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A restriction isomorphism for cycles of relative dimension zero

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We study the restriction map to the closed fiber of a regular projective scheme over an excellent henselian discrete valuation ring, for a cohomological version of the Chow group of relative zero-cycles. Our main result extends the work of Saito–Sato [SS10] to general perfect residue fields.

1. Introduction

Let A be an excellent henselian discrete valuation ring with perfect residue field k of exponential characteristic p. Let X/A be a regular connected scheme which is projective and flat over A, such that the reduced special fiber $Y \subset X$ is a simple normal crossings divisor. Let d be the dimension of Y.

Let Λ be a commutative ring. We denote by $H^{i,j}(Y) := H^i(Y_{\mathrm{cdh}}, \Lambda(j))$ the motivic cohomology group defined by Suslin–Voevodsky as the cdh-cohomology of the motivic complex $\Lambda(j)$ [SV02]. In particular, for Y/k smooth, there is an isomorphism $H^{2j,j}(Y) \cong \mathrm{CH}^j(Y)_\Lambda$ onto the Chow group of codimension j cycles with Λ-coefficients. Therefore one can view $H^{2d,d}(Y)$ as a cohomological version of the Chow group of zero-cycles up to rational equivalence, while for singular Y, the usual Chow groups form a Borel–Moore homology theory. Note that for $d = 1$ we have $H^{2,1}(Y) = \mathrm{Pic}(Y) \otimes \Lambda$.

We say that an integral one-cycle $Z \subset X$ is *in general position* if Z is flat over A and Z does not meet the singular locus of Y. Let $\mathrm{Z}_1^g(X)$ be the free $\mathbb{Z}$-module generated by these integral one-cycles in general position. An integral one-cycle in general position $Z \subset X$ can be restricted to a

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zero-cycle $[Y \cap Z] \in \mathrm{Z}_0(Y^{\mathrm{sm}})$ on the smooth locus Y^{sm} of Y. This yields a homomorphism of abelian groups

$$\tilde{\rho} : \mathrm{Z}_1^g(X) \to \mathrm{Z}_0(Y^{\mathrm{sm}}).$$

For a closed point $y \in Y^{\mathrm{sm}}$, there is a purity isomorphism $\Lambda \cong H_y^{2d,d}(Y)$ for motivic cohomology with support in y. These isomorphisms induce a homomorphism of Λ-modules $\mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda \to H^{2d,d}(Y)$. On the other hand, denoting by $\mathrm{CH}_1(X)$ the Chow group of one-cycles on X, one has canonical homomorphisms $\mathrm{Z}_1^g(X) \hookrightarrow \mathrm{Z}_1(X) \to \mathrm{CH}_1(X)$. We can now formulate the main theorem of this note.

Theorem 1.1. *Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p.*

1) If k is finite or algebraically closed, or $(d-1)!$ is prime to m, or A has equal characteristic, or X/A is smooth, then there is a unique Λ-module homomorphism ρ making the diagram

$$\begin{array}{ccc} \mathrm{Z}_1^g(X)_\Lambda & \xrightarrow{\tilde{\rho}} & \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda \\ \downarrow & & \downarrow \\ \mathrm{CH}_1(X)_\Lambda & \xrightarrow[\rho]{} & H^{2d,d}(Y) \end{array}$$

commutative.

2) If $\rho : \mathrm{CH}_1(X)_\Lambda \to H^{2d,d}(Y)$ is a homomorphism of Λ-modules making the above diagram commutative then it is an isomorphism.

Up to an explicit presentation of $H^{2d,d}(Y)$, established in Theorem 7.1 below, part 1) is discussed in Sections 3 and 8 and part 2) is Corollary 5.2.

In the case when k is finite or algebraically closed, Theorem 1.1 is equivalent to and reproves results of Saito–Sato [SS10] in view of the étale realization isomorphism, Proposition 9.1 below. In fact, in some sense, our arguments extend to general residue fields k the simplified approach to the results of Saito and Sato which is developed in [EW13, App.].

By the moving lemma of Gabber–Liu–Lorenzini [GLL13], $\mathrm{CH}_1(X)$ is generated by $\mathrm{Z}_1^g(X)$, so uniqueness in Theorem 1.1 is clear. Surprisingly, one of the main difficulties is the construction of ρ, which today can be performed only under the extra assumptions of 1). It is expected that the homomorphism ρ with the properties of part 1) exists in general for any ring of coefficients Λ. So far, we do not have the necessary motivic techniques at our disposal to show this.

The proof of part 1), in Section 8, rests, in equal characteristic, on the Gersten conjecture for Milnor K-theory due to the first author [Ker09]; when the residue field is finite, on the étale realization using the Kato conjecture established in [KS12]; when $(d-1)!$ is prime to m, on algebraic K-theory and the Grothendieck–Riemann–Roch theorem [BS98].

As for 2), the proof in Sections 4 and 5 relies on an explicit geometric presentation of $H^{2d,d}(Y)$, which is discussed in Sections 2 and 7, together with an idea of Bloch [EW13, App.] which enables one to construct an inverse to ρ in Section 5. Note that for 2) it is essential that $\Lambda = \mathbb{Z}/m\mathbb{Z}$ and that m is prime to p.

When X/A is smooth, the restriction map ρ factors as

$$\mathrm{CH}_1(X) \twoheadrightarrow \mathrm{CH}_0(X_K) \xrightarrow{sp} \mathrm{CH}_0(Y)$$

where sp denotes the specialization homomorphism [Ful84, 20.3.1]. Here we use the identification $H^{2d,d}(Y) = \mathrm{CH}_0(Y)_\Lambda$ for Y/k smooth and Λ arbitrary [MVW, Thm. 19.1]. So one deduces from Theorem 1.1

Corollary 1.2. *Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. If X/A is smooth, the specialization homomorphism $sp : \mathrm{CH}_0(X_K)_\Lambda \to \mathrm{CH}_0(Y)_\Lambda$ is an isomorphism.*

As an application of Theorem 1.1, we show, in Proposition 9.4, that for a formal Laurent power series field $K = k((\pi))$ with $[k : \mathbb{Q}_p] < \infty$, the Chow group $\mathrm{CH}_0(X_K)_\Lambda$ is finite for $\Lambda = \mathbb{Z}/m\mathbb{Z}$ assuming p is prime to m.

In the final section we state a conjecture describing $\mathrm{CH}_1(X)_\Lambda$ for the case $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m not necessarily prime to p. We also explain in which motivic bidegrees we expect the analogue of Theorem 1.1 to be true.

2. The cycle group $\mathcal{C}(Y)$ of a simple normal crossings variety over a perfect field

2.1. Simple normal crossings varieties

In this section we make precise the notion of simple normal crossings varieties we shall use throughout the article.

Let k be a perfect field of exponential characteristic p with algebraic closure $\bar{k}$. A d-dimensional k-scheme Y is a *normal crossings variety* (nc variety) if it is reduced, quasi-projective (which implies of finite type) and if for any closed point $y \in Y_{\bar{k}}$, the henselized local ring $\mathcal{O}^h_{Y_{\bar{k}},y}$ is $\bar{k}$-isomorphic to $\bar{k}[y_1, \dots, y_{d+1}]^h/(\prod_{i=1}^r y_i)$ ($1 \leq r \leq d+1$). The variety Y is a *simple normal crossings variety* (snc variety) if it is a nc variety and additionally every irreducible component of Y is smooth over k. Here the henselization $R[y_1, \dots, y_r]^h$ of a polynomial ring over a ring R means henselization in the ideal $(y_1, \dots, y_r)$.

By a *simple normal crossings divisor* (snc divisor) we denote what is called a strict normal crossings divisor in [dJ96, 2.4]. We use the notation of *normal crossings divisor* (nc divisor) as in *loc. cit.* We recall that if Y is a nc divisor on a scheme X, then by definition X is regular at the points of Y.

Let A be a henselian discrete valuation ring with residue field k. Let $\bar{A}$ be the integral unramified extension of A with residue field $\bar{k}$. For a scheme X/A we write $X_{\bar{A}}$ for the scheme $X \times_A \bar{A}$. We denote by π a fixed prime element of A (thus of $\bar{A}$).

Lemma 2.1. *Let X be a scheme which is flat and of finite type over A. If the reduced special fiber Y is a nc divisor, the henselized local ring $\mathcal{O}^h_{X_{\bar{A}},y}$ at a closed point y of $Y_{\bar{k}}$ is $\bar{A}$-isomorphic to*

$$\bar{A}[y_1, \dots, y_{d+1}]^h/(\prod_{i=1}^r y_i^{m_i} - \pi u), \tag{2.1}$$

where u is a unit of $\bar{A}[y_1, \dots, y_{d+1}]^h$ and the m_i are positive integers ($r \geq 1$).

Proof. Choose generators $y_1, \dots, y_{d+1}$ of the maximal ideal of $\mathcal{O}^h_{X_{\bar{A}},y}$ such that the product of the first r defines the snc divisor $Y_{\bar{k}}$ on $X_{\bar{A}}$ around y. The divisor of π on the regular scheme $\operatorname{Spec} \mathcal{O}^h_{X_{\bar{A}},y}$ is of the form $\operatorname{div}(y_1^{m_1} \cdots y_r^{m_r})$ for certain m_i. So the canonical surjection

$$\bar{A}[y_1, \dots, y_{d+1}]^h \to \mathcal{O}^h_{X_{\bar{A}},y}$$

factors through a ring of the form (2.1) with a unit u, since $\bar{A}[y_1, \dots, y_{d+1}]^h$ is regular. By dimension reasons this map is an isomorphism as the ring (2.1) is integral. □

Lemma 2.2. *Let X be a scheme which is flat and of finite type over A. The following conditions are equivalent:*

(i) the reduced special fiber Y/k of X is a nc variety and X is regular at the points of Y,

(ii) $Y \subset X$ is a nc divisor (thus X is regular at the points of Y).

This assertion remains true if one replaces nc with snc.

Proof. The only nontrivial part is to show that if Y is a nc divisor on X it is a nc variety over k. But this follows from Lemma 2.1. □

A 1-dimensional closed subscheme C of a snc variety Y is a *simple normal crossings subcurve* (snc subcurve) if its (scheme-theoretic) intersections with the irreducible components Y_i of Y are smooth and purely 1-dimensional, if its intersection with $Y_i \cap Y_j$ is reduced and purely 0-dimensional for all $i \neq j$ and if its intersection with $Y_i \cap Y_j \cap Y_\ell$ is empty for all $i \neq j \neq \ell \neq i$. We stress that according to this definition, if a smooth irreducible curve contained in Y_i is a snc subcurve of Y, then it must be disjoint from $Y_i \cap Y_j$ for every $j \neq i$.

Lemma 2.3. *A snc subcurve C of a snc variety Y is regularly immersed and is itself a one-dimensional snc variety.*

Proof. Consider a closed point y of $Y_{\bar{k}}$ lying on C. For $y \in Y^{\mathrm{sm}}$ the curve C is regular at y. For $y \in Y_1 \cap Y_2$ we choose an isomorphism

$$\widehat{\mathcal{O}}_{Y_{\bar{k}},y} \cong \bar{k}[[y_1, \dots, y_{d+1}]]/(y_1 y_2) =: \widehat{\mathcal{O}}.$$

Let $\widehat{\mathcal{I}}$ be the ideal defining C in $\widehat{\mathcal{O}}$. As $C \cap Y_1$ and Y_1 are regular the ideal $\widehat{\mathcal{I}} \cdot \widehat{\mathcal{O}}/(y_2)$ is generated by a regular sequence $z_3', \dots, z_{d+1}'$. Lift this sequence to $z_3, \dots, z_{d+1} \in \widehat{\mathcal{I}}$. Then

$$\widehat{\mathcal{O}} \cong \bar{k}[[y_1, y_2, z_3, \dots, z_{d+1}]]/(y_1 y_2)$$

and C is defined by the regular sequence $z_3, \dots, z_{d+1}$. □

2.2. Cycle group of a simple normal crossings variety

Throughout the article, we let Λ be a commutative ring with unity. All motivic cohomology groups have Λ-coefficients, while Chow groups have integral coefficients, unless otherwise specified.

Let us assume from now on that Y/k is a snc variety of dimension d. The abelian group $\mathcal{C}(Y)$ we define in this section can be thought of as a cohomological variant of the Chow group of zero-cycles of Y with Λ-coefficients. Recall that, in contrast, the usual Chow groups form a Borel–Moore homology theory. In fact, we will see in Section 7 that for p invertible in Λ and Y proper, the group $\mathcal{C}(Y)$ is isomorphic to motivic cohomology $H^{2d}(Y_{\mathrm{cdh}}, \Lambda(d))$ in the sense of Suslin–Voevodsky [SV02].

Let Y^{sm} be the smooth locus of Y over k and Y^{sing} its complement with the reduced subscheme structure. Let $\mathrm{Z}_0(Y^{\mathrm{sm}})$ be the free $\mathbb{Z}$-module generated by the integral zero-dimensional subschemes of Y^{sm}. The group $\mathcal{C}(Y)$ is defined by an exact sequence

$$0 \to \mathcal{R} \to \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda \to \mathcal{C}(Y) \to 0$$

where $\mathcal{R}$ is the sub-Λ-module of $\mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda$ generated by divisors $\mathrm{div}(g)$ associated with rational functions g on curves C of the following two types.

Type 1 data: A type 1 datum is a pair (C_1, g_1). Here C_1 is an integral one-dimensional closed subscheme $C_1 \subset Y$ which is not contained in Y^{sing}. Let η be the generic point of C_1 and let C_1^∞ be $\tilde{C}_1 \times_Y Y^{\mathrm{sing}}$ with the reduced subscheme structure, where $\tilde{C}_1$ is the normalization of C_1. The rational function g_1 is an element of $\ker(\mathcal{O}^\times_{C_1, C_1^\infty \cup \{\eta\}} \to \mathcal{O}^\times_{C_1^\infty})$, that is, it is a rational function on C_1, which is defined and equal to 1 over C_1^∞.

Type 2 data: A type 2 datum is a pair (C_2, g_2). Here $C_2 \subset Y$ is a snc subcurve on Y. Let C_2^∞ be the finite union of $C_2 \cap Y^{\mathrm{sing}}$ and the set of maximal points of C_2. The function g_2 is an element of $\mathcal{O}^\times_{C_2, C_2^\infty}$.

The group $\mathcal{C}(Y)$ is a variant of the cycle group studied in [LW85]. Note that if Y/k is smooth, then $\mathcal{C}(Y) = \mathrm{CH}_0(Y)_\Lambda$.

3. The restriction homomorphism

Let A be an excellent henselian discrete valuation ring with perfect residue field k of exponential characteristic p. Let X be a regular connected scheme which is projective and flat over A, such that the reduced special fiber Y of X is a snc divisor. So Y/k is a snc variety by Lemma 2.2. Let Λ be a commutative ring with unity.

By $\mathrm{Z}_1(X)$ we denote the free $\mathbb{Z}$-module generated by the integral closed subschemes $Z \subset X$ of dimension one. We write $[Z] \subset \mathrm{Z}_1(X)$ for the associated generator. The subgroup $\mathrm{Z}_1^g(X) \subset \mathrm{Z}_1(X)$ of 'generic' elements is by definition generated by those integral cycles $[Z]$ such that Z is flat over A and $Z \cap Y^{\mathrm{sing}} = \varnothing$.

To a zero-dimensional closed subscheme $S \subset Y^{\mathrm{sm}}$, one associates as usual the element

$$[S] = \sum_{z \in S} \mathrm{length}(\mathcal{O}_{S,z})[z] \in \mathcal{C}(Y).$$

Here length is the length of a ring as a module over itself.

One obtains the pre-restriction homomorphism

$$\tilde{\rho} : \mathrm{Z}_1^g(X) \to \mathrm{Z}_0(Y^{\mathrm{sm}}), \quad [Z] \mapsto [Z \cap Y]. \tag{3.1}$$

In Section 8 we prove the following theorem.

Theorem 3.1 (Restriction). *Assume that $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p and that additionally one of the following conditions holds:*

(i) k is algebraically closed,
(ii) k is finite,
(iii) $(d-1)!$ is prime to m,
(iv) A is equicharacteristic,
(v) X/A is smooth.

Then there is a unique restriction homomorphism $\rho : \mathrm{CH}_1(X)_\Lambda \to \mathcal{C}(Y)$ such that the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(X)_\Lambda & \xrightarrow{\ \rho\ } & \mathcal{C}(Y) \\ \uparrow & & \uparrow \\ \mathrm{Z}_1^g(X)_\Lambda & \xrightarrow[\ \tilde{\rho}\ ]{} & \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda \end{array}$$

is commutative.

Remark 3.2. Rather than proving Theorem 3.1 under the assumption (iii), we will see, more generally, that the restriction homomorphism exists for any ring Λ in which $p(d-1)!$ is invertible.

By [GLL13], the left vertical map in the diagram is surjective for any ring Λ, so uniqueness is clear in the theorem. A direct approach to the construction of ρ is to prove a moving lemma. Unfortunately, such Chow type moving lemmas are not available in mixed characteristic. We explain in Section 8 how to use the extra hypotheses to yield an indirect proof.

A general theory of derived mixed motives over base schemes, which satisfies standard properties, would show that the restriction map exists for any Λ in complete generality. Unfortunately, such a theory is not available at the moment.

4. Canonical lifting of zero-cycles

We use the notation of Section 3, in particular recall we considered the solid arrows in the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(X)_\Lambda & & \\ \uparrow & \nwarrow^{\tilde\gamma} & \\ \mathrm{Z}_1^g(X) & \xrightarrow[\tilde\rho]{} & \mathrm{Z}_0(Y^{\mathrm{sm}}). \end{array} \tag{4.1}$$

It is well known that the map $\tilde\rho$ is surjective, i.e. one can lift zero-cycles on Y^{sm} to flat one-cycles on X/A. As the idea for this construction is central for the arguments in this section and the next one, we recall it. Given a closed point $y \in Y^{\mathrm{sm}}$, let $(a_1, \dots, a_d) \in \mathcal{O}_{Y^{\mathrm{sm}},y}$ be a regular sequence generating the maximal ideal. Lift these parameters to elements $\hat a_1, \dots, \hat a_d \in \mathcal{O}_{X,y}$. Let $Z^{\mathrm{loc}} \subset \operatorname{Spec} \mathcal{O}_{X,y}$ be the associated closed subscheme of the ideal

$$\hat a_1 \mathcal{O}_{X,y} + \cdots + \hat a_d \mathcal{O}_{X,y} \subset \mathcal{O}_{X,y}$$

and let Z be the unique irreducible component of the closure of Z^{loc} in X which contains y. Then Z is finite, flat over A and $Z \cap Y$ is the integral scheme associated with the point $y \in Y$, so Z defines a lifting of y to $\mathrm{Z}_1^g(X)$.

Of course this lifting is not unique if $d = \dim(Y) > 0$. However under certain conditions the lift is unique in the Chow group, i.e. there exists a unique dashed map $\tilde\gamma$ making the diagram (4.1) commutative. The goal of this section is to explain this fact. We use the notation of Theorem 3.1.

Proposition 4.1. *Let X be a regular scheme, flat and projective over A, whose reduced special fiber Y is a snc divisor on X. Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. The group* $\ker\big(\tilde\rho : \mathrm{Z}_1^g(X)_\Lambda \to \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda\big)$ *is contained in* $\ker\big(\mathrm{Z}_1^g(X)_\Lambda \to \mathrm{CH}_1(X)_\Lambda\big)$. *In particular, there exists a unique homomorphism $\tilde\gamma : \mathrm{Z}_0(Y^{\mathrm{sm}}) \to \mathrm{CH}_1(X)_\Lambda$ making the diagram* (4.1) *commutative.*

Proof. We first assume $d = 1$. There is a commutative diagram involving étale cycle maps

$$\begin{array}{ccccc} \mathrm{Z}_1^g(X)_\Lambda & \longrightarrow & \mathrm{CH}_1(X)_\Lambda & \xrightarrow{c_X} & H^2(X_{\text{ét}}, \Lambda(1)) \\ \downarrow{\scriptstyle\tilde\rho} & & & & \downarrow{\scriptstyle\wr} \\ \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda & & \longrightarrow & & H^2(Y_{\text{ét}}, \Lambda(1)). \end{array}$$

The right vertical isomorphism is due to proper base change for étale cohomology [SGA4.5, IV, Thm. 1.2]. The cycle map c_X is injective as it is equal to the composite map

$$\mathrm{CH}_1(X)_\Lambda \cong \mathrm{Pic}(X)_\Lambda \hookrightarrow H^2(X_{\text{ét}}, \Lambda(1))$$

which is injective by Kummer theory. So the proposition follows in this case.

For d general fix a projective embedding $X \subset \mathbb{P}^N_A$. If k is finite, the assertion to be proved is invariant under extension of k to its maximal pro-ℓ-extension, for some prime ℓ prime to m. For this, one applies the flat pull-back and proper push-forward [Ful84, p. 394]. So we may assume k to be infinite.

Consider an element

$$\sum_{i=1}^{s} r_i[Z_i] \in \ker(\tilde{\rho})$$

with Z_i an integral scheme, flat over A, disjoint from Y^{sing}. Then $(Z_i \cap Y)_{\mathrm{red}}$ consists of a closed point z_i of Y^{sm}. Choose a lift Z'_i of z_i, i.e. an integral one-cycle $Z'_i \subset X$ which is finite, flat over A, and such that $Z'_i \cap Y = z_i$ in the sense of schemes. When choosing the lifts Z'_i we can make sure that $Z'_i = Z'_j$ whenever $z_i = z_j$. This condition implies that $\sum_{i=1}^{s} r_i n_i [Z'_i] = 0$ in $\mathrm{Z}^g_1(X)_\Lambda$, where n_i is the intersection multiplicity of Z_i and Y. Hence, in order to prove the proposition, it is enough to show that

$$\sigma([Z_i] - n_i[Z'_i]) = 0$$

for all $1 \leq i \leq s$. Here $\sigma : \mathrm{Z}^g_1(X)_\Lambda \to \mathrm{CH}_1(X)_\Lambda$ is the canonical map. So we need only consider an element of the form $[Z] - n[Z'] \in \ker(\tilde{\rho})$, i.e. Z', Z are one-dimensional integral, finite, flat schemes over A, with $Z' \cap Y$ reduced (in particular Z' is regular) and n is the intersection multiplicity of Z and Y.

We first assume that Z is regular. We prove this special case by induction on $d > 1$, using a Bertini type argument. Later we explain how to reduce to this special case. As by assumption, $H^0(Z, \mathcal{O}_Z)$ is a discrete valuation ring, the embedding dimension of $Z \cap Y$ is at most one. So by Bertini's theorem [AK79] there is an ample hypersurface section H_Y of Y such that $Z \cap Y$ is contained in H_Y and such that H_Y is a simple normal crossings subvariety of Y. If H_Y is chosen ample enough we can lift it to a hypersurface section H of X flat over A and containing Z, see [JS12, Thm. 1] and [SS10, Sec. 4]. Then H is regular and its reduced special fiber is a snc divisor.

Next we use Bertini's theorem (*loc. cit.*) to find a snc subvariety H'_Y of Y which satisfies

- H'_Y is of dimension one,
- $Y \cap Z'$ is contained in H'_Y,
- H'_Y is the intersection of $d-1$ ample hypersurface sections,
- $H'_Y \cap H_Y$ consists of reduced points in Y^{sm}.

If the hypersurface sections are chosen ample enough, we can lift H'_Y to a closed subscheme $H' \subset X$, regular, flat over A, such that $(H' \otimes k)_{\mathrm{red}}$ is a snc divisor on H' and such that H' contains Z'. The scheme $H \cap H'$ is finite, flat over A and its intersection with Y is reduced. In particular $H \cap H'$ is regular. Let Z'' be its component containing $Z' \cap Y$.

By the induction assumption, $[Z] - n[Z'']$ vanishes in $\mathrm{CH}_1(H)_\Lambda$, thus it also vanishes in $\mathrm{CH}_1(X)_\Lambda$, so $\sigma([Z] - n[Z'']) = 0$. By the $d = 1$ case of the proposition, $[Z'] - [Z'']$ vanishes in $\mathrm{CH}_1(H')_\Lambda$, thus it also vanishes in $\mathrm{CH}_1(X)_\Lambda$, so $\sigma([Z'] - [Z'']) = 0$. Finally, we obtain

$$\sigma([Z] - n[Z']) = \sigma([Z] - n[Z'']) + n\sigma([Z''] - [Z']) = 0.$$

To finish the proof we apply Bloch's idea in [EW13, App.] to treat the general case. Let $\bar{Z}$ be the normalization of Z. As A is excellent, $\bar{Z}$ is finite over Z, we can find a projective embedding $\bar{Z} \to \mathbb{P}^M_X$. Let $\bar{X}$ be $\mathbb{P}^M_X$ and $\bar{Y}$ be $\mathbb{P}^M_Y$. Choose a lift of the closed point $(\bar{Z} \cap \bar{Y})_{\mathrm{red}}$ to a closed subscheme $\bar{Z}' \subset \bar{X}$ which is étale over Z'. Let $\bar{n}$ be the intersection multiplicity of $\bar{Z}$ and $\bar{Y}$. Then the push-forward of $[\bar{Z}] - \bar{n}[\bar{Z}']$ along $\bar{X} \to X$ is equal to our old cycle $[Z] - n[Z']$.

The cycle $[\bar{Z}] - \bar{n}[\bar{Z}']$ is in the kernel of the restriction map

$$\bar{\tilde{\rho}} : \mathrm{Z}_1^g(\bar{X})_\Lambda \to \mathrm{Z}_0(\bar{Y}^{\mathrm{sm}})_\Lambda,$$

so it vanishes in $\mathrm{CH}_1(\bar{X})_\Lambda$ by what is shown above in the case of regular cycles. Finally, this implies that $\sigma([Z] - n[Z'])$ vanishes too. □

5. The inverse restriction map γ

Recall that in Section 4 we constructed a canonical homomorphism

$$\tilde{\gamma} : \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda \to \mathrm{CH}_1(X)_\Lambda.$$

Theorem 5.1. *Under the assumptions of Proposition 4.1, there is a unique homomorphism* $\gamma : \mathcal{C}(Y) \to \mathrm{CH}_1(X)_\Lambda$ *making the diagram*

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{\ \gamma\ } & \mathrm{CH}_1(X)_\Lambda \\ \uparrow & \overset{\tilde{\gamma}}{\nearrow} & \uparrow \sigma \\ \mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda & \xleftarrow[\ \tilde{\rho}\ ]{} & \mathrm{Z}_1^g(X)_\Lambda \end{array}$$

commutative. In other words, the map $\tilde{\gamma}$ *factors through* $\mathcal{C}(Y)$.

Proof. Uniqueness in the theorem is clear, as by construction, the left vertical arrow is surjective. In order to prove that the factorization γ of $\tilde{\gamma}$ exists, we have to show that for a type i datum (C_i, g_i) $(i \in \{1,2\})$ on Y we have

$$0 = \tilde{\gamma}(\mathrm{div}(g_i)) \in \mathrm{CH}_1(X)_\Lambda.$$

Type 2 datum:
Let (C_2, g_2) be a type 2 datum. The idea is first to lift the curve C_2 to a 'nice' flat curve $\hat{C}_2 \subset X$ over A. In a second step we lift g_2 to $\hat{g}_2 \in k(\hat{C}_2)^\times$ such that $\mathrm{div}(\hat{g}_2) \in \mathrm{Z}_1^g(X)$ has restriction $\mathrm{div}(g_2) \in \mathrm{Z}_0(Y^{\mathrm{sm}})$. Then the class of $\mathrm{div}(\hat{g}_2)$ in $\mathrm{CH}_1(X)_\Lambda$ is $\tilde{\gamma}(\mathrm{div}(g_2))$ and we are done.

We start with the construction of the lifted curve $\hat{C}_2$. The argument is similar to [GLL13, Lem. 2.5]. Let $\mathcal{I}$ be the ideal sheaf defining C_2 in Y. By Lemma 2.3, the $\mathcal{O}_{Y,C_2^\infty}/\mathcal{I}_{C_2^\infty}$-module $\mathcal{I}_{C_2^\infty}/\mathcal{I}^2_{C_2^\infty}$ has a basis $a_1, \dots, a_{d-1}$. Lift a_1 to an element $\hat{a}_1 \in \mathcal{O}_{X,C_2^\infty}$ which is a unit at the maximal points of Y^{sing}. Let $L(\hat{a}_1) \subset X$ be the Zariski closure of the closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,C_2^\infty}$ given by $\hat{a}_1 = 0$. Let $P(\hat{a}_1)$ be the set of maximal points of $L(\hat{a}_1) \cap Y^{\mathrm{sing}}$. Clearly, the points $P(\hat{a}_1)$ have height one in Y^{sing}. Lift a_2 to an element $\hat{a}_2 \in \mathcal{O}_{X,C_2^\infty}$ which is a unit at the points $P(\hat{a}_1)$. Let $L(\hat{a}_1, \hat{a}_2)$ be the Zariski closure of the closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,C_2^\infty}$ given by $\hat{a}_1 = \hat{a}_2 = 0$. Let $P(\hat{a}_1, \hat{a}_2)$ be the maximal points of $L(\hat{a}_1, \hat{a}_2) \cap Y^{\mathrm{sing}}$. Clearly, the points $P(\hat{a}_1, \hat{a}_2)$ have height two in Y^{sing}. We proceed like this.

In the end we get elements $\hat{a}_1, \dots, \hat{a}_{d-1} \in \mathcal{O}_{X,C_2^\infty}$. Let $\hat{C}_2^{\mathrm{loc}}$ be the closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,C_2^\infty}$ defined by the ideal

$$\hat{a}_1 \mathcal{O}_{X,C_2^\infty} + \cdots + \hat{a}_{d-1} \mathcal{O}_{X,C_2^\infty} \subset \mathcal{O}_{X,C_2^\infty}$$

and let $\hat{C}_2$ be the closure of $\hat{C}_2^{\mathrm{loc}}$ in X. By Nakayama's lemma the closed subschemes $\hat{C}_2 \cap Y$ and C_2 of Y coincide in a neighborhood of C_2^∞. The scheme $\hat{C}_2$ is regular around the points C_2^∞, because at each $y \in C_2^\infty$ the intersection of $\hat{C}_2$ with an irreducible component Y_1 of Y is regular and proper. This implies that $\hat{C}_2$ is flat over A. Furthermore, $\hat{C}_2 \cap Y^{\mathrm{sing}}$ is finite, because it has dimension zero or is empty.

Let $\hat{C}_2^\infty$ be the finite set of points of $\hat{C}_2$ consisting of the maximal points of $\hat{C}_2 \cap Y$ and the closed points $\hat{C}_2 \cap Y^{\mathrm{sing}}$. In a neighborhood of $\hat{C}_2^\infty$, the scheme $\hat{C}_2 \cap Y$ is the disjoint union of C_2 and a residual part. Thus, there is a direct product decomposition of rings

$$\mathcal{O}_{\hat{C}_2 \cap Y, \hat{C}_2^\infty} = \mathcal{O}_{C_2, C_2^\infty} \times R$$

where R is a semi-local ring of dimension one. It gives rise to an element $(g_2, 1) \in \mathcal{O}^\times_{\hat{C}_2 \cap Y, \hat{C}_2^\infty}$ which we lift to an element $\hat{g}_2 \in \mathcal{O}^\times_{\hat{C}_2, \hat{C}_2^\infty}$.

As $\hat{C}_2^\infty$ contains the points of $\hat{C}_2 \cap Y^{\mathrm{sing}}$, $\mathrm{div}(\hat{g}_2)$ is in $\mathrm{Z}_1^g(X)$. As restriction of rational functions commutes with taking associated divisors, we have $\tilde{\rho}(\mathrm{div}(\hat{g}_2)) = \mathrm{div}(g_2)$. So one has the commutative diagram

$$\begin{array}{ccccccc} & \nearrow^{\tilde{\gamma}} & \mathrm{CH}_1(X)_\Lambda & \qquad & & \nearrow^{\tilde{\gamma}} & 0 \\ & & \uparrow & & & & \uparrow \\ \mathrm{Z}_0(Y^{\mathrm{sm}}) & \xleftarrow[\tilde{\rho}]{} & \mathrm{Z}_1^g(X) & & \mathrm{div}(g_2) & \xleftarrow[\tilde{\rho}]{} & \mathrm{div}(\hat{g}_2). \end{array}$$

This finishes the proof in this case.

Type 1 datum:
Let (C_1, g_1) be a type 1 datum. We consider a morphism $\pi : \check{X} \to X$ which is a repeated blow-up of closed points of X, all closed points lying over Y^{sing}. We write $\check{Y}$ for the reduced special fiber of $\check{X}$, etc. Note that $\check{Y} \subset \check{X}$ is a snc divisor. By a careful choice of the closed points, i.e. the centers of the blow-ups, we can assume that the strict transform $\check{C}_1$ of C_1 in $\check{X}$ satisfies the following properties (see Jannsen's appendix to [SS10]):

(i) $\check{C}_1$ is smooth around $\pi^{-1}(Y^{\mathrm{sing}})$,
(ii) $\check{C}_1 \cap \check{Y}^{\mathrm{sing}}$ consists of reduced points,
(iii) all points $y \in \check{C}_1$ lie on at most two components of $\check{Y}$.

Let $\tilde{\check{\rho}} : \mathrm{Z}_1^g(\check{X}) \to \mathrm{Z}_0(\check{Y}^{\mathrm{sm}})$ and $\tilde{\check{\gamma}} : \mathrm{Z}_0(\check{Y}^{\mathrm{sm}}) \to \mathrm{CH}_1(\check{X})_\Lambda$ respectively denote the restriction map and the map given by Proposition 4.1 applied to $\check{X}$. There is a commutative diagram

$$\begin{array}{ccccc} & & \overset{\tilde{\check{\gamma}}}{\frown} & & \\ \mathrm{Z}_0(\check{Y}^{\mathrm{sm}}) & \xleftarrow{\tilde{\check{\rho}}} & \mathrm{Z}_1^g(\check{X}) & \longrightarrow & \mathrm{CH}_1(\check{X})_\Lambda \\ {\scriptstyle \pi_Y^*}\uparrow & & \uparrow & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Z}_0(Y^{\mathrm{sm}}) & \xleftarrow{\tilde{\rho}} & \mathrm{Z}_1^g(X) & \longrightarrow & \mathrm{CH}_1(X)_\Lambda \\ & & \underset{\tilde{\gamma}}{\smile} & & \end{array}$$

whose upward vertical arrows are the naive pull-back homomorphisms and whose downward vertical arrow is the canonical push-forward homomorphism $\pi_* : \mathrm{CH}_1(\check{X}) \to \mathrm{CH}_1(X)$ (see [Ful84, Sec. 20.1]).

We are going to show that $\tilde{\check{\gamma}}(\pi_Y^*(\mathrm{div}(g_1)))$ vanishes, which by the above commutative diagram will imply that $\tilde{\gamma}(\mathrm{div}(g_1))$ vanishes.

Let $\check{Y}_1$ be the irreducible component of $\check{Y}$ containing $\check{C}_1$ and let η be the generic point of $\check{C}_1$. From here we proceed similarly to type 2 data. Choose a basis $a_1, \dots, a_{d-1}$ of the $\mathcal{O}_{\check{Y}_1, \check{C}_1^\infty \cup \{\eta\}} / \mathcal{I}_{\check{C}_1^\infty \cup \{\eta\}}$-module $\mathcal{I}_{\check{C}_1^\infty \cup \{\eta\}} / \mathcal{I}^2_{\check{C}_1^\infty \cup \{\eta\}}$, where $\mathcal{I}$ is the ideal sheaf defining $\check{C}_1$ in $\check{Y}_1$.

Exactly as for type 2 data above we lift the a_i to elements

$$\hat{a}_1, \dots, \hat{a}_{d-1} \in \mathcal{O}_{\check{X}, \check{C}_1^\infty \cup \{\eta\}}$$

and obtain the closed subscheme $\hat{C}_1 \subset \check{X}$ as the Zariski closure of the closed subscheme of $\operatorname{Spec} \mathcal{O}_{\check{X}, \check{C}_1^\infty \cup \{\eta\}}$ defined by $\hat{a}_1 = \cdots = \hat{a}_{d-1} = 0$. Note that $\hat{C}_1$ is regular around $\check{C}_1^\infty \cup \{\eta\}$ and flat over A. Choosing the local parameters as for type 2 data, we can assume that $\hat{C}_1 \cap \check{Y}^{\mathrm{sing}}$ consists of only finitely many points.

Let $\hat{C}_1^\infty$ be the finite set of points consisting of the maximal points of $\hat{C}_1 \cap \check{Y}$ and the points $\hat{C}_1 \cap \check{Y}^{\mathrm{sing}}$. Let $\check{g}_1$ be the rational function on $\check{C}_1$ induced by g_1. There is a unique element $\overline{g}_1 \in \mathcal{O}^\times_{\hat{C}_1 \cap Y, \hat{C}_1^\infty}$ which restricts to $\check{g}_1$ on $\check{C}_1$ and to 1 on the other irreducible components of $\hat{C}_1 \cap Y$. We lift $\overline{g}_1$ to an element $\hat{g}_1 \in \mathcal{O}_{\hat{C}_1, \hat{C}_1^\infty}$.

We observe that $\mathrm{div}(\hat{g}_1)$ is in $\mathrm{Z}_1^g(\check{X})$ and that this element restricts, via $\tilde{\check{\rho}}$, to $\mathrm{div}(\check{g}_1) \in \mathrm{Z}_0(\check{Y}^{\mathrm{sm}})$. So $\mathrm{div}(\hat{g}_1) = \tilde{\check{\gamma}}(\mathrm{div}(\check{g}_1)) = \tilde{\check{\gamma}}(\pi_Y^*(\mathrm{div}(g_1))) \in \mathrm{CH}_1(\check{X})_\Lambda$. Hence $\tilde{\check{\gamma}}(\pi_Y^*(\mathrm{div}(g_1)))$ vanishes in $\mathrm{CH}_1(\check{X})_\Lambda$, as required. □

Corollary 5.2. *Under the conditions of Theorem 3.1, the restriction homomorphism* $\rho : \mathrm{CH}_1(X)_\Lambda \to \mathcal{C}(Y)$ *is an isomorphism.*

Indeed, γ is inverse to ρ.

6. Around the motivic Gysin homomorphism

The definition of the cycle group $\mathcal{C}(Y)$ of a snc variety Y over the perfect field k we gave in Section 2 is purely geometric. However we will see in the next section that under certain assumptions it has a canonical description as a motivic cohomology group. In order to construct such an isomorphism we have to use certain explicit descriptions of the motivic Gysin map for which we did not find a reference in the literature. In this section, we can only sketch the arguments, based on [Voe00], [Kel13] and [De12].

In the following, we work in Voevodsky's triangulated category of geometric motives $DM_{gm}(k) := DM_{gm}(k;\Lambda)$ with coefficients in Λ. Recall that Λ is our fixed commutative ring of coefficients. We say that condition (†) is satisfied if one of the following properties holds:

($\dagger_1$) $\Lambda = \mathbb{Z}[1/p]$ or

($\dagger_2$) $\Lambda = \mathbb{Z}$ and resolution of singularities holds over k in the sense of [FV00, Def. 3.4].

Under the condition (†), which in this section we will always assume to be satisfied, it is shown in [Voe00] and [Kel13] that the category $DM_{gm}(k)$ is a rigid tensor triangulated category. Furthermore, to any variety Z/k, one can associate a motive $M_{gm}(Z) \in DM_{gm}(k)$ and a motive with compact support $M^c_{gm}(Z) \in DM_{gm}(k)$ satisfying certain functorialities. For a smooth equidimensional variety Z/k of dimension d there is a canonical duality isomorphism

$$M_{gm}(Z)^* = M^c_{gm}(Z)(-d)[-2d].$$

For a closed immersion of varieties $Z_1 \to Z_2$ there is a canonical exact triangle

$$M^c_{gm}(Z_1) \to M^c_{gm}(Z_2) \to M^c_{gm}(Z_2 \setminus Z_1) \xrightarrow{\partial} M^c_{gm}(Z_1)[1]. \tag{6.1}$$

For a closed immersion of codimension c of smooth equidimensional varieties $Z_1 \to Z_2$ there is a dual Gysin exact triangle

$$M_{gm}(Z_2 \setminus Z_1) \to M_{gm}(Z_2) \xrightarrow{\mathrm{Gy}} M_{gm}(Z_1)(c)[2c] \xrightarrow{\partial} M_{gm}(Z_2 \setminus Z_1)[1]. \tag{6.2}$$

The following proposition is [De12, Prop. 1.19(iii)].

Proposition 6.1. *Consider a commutative square of varieties*

$$\begin{array}{ccc} Z_1' & \longrightarrow & Z_2' \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z_1 & \longrightarrow & Z_2 \end{array}$$

where the horizontal maps are closed immersions of codimension one between smooth varieties. We assume that in the sense of divisors $f^*(Z_1) = mZ_1'$. Then the square of Gysin maps

$$\begin{array}{ccc} M_{gm}(Z_2') & \xrightarrow{m\cdot \mathrm{Gy}} & M_{gm}(Z_1')(1)[2] \\ {\scriptstyle f_*}\downarrow & & \downarrow \\ M_{gm}(Z_2) & \xrightarrow[\mathrm{Gy}]{} & M_{gm}(Z_1)(1)[2] \end{array}$$

commutes. Here the upper horizontal arrow is m times the Gysin map.

Recall [Voe00, p. 197] that Suslin homology $h_j^S(Y)$ of a variety Y/k can be described in terms of motivic homology as

$$h_j^S(Y) = \mathrm{Hom}_{DM_{gm}(k)}(\Lambda[j], M_{gm}(Y)).$$

Lemma 6.2. *There is an exact sequence*

$$0 \to R \to \mathrm{Z}_0(Y)_\Lambda \to h_0^S(Y) \to 0,$$

where R is the Λ-module of zero-cycles generated by the divisors of rational functions g, on integral closed curves $C \subset Y$, which satisfy the following property: Let $\overline{C}$ be the compactification of C which is normal outside C and let $C^\infty \subset \overline{C}$ be the points not lying over C. We endow C^∞ with the reduced subscheme structure. Then we allow those g satisfying

$$g \in \ker\left(\mathcal{O}^\times_{\overline{C},C^\infty} \to \mathcal{O}^\times_{C^\infty}\right).$$

The lemma is deduced from the definition of Suslin homology, see [Sc07, Theorem 5.1]. Note that the rational functions in Lemma 6.2 are similar to the rational functions in type 1 data in Section 2.

For simplicity of notation, we write

$$\begin{aligned} H^{i,j}(Y) &= \mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(Y), \Lambda(j)[i]), \\ H_c^{i,j}(Y) &= \mathrm{Hom}_{DM_{gm}(k)}(M^c_{gm}(Y), \Lambda(j)[i]), \\ H_{i,j}(Y) &= \mathrm{Hom}_{DM_{gm}(k)}(\Lambda(j)[i], M_{gm}(Y)), \\ H^c_{i,j}(Y) &= \mathrm{Hom}_{DM_{gm}(k)}(\Lambda(j)[i], M^c_{gm}(Y)). \end{aligned}$$

Motivic cohomology has an explicit description in terms of a cdh-cohomology group

$$H^{i,j}(Y) = H^i(Y_{\mathrm{cdh}}, \Lambda(j)), \tag{6.3}$$

where $\Lambda(j) = C_*\Lambda_{tr}(\mathbb{G}_m^{\wedge j})[-j]$ is the bounded above complex of sheaves of Λ-modules defined in [SV02]. If Y/k is smooth we can replace the cdh-topology by the Nisnevich topology in equation (6.3).

Let now Y/k be a smooth integral curve with smooth compactification $\overline{Y}$. We write Y^∞ for the reduced closed subscheme $\overline{Y} \setminus Y$. The Gysin exact triangle (6.2) gives us a morphism $\partial : M_{gm}(Y^\infty)(1)[1] \to M_{gm}(Y)$. Note that there are canonical isomorphisms

$$\begin{aligned} \mathcal{O}(Y^\infty)^\times \otimes_{\mathbb{Z}} \Lambda &= \mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(Y^\infty), \Lambda(1)[1]) \\ &= \mathrm{Hom}_{DM_{gm}(k)}(\Lambda, M_{gm}(Y^\infty)(1)[1]), \end{aligned} \tag{6.4}$$

For the first equality use [MVW, Lec. 4]. Consider the diagram

$$\begin{array}{ccccc} \mathcal{O}^\times_{\overline{Y},Y^\infty} & \xrightarrow{\alpha} & \mathcal{O}(Y^\infty)^\times & \xrightarrow{\beta} & \mathrm{Hom}_{DM_{gm}(k)}(\Lambda, M_{gm}(Y^\infty)(1)[1]) \\ \downarrow{\scriptstyle \mathrm{div}} & & & & \downarrow{\scriptstyle \partial} \\ \mathrm{Z}_0(Y) & & \longrightarrow & & \mathrm{Hom}_{DM_{gm}(k)}(\Lambda, M_{gm}(Y)) = h_0^S(Y) \end{array} \tag{6.5}$$

where the map α is the restriction map to the product of the residue fields of the semi-local ring $\mathcal{O}_{\overline{Y},Y^\infty}$ and β is the map induced by (6.4).

Proposition 6.3. *The diagram* (6.5) *commutes.*

Proof. Let $V \subset \overline{Y}$ be an open neighborbood of Y^∞. We have the duality $h_0^S(Y) = \mathrm{Hom}_{DM_{gm}(k)}(M^c_{gm}(Y), \Lambda(1)[2])$ and the isomorphism

$$\mathcal{O}_V^\times(V) \otimes \Lambda = \mathrm{Hom}_{DM_{gm}}(M_{gm}(V), \Lambda(1)[1]).$$

Via these isomorphisms the two morphisms

$$\mathrm{Hom}_{DM_{gm}}(M_{gm}(V), \Lambda(1)[1]) \rightrightarrows \mathrm{Hom}_{DM_{gm}(k)}(M^c_{gm}(Y), \Lambda(1)[2])$$

from diagram (6.5) are by definition both induced by the boundary map of the homotopy cartesian square

$$\begin{array}{ccc} M_{gm}(V) & \longrightarrow & M_{gm}(\overline{Y}) \\ \downarrow & & \downarrow \\ M_{gm}(V/Y^\infty) & \longrightarrow & M^c_{gm}(Y). \end{array}$$

Here, for a morphism $Z_1 \to Z_2$, the relative motive $M_{gm}(Z_2/Z_1)$ is defined as the mapping cone of the morphism of complexes of Nisnevich sheaves with transfers $M_{gm}(Z_1) \to M_{gm}(Z_2)$; we view it as an object of $DM_{gm}(k)$. The motive with compact support $M^c_{gm}(Y)$ can be identified with $M_{gm}(\overline{Y}/Y^\infty)$, which explains the lower horizontal map in the square. The square is homotopy cartesian as the cones of the upper and of the lower horizontal maps are isomorphic to $M_{gm}(\overline{Y}/V)$.

One way to get the boundary map of the cartesian square is, via the above identification, as the composition of

$$M_{gm}(\overline{Y}/Y^\infty) \to M_{gm}(\overline{Y}/V) \to M_{gm}(V)[1],$$

which corresponds to the left/lower path in the diagram (6.5). Indeed, by the Gysin isomorphism we get

$$\mathrm{Hom}_{DM_{gm}}(M_{gm}(\overline{Y}/V), \Lambda(1)[2]) = \bigoplus_{y \in \overline{Y} \setminus V} \Lambda.$$

Another way is as the composition of

$$M^c_{gm}(Y) \to M_{gm}(Y^\infty)[1] \to M_{gm}(V)[1],$$

which corresponds to the upper/right path in the diagram (6.5). This is clear in view of the isomorphism (6.4). □

Let Y be equidimensional of dimension d and let $y \in Y^{\mathrm{sm}}$ be a closed point. The Gysin morphism and the description of motivic cohomology in terms of Milnor K-theory [MVW, Lec. 5] induces a morphism

$$K^M_{j-d}(k(y)) \xrightarrow{\iota_y} H^{j+d,j}(Y).$$

Here Milnor K-theory is taken with Λ-coefficients.

Proposition 6.4. *Let Y/k be a snc variety of dimension d and let $U \subset Y^{\mathrm{sm}}$ be a dense open subscheme.*

(i) For $i - j > d$, the group $H^{i,j}(Y)$ vanishes.
(ii) For $j \geq d$, the map

$$\oplus_y \iota_y : \bigoplus_{y \in U_{(0)}} K^M_{j-d}(k(y)) \to H^{j+d,j}(Y) \tag{6.6}$$

is surjective.

Proof. (i) This is clear from the calculation of $H^{i,j}(Y)$ in terms of cdh-cohomology as the complex $\Lambda(j)$ lies in $D^{\leq j}(Y;\Lambda)$ and the cdh-cohomological dimension of Y is d [SV02, App.].

(ii) We use a double induction on d and on the number r of irreducible components of Y. For $Y = Y_1$ smooth we can use the coniveau spectral sequence, see (9.1), to get an isomorphism

$$H^{j+d,j}(Y_1) \cong A_0(Y_1, j-d)$$

where the right-hand side is Rost's Chow group with coefficients in Milnor K-theory [R96]. An elementary moving technique shows that the canonical map

$$\bigoplus_{y \in U_{(0)}} K^M_{j-d}(k(y)) \to A_0(Y_1, j-d) \tag{6.7}$$

is surjective. In fact, for $\xi \in K^M_{j-d}(k(x))$ with x closed in $Y_1 \setminus U$, choose $x' \in U_{(1)}$ such that x lies in the regular locus of $W = \overline{\{x'\}}$. Choose $\xi' \in K^M_{j-d+1}(k(x'))$ such that the residue symbol of ξ' at x is ξ and such that the residue symbols of ξ' at the other points of $W \cap (Y_1 \setminus U)$ vanish. Now $[\xi] \in A_0(Y_1, j-d)$ is equal to $-\sum_{y \in (W \cap U)_0} [\partial_y(\xi')]$, which belongs to the image of (6.7).

For general Y with irreducible components $Y_1, \ldots, Y_r$, we consider the decomposition $Y = Y_1 \cup Y'$ with $Y' = Y_2 \cup \ldots \cup Y_r$. Part of the exact Mayer–Vietoris sequence for the cdh-covering $Y_1 \coprod Y' \to Y$ reads

$$H^{j+d-1,j}(Y_1 \cap Y') \xrightarrow{\alpha} H^{j+d,j}(Y) \to H^{j+d,j}(Y_1) \oplus H^{j+d,j}(Y').$$

By the induction assumption, we already know that the image of $\oplus_{y\in U_{(0)}}\iota_y$ maps surjectively onto the direct sum on the right-hand side. So we only have to show that the image of α also lies in the image of $\oplus_{y\in U_{(0)}}\iota_y$.

By the induction assumption, we know that the homomorphism

$$\iota_y : \bigoplus_{y\in(Y_1\cap Y')^{\mathrm{sm}}_{(0)}} K^M_{j-d+1}(k(y)) \to H^{j+d-1,j}(Y_1\cap Y') \tag{6.8}$$

induced by the Gysin map is surjective. We have to give an explicit calculation of its composition with α.

Claim 6.5. The image of the composition of (6.8) and α is contained in the image of the map (6.6).

Let y be a closed point in $(Y_1\cap Y')^{\mathrm{sm}}$. By the Bertini theorem, we can choose a snc subcurve C on Y containing y such that $C\cap U$ is dense in C (in the case of a finite base field k, use [P08]). Let $C_1 = C\cap Y_1$ and $C' = C\cap Y'$. In the triangulated category $DM_{gm}(k)$, we have the commutative diagram of Gysin maps

$$\begin{array}{ccccccc} M_{gm}(Y_1\cap Y') & \longrightarrow & M_{gm}(Y_1)\oplus M_{gm}(Y') & \longrightarrow & M_{gm}(Y) & \xrightarrow{\partial} & \\ \downarrow{\scriptstyle \mathrm{Gy}} & & \downarrow{\scriptstyle \mathrm{Gy}} & & \downarrow{\scriptstyle \mathrm{Gy}} & & \\ M_{gm}(C_1\cap C')(c)[2c] & \longrightarrow & (M_{gm}(C_1)\oplus M_{gm}(C'))(c)[2c] & \longrightarrow & M_{gm}(C)(c)[2c] & \longrightarrow & \end{array} \tag{6.9}$$

where $c = d-1$. These Gysin maps are constructed via the Gysin maps for smooth schemes by using the Cech simplicial scheme associated to the covering by irreducible components. This shows that we can assume $d = 1$ in the proof of the claim. For $d = 1$, one gets a commutative diagram

$$\begin{array}{ccc} K^M_j(\mathcal{O}_{Y_1,Y_1\setminus(U\cap Y_1)}) & \twoheadrightarrow & K^M_j(Y_1\cap Y') \\ \downarrow & & \downarrow \wr \\ H^{j,j}(\mathcal{O}_{Y_1,Y_1\setminus(U\cap Y_1)}) & \longrightarrow & H^{j,j}(Y_1\cap Y') \\ \partial\downarrow & & \downarrow\alpha \\ \bigoplus_{y\in(U\cap Y_1)_{(0)}} H^{j-1,j-1}(y) & \xrightarrow[\oplus \mathrm{Gy}_y]{} & H^{j+1,j}(Y). \end{array}$$

Here, the Milnor K-group of a ring is simply the quotient of the tensor algebra over the units by the Steinberg relations and the maps from Milnor

K-theory to motivic cohomology are as defined in [MVW, Lec. 5]. The lower square is commutative because for an open neighborhood V of $Y_1 \cap Y'$ in Y_1, the composition of ∂ from (6.9) with $M_{gm}(Y_1 \cap Y') \to M_{gm}(V)$ is equal to the composition of the maps

$$M_{gm}(Y) \to M_{gm}(Y/(V \cup Y')) \xleftarrow{\sim} M_{gm}(Y_1/V) \xrightarrow{\partial_{(Y_1,V)}} M_{gm}(V)[1].$$

We now explain the isomorphism in the middle: by Mayer–Vietoris applied to the covers $Y = Y_1 \cup Y'$ and $V \cup Y'$ ([Voe00, Prop. 4.1.3], [Kel13, Prop. 5.5.4]), one has a commutative diagram in which the rows are exact triangles

$$\begin{array}{ccccc} M_{gm}(V) & \longrightarrow & M_{gm}(V \cup Y') & \longrightarrow & M_{gm}(Y'/Y' \cap Y_1) \\ \downarrow & & \downarrow & & \| \\ M_{gm}(Y_1) & \longrightarrow & M_{gm}(Y_1 \cup Y') & \longrightarrow & M_{gm}(Y'/Y' \cap Y_1) \\ \downarrow & & \downarrow & & \\ M_{gm}(Y_1/V) & \xrightarrow{\sim} & M_{gm}(Y/V \cup Y'). & & \end{array}$$

In addition, by definition of $M_{gm}(Y_1/V)$ and $M_{gm}(Y/V \cup Y')$, the columns are exact. This implies the isomorphism. This finishes the proof of the claim and also of Proposition 6.4. □

Let now X/k be a connected smooth variety and $Y \subset X$ a snc divisor. For a closed point $y \in Y^{\rm sm}$, let

$$\lambda_y : k(y)^\times \to h_0^S(X \setminus Y) \tag{6.10}$$

be the composite homomorphism

$$k(y)^\times \xrightarrow{\sim} H^{1,1}(y) = H_{1,1}(y) \to H_{1,1}(Y^{\rm sm}) \xrightarrow{\partial} h_0^S(X \setminus Y).$$

Proposition 6.6. *The maps λ_y ($y \in Y^{\rm sm}$) are uniquely characterized by the following property. Let $C \subset X$ be an integral closed curve disjoint from $Y^{\rm sing}$ and not contained in Y. Let $C^\infty = (C \cap Y)_{\rm red}$ and let η be the generic point of C. For $y \in C^\infty$, let m_y be the intersection multiplicity of C and Y at y. Consider an element $g \in \mathcal{O}^\times_{C,\{\eta\}\cup C^\infty}$. Then we have*

$$\mathrm{div}(g) + \sum_{y \in C^\infty} m_y \lambda_y(g|_y) = 0 \quad \text{in} \quad h_0^S(X \setminus Y). \tag{6.11}$$

Proof. Uniqueness is clear: take a curve C intersecting Y transversally and a rational function g equal to 1 at all but one point y of C^∞, this determines the map λ_y. To prove (6.11), we first reduce to the case $\dim(X) = 1$ by applying Proposition 6.1 with $Z_1 = Y^{\mathrm{sm}}$, $Z_2 = X \setminus Y^{\mathrm{sing}}$ and Z_2' the normalization of C. The case $\dim(X) = 1$ is a consequence of Proposition 6.3. □

7. Motivic interpretation of $\mathcal{C}(Y)$

For the category $DM_{gm}(k)$ we use the same notation as in Section 6. In this section k is a perfect field, Y/k is a projective snc variety of dimension d and $\tau : Y \to \mathbb{N}$ is the function locally counting irreducible components. Let p be the exponential characteristic of k.

The localization exact triangle

$$M_{gm}(Y^{\mathrm{sing}}) \to M_{gm}(Y) \to M^c_{gm}(Y^{\mathrm{sm}}) \to M_{gm}(Y^{\mathrm{sing}})[1]$$

from [Voe00, Prop. 4.1.5], [Kel13, Prop. 5.5.5] induces an exact sequence

$$H^{2d-1,d}(Y^{\mathrm{sing}}) \to h_0^S(Y^{\mathrm{sm}}) \to H^{2d,d}(Y) \to 0. \tag{7.1}$$

In fact $h_0^S(Y^{\mathrm{sm}}) = H_c^{2d,d}(Y^{\mathrm{sm}})$ by duality. We remark that Proposition 6.4 implies that $H^{2d,d}(Y^{\mathrm{sing}}) = 0$ and that $H^{2d-1,d}(Y^{\mathrm{sing}})$ is spanned by the images of the groups $k(y)^\times \otimes_{\mathbb{Z}} \Lambda$ with $\tau(y) = 2$ via the Gysin maps. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y and assume that y is contained in the two components Y_1 and Y_2. The composition λ_y of the maps

$$k(y)^\times \to H^{2d-1,d}(Y^{\mathrm{sing}}) \to h_0^S(Y^{\mathrm{sm}}) = \oplus_{i=1}^r h_0^S(Y_i \setminus (Y_i \cap Y^{\mathrm{sing}}))$$

is equal to the sum of the two maps defined in (6.10) for the two components Y_1 and Y_2. This is clear by using functoriality along the map $Y_1 \coprod Y_2 \to Y$. Thus, the exact sequence (7.1) induces an exact sequence

$$\bigoplus_{\tau(y)=2} k(y)^\times \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\oplus_y \lambda_y} h_0^S(Y^{\mathrm{sm}}) \to H^{2d,d}(Y) \to 0.$$

By Lemma 6.2, we can identify the quotient of $\mathrm{Z}_0(Y^{\mathrm{sm}})_\Lambda$ by the sub-Λ-module generated by type 1 relations with $h_0^S(Y^{\mathrm{sm}})$ (see Section 2). Therefore, the cycle group $\mathcal{C}(Y)$ has the analogous presentation

$$\bigoplus_{(C_2,g_2)} \Lambda \xrightarrow{\mathrm{div}} h_0^S(Y^{\mathrm{sm}}) \to \mathcal{C}(Y) \to 0.$$

The direct sum on the left runs over all type 2 data.

The canonical surjective map

$$\theta : \bigoplus_{(C_2,g_2)} \Lambda \to \bigoplus_{\tau(y)=2} k(y)^\times \otimes_{\mathbb{Z}} \Lambda$$

which sends $1 \in \Lambda_{(C_2,g_2)}$ to $g_2|_{C_2\cap Y^{\mathrm{sing}}}$ gives rise to the left square in the diagram

$$\begin{array}{ccccccc} \bigoplus_{(C_2,g_2)}\Lambda & \xrightarrow{\mathrm{div}} & h_0^S(Y^{\mathrm{sm}}) & \longrightarrow & \mathcal{C}(Y) & \longrightarrow & 0 \\ \theta\downarrow & & \| & & \Gamma \downarrow \wr & & \\ \bigoplus_{\tau(y)=2} k(y)^\times \otimes_{\mathbb{Z}} \Lambda & \xrightarrow{\oplus_y \lambda_y} & h_0^S(Y^{\mathrm{sm}}) & \longrightarrow & H^{2d,d}(Y) & \longrightarrow & 0. \end{array}$$

In fact, the left square commutes by Proposition 6.6, so there is a unique isomorphism Γ making the diagram commutative. In summary, we have shown

Theorem 7.1. *Assume that the ring of coefficients Λ satisfies condition (†) from Section 6. For Y/k a projective snc variety, there is a unique isomorphism Γ making the diagram*

$$\begin{array}{ccc} \mathrm{Z}_0(Y^{\mathrm{sm}}) & & \\ \downarrow & \searrow & \\ \mathcal{C}(Y) & \xrightarrow{\Gamma} & H^{2d,d}(Y) \end{array}$$

commutative. The diagonal map is induced by the Gysin maps of closed points $y \in Y^{\mathrm{sm}}$ as in Proposition 6.4(ii) for $j = d$.

Remark 7.2. Theorem 7.1 remains true for $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. To see this, simply tensor the diagram taken with $\Lambda = \mathbb{Z}[1/p]$ coefficients with $\mathbb{Z}/m\mathbb{Z}$ and use cohomological dimension to show that

$$H^{2d,d}(Y, \mathbb{Z}[1/p]) \otimes_{\mathbb{Z}[1/p]} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} H^{2d,d}(Y, \mathbb{Z}/m\mathbb{Z})$$

is an isomorphism.

Remark 7.3. The natural homomorphism $\mathrm{Z}_0(U) \to \mathcal{C}(Y)$ is surjective for any dense Zariski open subset $U \subset Y^{\mathrm{sm}}$, as can be seen by moving zero-cycles on Y^{sm} by type 1 relations.

We expect that the condition of projectivity in Theorem 7.1 can be weakened to quasi-projectivity.

8. Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1, which we subdivide according to the extra assumptions. All constructions are based on the isomorphism $\mathcal{C}(Y) \cong H^{2d,d}(Y)$ of Theorem 7.1.

8.1. Proof of Theorem 3.1 (i) and (ii)

The assumption is that the (perfect) residue field k is separably closed (thus algebraically closed), in (i), or finite, in (ii).

By Corollary 9.2, the étale cycle map $\mathcal{C}(Y) \xrightarrow{\sim} H^{2d}(Y_{\text{ét}}, \Lambda(d))$ is an isomorphism. Define ρ such that the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(X)_\Lambda & \longrightarrow & H^{2d}(X_{\text{ét}}, \Lambda(d)) \\ {\scriptstyle\rho}\downarrow & & \downarrow\wr \\ \mathcal{C}(Y) & \xrightarrow{\sim} & H^{2d}(Y_{\text{ét}}, \Lambda(d)) \end{array}$$

is commutative. Here the upper horizontal map is the étale cycle map on X.

8.2. Proof of Theorem 3.1 (iv)

Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. Recall that the Milnor K-sheaf with Λ-coefficients is defined as the Nisnevich sheafification of the presheaf on affine schemes $\mathrm{Spec}\,(R)$ given as follows: take the quotient of the tensor algebra

$$R \mapsto \Lambda \otimes_{\mathbb{Z}} \bigoplus_{i \in \mathbb{N}} \underbrace{R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times}_{i \text{ times}}$$

by the two-sided ideal generated by the elements $a \otimes (1-a)$ with $a, 1-a \in R^\times$.

Rigidity for Milnor K-theory is the statement that the stalk of the Milnor K-sheaf in the Nisnevich topology at a point $y \in Y$ satisfies

$$(\mathcal{K}^M_{Y,j})_y \xrightarrow{\sim} K^M_j(k(y)).$$

The same rigidity property holds for motivic cohomology. More precisely, consider the morphism of sites $\epsilon : Y_{\mathrm{cdh}} \to Y_{\mathrm{Nis}}$ and the motivic complex in the Nisnevich topology defined as $\Lambda(j)_{Y,\mathrm{Nis}} = R\epsilon_* \Lambda(j)_Y$. Then, for $y \in Y$, the map $\mathcal{H}^i(\Lambda(j)_{Y,\mathrm{Nis}})_y \to H^{i,j}(y)$ is an isomorphism. For smooth Y, this follows from [HY07, Cor. 0.4]. For a general snc variety Y with irreducible components $Y_1, \dots, Y_r$, one reduces the statement to the smooth case by means of the descent spectral sequence of the cdh-covering

$$\coprod_{i=1}^{r} Y_i \to Y.$$

Thus, using the isomorphism between Milnor K-theory and motivic cohomology for fields [MVW, Lec. 5], we obtain

Proposition 8.1. *For $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p, there is a canonical isomorphism of Nisnevich sheaves*

$$S : \mathcal{K}^M_{Y,j} \xrightarrow{\sim} \mathcal{H}^j(\Lambda(j)_{Y,\mathrm{Nis}}).$$

For $i > j$, the sheaf $\mathcal{H}^i(\Lambda(j)_{Y,\mathrm{Nis}})$ vanishes.

Combining this proposition with Theorem 7.1 and cohomological dimension, we see that there is an isomorphism

$$\mathcal{C}(Y) \xrightarrow{\sim} H^d(Y_{\mathrm{Nis}}, \mathcal{K}^M_{Y,d}). \tag{8.1}$$

As A is equicharacteristic, the Gersten conjecture for Milnor K-theory [Ker09] implies that there is a canonical isomorphism

$$\mathrm{CH}_1(X)_\Lambda \xrightarrow{\sim} H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{X,d}).$$

Define ρ such that the diagram

$$\begin{array}{ccc} \mathrm{CH}_1(X)_\Lambda & \xrightarrow{\sim} & H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{X,d}) \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ \mathcal{C}(Y) & \xrightarrow{\sim} & H^d(Y_{\mathrm{Nis}}, \mathcal{K}^M_{Y,d}) \end{array}$$

is commutative. The right vertical arrow is the restriction homomorphism.

8.3. Proof of Theorem 3.1 (iii)

In view of Remark 3.2, we can assume that $\Lambda = \mathbb{Z}[1/(p(d-1)!)]$. As X is regular, its K-theory of vector bundles $K_0(X)$ is the same as its K-theory of coherent sheaves $K_0'(X)$ [Gi05, Sec. 2.3.2]. Let $F_0 \subset K_0'(X)$ be the subgroup spanned by classes of coherent sheaves supported in dimension 0.

Claim 8.2. We have $F_0 = 0 \subset K_0'(X)$.

Proof. By linearity, it is enough to see that the class of the skyscraper sheaf $i_{x,*}k(x)$ in $K_0'(X)$ is zero, where x is a closed point of Y and $i_x : x \to X$ is the immersion. For simplicity of notation, we write $k(x)$ for this coherent sheaf of $\mathcal{O}_X$-modules. Choose a regular connected closed subscheme $Z \subset X$ of dimension 1 with $x \in Z$ and Z/A flat, $Z = \operatorname{Spec} R$. Let $\pi_Z \in R$ be a prime element. The short exact sequence of coherent sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_Z \xrightarrow{\pi_Z} \mathcal{O}_Z \to k(x) \to 0$$

implies $[k(x)] = 0 \in K_0'(X)$. □

Let α be the composite homomorphism

$$\mathrm{CH}_1(X) \to K_0'(X)/F_0 = K_0'(X) \xleftarrow{\sim} K_0(X)$$

where the first map assigns to the cycle class of an integral curve the class of its structure sheaf [Gi05, Thm. 34]. As K-theory is contravariant, one has a restriction homomorphism $K_0(X) \to K_0(Y)$.

From the Grothendieck construction based on the projective bundle formula, one obtains a Chern class map $c_d : K_0(Y) \to H^{2d,d}(Y)$, see [W13, Sec. V.11]. Let ρ' be the map

$$\mathrm{CH}_1(X) \xrightarrow{\alpha} K_0(X) \to K_0(Y) \xrightarrow{c_d} H^{2d,d}(Y) \xrightarrow{\sim} \mathcal{C}(Y). \tag{8.2}$$

The following claim finishes the proof of Theorem 3.1(iii).

Claim 8.3. The map

$$\mathrm{Z}_1^g(X) \to \mathrm{CH}_1(X) \xrightarrow{\rho'} \mathcal{C}(Y) \tag{8.3}$$

is additive and is $(-1)^{d-1}(d-1)!$ times the map induced by the restriction homomorphism $\tilde{\rho} : \mathrm{Z}_1^g(X) \to \mathrm{Z}_0(Y^{\mathrm{sm}})$.

Claim 8.3 implies that the map ρ defined as ρ' tensored by Λ and divided by $(-1)^{d-1}(d-1)!$ satisfies the properties required by Theorem 3.1.

Proof of Claim 8.3. We extend the argument of [BS98, proof of Prop. 2]. First, we show additivity of the composite map (8.3). Then we prove the claim for an integral cycle in $\mathrm{Z}_1^g(X)$.

Let $F_0^{\mathrm{sm}} \subset K_0(Y)$ be the subgroup generated by the classes $[k(x)]$ with $x \in Y^{\mathrm{sm}}$ of dimension 0. Note that the coherent sheaf $k(x)$ of $\mathcal{O}_Y$-modules has finite projective dimension, so the element $[k(x)]$ is well-defined in $K_0(X)$. We prove that $c_i(w)$ vanishes in $H^{2i,i}(Y)$ for $w \in F_0^{\mathrm{sm}}$ and $i < d$. Indeed, there is an open subscheme $U \subset Y$ such that $Y \setminus U$ consists of finitely many closed points in Y^{sm} and such that $w|_U = 0$. Now the Gysin exact sequence

$$0 = H^{2i-2d,i-d}(Y \setminus U) \to H^{2i,i}(Y) \to H^{2i,i}(U)$$

and the vanishing of $c_i(w)|_U$ imply the vanishing of $c_i(w)$. It follows, by the Whitney sum formula, that $c_d|_{F_0^{\mathrm{sm}}}$ is additive. As the image of $\mathrm{Z}_1^g(X)$ in $K_0(Y)$ is contained in F_0^{sm}, the composite map (8.3) is additive as well.

Now we prove the claim for Z an integral cycle in $\mathrm{Z}_1^g(X)$. Let y be the point $(Z \cap Y)_{\mathrm{red}}$ and let n be the intersection multiplicity of Z and Y. Let Y_1 be the irreducible component of Y containing y.

The commutative diagram with exact columns (Gysin sequence; note that $H_y^{2d,d}(Y) = H_y^{2d,d}(Y_1) = \Lambda$ as y is a non-singular point)

$$\begin{array}{ccccccc}
 & & & & \Lambda & = & \Lambda \\
 & & & & \downarrow{\scriptstyle \mathrm{Gy}_y} & & \downarrow{\scriptstyle \mathrm{Gy}_y} \\
K_0(X) & \longrightarrow & K_0(Y) & \xrightarrow{c_d} & H^{2d,d}(Y) & \longrightarrow & H^{2d,d}(Y_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_0(X \setminus Z) & \longrightarrow & K_0(Y \setminus \{y\}) & \xrightarrow{c_d} & H^{2d,d}(Y \setminus \{y\}) & \longrightarrow & H^{2d,d}(Y_1 \setminus \{y\})
\end{array}$$

shows that the image γ of $Z \in \mathrm{Z}_1^g(X)$ in $H^{2d,d}(Y)$ satisfies $\gamma|_{Y \setminus \{y\}} = 0$. Thus $\gamma = \mathrm{Gy}_y(n')$ for a unique $n' \in \Lambda$. Note that the Gysin maps in the above diagram are injective. In fact, the composition of the Gysin map on the right with the degree map $H^{2d,d}(Y_1) \cong \mathrm{CH}_0(Y_1) \xrightarrow{\deg} \Lambda$ is multiplication by $[k(y) : k]$, which is injective (recall that $\Lambda = \mathbb{Z}[1/(p(d-1)!)]$).

In the final step, we give a different description of $\gamma|_{Y_1}$. The image of Z in $K_0(Y_1)$ is $n[k(y)]$ by [Gi05, p. 255]. We have a commutative diagram

$$\begin{array}{ccc}
K_0(Y) & \xrightarrow{c_d} & H^{2d,d}(Y) \\
\downarrow & & \downarrow \\
K_0(Y_1) & \xrightarrow{c_d} & H^{2d,d}(Y_1) \cong \mathrm{CH}_0(Y_1).
\end{array}$$

By the non-singular Grothendieck–Riemann–Roch theorem [Ful84, Sec. 15.3] the bottom Chern class satisfies

$$c_d([k(y)]) = \mathrm{Gy}_y((-1)^{d-1}(d-1)!)$$

in $H^{2d,d}(Y_1)$. As $nc_d([k(y)]) = \gamma|_{Y_1}$ we get $n' = n(-1)^{d-1}(d-1)!$. □

9. Special fields

Let Y/k be a proper snc variety of dimension d and let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. In this section, we calculate the group $\mathcal{C}(Y)$ in terms of étale cohomology when k is finite or algebraically closed. This calculation shows that for these special fields, our main result, Theorem 1.1, is equivalent to results of Saito–Sato and Bloch [SS10], [EW13, App.].

We write $H^{i,j}_{\text{ét}}(Y)$ for the étale cohomology group $H^i(Y_{\text{ét}}, \Lambda(j))$. Recall that there is a functorial étale realization homomorphism

$$R^{i,j}_{\text{ét}} : H^{i,j}(Y) \to H^{i,j}_{\text{ét}}(Y).$$

For smooth varieties, it is discussed in [MVW, Thm. 10.3]. For snc varieties, one can define it via the smooth Cech simplicial scheme corresponding to the covering of Y by its irreducible components.

Proposition 9.1. *Let k be a finite field or an algebraically closed field and let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. For a snc variety Y/k and $s \geq d$, $i \in \mathbb{Z}$, the realization map $R^{i,s}_{\text{ét}}$ is an isomorphism, except possibly when k is finite, $i = 2d+1$ and $s = d$.*

Proof. We give the proof for Y/k smooth first and then deduce the general case by a Mayer–Vietoris sequence.

Case Y/k smooth: Consider the coniveau spectral sequences, respectively in the Nisnevich and in the étale topology:

$$E_1^{i,j}(s) = \bigoplus_{y \in Y^{(i)}} H^{j-i,s-i}(y) \Rightarrow H^{i+j,s}(Y), \tag{9.1}$$

$$E_{1,\text{ét}}^{i,j}(s) = \bigoplus_{y \in Y^{(i)}} H^{j-i,s-i}_{\text{ét}}(y) \Rightarrow H^{i+j,s}_{\text{ét}}(Y). \tag{9.2}$$

Recall the following facts.

(i) (Beilinson–Lichtenbaum conjecture) For any field F of characteristic prime to m and any $i \le j$, the realization map

$$R^{i,j}_{\text{ét}} : H^{i,j}(F) \to H^{i,j}_{\text{ét}}(F)$$

is an isomorphism [Voe11].

(ii) $E_1^{i,j}(s) = 0$ for $j > s$, by the definition of the motivic complex [MVW, Lec. 3].

(iii) (Cohomological dimension) $E^{i,j}_{1,\text{ét}}(s) = 0$ for $j > d+1$ if k is finite, for $j > d$ if k is algebraically closed.

(iv) (Kato conjecture) For k finite and $i \neq d$, we have $E^{i,d+1}_{2,\text{ét}}(d) = 0$ [KS12].

Comparing the spectral sequences and using (i)-(iv), we get that the étale realization map induces an isomorphism

$$E^{i,j}_{\infty}(s) \xrightarrow{\sim} E^{i,j}_{\infty,\text{ét}}(s)$$

if $s \ge d$ except possibly if $s = i = d$ and $j = d+1$. This proves Proposition 9.1 for Y smooth.

Case Y/k snc variety: Let $Y = Y_1 \cup \cdots \cup Y_r$ be the decomposition into irreducible components and set $Y' = Y_2 \cup \cdots \cup Y_r$. We proceed by induction on r. The case $r = 1$ is shown above, so assume $r > 1$. Consider the étale realization morphism of exact Mayer–Vietoris sequences

$$\begin{array}{ccccc}
H^{i-1,s}(Y_1) \oplus H^{i-1,s}(Y') & \longrightarrow & H^{i-1,s}(Y_1 \cap Y') & \longrightarrow & H^{i,s}(Y) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
H^{i-1,s}_{\text{ét}}(Y_1) \oplus H^{i-1,s}_{\text{ét}}(Y') & \longrightarrow & H^{i-1,s}_{\text{ét}}(Y_1 \cap Y') & \longrightarrow & H^{i,s}_{\text{ét}}(Y) \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow H^{i,s}(Y_1) \oplus H^{i,s}(Y') & \longrightarrow & H^{i,s}(Y_1 \cap Y') \\
\downarrow & & \downarrow \\
\longrightarrow H^{i,s}_{\text{ét}}(Y_1) \oplus H^{i,s}_{\text{ét}}(Y') & \longrightarrow & H^{i,s}_{\text{ét}}(Y_1 \cap Y').
\end{array}$$

By the five lemma, the middle vertical map is an isomorphism, since, by the induction assumption, the other vertical maps are isomorphisms. □

For k algebraically closed, consider the composite map

$$\deg : \mathcal{C}(Y) \to \bigoplus_{1 \le i \le r} \mathrm{CH}_0(Y_i)_\Lambda \to \Lambda^r \tag{9.3}$$

in which the first map is induced by the canonical maps $\mathrm{Z}_0(Y^{\mathrm{sm}}) \to \mathrm{Z}_0(Y_i)$ and the second map is the direct sum of the usual degree maps of zero-cycles of the irreducible components of Y.

Combining Proposition 9.1 with Theorem 7.1, we get

Corollary 9.2. *For k finite or algebraically closed and Y/k a projective snc variety, the étale cycle map $\mathcal{C}(Y) \to H^{2d,d}_{\text{ét}}(Y)$ is an isomorphism.*

For k algebraically closed and Y/k a projective snc variety, the degree map (9.3) *is an isomorphism.*

As a further application, we discuss zero-cycles over certain 1-dimensional and 2-dimensional local fields. Over 1-dimensional local fields, this reproves a result of [SS10].

Proposition 9.3 (Saito–Sato)**.** *Let K be a local field with finite residue field k of characteristic p. Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. For any variety X_K/K, the group $\mathrm{CH}_0(X_K)_\Lambda$ is finite.*

Proof. By a simple resolution technique and the Gabber–de Jong alteration theorem [ILO, Thm. 0.3], we can assume, without loss of generality, that there is a regular scheme X which is projective and flat over A such that the reduced special fiber Y of X is a simple normal crossings divisor and such that the generic fiber of X is X_K. Here A is the ring of integers in K. The left-hand side map in

$$\mathrm{CH}_0(X_K)_\Lambda \twoheadleftarrow \mathrm{CH}_1(X)_\Lambda \xrightarrow{\sim} \mathcal{C}(Y)$$

is surjective and the right-hand side map is an isomorphism by Theorem 1.1. We conclude by the isomorphism $\mathcal{C}(Y) \xrightarrow{\sim} H^{2d}(Y_{\text{ét}}, \Lambda(d))$ from Corollary 9.2 and the finiteness of étale cohomology. □

Proposition 9.4. *Let K be the field of formal Laurent power series $k((\pi))$ with $[k : \mathbb{Q}_p] < \infty$, $A = k[[\pi]]$. For any variety X_K/K and for $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p, the group $\mathrm{CH}_0(X_K)_\Lambda$ is finite.*

Proof. We can assume, without loss of generality, that $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ for a prime ℓ. As the map $\mathrm{CH}_0(X_K)_\Lambda \to \mathrm{CH}_0(X_{X_{K(\mu_\ell)}})_\Lambda$ is injective, we can also assume that A contains a primitive ℓ-th root of unity, which we fix from here on. By a simple resolution technique and the Gabber–de Jong alteration theorem [ILO, Thm. 0.3], we can assume that there exists a regular scheme X, projective and flat over A, whose reduced special fiber Y/k is a simple normal crossings divisor and whose generic fiber is X_K.

The left-hand side map in

$$\mathrm{CH}_0(X_K)_\Lambda \twoheadleftarrow \mathrm{CH}_1(X)_\Lambda \xrightarrow{\sim} \mathcal{C}(Y)$$

is surjective and the right-hand side map is an isomorphism by Theorem 1.1.

Let $d = \dim(Y)$. Note that $\mathcal{C}(Y) \cong H^{2d,d}(Y)$ by Theorem 7.1. By a Mayer–Vietoris sequence argument for a closed covering similar to the proof of Proposition 9.1, we reduce to showing that $H^{d_W+d,d}(W)$ is finite for any smooth projective variety W/k of dimension $d_W \le d$.

The coniveau spectral sequence for W, see (9.1), degenerates to an isomorphism

$$H^{d_W+d,d}(W) \cong \operatorname{coker}\,[\oplus_{w\in W_{(1)}} K^M_{d-d_W+1}(k(w)) \to \oplus_{w\in W_{(0)}} K^M_{d-d_W}(k(w))],$$

from which we deduce:

(i) For $d_W < d-2$, we have $H^{d_W+d,d}(W) = 0$.

(ii) For $d_W = d-2$, there is an isomorphism

$$H^{d_W+d,d}(W) \cong H_0(KC^{(0)}(W))$$

onto Kato homology with Λ-coefficients, see e.g. [KS12].

(iii) For $d_W = d-1$, there is an isomorphism $H^{d_W+d,d}(W) \cong SK_1(W)_\Lambda$, see [Fo15].

(iv) For $d_W = d$, there is an isomorphism $H^{d_W+d,d}(W) \cong \mathrm{CH}_0(W)_\Lambda$.

Indeed, for a finite extension $F/\mathbb{Q}_p$ the Milnor K-group with finite coefficients $K^M_i(F)$ vanishes for $i > 2$ [W13, Prop. VI.7.1], hence (i). For (ii), we note that by the Bloch–Kato conjecture [Voe11] and the fixed ℓ-th root of unity, there is an isomorphism $K^M_i(F) \cong H^i(F_{\text{ét}}, \Lambda(i-1))$ for any field F/k.

It is known, see the introduction to [KS12] and [Fo15, Rmk. 5.2], that the groups $SK_1(W)_\Lambda$ and $H_0(KC^{(0)}(W))$ are finite. Indeed, for the latter, we can assume, without loss of generality (use [ILO], Theorem X.2.4), that there is a regular model $\mathcal{W}$ of W over $\mathcal{O}_k$ whose reduced closed fibre is a snc divisor. Then, by [KS12, Thm. 0.4], the group $H_0(KC^{(0)}(W))$ is isomorphic to a finitely generated combinatorial homology group which depends on the closed fibre of $\mathcal{W}$. The finiteness of $\mathrm{CH}_0(W)_\Lambda$ is shown in Proposition 9.3. □

10. Open problems

Let us keep the notation of Section 3 and consider the ring of coefficients $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m arbitrary, i.e. we allow $p|m$ in positive characteristic. Set

$Y_n = X \otimes_A A/(\pi^n)$, where π is a prime element of the discrete valuation ring A. Recall that K denotes the quotient field of A. The isomorphism (8.1) motivates us to define an ad hoc version of 'motivic cohomology' of Y_n in bidegree $(2d, d)$ as

$$\mathcal{C}(Y_n) = H^d(Y_{\mathrm{Nis}}, \mathcal{K}^M_{Y_n,d}),$$

where $K^M_{Y_n,d}$ is the Milnor K-sheaf with Λ-coefficients as defined in Section 8.

Assuming the Gersten conjecture for Milnor K-theory of X, we get a compatible system of homomorphism

$$\mathrm{CH}_1(X)_\Lambda \cong H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{X,d}) \to \mathcal{C}(Y_n) \qquad (n \geq 1). \tag{10.1}$$

It is not difficult to show that the pro system $(\mathcal{C}(Y_n))_n$ is pro constant if $\mathrm{ch}(K) = 0$. As a potential generalization of our main result, Theorem 1.1, we conjecture:

Conjecture 10.1. *For* $\mathrm{ch}(K) = 0$*, the map* (10.1) *is an isomorphism of pro abelian groups. Here we think of the group* $\mathrm{CH}_1(X)$ *as a constant pro group.*

Conjecture 10.1 is related to a conjecture of Colliot-Thélène [CT95]. In fact, we expect that for $\Lambda = \mathbb{Z}$ and $[K : \mathbb{Q}_p] < \infty$, there should be an isomorphism

$$\mathrm{CH}_0(X_K) \cong \mathbb{Z} \oplus \mathbb{Z}_p^w \oplus (\text{finite group}) \oplus (\text{divisible group}).$$

Here, the integer w should be the $\mathbb{Z}_p$-dimension of the Lie group of K-points of the Albanese variety of X_K. The decomposition is not canonical, but the individual summands are unique up to isomorphism.

In another direction, we expect that an analogue of Theorem 1.1 for "higher zero-cycles" holds. For a noetherian scheme S of finite Krull dimension, let $H^{i,j}(S)$ denote motivic cohomology with Λ-coefficients, defined for example using the Eilenberg–MacLane spectrum constructed in [Sp12]. We consider the restriction homomorphism of motivic cohomology groups

$$H^{i,j}(X) \to H^{i,j}(Y). \tag{10.2}$$

From here on, we assume that $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with m prime to p. Even with these coefficients, we cannot expect that the map (10.2) is an isomorphism for general i and j, as the following example shows.

Example 10.2 (Rosenschon–Srinivas)**.** In [RS07], an elliptic curve E over a p-adic local field K with finite residue field, such that $\mathrm{CH}^2(E^3)_\Lambda$ is infinite for certain m ($\Lambda = \mathbb{Z}/m\mathbb{Z}$), is constructed. By general conjectures, we expect that there exists a model X/A as above with $X_K \cong E^3$, that there is a surjection $H^{4,2}(X) \twoheadrightarrow \mathrm{CH}^2(X_K)_\Lambda$ and that $H^{4,2}(Y)$ is finite. Granting this, the restriction map (10.2) cannot be injective in this case.

Conjecture 10.3. *Under the condition $j = d$ (higher zero-cycles) or under the condition $i - j = d$ (zero-cycles with coefficients in Milnor K-theory), the map* (10.2) *is an isomorphism.*

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