. 11]. $\square$

Acknowledgments

It is a pleasure to thank Gerd Faltings for his interest in our work and for a discussion in May 2015 in Beijing, which helped us in proving Theorem 1.3. In addition, we thank Bhargav Bhatt, Johan de Jong, and Peter Scholze for discussions on isocrystals, notably on the question of whether they always carry a locally free lattice. Unfortunately, we still do not have a general answer to this basic question. Nikita Markaryan kindly sent us his unposted notes [31], which are aimed at constructing a crystalline Chern–Weil theory. Unfortunately the notes stop before this topic is addressed.

References

[1] Tomoyuki Abe, *Langlands correspondence for isocrystals and the existence of crystalline companions for curves*, J. Amer. Math. Soc. **31** (2018), no. 4, 921–1057, DOI 10.1090/jams/898. MR3836562

[2] Fabrizio Andreatta and Luca Barbieri-Viale, *Crystalline realizations of 1-motives*, Math. Ann. **331** (2005), no. 1, 111–172. MR2107442

[3] Masanori Asakura and Kanetomo Sato, *Chern class and Riemann-Roch theorem for cohomology theory without homotopy invariance*, arXiv:1301.5829 (2013).

[4] Pierre Berthelot, *Cohomologie cristalline des schémas de caractéristique $p > 0$* (French), Lecture Notes in Mathematics, Vol. 407, Springer-Verlag, Berlin-New York, 1974. MR0384804

[5] Pierre Berthelot, *$\mathcal{D}$-modules arithmétiques. I. Opérateurs différentiels de niveau fini* (French, with English summary), Ann. Sci. École Norm. Sup. (4) **29** (1996), no. 2, 185–272. MR1373933

[6] Pierre Berthelot, *Dualité de Poincaré et formule de Künneth en cohomologie rigide* (French, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 5, 493–498. MR1692313

[7] Pierre Berthelot, *Finitude et pureté cohomologique en cohomologie rigide* (French), with an appendix in English by Aise Johan de Jong, Invent. Math. **128** (1997), no. 2, 329–377. MR1440308

[8] Pierre Berthelot and Luc Illusie, *Classes de Chern en cohomologie cristalline* (French), C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1695-A1697; ibid. **270** (1970), A1750–A1752. MR0269660

[9] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978. MR0491705

[10] Shiing-shen Chern, *Differential geometry of fiber bundles*, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 397–411. MR0045432

[11] Bruno Chiarellotto and Bernard Le Stum, *Sur la pureté de la cohomologie cristalline* (French, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 8, 961–963. MR1649945

[12] Bruno Chiarellotto and Nobuo Tsuzuki, *Cohomological descent of rigid cohomology for étale coverings*, Rend. Sem. Mat. Univ. Padova **109** (2003), 63–215. MR1997987

[13] Richard M. Crew, *Etale p-covers in characteristic p*, Compositio Math. **52** (1984), no. 1, 31–45. MR742696

[14] Richard Crew, *F-isocrystals and p-adic representations*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 111–138. MR927977

[15] Pierre Deligne, *Théorie de Hodge. III* (French), Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77. MR0498552

[16] Hélène Esnault, *Characteristic classes of flat bundles*, Topology **27** (1988), no. 3, 323–352. MR963635

[17] Hélène Esnault, *Characteristic classes of flat bundles. II*, K-Theory **6** (1992), no. 1, 45–56. MR1186773

[18] Hélène Esnault, *Algebraic differential characters*, Regulators in analysis, geometry and number theory, Progr. Math., vol. 171, Birkhäuser Boston, Boston, MA, 2000, pp. 89–115. MR1724888

[19] Hélène Esnault and Atsushi Shiho, *Convergent isocrystals on simply connected varieties*, Ann. Inst. Fourier. (to appear)

[20] Eric M. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies, vol. 104, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982. MR676809

[21] Michel Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique* (French, with English summary), Mém. Soc. Math. France (N.S.) **21** (1985), 87 pp. MR844488

[22] Michel Gros, *Régulateurs syntomiques et valeurs de fonctions L p-adiques. I* (French), with an appendix by Masato Kurihara, Invent. Math. **99** (1990), no. 2, 293–320. MR1031903

[23] *Revêtements étales et groupe fondamental* (French), Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), dirigé par Alexandre Grothendieck, augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971. MR0354651

[24] Luc Illusie, *Complexe de de Rham-Witt et cohomologie cristalline* (French), Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 501–661. MR565469

[25] Nicholas M. Katz and William Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77. MR0332791

[26] Kiran S. Kedlaya, *Full faithfulness for overconvergent F-isocrystals*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 819–835. MR2099088

[27] Adrian Langer, *Semistable modules over Lie algebroids in positive characteristic*, Doc. Math. **19** (2014), 509–540. MR3218782

[28] Adrian Langer, *Bogomolov's inequality for Higgs sheaves in positive characteristic*, Invent. Math. **199** (2015), no. 3, 889–920. MR3314517

[29] Christopher Lazda, *Incarnations of Berthelot's conjecture*, J. Number Theory **166** (2016), 137–157. MR3486269

[30] Lasse Peter Mannisto, *Albanese and Picard 1-motives in positive characteristic*, Thesis (Ph.D.)–University of California, Berkeley, ProQuest LLC, Ann Arbor, MI, 2014. MR3295332

[31] N. Markaryan, *Crystalline characteristic classes via the Atiyah gerbe and generalized Chern–Weil construction*, unpublished.

[32] Arthur Ogus, *F-isocrystals and de Rham cohomology. II. Convergent isocrystals*, Duke Math. J. **51** (1984), no. 4, 765–850. MR771383

[33] Arthur Ogus, *F-crystals, Griffiths transversality, and the Hodge decomposition*, Astérisque **221** (1994), ii+183 pp. MR1280543

[34] Arthur Ogus and Vadim Vologodsky, *Nonabelian Hodge theory in characteristic p*, Publ. Math. Inst. Hautes Études Sci. **106** (2007), 1–138. MR2373230

[35] Bernard Saint-Donat, *Techniques de descente cohomologique*, Lecture Notes in Math., ch. Vbis, Springer Verlag, 1972.

[36] Igor Shafarevitch, *On p-extensions* (Russian, with English summary), Rec. Math. [Mat. Sbornik] N.S. **20(62)** (1947), 351–363. MR0020546

[37] Bernard Le Stum, *Rigid cohomology*, Cambridge Tracts in Mathematics, vol. 172, Cambridge University Press, Cambridge, 2007. MR2358812

[38] John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144. MR0206004

Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany
Email address: esnault@math.fu-berlin.de

Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
Email address: shiho@ms.u-tokyo.ac.jp