SIMPLY CONNECTED VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract. We show that there are no non-trivial stratified bundles over a smooth simply connected quasi-projective variety over an algebraic closure of a finite field, if the variety admits a normal projective compactification with boundary locus of codimension ≥ 2.

1. Introduction

On a smooth quasi-projective variety X over the field k of complex numbers, the theorem of Malčev-Grothendieck ([Mal40], [Gro70]) shows that the étale fundamental group $\pi_1^{\text{ét}}(X)$ controls the regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules, that is the regular singular flat connections: they are trivial if $\pi_1^{\text{ét}}(X) = 0$. The aim of this article is to show some variant if the ground field k is algebraically closed of characteristic $p > 0$.

Over the field $\mathbb{C}$ of complex numbers, the proof is very easy, but it makes a crucial use of the topological fundamental group $\pi_1^{\text{top}}(X)$ and the fact that it is finitely generated. Indeed, a complex linear representation has values in $GL(r, A)$ for a ring A of finite type over $\mathbb{Z}$, and if it is non-trivial, it remains non-trivial after specializing to some closed point of A. If k has characteristic $p > 0$, we no longer have this tool at our disposal.

All we know is that the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is Tannakian, neutralized by a rational point on X, and that its profinite completion is, according to dos Santos, the étale fundamental group ([dSan07, Cor. 12]).

Nonetheless, as conjectured by Gieseker [Gie75], the same theorem holds true under the extra assumption that X is projective ([EsnMeh10]): triviality of $\pi_1^{\text{ét}}(X)$ implies triviality of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.

On the other hand, it is shown in [Kin14, Thm.1.1] that dos Santos' theorem *loc. cit.* may be refined in the following way. The category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules has

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Date: May 5, 2014.

2010 *Mathematics Subject Classification.* 11G99, 14B20, 14G17, 14G99, 14F35.

Key words and phrases. Stratified bundles, fundamental groups, formal and algebraic geometry.

The first author is supported by the Einstein program and the ERC Advanced Grant 226257, the second author is supported by the Humboldt Foundation, by a J.C. Bose Fellowship, and by a membership at IAS, Princeton.

a full subcategory of regular singular ones. The profinite completion of its Tannaka group is then the tame fundamental group.

This enables one to ask (see [Esn12, Questions 3.1]):

i) If $\pi_1^{\text{ét}}(X) = 0$, are all $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules trivial?
ii) If $\pi_1^{\text{ét,tame}}(X) = 0$, are all regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules trivial?

In this article, we address Question i). Fundamental groups of quasi-projective non-projective smooth varieties in characteristic $p > 0$ are not well undersood. We give in Section 5 some non-trivial examples where we can obtain some reasonable structure.

Our main theorem 3.2 asserts that Question i) has a positive answer if X has a normal compactification with boundary of codimension ≥ 2, and if k is an algebraic closure of the prime field $\mathbb{F}_p$.

The arithmetic assumption on the ground field comes from the application of Hrushovski's main theorem [Hru04, Cor. 1.2] and the fact that we cannot, in general, define a surjective specialization homomorphism on the étale fundamental group (we do not know, however, if this can be done for the smooth loci of a family of normal projective varieties). See Section 4.

The geometric assumption on the existence of a good compactification with small boundary enables us to show a strong boundedness theorem 2.1. The main issue, if we drop the assumption, is to define a suitable family of extensions to a particular normal projective compactification of X of the F-divided vector bundles associated to a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module via Frobenius descent, in such a way that they form a bounded family of sheaves.

Under the geometric assumption in our result, the reflexive extension of the bundles does it. See Remark 3.10. In general, we do not know. Over the field of complex numbers, if we assume in addition that the $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is regular singular, then we have Deligne's canonical extension at our disposal. Indeed, Deligne shows in [Del14] boundedness for those. His non-algebraic proof relies on the fact that the topological fundamental group is finitely generated.

We use the existence of a projective ample complete intersection curve to reduce the problem to the case where the underlying F-divided bundles E_n of a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module are stable of slope 0, and of given Hilbert polynomial. This comes from the Lefschetz theorem Theorem 3.5 for stratified bundles, which ultimately relies on Bost's most recent generalization of the Grothendieck **Lef** conditions, see Appendix A. Then the proof follows the line of the proof of the main theorem in [EsnMeh10].

In Section 5, relying again on the Lefschetz theorem, we are able to give a non-trivial family of examples with trivial étale fundamental group for which one can also conclude that $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are trivial, this time over any field, as we can argue without using Hrushovski's theorem *loc. cit.*.

Acknowledgements: It is a pleasure to thank Adrian Langer for a discussion his boundedness theorems, which play a very important role in this article. We also thank Alexander Schmitt for discussions. We thank Jean-Benoît Bost for his interest, and for contributing in the form of Appendix A to this article. We thank the referees for their precise work which helped us to improve the exposition of our text, in particular, it led us to treat the Verschiebung map in a clearer way in the proof of Theorem 3.2.

2. Boundedness

We fix the notations for this section. Let X be a projective normal irreducible variety of dimension $d \geq 1$ over an algebraically closed field k of characteristic $p > 0$. Let $j : U \to X$ be the open embedding of the regular locus. Recall that if E is a vector bundle, i.e., a locally free coherent sheaf, on U, then j_*E is a coherent sheaf on X, and is the reflexive hull of E. Indeed by normality, one has $j_*j^*\mathcal{O}_X = j_*\mathcal{O}_U = \mathcal{O}_X$, which implies that $j_*j^*\mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = \mathcal{H}om(\mathcal{F}, j_*\mathcal{O}_U) = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ for any $\mathcal{O}_X$-coherent sheaf $\mathcal{F}$, which is thus coherent. In addition, $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = j_*j^*\mathcal{F}^\vee$ if $\mathcal{F}$ a vector bundle, and contains any other torsion free extension to X of $j_*\mathcal{F}^\vee$. We apply this to $j^*\mathcal{F} = E^\vee$.

We fix an ample Cartier divisor $Y \to X$. For any coherent sheaf $\mathcal{E}$ on X, we write $\mathcal{E}(mY)$ for $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mY)$. For a vector bundle E on U, we define the Hilbert polynomial $\chi(X, j_*E(mY)) \in \mathbb{Q}[m]$. Here χ stands for the Euler characteristic of the coherent sheaf $j_*E(mY)$, which is equal to $j_*(E(mY|_U))$ by the projection formula. We recall the definition of an F-divided sheaf in Definition 3.1.

This section is devoted to the proof of the following statement.

Theorem 2.1. *We fix $r \in \mathbb{N} \setminus \{0\}$. There are finitely many polynomials $\chi_i(m) \in \mathbb{Q}[m]$, $i \in I = \{1, \ldots, N\}$ such that for any F-divided sheaf $E = (E_n, \sigma_n)_{n \in \mathbb{N}}$ of rank r on U, there is a $n_0(E) \in \mathbb{N}$ such that for all $n \geq n_0(E)$, $\chi(X, j_*E_n(mY)) \in \{\chi_1(m), \ldots, \chi_N(m)\}$.*

If $\dim(X) = 1$, then X is smooth projective, so E has degree 0 ([EsnMeh10, Cor. 2.2]), so one has $\chi(X, E(m)) = r\chi(X, \mathcal{O}_X(m))$ by Riemann-Roch on curves. So we may assume $\dim X \geq 2$.

In general, we reduce to the 2-dimensional case as follows. Let $d = \dim(X) \geq 3$. We choose a Noether normalization $h : X \to \mathbb{P}^d$ over k. This defines the open $\mathcal{U}$ of the product of the dual projective spaces $((\mathbb{P}^d)^\vee)^{d-2}$, whose points $x \in \mathcal{U}$ parametrize intersections of hyperplanes $H_x := (H_{1,x} \cap \ldots \cap H_{d-2,x}) \to \mathbb{P}^d$ such that $h^*H_x =: Y_x \subset X \otimes_k \kappa(x)$ intersects $X \setminus U$ properly. Here $\kappa(x)$ is the residue field of x. The open $\mathcal{U}$ is non-empty, and irreducible.

Lemma 2.2. *Let* $\operatorname{Spec} K \xrightarrow{x} \mathcal{U}$ *be a geometric generic point, $i_x : Y_x \to X \otimes_k K$ be the corresponding closed embedding, $j_{Y_x} : V_x = (U \otimes_k K) \cap Y_x \to Y_x$. Then*

i) *Y_x is a projective normal irreducible 2-dimensional variety;*

ii) *For any vector bundle E on U, $i_x^* j_* E = (j_{Y_x})_* (i_x|_{U \otimes_k K})^* E$.*

Proof. For a vector bundle E, the sheaf $h_* j_* E$ is reflexive, thus is locally free outside of codimension ≥ 3 as $\mathbb{P}^d$ is smooth. So there is a non-empty open $\mathcal{U}_E \subset \mathcal{U}$ defined over k, such that for any $x \in \mathcal{U}_E$, Y_x is smooth in codimension 1 and $h_*(j_* E|_{Y_x}) = (h_* j_* E)|_{H_x}$ is locally free. In particular, $(j_* E)|_{Y_x} = (j|_{Y_x})_* E|_{Y_x \cap U}$ is a reflexive sheaf on Y_x. Applied to $\mathcal{O}_U$, this yields $\mathcal{O}_{Y_x} = (j|_{Y_x})_* \mathcal{O}_U|_{Y_x \cap U}$, so Y_x is normal, by Serre's criterion (see [Sei50, Thm. 7] for the original proof). By Artin vanishing theorem [SGA4, Cor. 3.5], Y_x is connected. As Y_x is normal, Y_x is irreducible. This shows i). One does not need here x to be generic, a closed point x in $\mathcal{U}_{\mathcal{O}_U}$ also does it. On the other hand, the K-point $x : \operatorname{Spec} K \to \mathcal{U}$ factors through $x : \operatorname{Spec} K \to \mathcal{U}_E \cap \mathcal{U}_{\mathcal{O}_U} \to \mathcal{U}$ for any E. This implies ii).

□

Corollary 2.3. *Theorem 2.1 is true if and only if it is true for $d = 2$ over any algebraically closed field k.*

Proof. We apply A. Langer's boundedness theorem [Lan04, Thm. 4.4]: let X be a projective variety of dimension ≥ 3 over an algebraically closed field k, let $S \to X$ be a dimension 2 complete intersection of very ample divisors Y_i, $2 \leq i \leq \dim(X)$, of some linear system $\mathcal{O}_X(1) = \mathcal{O}_X(Y)$. Then fixing some polynomial $\chi(m) \in \mathbb{Q}[m]$ and some positive number μ, there are finitely many polynomials $p_i(m) \in \mathbb{Q}[m]$ such that the Hilbert polynomial $\chi(X, E(m))$ of any pure sheaf E on X verifying $\chi(S, E|_S(m)) = \chi(m)$ and $\mu_{\max}(E) \leq \mu$ is equal to one of the $p_i(m)$. Here $\mu_{\max}$ denotes the slope of a maximal destabilizing subsheaf (lowest term in the Harder-Narasimhan filtration) of E. So it remains to show: there exits a constant μ (depending only on $(X, \mathcal{O}_X(1))$ and the rank), such that for any given $(E_n)_n$ as in Theorem 2.1, $\{\mu_{\max}(E_n)\}$ is bounded above by μ for all but finitely many n.

Given a constant $\mu \in \mathbb{R}$, in order to show that $\mu_{\max}(E_n) \leq \mu$ for all $n \geq n_0$ (for some n_0 depending on $\{E_n\}$), it is enough to do it after base change $X_K \to X$ over an algebraically closed field $K \supset k$.

As in the situation of Lemma 2.2, we may consider the open $\mathcal{U}'$ in the product $((\mathbb{P}^d)^\vee)^{d-1}$ corresponding to complete intersection curves in X of the form $\cap_{i=1}^{d-1} Y_i$, where $Y_i \in |\mathcal{O}_X(1)|, s = 1, \ldots, d$. Let η be the generic point of $\mathcal{U}'$, and C_η be the corresponding curve in X_η. Then C_η is a projective nonsingular curve contained entirely in U_η. (See e.g. [Jou83, Thm. 6.10]). We choose K to be an algebraic closure of $k(\eta) = k(\mathcal{U}')$. Note that the resulting k-morphisms $C_\eta \to X$ and $C_K \to X$ are flat, since they are obtained from the flat morphism from $I \to X$, where $I \subset X \times_k \mathcal{U}'$ is the incidence variety.

Hence, if E is any reflexive sheaf on X, any injective morphism of coherent sheaves $E_1 \to E$ remains so after pull-back to C_K, which we may view as obtained by first pulling back to X_K, and then restricting to the closed subscheme $C_K \subset X_K$, which is a smooth complete intersection curve in X_K. In particular, if $E_1 \subset E$ is the maximal destabilizing subsheaf, then $E_1 \mid_{C_K} \hookrightarrow E \mid_{C_K}$ is a destabilizing subsheaf (perhaps not maximal) of the same degree. By definition of $\mu_{\max}$, it is thus enough to show that $\{\mu_{\max}(E_n|_{C_K})\}$ is bounded above by some μ, depending only on X, after possibly omitting finitely many values of n.

Since for any torsion free sheaf V on C_K, the Riemann-Roch theorem implies that the degree of V is bounded above by the dimension of $H^0(C_K, V)$ plus some constant depending only on C_K and the rank of V, thus only on C_K and r for $V \subset E_n|_{C_K}$, it is enough to bound above $H^0(C_K, V)$ by a constant μ, for any such $V \subset E_n \mid_{C_K}$, after possibly omitting finitely many values of n.

The maximal trivial sub F-divided sheaf of $(E_n|_{C_K})_n$ is by definition $(V \otimes \mathcal{O}_{C_K})_n$ with $F^*(V \otimes \mathcal{O}_{C_K}) = V \otimes \mathcal{O}_{C_K}$, where V is a finite dimensional K-vector space of rank $\leq r$. The decreasing sequence $(H^0(C_K, E_n|_{C_K}))_n$, where the inclusion $H^0(C_K, E_n|_{C_K}) \to H^0(C_K, E_{n-1}|_{C_K})$ is defined by the pull-back by Frobenius, thus stabilizes to V, for $n \geq n_0$ for some natural number n_0. This finishes the proof.

□

Proof of Theorem 2.1. By Corollary 2.3, we may assume that X has dimension 2. We use below various standard facts about normal projective surfaces; details may for example be found in the book [Ba01]; for finite generation of the Néron-Severi group of a surface, see [Mi80, V, Thm.3.25].

We denote by $i : Y \to X$ a smooth projective irreducible ample curve, by $\Sigma = X \setminus U$ the singular locus of X, which thus consists of finitely many closed points (outside Y). Let $\pi : X' \to X$ be a desingularization such that $\pi^{-1}\Sigma$ is a strict normal crossings divisor $D = \cup_{x \in \Sigma} D_x,\ D_x = \cup_i D_{x,i}$ the irreducible components. We may thus identify U with $X' \setminus D$. We note that $NS_\Sigma(X') := \oplus_{x,i} \mathbb{Z} \cdot D_{x,i} \subset NS(X')$, and the intersection form of $NS(X')$ has negative definite restriction to this subgroup. We set $NS(X) = \text{coker } (\oplus_{x,i} \mathbb{Z} \cdot D_{x,i} \to NS(X'))$ so that $NS(X)$ may be identified with the Néron-Severi group of X (of Weil divisor classes modulo algebraic equivalence). This is a finitely generated abelian group; further, the orthogonal complement $NS_\Sigma(X')^\perp$ of $NS_\Sigma(X')$ naturally maps to $NS(X)$, such that $NS_\Sigma(X')^\perp \to NS(X)$ is injective with finite cokernel. We also note that there is a natural, surjective homomorphism $\text{Pic}(U) \to NS(X)$ induced by the surjection $\text{Pic}(X') \to \text{Pic}(U)$ obtained by restriction of line bundles.

For each bundle V^U on U, we define $V = j_* V^U,\ V' = (\pi^* V)^{\vee\vee}$. Then V is a coherent reflexive sheaf on X, and V' is a vector bundle on X'. For any vector bundle V' on X', we define $c_1(V') \in NS(X')$ and $c_2(V') \in \mathbb{Z}$ as the images of the corresponding algebraic Chern classes in $CH_*(X')$.

With those notations, it is enough to show

i) the subset $\{c_1(V_n')\} \subset NS(X')$ is finite when $\mathcal{V}^U = (V_n^U, \sigma_n)$ runs through all rank r F-divided sheaves;

ii) there is an $M > 0$ such that for each such $\mathcal{V}^U$, there is an $n_0(\mathcal{V}^U) \geq 0$ such that for all $n \geq n_0(\mathcal{V}^U)$, we have $|c_2(V'_n)| \leq M$ in $\mathbb{Z}$.

Indeed, if i) and ii) hold, then by the Riemann-Roch theorem on X', there is a finite set of polynomials $S \subset \mathbb{Q}[m]$ such that

$$\text{for all } \mathcal{V}^U, \chi(X', V'_n(m\pi^*Y)) \in S \subset \mathbb{Q}[m], \text{ for all } n \geq n_0(\mathcal{V}^U).$$

One has

$$\chi(X', V'_n(m\pi^*Y)) = \chi(X, \pi_*V'_n(mY)) - \dim_k H^0(X, R^1\pi_*V'_n).$$

But $\pi_*V'_n = V_n$, and since V_n is generated along Σ by its sections defined on a neighbourhood of Σ, so is π^*V_n generated by its sections on a neighbourhood of $\pi^{-1}(\Sigma)$. Thus there is an $\mathcal{O}_{X'}$-linear homomorphism $\oplus_1^{r+1}\mathcal{O}_{X'} \to V'_n$ defined in a neighbourhood of D, which is surjective outside of codimension ≥ 2. Thus one has a surjection

$$\oplus_1^{r+1} R^1\pi_*\mathcal{O}_{X'} \twoheadrightarrow R^1\pi_*V'_n$$

locally near Σ, and both sheaves are supported within Σ, which bounds the dimension over k of $H^0(X, R^1\pi_*V'_n)$. This shows then Theorem 2.1, assuming i) and ii); we now prove these. Note that our proof of boundedness of $\dim_k H^0(X, R^1\pi_*V'_n)$ did not use i) and ii), and so we may make use of this in the proof of i) and ii).

We denote by $c_1(V)$ the image of $c_1(V')$ in $NS(X)$. For a rank r F-divided sheaf $\mathcal{V}^U = (V_n^U, \sigma_n)_{n\in\mathbb{N}}$ on U, we first note that the sequence $c_1(V_n)$ in the finitely generated abelian group $NS(X)$ satisfies $p^n c_1(V_n) = c_1(V_0)$ for all $n \geq 0$, since a similar relation holds between the determinant line bundles of V_n^U and V_0^U. This implies that $c_1(V_n)$ is torsion for all n. It also implies that for some positive integer δ depending only on (X, Σ), $\delta \cdot c_1(V'_n) \in NS_\Sigma(X') \subset NS(X')$. Hence, for any V_n^U, we may uniquely write $\delta \cdot c_1(V'_n) = \sum m_{x,i}(V'_n)D_{x,i}$ where $m_{x,i} \in \mathbb{Z}$, So the assertion for c_1 is equivalent to saying that *the* $m_{x,i}(V'_n)$ *are all bounded.*

The matrix $(D_{x,i} \cdot D_{x,j})_{ij}$ is negative definite for any x, so boundedness of the $m_{x,i}(V'_n)$ is equivalent to the statement that the subset $\{c_1(V'_n) \cdot D_{x,i},\ \forall x, i\} \subset \mathbb{Z}$ is bounded. We adapt Langer's argument from [Lan00, Prop. 4.6].

We first write $D = \sum_{x,i} D_{x,i}$ also for the corresponding reduced divisor. Now, consider the exact sequence

$$0 \to V'_n \to V'_n(D) \to V'_n(D)|_D \to 0.$$

Since $\pi_*V'_n = V_n$ is reflexive, so that $\pi_*V'_n = \pi_*V'_n(D)$, we deduce that the boundary homomorphism

$$H^0(D, V'_n(D)|_D) \to H^0(X, R^1\pi_*V'_n)$$

is injective. We had seen earlier that

$$\dim H^0(X, R^1\pi_*V'_n) \leq (r+1)\dim H^0(X, R^1\pi_*\mathcal{O}_{X'}).$$

Thus

$$\chi(V_n'(D)|_D) \le h^0(V_n'(D)|_D) \le C_1$$

for a constant $C_1 > 0$ depending only on X and r.

On the other hand, $\deg(V_n'|_{D_{x,i}}) := c_1(V_n') \cdot D_{x,i} \ge 0$ for all x, i. Indeed, on a neighbourhood of D, V_n' is spanned by $(r+1)$ global section outside of codimension ≥ 2, and $\det(V_n')$ has a non-zero section on this neighbourhood, the divisor of which intersects D in dimension ≤ 0. One thus has

$$\begin{aligned} &C_1 - rD^2 - r\chi(\mathcal{O}_D) \\ &\ge \chi(V_n'(D)|_D) - rD^2 - r\chi(\mathcal{O}_D) = \chi(V_n'|_D) - r\chi(\mathcal{O}_D) = \deg(V_n'|_D) \ge 0. \end{aligned} \tag{2.1}$$

Since $\deg(V_n'|_D) = \sum_{x,i} \det(V_n') \cdot D_{x,i}$ where each term $\det(V_n') \cdot D_{x,i} \ge 0$, we conclude that there is a constant $C_2 > 0$ such that

$$0 \le c_1(V_n') \cdot D_{x,i} \le C_2 \ \forall x, i.$$

Thus there is a constant $C_3 > 0$ such that

$$|m_{x,i}(V_n')| \le C_3 \ \forall x, i.$$

This finishes the proof for c_1.

We show the statement for c_2. The isomorphism $(F^n)^* V_n^U \to V_0^U$ extends to an injective $\mathcal{O}_{X'}$-linear map $(F^n)^* V_n' \to V_0'$. Thus $\det(V_0') = p^n \det(V_n') + A_n$ where $A_n = \sum a_{x,i}(n) D_{x,i},\ a_{x,i}(n) \in \mathbb{N}$. Thus (2.1) implies $0 \le a_{x,i}(n) \le (p^n+1)C_3$. On the other hand, the ideal sheaf of A_n on X' annihilates $Q_n = V_0'/(F^n)^* V_n$. Thus one has induced surjections

$$V_0' \twoheadrightarrow V_0'|_{A_n},\ q_n : V_0'\ |_{A_n} \twoheadrightarrow Q_n, \quad (F^n)^* V_n' \twoheadrightarrow K_n, \text{ with } K_n := \mathrm{Ker}(q_n).$$

Let $\mathcal{F}$ denote any of the sheaves $Q_n, K_n, V_0'|_{A_n}$, each of which is a quotient of a vector bundle of rank r on A_n. All three sheaves are generated by their global sections outside a set of dimension ≤ 0 supported in D. So one has maps $\oplus_1^r \mathcal{O}_{A_n} \to \mathcal{F}$ with cokernel supported in dimension ≤ 0, and thus

$$h^1(\mathcal{F}) \le r h^1(\mathcal{O}_{A_n})$$

for each of the sheaves $\mathcal{F}$, and further one has

$$H^0(X, R^1\pi_* \mathcal{O}_X) \twoheadrightarrow H^1(\mathcal{O}_{A_n}).$$

So, one obtains from Riemann-Roch for $\mathcal{O}_{A_n}$ the existence of a constant $C_4 > 0$ such that $h^0(\mathcal{O}_{A_n}) = \chi(\mathcal{O}_{A_n}) + h^1(\mathcal{O}_{A_n}) \le p^{2n} C_4$. One also has $h^0(V_0'|_{A_n}) = \chi(V_0'|_{A_n}) + h^1(V_0'|_{A_n}) = \deg(V_0'|_{A_n}) + r\chi(\mathcal{O}_{A_n}) + h^1(V_0'|_{A_n})$. On the other hand, one has the exact sequence

$$0 \to H^0(K_n) \to H^0(V_0'|_{A_n}) \to H^0(Q_n) \to H^1(K_n).$$

Thus, using that $\deg(V_0'|_D)$ is bounded, there is a constant $C_5 > 0$ such that $h^0(\mathcal{F}) \le p^{2n} C_5$ for any of the choices of $\mathcal{F}$. This, for $\mathcal{F} = Q_n$, and boundedness of $h^1(\mathcal{F})$ imply, via the Riemann-Roch formula $\chi(Q_n) = \frac{1}{2}c_1(X)\cdot ch_1(Q_n) + ch_2(Q_n)$, that here is a constant $C_6 > 0$ such that

$$|ch_2(Q_n)| \le p^{2n} C_6.$$

Here we use the "numerical Chern character" $ch(Q_n) = ch_1(Q_n) + ch_2(Q_n) \in NS(X') \oplus \frac{1}{2}\mathbb{Z}$. Now we use the definition of Q_n to conclude $|ch_2(V'_0) - p^{2n}ch_2(V'_n)| \le p^{2n}C_6$. Thus

$$|ch_2(V'_n)| \le \frac{1}{p^{2n}}|ch_2(V'_0)| + C_6.$$

This shows boundedness for $ch_2(V_n)$. The statement for c_1 (i.e., finiteness for possible c_1) shows now the statement for c_2. This finishes the proof of the theorem.

□

3. First main theorem

Definition 3.1. Let C be a connected scheme of finite type defined over an algebraically closed field k. An F-divided sheaf $\mathcal{E}$ is a sequence $(E_0, E_1, \ldots, \sigma_0, \sigma_1, \ldots)$ of $\mathcal{O}_C$-coherent sheaves E_n on C, with $\mathcal{O}_C$-isomorphisms $\sigma_n : E_n \to F^*E_{n+1}$, where F is the absolute Frobenius of C. The category of F-divided sheaves is constructed by defining $\mathrm{Hom}(\mathcal{E}, \mathcal{V})$ as usual: one replaces E_n be its inverse image E'_n on the n-th Frobenius twist $C^{(n)}$ of C, the $\mathcal{O}_C$-isomorphism σ_n by $\sigma'_n; E'_n \to F_n^* E'_{n+1}$, where $F_n : C^{(n)} \to C^{(n+1)}$, and then $f \in \mathrm{Hom}(E, V)$ consists of $f_n \in \mathrm{Hom}_{C^{(n)}}(E'_n, V'_n)$ commuting with the σ'_n.

(We may abuse notation and write $\mathcal{E} = (E_0, E_1, \ldots)$ to denote an F-divided sheaf, suppressing the maps in the notation.)

If C is smooth, by Katz' theorem [Gie75, Thm. 1.3], the category of F-divided sheaves is equivalent to the category of $\mathcal{O}$-coherent $\mathcal{D}$-modules. The category is then k-linear Tannakian.

We note that, even without smoothness, an F divided sheaf $\mathcal{E} = (E_0, E_1, \ldots)$ has component sheaves E_i which are locally free. This follows easily from an argument with Fitting ideals, as in [dSan07, Lem. 6] (regularity is not needed for the argument there, attributed to Shepherd-Barron).

The aim of this section is to prove the

Theorem 3.2. *Let X be a normal projective variety defined over $k = \bar{\mathbb{F}}_p$. Let U be the regular locus. If $\pi_1^{\text{ét}}(U) = 0$, then there are no non-trivial stratified bundles on U.*

Proposition 3.3. *Let C be as in Definition 3.1.*

i) *The category of F-divided sheaves on C is k-linear, Tannakian.*
ii) *If $f : C' \to C$ is a universal homeomorphism, i.e. an integral surjective and radical morphism, then f^* induces an equivalence of categories between F-divided sheaves on C and on C'.*

Proof. We prove ii). A fixed power of the absolute Frobenius F_C of C factors through C', so $(F_C)^N : C \to C' \xrightarrow{f} C$. Given $(V_0, V_1, \ldots, \tau_0, \tau_1, \ldots)$ on C, one defines $E_n = f^*V_n, \sigma_n = f^*\tau_n$.

We prove i). We refer to [Saa72] when C is smooth. By ii), we may assume that C is reduced.

We show that the category is abelian. To this aim, we first assume that C is irreducible. Let $u : (V_0, V_1, \ldots) \to (W_0, W_1, \ldots)$ be a morphism, and let r be the generic rank of the image. Set $L_n = \mathrm{Im} \bigwedge^r V_n \subset P_n = \bigwedge^r W_n$; then L_n is a non-zero subsheaf of rank 1 of the vector bundle P_n.

To say that $L_0 \subset P_0$ is a subbundle (which is equivalent to $\mathrm{im}\, V_0 \to W_0$ being a subbundle of rank r), is equivalent to saying that for any point $x \in C$, with local ring $\mathcal{O}_x$, maximal ideal $\mathfrak{m}_x$, and residue field $k(x) = \mathcal{O}_X/\mathfrak{m}_x$, we have that the map on fibres $L_x \otimes_{\mathcal{O}_x} k(x) \to P_x \otimes_{\mathcal{O}_x} k(x)$ is non-trivial. If it was trivial, then for all n, $(L_n)_x \subset \mathrm{Im}(\mathfrak{m}_x \otimes (P_n)_x)$ thus $(L_0)_x \subset \mathrm{Im}(\,\mathfrak{m}_x^{p^n} \otimes_{\mathcal{O}_x} (P_0)_x)$ for arbitrarily large n, which is impossible. Hence $\mathrm{Im}(\,V_0 \to W_0)$ is a sub-bundle, and similarly for $V_n \to W_n$ for any n. (Note that this is similar to the argument, alluded to above, in [dSan07, Lem. 6], using Fitting ideals.)

So the image of u is a sub F-divided sheaf while restricted to the components of C. In particular, at the intersections points of two components, the ranks are equal. Since C is connected, all the ranks are equal. This shows that the category is abelian.

We show that the category of F-divided sheaves is k-linear, that is that the Homs are finite dimensional k-vector spaces. It is enough to show that if $i : C_{\mathrm{reg}} \hookrightarrow C$ is the inclusion of the smooth locus, the restriction homomorphism $\mathrm{Hom}_C(\mathcal{V}, \mathcal{W}) \to \mathrm{Hom}_{C_{\mathrm{reg}}}(i^*\mathcal{V}, i^*\mathcal{W})$ is injective, which is true as it is already injective in the vector bundle category. This finishes the proof.

□

Choosing a rational point $c \in C(k)$ defines a neutralization $E \mapsto E_0|_c$ of the category, and thus a Tannaka group scheme $\pi^{\mathrm{strat}}(C)$ over k. (We drop the base point c from the notations).

We now compare this definition with the definition of stratified modules given by Saavedra, see [Saa72, Chap. VI, 1.2] and references in there. As this definition is not used elsewhere in the paper, we use the terminology of "stratified module in the sense of Saavedra" to refer to it.

Proposition 3.4. *An F-divided sheaf $\mathcal{E} = (E_0, E_1, \ldots)$ has the property that for all i, E_i is a stratified module in the sense of Saavedra.*

Proof. it is enough to show it for E_0. Let $C(n) = C^n_\Delta$ be the formal completion of C^n along the diagonal. Then $C(n) = \varinjlim_{m \geq 1} C(n)_m = \varprojlim_{m \geq 0} C(n)_{p^m}$ where $_m$ means modulo the m-th power of the ideal of the diagonal. The Frobenius $F^m : C^n \to C^n$ which on functions raises a function to its p^m-th power, yields a factorization F^m :

$C \to C(n)_{p^m} \xrightarrow{\Phi(n)_{p^m}} C$. So if $\mathcal{E}$ is F-divided in the sense of Proposition 3.3, one defines the value of $\mathcal{E}$ on $C(n)_{p^m}$ as $(\Phi(n)_{p^m})^* E_m$.

□

The normality assumption on X in Theorem 3.2 is reflected in the following Theorem 3.5. In order to prove Theorem 3.2, we need only the case when C is smooth, which is much easier, but we will use the full strength of Theorem 3.5 in Section 5.

Theorem 3.5. *Let U be a smooth quasi-projective variety of dimension $d \geq 1$ over an algebraically closed field k, let $\iota : C \hookrightarrow U$ be a projective curve, which is a complete intersection of $(d-1)$-hypersurface sections. Then $\iota_* : \pi^{\mathrm{strat}}(C) \to \pi_1^{\mathrm{strat}}(U)$ is surjective.*

Proof. By [DM, Prop. 2.21], one has to show that ι^* is fully faithful and that if $\mathcal{V} = (V_n)_n$ is a F-divided sheaf on U, and $\mathcal{W} = (W_n)_n \subset \iota^*\mathcal{V}$ is an F-divided sheaf on C, then $\mathcal{W} = \iota^*\tilde{\mathcal{W}}$ for some sub-F-divided sheaf $\tilde{\mathcal{W}} \subset \mathcal{V}$. Full faithfulness follows from the unicity up to isomorphism of $\tilde{\mathcal{W}} \subset \mathcal{V}$ with the property that ι^* of this inclusion is $\mathcal{W} \subset \iota^*\mathcal{V}$.

By definition of a F-divided sheaf as a crystal in the infinitesimal site of C , the inclusion $\mathcal{W} \subset \iota^*\mathcal{V}$ lifts to $\hat{U}_C$, the completion of U along C. By an improved **Lef** condition, as formulated and proven in the Appendix A, Theorem 9, it implies that each subbundle $W_n \hookrightarrow \iota^* V_n$ lifts to a sub-bundle $\mathcal{W}'_n \hookrightarrow j_n^* V_n$ on some non-trivial open $j_n : U_n \hookrightarrow U$ which contains C. Thus $U \setminus U_n$ consists of a closed subset of codimension ≥ 2, and $\mathcal{W}'_n|_C = W_n$. Then, by the usual **Lef** condition ([SGA2, Exp. 10, Ex. 2.1 (1)]), the section of $(F^*W_{n+1})^\vee \otimes W_n$ lifts to a uniquely defined section of $(F^*\mathcal{W}'_{n+1})^\vee \otimes \mathcal{W}'_n$ on the open $U_{n+1} \cap U_n$ which is also the complement of a closed subset of codimension ≥ 2 in U. We define $\widetilde{W_n} := j_{n*}\mathcal{W}'_n$. It is a torsion-free coherent subsheaf of V_n. Moreover, since, as U is smooth thus F is faithfully flat, $F^* j_* = j_* F^*$ for any open embedding $j : U \setminus \Sigma \to U$, where Σ is any closed subset (here taken to be $\Sigma = U \setminus U_n \cap U_{n+1}$), we deduce that $\widetilde{W_n} \subset V_n$ defines a sub-F-divided sheaf (and is locally free). □

Lemma 3.6. *Let X be a projective normal variety defined over an algebraically closed field k, let $j : U \to X$ be the open embedding of its regular locus.*

i) *If $\mathcal{E} = (E_0, E_1, \ldots, \sigma_0, \sigma_1, \ldots)$ and $\mathcal{E}' = (E'_0, E'_1, \ldots, \sigma'_0, \sigma'_1, \ldots)$ are two stratified bundles on U with isomorphic underlying vector bundles E_n, E'_n for all n, then the two F-divided sheaves are isomorphic. In particular, $\mathcal{E}$ is trivial if and only if the vector bundles E_n are trivial.*
ii) *Let $\mathbb{I} = (\mathcal{O}, \mathcal{O}, \ldots, \mathrm{id}, \mathrm{id}, \ldots)$ be the trivial object on U. Then the Ext group in the category of stratified bundles fulfills:* $\mathrm{Ext}^1(\mathbb{I}, \mathbb{I}) \cong H^1_{\text{ét}}(U, \mathbb{Z}/p) \otimes_{\mathbb{F}_p} k$.

Proof. We prove i). Following [Gie75, Prop. 1.7], one just has to see that for any vector bundle V on U, $H^0(U, V)$ is a finite dimensional k-vector space, which is fulfilled as

$H^0(U, V) = H^0(X, j_*V)$ and j_*V is a coherent sheaf. Then $\mathrm{Hom}_U(E_n, E'_n)$ satisfies the Mittag-Leffler condition, and the proof given in [Gie75, Prop. 1.7] applies.

sections Then over k.

We prove ii). By i) and [dSan07, (9)], one has $\mathrm{Ext}^1(\mathbb{I}, \mathbb{I}) \subset H^1_{\text{ét}}(U, \mathbb{Z}/p) \otimes k$. On the other hand, a $\mathbb{Z}/p$-torsor $h : V \to U$ defines the stratified bundle $h_*\mathcal{O}_V$, which is a successive extension of $\mathbb{I}$ by $\mathbb{I}$. The bottom sub of rank 2 defines a class in $\mathrm{Ext}^1(\mathbb{I}, \mathbb{I})$ with image h. □

Proof of Theorem 3.2. Let $\iota : C \to U$ be a curve which is a complete intersection of $(\dim X - 1)$ hyperplane sections Y_i for the same very ample linear system $|Y|$ on X of very high degree. As argued in the proof of Corollary 2.3, C is contained in U, and is smooth by [Poo04, Thm. 1.1] (which applies also over a finite field and to quasi-projective varieties.)

Let $\mathcal{E}$ be a non-trivial stratified bundle on U. By [EsnMeh10, Prop. 2.3] and [EsnMeh10b], there is a $n_0 \geq 0$ such that the stratified bundle

$$(\iota^* E_{n-n_0}, \iota^* \sigma_{n-n_0})_{(n-n_0)\geq 0}$$

is a successive extension of stratified bundles $(U_n, \sigma_n)_{n\geq 0}$ with the property that all U_n are μ-stable bundles of slope 0 on C. By Theorem 3.5, there are $(V_n, \tau_n)_{n\geq 0}$ on U such that $(U_n, \sigma_n)_{n\geq 0} = \iota^*(V_n, \tau_n)_{n\geq 0}$, and those are irreducible objects. Then by Lemma 3.6 ii), if (V_n, τ_n) is trivial, so is (E_n, σ_n). So we may assume that $\mathcal{E}$ is irreducible and that $\iota^* E_n$ is μ-stable for all $n \geq 0$.

By Theorem 2.1, there are finitely many polynomials $\chi_1(m), \ldots, \chi_N(m) \in \mathbb{Q}[m]$ describing the Hilbert polynomials $\chi(X, j_*E_n(mY))$. Let $\mathcal{P} \subset \mathbb{Q}[m]$ denote this set of polynomials. Let M be the disjoint union of the moduli $M_i = M(\chi_i), i = 1, \ldots, N$ of χ_i-stable (in fact the moduli points of interest for us are even μ-stable, thus χ_i-stable, see [HuyLeh97, Lem. 1.2.13]) torsion-free coherent sheaves on X, of Hilbert polynomial $\chi_i(m)$ [Lan04, Thm. 0.2]. Recall that $M(\chi_i)$ uniformly corepresents the functor from the category of schemes over k to the one of sets, which assigns to any T/k the set $\mathcal{M}_X(\chi_i)(T)$ of isomorphism classes of coherent sheaves E on $X \times_k T$, flat over T, which are pure and χ_i-stable on all geometric fibers of the projection $X \times_k T \to T$. It is a quasi-projective scheme. The sheaves j_*E_n define moduli points $[j_*E_n] \in M(k)$. Let $N_t \subset M$ be the reduced subschemes defined as the Zariski closure of the subset of closed points $[j_*E_n], n \geq t$. By the noetherian property, the decreasing sequence of subschemes N_t stabilizes to N say.

Let us denote by $\sigma : \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ the Frobenius homomorphism of the ground field k; by $X^{(1)} = X \otimes_\sigma k$, $U^{(1)} = U \times_\sigma k$, $Y^{(1)} = Y \otimes_\sigma k$, $C^{(1)} = C \otimes_\sigma k$ the Frobenius twists of X, U, Y and C respectively; by $W : X^{(1)} \to X$ the base change morphism over σ, that is the canonical projection, which is an isomorphism of schemes; by $j^{(1)} : U^{(1)} \to X^{(1)}$ the open embedding, which is the base change by σ of j. One has

$$j^{(1)}_* W|^*_{U^{(1)}} E_n = W^* j_* E_n,$$

thus $\chi(X, j_*E_n(mY)) = \chi(X^{(1)}, j_*^{(1)}W|^*_{U^{(1)}}E_n(mY^{(1)}))$, and $\deg(E_n|_C) = \deg(W^*E_n)|_{C^{(1)}}$. So stability is preserved. Let $M_i^{(1)}$ and $M^{(1)}$ be the corresponding moduli schemes. Then W^* is a transformation from the moduli functor for coherent sheaves on X to the one for coherent sheaves on $X^{(1)}$, which is invertible. Thus it defines a morphism $M_i \to M_i^{(1)}$ over σ, which is an isomorphism of schemes, in particular a bijection on k-points. One defines $N_t^{(1)}$ and $N^{(1)}$ associated to the moduli points $[W^*j_*E_n],\ n \in \mathbb{N}$, as reduced subschemes. Then $W^* : N \to N^{(1)}$ induces an isomorphism of schemes. We denote by $(W^*)^{-1} : N^{(1)} \to N$ its inverse.

Our next goal is to construct a rational map $\Phi : N^{(1)} \dashrightarrow N$ which is defined on a neighbourhood of a dense set of the closed points $[j_*E_n]$ as above, and such that $\Phi([W^*j_*E_n]) = [j_*F^*E_n]$.

If we construct a rational map $\tilde{\Phi} : N^{(1)} \dashrightarrow M = \coprod M_i$ which is a morphism on a neighbourhood of a dense set of the closed points $[j_*E_n]$ as above, and satisfies $\Phi([W^*j_*E_n]) = [j_*F^*E_n]$, then in fact $\tilde{\Phi}$ factors through N, and has dense image in N.

If $\mathcal{H}$ is a coherent sheaf on a product variety $T \times_k X$, one sets $\mathcal{H}_t = \mathcal{H} \otimes_{(T\times_k X)} (t \times_k X)$ for the pull-back coherent sheaf for any morphism of schemes $t \to T$. We use in the sequel a variant of [Har80, Prop. 1.1]:

Lemma 3.7. *Let X be a projective normal variety over k, T be a smooth quasi-projective variety over k, $\mathcal{H}$ be a reflexive sheaf over $T \times_k X$. Then the locus of points $t \in T$ such that $\mathcal{H}_t$ is reflexive is a constructible subset of T.*

Proof. On a normal quasi-projective scheme Z, a coherent sheaf $\mathcal{E}$ is reflexive if and only if there is an exact sequence $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{F} \to 0$ where $\mathcal{L}$ is locally free and $\mathcal{F}$ is torsion free. [To obtain this sequence, one uses quasi-projectivity to write an exact sequence $\mathcal{L}'^\vee \to \mathcal{L}^\vee \to \mathcal{E}^\vee \to 0$, where $^\vee$ denotes the dual, and where $\mathcal{L}, \mathcal{L}'$ are locally free, then normality of Z and reflexivity of $\mathcal{E}$ to obtain the desired exact sequence using that the natural morphism $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$ is an isomorphism. Conversely, if one has such an exact sequence, $\mathcal{E}$ is torsion free and the cokernel of $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$ lies in $\mathcal{F}$, thus has to be zero as $\mathcal{F}$ is torsion free.]

Applying this to $Z = T \times_k X$ and $Z = t \times_k X$, we see that $\mathcal{H}_t$ is reflexive if and only if the sheaves $\mathcal{H}_t$ and $\mathcal{H}'_t$ on $t \times_k X$ are torsion free. Indeed, if $\mathcal{H}'_t$ is torsion free, then the criterion implies that $\mathrm{Ker}(\mathcal{L}_t \to \mathcal{H}'_t)$ is reflexive. If in addition $\mathcal{H}_t$ is torsion free, then the surjective morphism $\mathcal{H}_t \to \mathrm{Ker}(\mathcal{L}_t \to \mathcal{H}'_t)$ is injective, thus an isomorphism. Vice-versa, if $\mathcal{H}_t$ is reflexive, then it is torsion free thus $\mathcal{H}_t \to \mathcal{L}_t$ has to be injective, and thus the surjective morphism $\mathcal{H}_t \to \mathrm{Ker}(\mathcal{L}_t \to \mathcal{H}'_t)$ has to be an isomorphism, thus one has an exact sequence $0 \to \mathcal{H}_t \to \mathcal{L}_t \to \mathcal{H}'_t \to 0$, and taking again double duals, one concludes that $\mathcal{H}'_t \to ((\mathcal{H}'_t)^\vee)^\vee$ is injective, thus $\mathcal{H}'_t$ is torsion free. On the other hand, the locus of points $t \in T$ for which $\mathcal{H}_t$ and $\mathcal{H}'_t$ are torsion free is constructible ([EGAIV, Prop. 9.4.8]).

∎

Restricting the Quot-scheme construction [HuyLeh97, Thms. 4.3.3, 4.3.4] of M to N yields the existence of a quasi-projective scheme T over k, a $\mathcal{G} := PGL(n)$ free action on T, for some natural number n, such that N is the categorial quotient $N = T/\mathcal{G}$ of T by $\mathcal{G}$, together with a $\tilde{\mathcal{G}} = GL(n)$-linearized coherent sheaf $\mathcal{E}$ on $T \times_k X$, flat over T, such that for all geometric points $t \in T$, the point $t \in N$ is the moduli point of the restriction $\mathcal{E}_t$ of $\mathcal{E}$ to $t \times_k X$, which is stable and of Hilbert polynomial χ.

We define the sheaf $\mathcal{E}' = (1 \times j)_*(1 \times j)^*\mathcal{E}$. Then $\mathcal{E}'$ is coherent, $\tilde{\mathcal{G}}$-linearized and one has a $\tilde{\mathcal{G}}$-morphism $\rho : \mathcal{E} \to \mathcal{E}'$. For a subscheme $T' \subset T$, we denote by $\rho_{T'} : \mathcal{E}_{T'} \to \mathcal{E}'_{T'}$ the restriction of ρ to $T' \times_k X$.

Claim 3.8. There is a smooth dense open $\mathcal{G}$-invariant subscheme $T' \subset T$ such that the restriction $\rho_{T'}$ of ρ is an isomorphism, and such that for all geometric points $t \in T'$, the sheaf $\mathcal{E}_t|_{t \times_k U}$ is locally free and $\rho_t : \mathcal{E}_t \to \mathcal{E}'_t$ and $\mathcal{E}_t \to j_{t*}j_t^*\mathcal{E}_t$ are isomorphisms.

Proof. The connected components of the smooth locus of T are clearly $\mathcal{G}$-stable, and so we may replace T by any connected component of its smooth locus, for the purposes of this claim. So we may assume here that T is smooth and connected.

Let S denote the (dense) set of points $[j_*E_n] \in N$, and let $S^1 = S \times_N T$. For any closed point $s \in S^1$, the local ring $\mathcal{O}_{s,T}$ is a regular local ring, so that the ideal sheaf of the closed subscheme $\{s\} \times X \subset T \times X$ is everywhere locally generated by a regular sequence. Further, X is normal, and $\mathcal{E}_s = j_*j^*\mathcal{E}_s$, where $j^*\mathcal{E}_s$ is locally free, and so $\mathcal{E}_s$ satisfies Serre's property (S_2). By repeatedly applying [EGAIV, Prop. 5.12.2] we conclude that $\mathcal{E}$ satisfies Serre's property (S_2) on $T \times X$ at all points of $\{s\} \times X$, for each $s \in S^1$. Hence there is an open subscheme $T \times_k X$ containing S^1 on which $\mathcal{E}$ satisfies (S_2) (namely, the open where $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$ is an isomorphism). This is $\mathcal{G}$-invariant, and has a maximal open subset of the form $T^1 \times X$ with $T^1 \subset T$ an open subscheme containing S^1; clearly T^1 is also $\mathcal{G}$-invariant.

On $T^1 \times_k X$, $\rho_{T^1} : \mathcal{E}_{T^1} \to \mathcal{E}'_{T^1} = (1_{T^1} \times j)_*(1_{T^1} \times j)^*\mathcal{E}_{T^1}$ is an isomorphism, since $\mathcal{E}_{T_1}$ atisfies (S_2) and ρ_{T^1} is an isomorphism on a dense open subscheme with complement of codimension ≥ 2. We slightly abuse notations, and write $\mathcal{E}_{T^1} = \mathcal{E}'_{T^1}$. The set S^1 lies in T^1 and is dense.

On the other hand, for all $s \in S^1$, $\mathcal{E}|_{s \times_k U}$ is locally free. Thus the largest open subscheme of $T^2 \times_k U$ on which $\mathcal{E}$ is locally free is not empty. It is $\mathcal{G}$-invariant. Its projection to T^2 is a constructible subset, which contains the dense subset S^1. Thus it contains a dense open subscheme. The maximal such open T^2 is $\mathcal{G}$-invariant, dense in T^1, and by definition, $\mathcal{E}|_{t \times_k U}$ is locally free for all geometric $t \in T^2$.

Finally we apply Lemma 3.7, to conclude that there is a constructible subset $T^3 \subset T^2$, which is $\mathcal{G}$-invariant, and contains the dense subset S^1, consisting of the points $t \in T^2$ such that $\mathcal{E}_t$ is reflexive. The largest open subset $T' \subset T^3$ is then also $\mathcal{G}$-invariant, and $S^1 \cap T'$ is dense in it. For $t \in T'$, we then have that $\mathcal{E}_t \to j_{t*}j_t^*\mathcal{E}_t$ is an isomorphism, since $\mathcal{E}_t$ is reflexive.

This finishes the proof. $\square$

We abuse notations, denote by T this open, by $S \subset T$ the dense set of points, and simply by $\mathcal{E} \to \mathcal{E}'$ the isomorphism. So $T/\mathcal{G}$ is a dense open subscheme of N, in which $S \cap (T/\mathcal{G})$ is dense.

We perform the Quot-scheme construction on $X^{(1)}$, $N^{(1)}$, defining the smooth $\mathcal{G}$-invariant scheme $T^{(1)}$, with free $\mathcal{G}$-action, such that $T^{(1)}/\mathcal{G} \hookrightarrow N^{(1)}$ is open dense, contained in the smooth locus, contains a dense open set of k-points from $S^{(1)}$, and defining the reflexive sheaf $\mathcal{E}^{(1)}$ on $T^{(1)} \times_k X^{(1)}$, locally free on $T^{(1)} \times_k U^{(1)}$, flat over $T^{(1)}$, χ-stable and reflexive on closed fibers. We denote by $F_{/k} : X \to X^{(1)}$ the relative Frobenius morphism.

Claim 3.9. For each connected component T_0 of T, there are a polynomial $\chi'(m) \in \mathbb{Q}[m]$, with $\chi'(m) \in \mathcal{P}$, and a smooth dense open $\mathcal{G}$-invariant subscheme $\mathcal{T}^{(1)} \subset T_0^{(1)}$, such that $S^{(1)} \times_{N^{(1)}} \mathcal{T}^{(1)} \subset \mathcal{T}^{(1)}$ is dense, and such that for all geometric points $t \in \mathcal{T}^{(1)}$, $(1_t \times j)_*(1_t \times F_{/k})^*(1_t \times j^{(1)})^*\mathcal{E}_t^{(1)}$ is χ'-stable.

Proof. We define $\mathcal{V} = (1_{T_0} \times j)_*(1_{T_0} \times F_{/k})^*(1_{T_0} \times j)^*\mathcal{E}^{(1)}$, which is $\tilde{\mathcal{G}}$-invariant, and apply for $\mathcal{V}$ a similar argument as for $\mathcal{E}$. We conclude that there is a maximal smooth open subscheme $\mathcal{T}^{(1)}$ of $T_0^{(1)}$, which is $\mathcal{G}$-invariant, on which $\mathcal{V}$ is $\mathcal{T}^{(1)}$-flat, thus of constant Hilbert polynomial $\chi'(m) \in \mathbb{Q}[m]$ on geometric fibers, and stable and reflexive on geometric fibers, of value $(1_t \times j)_*(1_t \times F_{/k})^*(1_t \times j^{(1)})^*\mathcal{E}_t^{(1)}$. And by construction, $S^{(1)} \times_{N^{(1)}} \mathcal{T}^{(1)} \subset \mathcal{T}^{(1)}$ is dense. $\square$

Let $M(\chi')$ be the moduli scheme of χ'-stable sheaves on X. The sheaf $\mathcal{V}$ induces a morphism $\mathcal{T}^{(1)} \to M(\chi')$ which sends a geometric point t to the moduli point of $(1_t \times j)_*(1_t \times F_{/k})^*(1_t \times j^{(1)})^*\mathcal{E}_t^{(1)}$. This morphism is $\mathcal{G}$-invariant, thus induces a morphism $\mathcal{U}_0^{(1)} = \mathcal{T}^{(1)}/\mathcal{G} \to M(\chi')$. Let $\mathcal{U}_0 = (W^*)^{-1}(\mathcal{U}_0^{(1)})$, and let $\mathcal{U}$ be the (disjoint) union of the $\mathcal{U}_0$. Then $\mathcal{U} \subset N$ is a dense open subscheme contained in the smooth locus of N. There is thus a well-defined morphism $\mathcal{U}^{(1)} \to N$ and it may be viewed as a rational dominant map $\Phi : N^{(1)} \dashrightarrow N$, called the Verschiebung map (see [Oss06, App. A]).

Let Γ' be the closure of the graph of Φ in $\mathcal{U}^{(1)} \times_k \mathcal{U}$. Then $\Gamma' \subset \mathcal{U}^{(1)} \times_k \mathcal{U}$ is a closed reduced subscheme, which is birational to $\mathcal{U}^{(1)}$ via the first projection, and dominates $\mathcal{U}$ via the second projection. We define $\Gamma \subset \mathcal{U} \times_k \mathcal{U}$ by base change via $(W^*)^{-1}$ on $\mathcal{U}^{(1)}$. This defines an isomorphism of schemes $\Gamma \to \Gamma'$, and Γ is birational to $\mathcal{U}$ via the first projection and dominant onto $\mathcal{U}$ via the second projection. Thus Γ is a rational dominant map. As $\mathcal{U}$ has finitely many components, a power of this rational map stabilizes them. We apply [EsnMeh10, Thm. 3.14], an application of Hrushovski's theorem [Hru04, Cor. 1.2], to conclude that $\mathcal{U}$ contains k-points $[j_*E]$, for $[j_*E] \in \mathcal{U}(k)$, such that $(F^a)^*E = E$ for some $a \in \mathbb{N} \setminus \{0\}$, and E defines a F-divided sheaf. By [LS77, Satz 1.4], this equation defines a finite étale cover $\tau : U' \to U$, restricting to $\tau_C : C' \to C$ to C. As U is simply connected, τ is trivial, so is τ_C. Thus $(\tau_C)_*\mathcal{O}_{C'}$ is trivial, and so is $\iota^*E \subset (\tau_C)_*\mathcal{O}_{C'}$. Thus by Theorem 3.5, E is trivial as well. This is a contradiction if $r \geq 2$. For $r = 1$, we use in addition that the points $[j_*E]$ constructed are dense in $\mathcal{U}$, and therefore $\mathcal{U} = [\mathcal{O}_X]$. Thus $E_n = \mathcal{O}_U$ for infinitely many n, thus by Lemma 3.6 i), the F-divided sheaf $(E_n)_n$ is trivial. This finishes the proof.

□

Remark 3.10. In Theorem 3.2 one has a geometric assumption, X being normal, and an arithmetic one, $k = \bar{\mathbb{F}}_p$.

The geometric assumption is necessary to define, for each stratified bundle $E = (E_n, \sigma_n)_{n\geq 0}$ on U, an extension j_*E_n on X for which one can show the boundedness theorem 2.1. If U does not admit a normal compactification with boundary of codimension ≥ 2, we do not know how to bound the family of E_n, even if we assume that E is regular singular. The analogous question in complex geometry is interesting. We asked P. Deligne whether over $\mathbb{C}$, given a smooth compactification $j : U \to X$ such that the boundary $X \setminus U$ is a normal crossings divisor, the set of all Deligne canonical extensions (E_X, ∇_X) of regular singular algebraic flat connections (E, ∇) on U of bounded rank fulfills: the set $\{c_i(E_X)\}$ in the Betti cohomology algebra $\oplus_i H^{2i}(U, \mathbb{Z})$ is bounded. The answer is yes [Del14]; the proof, which is non-algebraic, uses as a key tool, that the topological fundamental group is finitely generated, and thus there is an affine Betti moduli space, etc. Our aim in this article is precisely to overcome the lack of such a finitely generated abstract group which controls $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.

As for the arithmetic assumption, we could drop it, if we had a specialization homomorphism on the étale fundamental group with suitable properties. We discuss this in Section 4.

4. Specialization

In [SGA1, X], Grothendieck shows the existence of a continuous specialization homomorphism $\pi_1^{\text{ét}}(X_{\bar{K}}) \to \pi_1^{\text{ét}}(X_{\bar{k}})$ for $f : X \to S$ a proper morphism, with geometric connected fibers, that is $f_*\mathcal{O}_X = \mathcal{O}_S$, S integral, K the function field of S and $\operatorname{Spec} k \to S$ a closed point ([SGA1, p. 207]) and he shows in [SGA1, Cor. 2.3] that this morphism is surjective if f is separable. Moreover it is shown in [SGA1, XIII, Thm. 2.4] that the existence of the specialization homomorphism extends to the tame fundamental group $\pi_1^{\text{ét,t}}(U_{\bar{K}}) \to \pi_1^{\text{ét,t}}(U_{\bar{k}})$ if $U \to S$ is the complement of a relative normal crossings divisor in $f : X \to S$, satisfying the previous assumptions, proper with $f_*\mathcal{O}_X = \mathcal{O}_S$. This specialization homomorphism is an isomorphism on the prime to p-quotient. That the tameness is necessary is of course visible already for $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $S = \operatorname{Spec} \mathbb{Z}$ as here $\pi_1^{\text{ét,t}}(U_{\bar{K}}) = \pi_1^{\text{ét}}(U_{\bar{K}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}})$ is as huge as Abhyankar's conjecture predicts. The aim of this section is to show the existence of examples in pure characteristic $p > 0$ over a base, for which over the geometric generic fiber, the fundamental group has no $\mathbb{Z}/p$-quotient, that is the variety has no Artin-Schreier covering, while for all geometric fibers of it over $\bar{\mathbb{F}}_p$, it does have such non-trivial coverings.

Let $k = \bar{\mathbb{F}}_p$, $C' \subset \mathbb{P}^2$ be a smooth elliptic curve, K be $k(C')$, $x_i : \operatorname{Spec} K \to C'$, $i = 1, \ldots, 9$ be 9 K-valued points such that if $X \to \mathbb{P}^2_K$ is the blow up of those points, with strict transform C_K of C'_K, then $\mathcal{O}_{C_K}(C_K) \in \operatorname{Pic}^0(C_K)$ is not torsion (this is arranged by choosing 8 points to be distinct k-points, base-changed, and the ninth to be a k-generic point). Here X is defined over K, and so is $U = X \setminus C_K$. We take a model $X_R \to \operatorname{Spec} R$ of $X \to \operatorname{Spec} K$, where $\operatorname{Spec} R$ is a non-empty open in $C'_0/\mathbb{F}_q$ where $C'_0/\mathbb{F}_q$ descends C'/k, over which the x_i are rational, and are disjoint sections. So $C \subset X$ over K has a model $C_R \subset X_R$ over R and $U_R := X_R \setminus C_R \to \operatorname{Spec} R$ is the complement of a strict relative normal crossings divisor.

Lemma 4.1. *For any closed point $a : \operatorname{Spec} \mathbb{F}_{q^m} \to \operatorname{Spec} R$,*

i) $\Gamma(U_a, \mathcal{O})$ *has transcendence degree* 1 *over* $\mathbb{F}_{q^m}$*;*
ii) $H^1_{\text{ét}}(U_x \otimes_{\mathbb{F}_{q^m}} k, \mathbb{Z}/p) \neq 0$*;*
iii) $\pi_1^{\text{ét}}(U_a \otimes_{\mathbb{F}_{q^m}} k) \neq 0$.

Proof. Clearly i) implies ii) as the residue class of a transcendental element in

$$\Gamma(U_x \otimes_{\mathbb{F}_{q^m}} k, \mathcal{O})/\mathrm{Im}(F-1) \subset H^1_{\text{ét}}(U \otimes_{\mathbb{F}_{q^m}} k, \mathbb{Z}/p)$$

is non-trivial, and ii) implies iii) by definition.

We show i). One has

$$H^0(U_a, \mathcal{O})/\mathbb{F}_{q^m} = \varinjlim_{n \in \mathbb{N}} H^0(\mathcal{O}_{nC_a}(nC_a)) = \varinjlim_{n \in \mathbb{N}} H^0(\mathcal{O}_{rnC_a}(rnC_a))$$

where r is the torsion order of $\mathcal{O}_{C_a}(C_a)$. One easily computes that $\dim_k H^0(\mathcal{O}_{rnC_a}(rnC_a)) = (n+1)$, and that, as a $\mathbb{F}_{q^m}$-algebra, $H^0(U_a, \mathcal{O})$ is spanned by any element t whose image spans $H^0(\mathcal{O}_{rC}(rC))$. Thus $\Gamma(U_a, \mathcal{O}) = \mathbb{F}_{q^m}[t]$.

□

Lemma 4.2. *One has*

i) $\Gamma(U_K, \mathcal{O}) = K$*;*
ii) $H^1_{\text{ét}}(U_{\bar{K}}, \mathbb{Z}/p) = 0$.

Proof. The condition on $\mathcal{O}_C(C) \in \operatorname{Pic}^0(C)$ not being torsion implies $H^i(\mathcal{O}_{nC}(nC)) = 0$ for all $n \geq 1$ and $i = 0, 1$. This immediately implies that i), and also ii), as $H^1_{\text{ét}}(U_{\bar{K}}, \mathbb{Z}/p) = \big(\varinjlim_{n \in \mathbb{N}} H^1(\mathcal{O}_{nC}(nC))\big)^{F=1}$.

□

Remark 4.3. It would be desirable to understand whether or not $\pi_1^{\text{ét}}(U_{\bar{K}}) = 0$. This is equivalent to saying that for any finite étale map $h : V \to U$, $H^1_{\text{ét}}(V, \mathbb{Z}/p) = 0$. For any finite flat map $h : V \to U$, such that V is integral, one has $H^0(V, \mathcal{O}_V) = k$. Indeed, $H^0(V, \mathcal{O})$ is then a finite $k = H^0(U, \mathcal{O})$-algebra, thus is an artinian k-algebra, and is integral, thus is equal to k. But this is not enough to conclude that $H^1_{\text{ét}}(V, \mathbb{Z}/p) = 0$ and we have not succeeded in computing it.

5. Some theorems over any field

5.1. **A purely geometric example.** As $\mathbb{A}^1$ has a huge fundamental group in positive characteristic, it is not easy to find examples of smooth non-proper simply connected varieties which are not simply the complement of a codimension ≥ 2 subscheme in a smooth projective simply connected variety. We construct in this section a simple example of a normal surface for which the maximal smooth open subvariety is simply connected.

Let Y be a projective smooth variety over an algebraically closed field k of characteristic p, and let $L = \mathcal{O}_Y(\Delta)$ be a line bundle, where Δ is a non-empty effective divisor. We define

$$U = \mathbb{P}(\mathcal{O}_Y \oplus L) \setminus \infty\text{-section},$$

so $f : U \to Y$ is the total space of the geometric line bundle L. Note that if L is globally generated and ample, then U also appears as the complement of the vertex of the corresponding normal projective cone X over Y (which is Proj $\oplus_{n\geq 0}$ $H^0(Y, Sym^n(\mathcal{O}_Y \oplus L)))$. This also gives concrete examples as in Theorem 3.2, where we can get the conclusion in an alternate way, because of the special geometry.

Proposition 5.1. i) *$f_* : \pi_1^{\text{ét}}(U) \to \pi_1^{\text{ét}}(Y)$ is an isomorphism, and f^* induces an isomorphism on the category of stratified bundles.*
ii) *If $\pi_1^{\text{ét}}(Y) = 0$, there are no non-trivial stratified bundles on U.*

Proof. ii): we apply i) and [EsnMeh10, Thm. 1.1] to Y. We are left with i): the section $\sigma_0 : \mathcal{O}_Y(-\Delta) \to \mathcal{O}_Y$ induces Sym $\mathcal{O}_Y(-\Delta) \to$ Sym $\mathcal{O}_Y$ thus a morphism $\sigma : Y \times_k \mathbb{A}^1 \to U$ which is an isomorphism on $U \setminus f^{-1}(\Delta)$, is birational dominant. We fix base points $(y_0 \in Y,\ 0 \in \mathbb{A}^1)$, where $y_0 \in \text{supp}(\Delta)$, and set $\sigma(y_0, 0) = x_0$. Thus $\sigma_* : \pi_1^{\text{ét}}(Y \times_k \mathbb{A}^1, (y_0, 0)) \to \pi_1^{\text{ét}}(U, x_0)$ is surjective. One has the Künneth formula $\pi_1^{\text{ét}}(Y \times_k \mathbb{A}^1, (y_0, 0)) = \pi_1^{\text{ét}}(Y, y_0) \times \pi_1^{\text{ét}}(\mathbb{A}^1, 0)$ as Y is proper [SGA1, Exp. X, Cor. 1.9]. Furthermore, $\sigma(y_0 \times \mathbb{A}^1) = x_0$, so that the composition $\{y_0\} \times \mathbb{A}^1 \to U$ induces the trivial homomorphism on fundamental groups. Hence the morphism $Y \times \{0\} \to U$, induced by restricting σ, is a surjection on fundamental groups, which must be an isomorphism, as seen by composing with the projection $f : U \to Y$. Hence f induces an isomorphism on fundamental groups as well.

On the other hand, the Künneth formula holds for the stratified fundamental group [Gie75, Prop. 2.4]. Thus a very similar argument shows that if E is stratified on U, $\sigma^*(E) = p_Y^*(E')$ for some E' on Y, $p_Y : Y \times_k \mathbb{A}^1 \to Y$. So E and $f^*(E')$ are isomorphic on a non-trivial open of U, thus are isomorphic. Indeed $\pi^{\text{strat}}(U) \to \pi^{\text{strat}}(X)$ is surjective ([Phu12, Comment before Thm.3.5] and [Kin14, Lem. 2.5]). Again restricting to the 0-section shows E' is unique.

□

5.2. **Second main theorem.** The aim of this section is to give a general statement which does not require k to be isomorphic to the algebraic closure of a finite field. The proof is a simple application of Theorem 3.5. It is nonethelss worth being mentioned as it yields new concrete examples, see 5.4.

Theorem 5.2. *Let X be a normal projective variety of dimension $d \geq 1$, defined over an algebraically closed field k. Let U be its regular locus. If $\pi_1^{\text{ét}}(U) = 0$, and U contains a projective curve C, intersection of $(d-1)$-hyperplane sections, such that $\pi_1^{\text{ét}}(C)$ is abelian, then there are no non-trivial $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module on U.*

Proposition 5.3. *Let $C \to U$ be a proper subscheme of a smooth quasi-projective variety defined over an algebraically closed field k of characteristic $p > 0$. If $\pi^{\text{strat}}(C) \to \pi^{\text{strat}}(U)$ is surjective, then any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module on U which is an extension of $\mathbb{I}$ by itself has finite monodromy.*

Proof. As $\mathrm{Ext}^1_C(\mathbb{I}, \mathbb{I}) = H^1_{\text{ét}}(C, \mathbb{Z}/p) \otimes_{\mathbb{F}_p} k$ ([dSan07, (9)]), restriction to C of any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module on U which is an extension of $\mathbb{I}$ by itself has finite monodromy, thus by the surjectivity $\pi^{\text{strat}}(C) \to \pi^{\text{strat}}(U)$, any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module on U which is an extension of $\mathbb{I}$ by itself has finite monodromy as well.

□

Proof of Theorem 5.2. By Proposition 3.3 ii), we may assume that C is semi-normal. This implies that

i) either all components of C are smooth rational curves and the homology of the dual graph of C is 0 (so the graph is a tree) or 1-dimensional,
ii) or else C is the union $C' \cup C''$, where C' is a tree of smooth rational curves, and C'' is irreducible, so is either a smooth elliptic curve or an irreducible rational with one node (called 1-nodal curve in the sequel), and $C' \cap C''$ consists of one point.

For any smooth irreducible component C_0 of C, $\pi^{\text{strat}}(C_0)$ is trivial if C_0 is rational, and abelian if C_0 is an elliptic curve ([Gie75, p. 71]). If the graph is a tree, then the bundles E_n of a F-divided sheaf $E = (E_n, \sigma_n)_{n\in\mathbb{N}}$ on C are uniquely determined by their restriction to the components E_i, so $\pi^{\text{strat}}(C)$ is 0 if all components are rational, else (case (ii) above) is equal to $\pi^{\text{strat}}(C'')$.

Assume C is a graph of rational curves with 1-dimensional homology. One shows easily that any indecomposable vector bundle of rank r on C, whose pull-back to the normalization is trivial, must be isomorphic to $L \otimes E_r$, where L is a line bundle, and E_r is the unique indecomposable vector bundle of rank r which is unipotent (multiple extension of $\mathcal{O}_C$). Indeed, we may write $C = C_1 \cup C_2$ where C_1 is a tree of smooth rational curves, C_2 an irreducible rational curve, and $C_1 \cap C_2$ consists of two points. Then trivializing the restriction of the bundle to C_1 and C_2 in a compatible manner at one of the intersection points of C_1 and C_2, the bundle is determined up to isomorphism by the conjugacy class of the element of $GL_r(k)$ given by the glueing data at the second point of $C_1 \cap C_2$. Using the Jordan canonical form we deduce the description of bundles.

A similar description holds when $C = C' \cup C''$, where C' is a tree of smooth rational curves, and C'' is 1-nodal. In both of these cases, we note the resemblance to the Atiyah classification of indecomposable bundles on elliptic curves.

The proof in [Gie75, p. 71] for elliptic curves, that $\pi^{\rm strat}$ is abelian, (since any irreducible $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is of rank 1, which results from the explicit description using Atiyah's classification), adapts to the other two remaining cases. It shows that in the situation where the homology of the graph is 1-dimensional, and where one component is 1-nodal, $\pi^{\rm strat}$ is again abelian. There is a simplification resulting from the fact that Pic^0 has no p-torsion, so that line bundles which admit an$\mathcal{O}_X$-coherent $\mathcal{D}_X$-module admit only one.

By Theorem 3.5, this implies that $\pi^{\rm strat}(U)$ is abelian, so all irreducible objects have rank 1. Since $\pi_1^{\rm ét}(U) = 0$ implies in particular that $\mathrm{Pic}\,(U)$ is finitely generated, the rank 1 objects are trivial, so all we have to understand are extensions of $\mathbb{I}$ by itself when $\pi_1^{\rm ét}(C)$ is not-trivial. We apply Proposition 5.3 to conclude. $\square$

Example 5.4. Assume char. $k \neq 2, 3$. A non-trivial example of Theorem 5.2 is as follows. Let $X \subset \mathbb{P}^3$ be given by an equation

$$f(x,y,z)w + g(x,y,z) = 0$$

where $f = 0$ defines a smooth plane cubic, and $g = 0$ is a plane quartic which (i) intersects the plane cubic $f = 0$ transversally, and (ii) has abelian étale fundamental group, so that it is singular. Thus the semi-normalization of $(g = 0)$ is as described in the proof of Theorem 5.2.

It is easy to see that X has a unique singular point, given by $P = \{x = y = z = 0, w = 1\}$, whose complement U contains the ample curve given by $w = 0$, which is the plane curve $g = 0$. One also sees that the blow up $\pi : \tilde{X} \to X$ of X at its singular point is identified with the blow up of $\mathbb{P}^2$ at the 12 points given by $f = g = 0$, and the exceptional divisor $\pi^{-1}(P)$ is identified with the strict transform of the plane cubic $f = 0$ under the blow-up of $\mathbb{P}^2$.

The surface U satisfies $\pi_1^{\rm ét}(U) = 0$, because (i) the existence of the ample curve C with abelian fundamental group implies that the fundamental group of U is also abelian (ii) using that $U = X \setminus \{P\}$ is identifed with an open subset of the blow-up of $\mathbb{P}^2$ obtained by removing the strict transform of the cubic $f = 0$, we see that the tame fundamental group of U is trivial (iii) a direct computuation now shows also that $H^1_{\rm ét}(U, \mathbb{Z}/p\mathbb{Z}) = 0$ by considering the Frobenius action on $H^1(U, \mathcal{O}_U)$.

Here are some explicit examples: assume char.$k \neq 2, 3$; take $f = x^3 + y^3 + z^3$. The following choices of g satisfy the conditions above:
(i) $g = x^4 + ay^3z$, where $a = 1$ or -1 depending on p (here $g = 0$ is simply connected)
(ii) $g = (x^2 + ay^2 - 4z^2)(x^2 + by^2 - 4z^2)$, where $a \neq b$ are suitably chosen, depending on p (here $g = 0$ consists of 2 conics tangent at 2 points, so has abelian fundamental group); a variant is obtained by taking $b = 0$, so that the quartic $g = 0$ is the union of a conic and two tangent lines.

Remark 5.5. In the above example, X is a normal quartic surface with a unique singular point, whose complement U is simply connected, when U contains a "special" singular hyperplane section. It seems plausible to guess that U is simply connected for any such quartic with a triple point, even if no such special hyperplane section exists.

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Appendix A. Algebraization of formal morphisms and formal subbundles, by Jean-Benoît Bost

In this Appendix, we establish the algebraicity criterion for vector bundles on which the proof of Theorem 3.5 relies.

Our derivation of this criterion relies on two classical results: an *algebraicity criterion* for formal subschemes, which already appears in [Bos01] in a quasi-projective setting, and actually is a straightforward consequence of the study by Hartshorne ([Har68]) of the fields of meromorphic functions on some formal schemes, and a *connectedness theorem à la* Bertini–Fulton–Hansen ([FH79]).

In the first two sections of this Appendix, we recall these two results in a form suitable for our purposes. Then we prove an algebraicity criterion concerning formal morphisms, from which we finally deduce algebraicity results for formal vector bundles.

Let us also mention that the type of algebraicity results discussed in this Appendix originates in Grothendieck's seminar SGA2 ([Gro68]) and in the work of Hironaka and Matsumura ([Hir68], [HM68]).

We denote by k an algebraically closed field. All k-schemes are assumed to be separated, of finite type over k.

In this Appendix, we could have worked with quasi-projective k-schemes only. Our results still hold in this quasi-projective setting[1], which would be enough for the proof of Theorem 3.5. Actually several of our arguments become more elementary in this setting; this is for instance the case of the proofs of Theorem 2 and Proposition 5.

The exposition of the results in this Appendix has benefited from the suggestions of an anonymous referee, to whom I am grateful.

A.1. **Algebraization of formal subschemes.** Let Z be a k-scheme over k, and Y a closed subscheme of Z. We shall denote by $i : Y \hookrightarrow Z$ the inclusion morphism, and by $\widehat{Z}_Y$ the formal completion of Z along Y.

Let $\widehat{V}$ be a closed k-formal subscheme of $\widehat{Z}_Y$, which admits Y as scheme of definition[2] and which is formally *smooth* over k (or equivalently is regular), of pure dimension d. The *Zariski closure* $\overline{\widehat{V}}$ is the smallest closed subscheme of Z which, considered as a formal scheme, contains $\widehat{V}$, or equivalently, the smallest closed subscheme of Z which contains Y and whose formal completion along Y contains $\widehat{V}$. The smoothness of $\widehat{V}$ implies that $\overline{\widehat{V}}$ is reduced, and is irreducible if Y is connected. The dimension of $\overline{\widehat{V}}$ is at least d.

Proposition 1. *With the above notation, the following conditions are equivalent:*

(1) *The Zariski closure $\overline{\widehat{V}}$ of $\widehat{V}$ in Z has dimension d.*

[1] This is formal, but for the implication (1) $\Rightarrow$ (2) in Proposition 1; however, the validity of the quasi-projective version of this implication directly follows from its proof below.

[2] In other words, it contains Y and its underlying topological space $|\widehat{V}|$ coincides with $|\widehat{Z}_Y| = |Y|$.

(2) *There exists a smooth k-scheme M and morphisms of k-schemes $j : Y \longrightarrow M$ and $\mu : M \longrightarrow Z$ such the diagram*

(A.1)
$$\begin{tikzcd} & M \arrow[d, "\mu"] \\ Y \arrow[ur, "j"] \arrow[r, "i"'] & Z \end{tikzcd}$$

is commutative and the associated morphism of formal completion $\hat{\mu} : \widehat{M}_{j(Y)} \longrightarrow \widehat{Z}_Y$ maps $\widehat{M}_{j(Y)}$ isomorphically onto $\widehat{V}$:

(A.2)
$$\hat{\mu} : \widehat{M}_{j(Y)} \xrightarrow{\sim} \widehat{V}.$$

(Observe that the commutativity of (A.1) implies that j is a closed immersion.)

When the equivalent conditions in Proposition 1 conditions are satisfied, we shall say that *the smooth formal scheme $\widehat{V}$ is algebraic.*

Proof. To prove the Proposition, we may assume that Y is connected.

Suppose that (1) is satisfied, and consider the normalization

$$\nu : \overline{\widehat{V}}^{\mathrm{nor}} \longrightarrow \overline{\widehat{V}}$$

of the integral scheme $\overline{\widehat{V}}$. We may consider the completion $\overline{\widehat{V}}^{\mathrm{nor}\wedge}_{\nu^{-1}(Y)}$ of $\overline{\widehat{V}}^{\mathrm{nor}}$ along $\nu^{-1}(Y)$. By the "analytic normality of normal rings", it is a normal formal scheme [3] and there is a unique morphism of formal scheme

$$\hat{j} : \widehat{V} \longrightarrow \overline{\widehat{V}}^{\mathrm{nor}\wedge}_{\nu^{-1}(Y)}$$

such that the following diagram is commutative:

(A.3)
$$\begin{tikzcd} & \overline{\widehat{V}}^{\mathrm{nor}\wedge}_{\nu^{-1}(Y)} \arrow[d, "\hat{\nu}"] \\ \widehat{V} \arrow[ur, "\hat{j}"] \arrow[r, "\iota"'] & \overline{\widehat{V}}^{\wedge}_{Y}, \end{tikzcd}$$

where $\iota : \widehat{V} \to \overline{\widehat{V}}^{\wedge}_{Y}$ denotes the inclusion morphism. The morphism $\hat{j}$ is a closed immersion, and at the level of local rings, defines surjections of integral domains

[3]Namely, for any affine connected open subscheme U of $|\overline{\widehat{V}}^{\mathrm{nor}\wedge}_{\nu^{-1}(Y)}| = |\nu^{-1}(Y)|$, the ring $\hat{A}$ of sections over U of the structure sheaf of $\overline{\widehat{V}}^{\mathrm{nor}\wedge}_{\nu^{-1}(Y)}$ is a normal domain. Actually, if V denotes a connected affine open subscheme of $\overline{\widehat{V}}^{\mathrm{nor}}$ such that $U = V \cap |\nu^{-1}(Y)|$, the ring $\hat{A}$ may be identified with the completion of $A := \Gamma(V, \mathcal{O}_{\overline{\widehat{V}}^{\mathrm{nor}}})$ in the I-adic topology, where I denote the ideal of A defining the closed subscheme U of V. The fact that this completion is normal easily follows from the fact that, for every closed point x of V, the completion $\hat{A}_x$ of A at x is itself a normal domain: this last fact is precisely the content of the "analytic normality" of the normal ring A, as established for instance in [ZS75], p. 313-320, or [Nag62], Section 37.5. See for instance [Bǎd04] p. 102 for more details.

of the same dimension, which are therefore isomorphisms. (See [Gie77], Proof of Theorem 4.3, p. 1148-1149, and [Băd04], Proof of Theorem 9.16, p. 100-103, for similar arguments.)

In other words, if we let

$$j := \hat{j}_{|Y},$$

then j defines a closed immersion of Y in $\overline{\widehat{V}}^{\mathrm{nor}}$, its image is a connected component of $\nu^{-1}(Y)$, and $\hat{j}$ establishes an isomorphism from $\widehat{V}$ to the formal completion $\overline{\widehat{V}}^{\mathrm{nor}\wedge}_{j(Y)}$.

This shows that $\overline{\widehat{V}}^{\mathrm{nor}}$ is smooth on some open neighborhood of $j(Y)$, and Condition (2) is satisfied by the regular locus $M := \left(\overline{\widehat{V}}^{\mathrm{nor}}\right)_{\mathrm{reg}}$ of $\overline{\widehat{V}}^{\mathrm{nor}}$ and $\mu := \nu_{|M}$.

Conversely, when Condition (2) holds, we may assume that M is connected, after possibly replacing it by its connected component containing $j(Y)$. The isomorphism (A.2) shows that M, like $\widehat{V}$, has dimension d. Consequently $\overline{\mu(M)}$ has dimension at most d, and its subscheme $\overline{\widehat{V}}$ also. □

Theorem 2. *With the above notation, if Y is proper over k and a l.c.i. in $\widehat{V}$, with all components of dimension ≥ 1, and if its normal bundle $N_Y\widehat{V}$ in $\widehat{V}$ is ample, then $\widehat{V}$ is algebraic.*

This theorem may be seen as an avatar, valid in algebraic geometry (possibly over a field of positive characteristic) of the classical theorem of Andreotti about the algebraicity of pseudo-concave complex analytic submanifolds of the complex projective spaces ([And63], Théorème 6).

The proof below relies on some upper bound, established by Harshorne ([Har68]), on the transcendence degree of the field of meromorphic functions — the "formal-rational functions" investigated by Hironaka and Matsumura ([Hir68], [HM68]) — on the formal scheme $\widehat{V}$. We refer the reader to [Gie77], Section 4, for geometric applications of Hartshorne's results in a similar spirit.

In a quasi-projective setting, Theorem 2 admits a simpler proof, inspired by techniques of Diophantine approximation, which avoids the use of formal meromorphic functions ([Bos01], Section 3.3). Actually, as shown in [Che12], these techniques lead to algebraization results valid under assumptions significantly weaker that the ampleness of $N_Y\widehat{V}$ when $\dim Y > 1$.

Proof. We may assume that $\widehat{V}$ (or equivalently Y) is connected. Then we may consider the field $k(\widehat{V})$ of meromorphic functions on $\widehat{V}$, as introduced in [Hir68], p. 600, and [HM68], §1. Recall that, as a ring, it is defined as the space of global sections $\Gamma(\widehat{V}, \mathcal{M}_{\widehat{V}})$ of the sheaf $\mathcal{M}_{\widehat{V}}$ on $\widehat{V}$ defined as the associated sheaf of the presheaf $\mathcal{M}^0_{\widehat{V}}$ defined by sending an arbitrary open subset U of $\widehat{V}$ (or equivalently of Y) to the total ring of fractions $\mathcal{M}^0_{\widehat{V}}(U)$ of $\mathcal{O}_{\widehat{V}}(U)$. In the present situation where $\widehat{V}$ is assumed connected, $k(\widehat{V})$ is actually a field (for instance, by a simple variant of the discussion in [Har70], pp. 189–190; see also [Băd04], Chapter 9, for an exposition of the basic properties of formal rational functions).

According to [Har68], Theorem 6.7, the transcendence degree $\deg.\mathrm{tr}_k k(\widehat{V})$ of the field $k(\widehat{V})$ is at most d.

Besides, the k-morphism $\widehat{V} \hookrightarrow \overline{\widehat{V}}$ induces a morphism of extensions of k:

$$k(\overline{\widehat{V}}) \longrightarrow k(\widehat{V}).$$

Indeed, by the very definition of $\overline{\widehat{V}}$, for any open subscheme U of $\overline{\widehat{V}}$ which meets $|Y|$, the restriction morphism

$$\mathcal{O}_{\overline{\widehat{V}}}(U) \longrightarrow \mathcal{O}_{\widehat{V}}(U \cap |Y|)$$

is injective.

This implies that

$$\dim \overline{\widehat{V}} = \deg.\mathrm{tr}_k k(\overline{\widehat{V}}) \leq \deg.\mathrm{tr}_k k(\widehat{V}) \leq d,$$

and establishes that $\widehat{V}$ is algebraic. □

A.2. **Connectedness and étale neighborhoods of intersections of ample hypersurfaces.**

A.2.1. *A connectedness theorem.* The following theorem is an avatar of the classical connectedness theorem of Fulton-Hansen [FH79]. Actually, its special case when the inclusion morphism $H_1 \cap \ldots \cap H_e \hookrightarrow X$ is a regular imbedding of codimension e, is alluded to by Fulton and Hansen in [FH79], p. 161. The general case is essentially established by Debarre in [Deb96].

Theorem 3. *Let X be an integral projective k-scheme, e a positive integer, and $H_1, \ldots, H_e$ ample effective Cartier divisors on X.*

For any integral proper k-scheme X' and any k-morphism $f : X' \longrightarrow X$ such that

$$\dim f(X') > e,$$

the inverse image $f^{-1}(H_1 \cap \ldots \cap H_e)$ is a non-empty connected *subscheme of X'.*

For every $i \in \{1, \ldots, e\}$, we may choose a positive integer D_i such that the Cartier divisor $D_i H_i$ is very ample. In other terms, there exists a projective embedding $\iota_i : X \longrightarrow \mathbb{P}_k^{N_i}$ and an hyperplane L_i in $\mathbb{P}_k^{N_i}$, which does not contain $\iota_i(X)$, such that the following equality of divisors hold:

$$D_i H_i = \iota_i^{-1}(L_i).$$

Then, if we let

$$L := \prod_{1 \leq i \leq e} L_i,$$

we have

$$\begin{aligned}|H_1 \cap \cdots \cap H_e| &= |D_1H_1 \cap \cdots \cap D_eH_e| \\ &= |\iota_1^{-1}(L_1) \cap \cdots \cap \iota_e^{-1}(L_e)| \\ &= |(\iota_1, \ldots, \iota_e)^{-1}(L)|.\end{aligned}$$

Consequently the support of $f^{-1}(H_1 \cap \ldots \cap H_e)$ coincides with the one of the inverse image of L by the composite morphism

$$X' \xrightarrow{f} X \xrightarrow{(\iota_1,\ldots,\iota_e)} \prod_{1\leq i\leq e} \mathbb{P}_k^{N_i}.$$

Theorem 3 is easily seen to follow from [Deb96], Théorème 1.4, 2) a), applied to this morphism and to the subspace L of $\mathbb{P} := \prod_{1\leq i\leq e} \mathbb{P}_k^{N_i}$.

For the sake of completeness, we sketch a proof of Theorem 3 from "basic results" in algebraic geometry, at the level of Harsthorne's textbook [Har83] and Jouanolou's monograph [Jou83]. We also refer the reader to [FL81] for a beautiful discussion of related connectedness results. (Actually, the proof of Théorème 1.4 in [Deb96] relies on results and techniques presented in [FL81].)

Proof. By considering the Stein factorization of f (see [Gro61], Section III.4.3), we may assume that f is a *finite* morphism. Then, for every $i \in \{1, \ldots, e\}$, the following alternative holds: either $f(X') \subset H_i$, or $f^*(H_i)$ defines an ample effective divisor in X'. Moreover, if I denotes the set of elements i in $\{1, \ldots, e\}$ such that the second case arises, we clearly have:

$$f^{-1}(H_1 \cap \ldots \cap H_e) = \bigcap_{i\in I} f^{-1}(H_i).$$

These observations show that, to establish Theorem 3 in full generality, it is enough to establish the following assertion, which is actually the special case of Theorem 3 when $X' = X$ and $f = Id_X$:

Let X be an integral projective k-scheme, e a positive integer, and $H_1, \ldots, H_e$ ample effective Cartier divisors on X. If $\dim X > e$, then $H_1 \cap \ldots \cap H_e$ is a connected subscheme of X.

To establish this assertion, let us choose, for every $i \in \{1, \ldots, e\}$, a positive integer D_i such that the line bundle $\mathcal{O}_X(D_iH_i)$ is very ample, and let us consider the associated projective embedding[4]

$$f_i : X \longrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}_X(D_iH_i)) \simeq \mathbb{P}_k^{N_i}$$

(where $N_i := \dim_k \Gamma(X, \mathcal{O}_X(D_iH_i) - 1)$ and the product imbedding

$$f := (f_1, \ldots, f_e) : X \longrightarrow \mathbb{P} := \prod_{1\leq i\leq e} \mathbb{P}(\Gamma(X, \mathcal{O}_X(D_iH_i)) \simeq \prod_{1\leq i\leq e} \mathbb{P}_k^{N_i}.$$

[4]We use Grothendieck's notation: the projective space $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_iH_i))$ parametrizes hyperplanes in $\Gamma(X, \mathcal{O}_X(D_iH_i))$.

As usual, we denote $|D_iH_i|$ the projective space dual to $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_iH_i))$. We also consider the incidence correspondences

$$\begin{tikzcd} & I_i \arrow[dl, "p_i"'] \arrow[dr, "q_i"] & \\ \mathbb{P}_k^{N_i} & & |D_iH_i| \end{tikzcd}$$

and their product

$$\begin{tikzcd} & \prod_{i=1}^e I_i \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P} & & \prod_{i=1}^e |D_iH_i|. \end{tikzcd} \tag{A.4}$$

By "restriction" to X of the correspondence (A.4) by means of the imbedding $f : X \hookrightarrow \mathbb{P}$, we get a correspondence:

$$\begin{tikzcd} & Z := X \times_{\mathbb{P}} \prod_{i=1}^e I_i \arrow[dl, "p_Z"'] \arrow[dr, "q_Z"] & \\ X & & \prod_{i=1}^e |D_iH_i|. \end{tikzcd}$$

The morphism p, and consequently p_Z, is a "Zariski locally trivial fibration", the fibers of which are isomorphic to $\prod_{i=1}^e \mathbb{P}_k^{N_i-1}$. Therefore Z is an integral projective scheme of dimension

$$\dim Z = \dim X + \sum_{i=1}^{e} N_i - e.$$

Moreover the morphism q_Z is surjective; indeed $\dim X \geq e$, and therefore any e-tuple of ample divisors on X has a non-empty intersection. Since

$$\dim \prod_{1 \leq i \leq e} |D_iH_i| = \sum_{i=1}^{e} N_i,$$

the generic fiber of q_Z has dimension $\dim X - e$.

Observe that, if ξ_i^0 denotes the point of $|D_iH_i|(k)$ defined by the Cartier divisor D_iH_i, the intersection $D_1H_1 \cap \ldots \cap D_eH_e$ coincides, as a scheme, with the fiber $q_Z^{-1}(\xi_1^0, \ldots, \xi_e^0)$. The connectedness of $H_1 \cap \ldots \cap H_e$ will therefore follow from the connectedness of the fibers of q_Z, that we shall establish by means of the following classical result, basically due to Zariski (see for instance [Jou83], Part I, Section 4 and proof of Théorème 7.1; recall that, in this Appendix, the base field k is algebraically closed):

Lemma 4. *Let $\phi : X_1 \longrightarrow X_2$ be a dominant morphism of integral k-schemes.*

a) The following conditions are equivalent:

(1) *the generic fiber of ϕ is geometrically irreducible;*

(2) *there exists a dense open subscheme U of X_2 such that, for any P in $U(k)$, the fiber $\phi^{-1}(P)$ is irreducible;*
(3) *there exists a Zariski dense subset $\mathcal{D}$ of $X_2(k)$ such that, for any P in $\mathcal{D}$, the fiber $\phi^{-1}(P)$ is irreducible.*

b) Assume moreover that ϕ is proper and surjective and that X_2 is normal. Then, when the conditions (1)–(3) above are satisfied, the fiber $\phi^{-1}(P)$ is connected for every P in $X_2(k)$.

According to Lemma 4, to complete the proof of the connectedness of the fibers of q_Z, it is enough to show the existence of a Zariski dense set of points $(\xi_1, \dots, \xi_e)$ in $\prod_{1\leq i\leq e} |D_i H_i|(k)$ such that the fibers $q_Z^{-1}(\xi_1, \dots, \xi_e)$ are irreducible, or equivalently, such that the schemes

$$X \cap f_1^{-1}(\Xi_1) \cap \ldots \cap f_e^{-1}(\Xi_e)$$

— where Ξ_i denotes the hyperplane in $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_i H_i))$ parametrized by ξ_i — are irreducible.

This follows by applying e-times the usual Theorem of Bertini (see for instance [Jou83], Part I, Théorème 6.3, 4), Théorème 6.10, 3) and Corollaire 6.11, 3)) to construct successively $\xi_1, \dots, \xi_e$ such that, for every $i \in \{1, \dots, e\}$, the intersection scheme

$$X_i := X \cap f_1^{-1}(\Xi_1) \cap \ldots \cap f_i^{-1}(\Xi_i)$$

is geometrically irreducible of dimension $\dim X - i$. (Observe that, if $i \leq e - 1$, then $\dim X - i \geq 2$. This allows us to apply the usual Theorem of Bertini to X_i projectively imbedded by f_{i+1}, and, the points $\xi_1, \dots, \xi_i$ being already constructed, to find a Zariski dense set of points ξ_{i+1} in $|D_{i+1} H_{i+1}|(k)$ such that $X_i \cap f_{i+1}^{-1}(\Xi_{i+1})$ is irreducible.)

□

A.2.2. *Application to étale neighborhoods of intersections of ample hypersurfaces.* Let Y be closed subscheme of some k-scheme X, and denote $i : Y \hookrightarrow X$ the inclusion morphism. By definition, an *étale neighborhood* of Y in X is a commutative diagram of k-schemes:

(A.5)
$$\begin{tikzcd} & \tilde{X} \arrow[d, "\nu"] \\ Y \arrow[ur, "j"] \arrow[r, "i"'] & X, \end{tikzcd}$$

with ν étale at every point of $j(Y)$.

Observe that the commutativity of the diagram (A.5) implies that j is a closed immersion. Moreover ν is étale at every point of $j(Y)$ if and only if the morphism of formal completions induced by ν,

(A.6)
$$\hat{\nu} : \widehat{\tilde{X}}_{j(Y)} \longrightarrow \widehat{X}_Y,$$

is an isomorphism of formal schemes. (See for instance [Gie77], Section 4.)

Besides, after replacing $\tilde{X}$ by some open Zariski neighborhood of $j(Y)$, we may assume that ν is an étale morphism. If moreover X and $\tilde{X}$ are integral, furthermore X and $\tilde{X}$ are connected (hence integral), then then ν is birational iff it is an open immersion (*cf.* [Gro67], §18, Lemme 18.10.18, p. 166).

Observe also that étale neighborhoods in X of Y and of the underlying reduced scheme $|Y|$ may be identified.

Proposition 5. *Let X be an integral proper k-scheme of dimension d, and let $H_1, \dots,$ H_e, $1 \leq e \leq d-1$, be ample effective Cartier divisors in X. Let us assume that X is normal at every point of $H_1 \cap \dots \cap H_e$.*

Then, for any étale neighborhood of $H_1 \cap \dots \cap H_e$ in X

(A.7)
$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{j}{\nearrow} & \downarrow{\scriptstyle \nu} \\ H_1 \cap \dots \cap H_e & \xrightarrow{i} & X, \end{array}$$

there exists a Zariski open neighborhoods U of $H_1 \cap \dots \cap H_e$ in X (resp. $\tilde{U}$ of of $j(H_1 \cap \dots \cap H_e)$ in $\tilde{X}$) between which ν establishes an isomorphism:

$$\nu_{\tilde{U}} : \tilde{U} \xrightarrow{\sim} U.$$

In brief, Proposition 5 asserts that an étale neighborhood of the intersection of at most $d-1$ ample hypersurfaces in a normal projective variety of dimension d "is" actually a Zariski neighborhood.

When $d = 2$ and $e = 1$, Proposition 5 is essentially Proposition 2.2 in [BCL09]. In that case, the connectedness result on which its proof relies (namely, the special case of Theorem 3 with f a dominant morphism between two projective surfaces X' and X and $e = 1$) directly follows from Hodge Index Theorem, by Ramanujam's argument which shows that an effective, nef and big divisor on a surface is numerically connected ([Ram72]).

Proof. The normality assumption on X and the existence of the isomorphism (A.6) imply that $\tilde{X}$ is normal on some Zariski neighborhood of $j(Y)$. Therefore, after possibly shrinking $\tilde{X}$, we may assume that $\tilde{X}$ is a normal scheme. According to Theorem 3, the scheme

$$Y := H_1 \cap \dots \cap H_e$$

is connected. Consequently, its image $j(Y)$ lies in a unique component of the normal scheme $\tilde{X}$. We may therefore assume that $\tilde{X}$ is integral.

By Nagata's compactification theorem, we may assume that $\tilde{X}$ is an open subscheme of some integral proper k-scheme $\overline{\tilde{X}}$. After replacing $\overline{\tilde{X}}$ by the closure in $\overline{\tilde{X}} \times X$ of the graph of ν, we may also assume that ν extends to a morphism $\overline{\nu} : \overline{\tilde{X}} \to X$.

So we may — and will — assume that, in (A.5), the scheme $\tilde{X}$ is *integral and proper* over k.

The intersection $H_1 \cap \cdots \cap H_e$ is not empty, and ν is therefore étale at some point of $\tilde{X}$. The morphism ν is therefore dominant and its image has dimension $d > e$. Therefore, according to Theorem 3, the closed subscheme $\nu^*(Y) = \nu^*(H_1 \cap \cdots \cap H_e)$ of $\tilde{X}$ is connected.

The commutativity of the diagram (A.7) shows that the subscheme $\nu^*(Y)$ of $\tilde{X}$ contains $j(Y)$. Moreover, since ν defines an isomorphism

$$\hat{\nu} : \widehat{\tilde{X}}_{j(Y)} \xrightarrow{\sim} \widehat{X}_Y,$$

the formal subschemes of $\widehat{\tilde{X}}_{j(Y)}$ defined by completing $\nu^*(Y)$ along $j(Y)$ is nothing but $j(Y)$.

This implies that the trace of $\nu^*(Y)$ on some Zariski open neighborhood of $j(Y)$ in $\tilde{X}$ coincides with $j(Y)$. (This follows form the basic faithfulness properties of the functor of completion along a closed subscheme in a some noetherian scheme; see for instance [GD71], Propositions 10.8.8 and 10.8.11 and their corollaries.) In other words, $\nu^*(Y)$ may be written as a disjoint union

$$\nu^*(Y) = j(Y) \amalg R,$$

for some closed subscheme R of $\tilde{X}$.

Together with the connectedness of $\nu^*(Y)$, this shows that

$$\nu^*(Y) = j(Y).$$

Consider the set F of points of $\tilde{X}$ where ν is not étale. It is closed in $\tilde{X}$, and disjoint of

$$\nu^{-1}(Y) = |j(Y)|.$$

Therefore

$$U := X \setminus \nu(F)$$

is an open Zariski neighborhood of Y in X, and the restriction morphism

$$\nu_{|\nu^{-1}(U)} : \nu^{-1}(U) \longrightarrow U$$

is proper, étale, and an isomorphism over the non-empty subscheme Y, hence an isomorphism. □

A.3. **Algebraization of formal morphisms.** The following theorem provides criteria for a *formal* morphism, defined on the completion $\widehat{X}_Y$ along a closed subscheme Y of a proper k-scheme X with range some k-scheme, to be defined by some morphism of k-schemes defined on some étale or Zariski neighborhood of Y in X.

Theorem 6. *Let X be an integral projective k-scheme of dimension d, Y a closed subscheme of X, T a k-scheme, and $\psi : \widehat{X}_Y \to T$ a morphism of k-formal schemes*[5].

[5]Recall that k-schemes may be identified to k-formal schemes, which actually are noetherian, and separated over k. In particular, we may consider a morphism from a k-formal scheme, defined for instance as a completion, to some k-scheme: it is simply a morphism in the category of k-formal schemes. Such a morphism is nothing but a morphism in the category of k-locally ringed spaces (the continuity conditions are automaticaly satisfied, since the topological sheaf of ring of a scheme

Let us assume that Y is a local complete intersection contained in the regular locus X_{reg} of X.

1) If the normal bundle $N_Y X$ of Y in X is ample on Y and if every component of Y has dimension ≥ 1, then there exist an étale neighborhood of Y in X_{reg}

(A.8)
$$\begin{tikzcd} & \tilde{X} \arrow[d, "\nu"] \\ Y \arrow[ur, "j"] \arrow[r, "i"] & X_{\mathrm{reg}}, \end{tikzcd}$$

(where i denotes the inclusion morphism) and a morphism of k-schemes

$$\phi : \tilde{X} \longrightarrow T$$

such that the k-morphisms of formal schemes induced by ν and ϕ,

$$\hat{\nu} : \widehat{\tilde{X}}_{j(Y)} \xrightarrow{\sim} \widehat{X}_Y$$

and

$$\hat{\phi} : \widehat{\tilde{X}}_{j(Y)} \longrightarrow T,$$

make the following diagram commutative:

(A.9)
$$\begin{tikzcd} \widehat{\tilde{X}}_{j(Y)} \arrow[d, "\hat{\nu}", "\simeq"'] \arrow[dr, "\hat{\phi}"] & \\ \widehat{X}_Y \arrow[r, "\psi"] & T. \end{tikzcd}$$

2) If there exist some ample effective Cartier divisors $H_1, \ldots, H_e$, $1 \leq e \leq d-1$, in X such that Y has dimension $d-e$ and may be written as the complete intersection

$$Y = H_1 \cap \cdots \cap H_e,$$

then there exists an open Zariski neighborhood U of Y in X and a morphism of k-schemes

$$\phi : U \longrightarrow T$$

such that the morphism of k-formal schemes induced by ϕ,

$$\hat{\phi} : \widehat{X}_Y = \widehat{U}_Y \longrightarrow T,$$

coincides with ψ.

Proof. Let us assume that the hypotheses of 1) are satisfied. Then we may introduce the k-scheme

$$Z := X \times_k T,$$

interpreted as a formal scheme is discrete). In turn, a k-morphism from the completion $\widehat{X}_Y$ to the k-scheme T may be identified with a compatible system of morphisms of k-schemes from the successive infinitesimal neighborhoods Y_i of Y in X to T.

and consider Y as a subscheme of Z, by means of the closed immersion

$$i' := (i, \psi_{|Y}) : Y \longrightarrow Z.$$

The graph of ψ, seen as a k-morphism of formal schemes from $\widehat{X}_Y$ to $\widehat{T}_{\psi(Y)}$, defines a closed formal subscheme $\widehat{V}$ of $\widehat{Z}_{i'(Y)}$ admitting $i'(Y)$ as scheme of definition. The projections

$$\mathrm{pr}_1 : Z \longrightarrow X \quad \text{and} \quad \mathrm{pr}_2 : Z \longrightarrow T$$

define, after completion, morphisms of formal schemes

$$\widehat{\mathrm{pr}}_1 : \widehat{Z}_{i'(Y)} \longrightarrow \widehat{X}_Y \quad \text{and} \quad \widehat{\mathrm{pr}}_2 : \widehat{Z}_{i'(Y)} \longrightarrow \widehat{T}_{\psi(Y)}.$$

By restriction to $\widehat{V}$, they define an isomorphism of formal schemes

$$\widehat{\mathrm{pr}}_{1|\widehat{V}} : \widehat{V} \xrightarrow{\sim} \widehat{X}_Y$$

and a morphism

$$\widehat{\mathrm{pr}}_{2|\widehat{V}} : \widehat{V} \longrightarrow \widehat{T}_{\psi(Y)}$$

such that

$$\widehat{\mathrm{pr}}_{2|\widehat{V}} = \psi \circ \widehat{\mathrm{pr}}_{1|\widehat{V}}.$$

Clearly the restriction of $\widehat{\mathrm{pr}}_{1|\widehat{V}}$ to the scheme of definition $i'(Y)$ of $\widehat{V}$ is the first projection

$$\mathrm{pr}_{1|i'(Y)} : i'(Y) \xrightarrow{\sim} Y. \tag{A.10}$$

Consequently $\widehat{V}$ is smooth and $i'(Y)$ is a l.c.i. in $\widehat{V}$, and through the isomorphism (A.10), the normal bundle $N_{i'(Y)}\widehat{V}$ gets identified to $N_Y\widehat{X}_Y \simeq N_Y X$, hence is ample.

According to Theorem 2, the formal subscheme $\widehat{V}$ is algebraic. By Proposition 1, there exists a commutative diagram of k-schemes

$$\begin{tikzcd} & M \arrow[d, "\mu"] \\ Y \arrow[ur, "j"] \arrow[r, "(i,\psi_{|Y})"'] & X \times_k T \end{tikzcd}$$

with M smooth over k, such that, after completing along $j(Y)$, μ becomes an isomorphism:

$$\hat{\mu} : \widehat{M}_{j(Y)} \xrightarrow{\sim} \widehat{V} \,(\hookrightarrow \widehat{Z}_{i'(Y)}).$$

Let us consider the open subscheme of M:

$$\tilde{X} := (\mathrm{pr}_1 \circ \mu)^{-1}(X_{\mathrm{reg}}).$$

It is straightforward that $j(Y)$ lies in $\tilde{X}$ and that the diagram

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\nu:=\mathrm{pr}_1\circ\mu"] \\ Y \arrow[ur, "j"] \arrow[r, "i"] & X_{\mathrm{reg}} \end{tikzcd}$$

is an étale neighborhood of Y in X_{reg} which satisfies the conclusion of 1).

Let us now assume that the hypotheses of 2) are satisfied. Then the normal bundle of X in Y is isomorphic to $\bigoplus_{j=1}^{e} \mathcal{O}_X(H_j)_{|Y}$, hence is ample. According to Part 1) of the Theorem, there exists an étale neighborhood (A.8) of Y in X and a morphism of k-schemes $\phi : \tilde{X} \to Z$ such that the diagram (A.9) is commutative. Proposition 5 shows that, after possibly shrinking $\tilde{X}$, we may assume that ν is an open immersion. It may be identified with the inclusion morphism in X_{reg} of some open Zariski neighborhood U of Y, and the commutativity of (A.9) precisely asserts that $\hat{\phi} = \psi$.

□

A.4. **Algebraization of formal bundles and subbundles.** In this Section, we place ourselves under the assumptions of Theorem 6, 2). Namely, we denote by X a projective k-scheme of dimension d, by e a positive integer $\leq d-1$, and by $H_1, \ldots, H_e$ ample effective Cartier divisors in X such that

$$Y := H_1 \cap \cdots \cap H_e$$

has (necessarily pure) dimension $d - e$ and is contained in the regular locus X_{reg} of X.

Besides, we choose an ample line bundle $\mathcal{O}_X(1)$ over X.

We shall say that a vector bundle $\widehat{E}$ over $\widehat{X}_Y$ is *algebraizable* if there exists a coherent sheaf of $\mathcal{O}_X$-modules E over X (necessarily locally free on some open Zariski neighborhood of Y) such that $\widehat{E} \simeq E_{|\widehat{X}_Y}$, or equivalently, if there exists a vector bundle on some open Zariski neighborhood of Y in X such that $\widehat{E} \simeq E_{|\widehat{X}_Y}$.

The category of algebraizable vector bundles over $\widehat{X}_Y$ is clearly stable under elementary tensor operations, like directs sums, tensor products, and dual.

Theorem 7. *1) For any formal vector bundle $\widehat{E}$ over $\widehat{X}_Y$, the space of global sections $\Gamma(\widehat{X}_Y, \widehat{E})$ is a finite dimensional k-vector space.*

2) For any open Zariski neighborhood V of Y in X included in the open subscheme X_{nor} of normal points of X and any vector bundle E over V, the restriction map

$$\Gamma(V, E) \longrightarrow \Gamma(\widehat{X}_Y, E_{|\widehat{X}_Y}) \tag{A.11}$$

is an isomorphism.

Part 1) of Theorem 7 is a special case of [Har68], Theorem 6.2. Actually, we will not really use this result in this Appendix.

In the terminology of [Gro68], X.2, and [Har70], IV.1, Part 2) asserts that the pair (X_{nor}, Y) satisfies the *Lefschetz property* $\mathbf{Lef}(X_{\mathrm{nor}}, Y)$. When $e = 1$, it is a special case of [Gro68], XII.2, Corollaire[6] 2.4. When X is smooth over k, it is proved in [Har70], X.1, Corollary 1.2.

Proof. We are left to prove 2) in our setting. It will follow from the the algebraization criterion for formal morphisms established in the previous section.

Let us consider V and E as in 2), and let us introduce the "total space"

$$\mathbb{V}(E^\vee) := \operatorname{Spec}_V \operatorname{Sym} E^\vee$$

of the vector bundle E, and its structural morphism

$$p : \mathbb{V}(E^\vee) \longrightarrow V.$$

For any open subscheme U of V, the elements of $\Gamma(U, E)$ may be identified with the morphisms of k-schemes

$$s : U \longrightarrow \mathbb{V}(E^\vee)$$

which are sections of p over U (that is, which satisfy $p \circ s = Id_U$). Similarly, the elements of $\Gamma(\widehat{X}_Y, E_{|\widehat{X}_Y})$ may be identified with the k-morphisms of formal schemes

$$t : \widehat{X}_Y \longrightarrow \mathbb{V}(E^\vee)$$

which are sections of p over $\widehat{X}_Y$.

Observe also that, for any morphism of k-schemes

$$\phi : U \longrightarrow \widehat{V}(E^\vee)$$

defined on some open Zariski neighborhood of U of Y in X, if the induced morphism of formal schemes

$$\hat{\phi} : \widehat{X}_Y = \widehat{V}_Y \longrightarrow \mathbb{V}(E^\vee)$$

is a section of p over $\widehat{X}_Y$, then ϕ is a section of p over U (indeed Y is non-empty, and X — hence U — is an integral scheme).

The injectivity of the restriction morphism (A.11) is straightforward. Let us show that it is surjective.

Together with the remarks above, Theorem 6, 2) applied with $T := \mathbb{V}(E^\vee)$ establishes that, for any formal section t in $\Gamma(\widehat{X}_Y, E_{|\widehat{X}_Y})$, there exists an open Zariski neighborhood U of Y in V and a section s in $\Gamma(U, E)$ such that

$$s_{|\widehat{X}_Y} = t.$$

Observe that the dimension $\dim I$ of any closed integral subscheme I of X disjoint of Y satisfies

$$\dim I < e.$$

[6]Actually, the statement of Corollaire 2.4 in *loc. cit.* does not exactly cover the situation we deal with here, but require slightly stronger hypotheses, satisfied for instance when X itself is normal. By working on the normalization of X, one may actually assume that these hypotheses are satisfied.

(Otherwise the intersection number

$$\mathcal{O}_X(1)^{\dim I - e}.H_1.\dots.H_e.I$$

would vanish, in contradiction to the ampleness of $\mathcal{O}_X(1)$ and of $H_1, \dots, H_e$.) Consequently the codimension in X of any component of $X \setminus U$ is at least 2. This implies that the depth of $\mathcal{O}_X$ at every point of $X_{\mathrm{nor}} \setminus U$ is at least 2 and that the restriction morphism

$$\Gamma(V, E) \longrightarrow \Gamma(U, E)$$

is an isomorphism.

This shows that the section s of E over U uniquely extends to a section over V, and completes the proof.

□

The following two theorems are closely related — each of them may easily be deduced from the other one — and will be established together.

Theorem 8. *For any vector bundle $\widehat{E}$ over $\widehat{X}_Y$, the following conditions are equivalent:*

(1) *The vector bundle $\widehat{E}$ is algebraizable.*
(2) *For any large enough positive integer D, the vector bundle over $\widehat{X}_Y$*

$$\widehat{E}(D) := \widehat{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

is generated by its global sections over $\widehat{X}_Y$.
(3) *For some integer D, the vector bundle $\widehat{E}(D)$ is generated by its global sections over $\widehat{X}_Y$.*

With the notation of Theorem 8, observe that a morphism of coherent $\mathcal{O}_{\widehat{X}_Y}$-modules $\hat{\varphi} : \hat{\mathcal{F}} \longrightarrow \hat{\mathcal{G}}$ is onto iff its restriction $\hat{\varphi}_{|Y} : \hat{\mathcal{F}}_{|Y} \longrightarrow \hat{\mathcal{G}}_{|Y}$ to Y is onto (this directly follows from Nakayama's Lemma). Consequently, the vector bundle $\widehat{E}(D)$ is generated by its global sections over $\widehat{X}_Y$ iff the vector bundle over Y

$$\widehat{E}(D)_{|Y} \simeq \widehat{E}_{|Y} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

is generated by its global sections in the image of the restriction morphism

$$\Gamma(\widehat{X}_Y, \widehat{E}(D)) \longrightarrow \Gamma(Y, \widehat{E}(D)_{|Y}). \tag{A.12}$$

Theorem 9. *If a vector bundle $\widehat{E}$ over $\widehat{X}_Y$ is algebraizable, then any quotient vector bundle and any sub-vector bundle*[7] *of $\widehat{E}$ is algebraizable.*

Proof. We begin by the proof of Theorem 9.

Let E be a vector bundle over some open Zariski neighborhood V of Y in X. We are going to show that any quotient bundle of $E_{\widehat{X}_Y}$ over $\widehat{X}_Y$ is algebraizable. By a

[7]Namely, any subsheaf of $\mathcal{O}_{\widehat{X}_Y}$-modules which is locally a direct summand.

straightforward duality argument, this will also prove that any sub-vector bundle of an algebraizable vector bundle over $\widehat{X}_Y$ is algebraizable.

Let

$$p : \mathbf{Gr} := \coprod_{0 \leq n \leq \mathrm{rk}\, E} \mathbf{Grass}_n(E) \longrightarrow V$$

be the Grasmannian scheme of the vector bundle E over V, which classifies locally free quotients of E over varying V-schemes; *cf.* [GD71], Section 9.7. (It is a "Zariski locally trivial bundle over V", with a fiber isomorphic to the disjoint union of the classical Grassmann varieties, classifying quotients of rank n of $k^{\mathrm{rk}\, E}$, for $0 \leq n \leq \mathrm{rk}\, V$.)

The vector bundle p^*E over $\mathbf{Gr}$ is equipped with a canonical quotient bundle

$$\mathbf{q} : p^*E \longrightarrow \mathbf{Q},$$

and, by the very construction of $\mathbf{Gr}$, for any subscheme S of X, the map which sends a section σ of

$$p_S : p^{-1}(S) \longrightarrow S$$

to the quotient vector bundle

$$\sigma^*\mathbf{q} : E_{|S} \simeq \sigma^* p^* E \longrightarrow \sigma^*\mathbf{Q}$$

establishes a bijection between the set of sections of p_S and the set of (isomorphisms classes) of quotient vector bundles of $E_{|S}$ over S.

By taking for S the successive thickenings Y_i of Y in X, this bijective correspondence extends between (formal) sections of p over $\widehat{X}_Y$ and quotient vector bundles of $\widehat{E} := E_{|\widehat{X}_Y}$. In other words, for any quotient vector bundle

$$\hat{q} : \widehat{E} \longrightarrow \widehat{Q}, \tag{A.13}$$

we may consider its "classifying map"

$$\psi : \widehat{X}_Y \longrightarrow \mathbf{Gr},$$

which is a section of p over $\widehat{X}_Y$: the quotient vector bundle

$$\psi^*\mathbf{q} : E_{|\widehat{X}_Y} \longrightarrow \psi^*\mathbf{Q}$$

is isomorphic to (A.13).

According to Theorem 6 applied to $T = \mathbf{Gr}$, the formal morphism ψ is induced by some morphism of k-schemes

$$\phi : U \longrightarrow \mathbf{Gr}$$

defined on some open Zariski neighborhood U of Y in X, which we may assume to lie in V. It is straightforward that ϕ, like ψ, is a section of p. Moreover the corresponding quotient bundle

$$\phi^*\mathbf{q} : E_{|U} \longrightarrow \phi^*\mathbf{Q}$$

becomes isomorphic to (A.13) after restriction to $\widehat{X}_Y$ (that is, after completing along Y).

In particular, $\widehat{Q}$ is isomorphic to the restriction to $\widehat{X}_Y$ of the vector bundle $\phi^*\mathbf{Q}$ on V, and is therefore algebraizable.

We now turn to the proof of Theorem 8.

To prove the implication (1) $\Rightarrow$ (2), observe that, for any coherent sheaf of $\mathcal{O}_X$-modules E over X such that $\widehat{E} \simeq E_{|\widehat{X}_Y}$, the image of the restriction morphism

$$\Gamma(X, E(D)) \longrightarrow \Gamma(Y, E(D)_{|Y}) \simeq \Gamma(Y, \widehat{E}(D)_{|Y}) \tag{A.14}$$

is contained in the image of (A.12), and that, for any large enough positive integer D, the morphism (A.14) is surjective and $E(D)_{|Y}$ is generated by its global sections over Y.

The implication (2) $\Rightarrow$ (3) is trivial.

Finally, assume that Condition (3) is satisfied, and consider the "tautological" morphism of vector bundles over $\widehat{X}_Y$:

$$p : \Gamma(\widehat{X}_Y, \widehat{E}(D)) \otimes_k \mathcal{O}_{\widehat{X}_Y} \longrightarrow \widehat{E}(D). \tag{A.15}$$

(Recall that, according to Theorem 7, 1), the k-vector space $\Gamma(\widehat{X}_Y, \widehat{E}(D))$ is finite dimensional. We could easily avoid to rely on this result by replacing $\Gamma(\widehat{X}_Y, \widehat{E}(D))$ by a "sufficiently large" finite dimensional sub-vector space.) By hypothesis, it is surjective, and $\widehat{E}(D)$ is therefore identified with a quotient of the "trivial" vector bundle $\Gamma(\widehat{X}_Y, \widehat{E}(D)) \otimes_k \mathcal{O}_{\widehat{X}_Y}$ over $\widehat{X}_Y$, which is clearly algebraizable. According to Theorem 9, $\widehat{E}(D)$ is therefore algebraizable. Finally, the vector bundle

$$\widehat{E} \simeq \widehat{E}(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$$

is algebraizable. This completes the proof of (3) $\Rightarrow$ (1).

$\square$

A.5. **Comments and examples.**

A.5.1. Under the assumptions of Theorem 6, Part 1), the conclusion of Part 2) does *not* hold in general. In other words, under the mere assumption of ampleness of the normal bundle $N_Y X$, the étale neighborhood of Y in X onto which ψ extends cannot be chosen to be a Zariski neighborhood.

This is demonstrated by a classical example of Hironaka (see [Hir68], p. 588, and [Har70], Chapter V), which in its simplest form may be presented as follows.

Consider a smooth, connected, projective threefold over k whose algebraic fundamental group is not trivial, and choose a finite étale covering

$$\nu : \tilde{X} \longrightarrow X,$$

with $\tilde{X}$ connected and (necessarily) projective. Define $\tilde{Y}$ as a "general" intersection of two effective divisors in the linear system defined by some large multiple of an ample divisor on $\tilde{X}$. Then $\tilde{Y}$ is smooth, connected, its normal bundle $N_{\tilde{Y}}\tilde{X}$ is ample, and the map

$$\nu_{|\tilde{Y}}\tilde{Y} \longrightarrow Y := \nu(\tilde{Y})$$

is an isomorphism.

Since ν is étale, it induces an isomorphism between completions

$$\hat{\nu} : \widehat{\tilde{X}}_{\tilde{Y}} \xrightarrow{\sim} \widehat{X}_Y$$

and an isomorphism of normal vector bundles:

$$N_{\tilde{Y}}\tilde{X} \xrightarrow{\sim} \nu^*_{|\tilde{Y}} N_Y X.$$

In particular, $N_Y X$ also is ample.

If we let

$$T := \tilde{X} \quad \text{and} \quad \psi := \hat{\nu}^{-1} : \widehat{X}_Y \longrightarrow \widehat{\tilde{X}}_{\tilde{Y}} \hookrightarrow \tilde{X},$$

then it is straightforward that ψ cannot be realized as the restriction to $\widehat{X}_Y$ of some morphism of k-schemes from some open Zariski neighborhood of Y in X to $\tilde{X}$.

A.5.2. The results of Section A.4 may be rephrased in terms of functors between categories of vector bundles.

Let indeed V be an open Zariski neigborhood of Y in X_{nor}. We may introduce the k-linear categories $\mathbf{Bun}(V)$ of vector bundles on V, $\mathbf{Bun}(X_Y)$ of germs of vector bundles on X along Y, and $\mathbf{Bun}(\widehat{X}_Y)$ of vector bundles on $\widehat{X}_Y$, and the obvious restriction (or completion) functors

$$\mathbf{Bun}(V) \xrightarrow{F^V_{X_Y}} \mathbf{Bun}(X_Y) \xrightarrow{F^{X_Y}_{\widehat{X}_Y}} \mathbf{Bun}(\widehat{X}_Y).$$

Theorem 7, 2), and its proof show that the functors $F^V_{X_Y}$ and $F^{X_Y}_{\widehat{X}_Y}$ are fully faithfull.

When V is smooth, the functor $F^V_{X_Y}$ is easily seen to be essentially surjective iff $d = 2$ (and consequently $e = 1$).

The functor $F^{X_Y}_{\widehat{X}_Y}$ is *not* essentially surjective when

$$\dim Y (= d - e) = 1.$$

Indeed, the following Proposition — which is a straightforward consequence of Theorem 7, 2) — allows one to construct non-algebraizable formal vector bundles on $\widehat{X}_Y$ when $\dim Y = 1$.

Proposition 10. *Let us keep the notation of Section A.4, and assume that* $\dim Y = 1$.

Let p be a point of $Y(k)$ and let f be an element of the local ring $\mathcal{O}_{\widehat{X}_Y,P}$ such that $f(P) = 0$ and $f_{|Y} \in \mathcal{O}_{Y,P}$ is invertible on $\operatorname{Spec} \mathcal{O}_{Y,P} \setminus \{P\}$. *Then the following two conditions are equivalent:*

1. *The line bundle on $\widehat{X}_Y$ defined by the divisor* $\operatorname{div} f$ *is algebraizable.*
2. *There exists an effective (Cartier) divisor on X_{reg} whose completion along Y coincides with* $\operatorname{div} f$.

The essential surjectivity of the functor $F^{X_Y}_{\widehat{X}_Y}$ is precisely the *effective Lefschetz property* $\mathbf{Leff}(X, Y)$ considered in [Gro68], X.2, and [Har70], IV.1. According to [Gro68], XII, Corollaire 3.4 and to [Har70], IV, Theorem 1.5, it holds under the assumptions of Section A.4, when moreover X is smooth, some positive multiples of the H_j's lie in the same linear system, and $\dim Y \geq 2$.

It appears very likely that, under the assumptions of Section A.4, this last condition $\dim Y \geq 2$ would be enough to ensure the validity of $\mathbf{Leff}(X, Y)$.

This discussion shows that Theorem 9, which asserts the stability under quotients of the essential image of $F^{X_Y}_{\widehat{X}_Y}$, is significant mostly when $\dim Y = 1$.

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