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Logarithmic De Rham complexes and vanishing theorems

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Classically the vanishing of cohomology groups of a compact complex Kähler manifold X with values in certain locally free sheaves $\mathcal{M}$ is proved by studying positivity properties of the curvature form of a differentiable connection on $\mathcal{M}$ compatible with the complex structure of X (e.g. [7]). If the Chern classes of $\mathcal{M}$ are non trivial, the connection is neither holomorphic nor integrable. Therefore, trying to replace the differentiable connection with non trivial curvature by an integrable holomorphic connection ∇, one has at least to allow ∇ to have poles along a "boundary divisor" D. We even will assume D to be a normal crossing divisor and ∇ to have at most logarithmic poles along D. Since ∇ is non singular and integrable on $U = X - D$, the positivity properties have to be replaced by topological properties of U together with conditions on the boundary behaviour of $(\mathcal{M}, \nabla)$.

Because of the restriction made on the poles of ∇ along D one has at disposal the theory of P. Deligne on differential equations with regular singular points [3] and in fact his Lecture Notes was the main source of inspiration of our work:

Let V be the local constant system on U defined by sections of $\mathcal{M}|_U$, flat with respect to ∇ and $j: U \hookrightarrow X$ be the inclusion. $(\mathcal{M}, \nabla)$ is equipped with its logarithmic De Rham complex $DR_D\mathcal{M} = \Omega^{\bullet}\langle D\rangle \otimes_{\mathcal{O}_X} \mathcal{M}$, which is over U quasi-isomorphic to V. If the monodromies of V around the components of D do not have 1 as eigenvalue the complexes $j_! V$, $Rj_* V$ and $\mathrm{DR}_D\mathcal{M}$ are all quasi-isomorphic and the hypercohomology of $DR_D\mathcal{M}$ is the same as the cohomology or as the cohomology with compact support of U with values in V. The spectral sequence $E_1(\mathcal{M})$ associated to the "filtration bête" of the logarithmic De Rham complex describes the hypercohomology in terms of the cohomology of the coherent $\mathcal{O}_X$-modules $\Omega^p\langle D\rangle \otimes \mathcal{M}$. If in addition the spectral sequence degenerates in E_1, topological vanishing theorems on U imply global coherent vanishing theorems on X.

In general it is quite difficult to decide when the spectral sequence degenerates (see (2.6)). Using Deligne's theory of mixed Hodge structures [4], this

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is true however for sheaves arising from finite covers of X, branched along D. The main examples of sheaves arising by this construction are invertible sheaves $\mathscr{L}^{-1}$, where $\mathscr{L}$ is ample or more generally related to integral parts of effective $\mathbb{Q}$-divisors with support in D. If $U=X-D$ is affine, the degeneration of $E_1(\mathscr{L}^{-1})$ implies immediately a general global vanishing theorem containing as special cases:

- Kodaira-Nakano's vanishing theorem (2.10)
- Bogomolov-Sommese's vanishing theorem (2.11) and
- Grauert-Riemenschneider's vanishing theorem (see (2.14, a)) as well as its generalization due to Y. Kawamata and the second author ((2.12) and (2.13)).

If one drops the assumption on U, the degeneration of $E_1(\mathscr{M})$ implies the vanishing of certain natural restriction maps in cohomology. Applied to the sheaves $\mathscr{L}^{-1}$ considered above one obtains the vanishing of the restriction maps of twisted differential forms in the cohomology ((3.2) and (3.3)). Especially one gets an improvement of the Kollár-Tankeev vanishing theorems (3.5) as a direct interpretation of the degeneration of the spectral sequence.

In § 1 we recall properties of sheaves with logarithmic connections and their De Rham complexes. The condition that the monodromies of V do not have 1 as eigenvalue implies that the minimal and the maximal extensions of V coincide, as we prove in (1.6).

In § 2 we give the cohomological interpretation of (1.6), provided the spectral sequence $E_1(\mathscr{M})$ degenerates and $U=X-D$ is affine. We discuss examples where all three assumptions hold and state and prove the vanishing theorems mentioned.

§ 3 contains the applications to the cohomology of restriction maps, useful if U is not affine. The main observation is that the conditions posed on the monodromy of V imply that the residue maps obtained from ∇ are surjective on each component of D and can be identified with the natural restriction map.

In order to recover the positivity properties of differentiable connections in terms of the logarithmic connection ∇, one should at least be able to define the Chern classes in the De Rham cohomology. That is done in Appendix B by describing the Atiyah class as the image of the residue of ∇, a construction due to J.L. Verdier and independently to the first author (and probably to others, too). We believe that the language of $\mathcal{O}_X$-coherent logarithmic $\mathscr{D}$-modules is a quite adequate tool in algebraic geometry to describe properties of certain sheaves and we hope that the methods used in this article are useful for different applications as well.

Appendix A contains an algebraic proof that the Verdier dual of the complex $DR_D\mathscr{M}$ is quasi-isomorphic to $DR_D(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X)\otimes\mathcal{O}_X(-D_{\text{red}}))$. The statement (1.6) on the extensions of V is an immediate consequence. The corresponding duality for $\mathscr{D}$-modules is one of the key points in the proof of the Riemann-Hilbert correspondence for $\mathscr{D}$-modules (M. Kashiwara, Z. Mebkhout, see for example [2]).

For Appendix C we just worked out notes given to us by P. Deligne on the classification of free logarithmic connections on a polycylinder in terms of "splittable" filtrations of local constant systems.

The vanishing theorems of §2 and §3 are based on the degeneration of the E_1 spectral sequence and the conditions on the monodromy. For the applications given it is enough to verify those conditions for connections coming from finite representations of the fundamental group of U. In Appendix D K. Timmerscheidt proves that for a special extension $\mathscr{M}$ of a local constant system, coming from a unitary representation of $\pi_1(U)$, the monodromy condition implies the degeneration of the E_1 spectral sequence. In fact, S. Zucker proved in [12] similarly results for arbitrary variations of polarised Hodge-structures over curves, and the methods used by K. Timmerscheidt are based on [12].

After finishing a first version of this paper we learned that our approach towards global vanishing theorems is close in spirit to methods used by J. Kollár [10] and related to results by M. Saito (see also (2.6)).

Acknowledgements. The theory of $\mathscr{D}$-modules and perverse sheaves, even if they do not appear explicitly, strongly inspired this article. We had a small seminar on those subjects at the MPI at Bonn and we thank our friends for their active participation which certainly helped us to understand the point of view explained here.

J.L. Verdier considered independently and for different purposes geometric applications of logarithmic $\mathscr{D}$-modules. It is a pleasure for both of us to thank him for telling us about his ideas. We are grateful to M. Saito for answering question on $\mathscr{D}$-modules and K. Timmerscheidt for answering those on the analytic part of classical Hodge theory and for contributing to this article in form of Appendix D.

We cordially thank P. Deligne. Not only the article is based on his theory of logarithmic connections, but also he helped us to understand it in the right way and corrected several ambiguities in the first version of this article.

§1. Logarithmic De Rham complexes and extensions of local constant systems

In this section we recall the basic properties of locally free sheaves with logarithmic connections, their monodromy and the extensions of the corresponding local constant systems, as developed by Deligne. Most statements given in this section are due to him, and all proofs omitted can be found in [3], especially the Riemann-Hilbert correspondence. The reader mainly interested in the proof of the global vanishing theorems is invited to read just up to (1.5) and then to read §2.

(1.1) Throughout this article X denotes either a connected algebraic manifold over $\mathbb{C}$ or a connected complex analytic manifold of dimension n. If not stated otherwise X is assumed to be compact. $\mathcal{O}_X$ denotes the sheaf of algebraic functions or the sheaf of analytic functions in the second case. In §1 – starting from (1.2) – and in Appendix A we have to restrict ourselves to the complex analytic case. Let $D=\sum_{i=1}^{s} v_i D_i$ be an effective normal crossing divisor on X, i.e. an effective divisor locally (in the analytic topology) with nonsingular components meeting transversally. We write $j\colon U=X-D_{\text{red}}\to X$ for the open embedding. We consider locally free sheaves $\mathscr{M}$ of $\mathcal{O}_X$-modules endowed with a holomorphic connection ∇ with logarithmic poles along D

$$\nabla: \mathscr{M} \to \Omega_X^1\langle D\rangle \otimes \mathscr{M}$$

as defined by Deligne ([3], II §3). Such a pair $(\mathscr{M}, \nabla)$ will be called a *logarithmic connection* along D. It induces

$$\nabla_p: \Omega_X^p\langle D\rangle \otimes \mathscr{M} \to \Omega_X^{p+1}\langle D\rangle \otimes \mathscr{M}$$

by the rule $\nabla_p(\omega \otimes m) = d\omega \otimes m + (-1)^p \cdot \omega \wedge \nabla m$. We assume ∇ to be integrable, i.e.: $\nabla_p \circ \nabla_{p+1} = 0$. The complex $\Omega_X^{\bullet}\langle D\rangle \otimes \mathscr{M}$ obtained is denoted by $DR_D\mathscr{M}$ and called the *logarithmic De Rham complex* of $(\mathscr{M}, \nabla)$.

In the literature, $DR_D\mathscr{M}$ is often denoted by $\Omega_X^{\bullet}(\log D)(\mathscr{M})$. Of course, one has $\Omega_X^p\langle D\rangle = \Omega_X^p\langle D_{\text{red}}\rangle$ and $DR_D\mathscr{M} = DR_{D_{\text{red}}}\mathscr{M}$. However in the covering constructions of §2, the divisor D is naturally equipped with multiplicities.

(1.2) From now on all the sheaves are considered in the classical topology and $\mathscr{O}_X$ denotes the sheaf of holomorphic functions. The following statements are local in X and whenever it is convenient we may assume that X is a polydisk Δ^n and $D = \bigcup_1^l pr_i^{-1}(0)$.

a) For a local constant system V on U we consider the direct image Rj_*V and the extension by zero $j_!V$. The functor $j_!$ is exact. Of course $\mathbb{H}^k(X, Rj_*V) = H^k(U, V)$ and (for X compact) $\mathbb{H}^k(X, j_!V) = H_c^k(U, V)$, the cohomology with compact supports. The fundamental group $\pi_1(U)$ operates on the fibres of the local system V. If T_i denotes a loop around D_i with base point x_0 then one obtains an automorphism γ_i of V_{x_0} called the monodromy of V around D_i. Since we are only interested in the eigenvalues of γ_i, we do not refer to the base point.

b) The complex $DR_D\mathscr{M}|_U$ is exact at $p > 0$ and the flat sections form a local constant system $V = \operatorname{Ker} \nabla_{|U}$. In other words, the inclusion $V \to DR_D\mathscr{M}_{|U}$ is a quasi-isomorphism of complexes. The operation of T_i around D_i can be extended to the boundary D by

c) **Proposition** (Deligne). *Let $X = \Delta^n$, $D = \bigcup_1^l pr_i^{-1}(0)$ and $\mathscr{M}$ be a free sheaf of rank r with a logarithmic connection along D. Then the operation of $\pi_1(X - D)$ extends to $\mathscr{M}$.*

Proof. Let $p: X \to S = \Delta^{n-1}$ be the projection where one forgets the first coordinate. The connection ∇ induces a relative one

$$\nabla_{\text{rel}}: \mathscr{M} \to \Omega_{X/S}^1\langle D_1\rangle \otimes \mathscr{M},$$

where

$$\Omega_{X/S}^1\langle D_1\rangle = \Omega_X^1\langle D\rangle / p^*\Omega_S^1\left\langle \bigcup_2^l pr_i^{-1}(0)\right\rangle = \Omega_X^1\langle D_1\rangle / p^*\Omega_S^1.$$

The kernel V_{rel} of $\nabla_{\text{rel}|X-D_1}$ is a relative constant system, isomorphic to $V \otimes_{\mathbb{C}} p^{-1}\mathscr{O}_S$ over U. Since V_{rel} is locally on $X - D_1$ the sheaf-theoretic inverse image of a free $\mathscr{O}_S$-module, the operation of T_1 extends to V_{rel}. If we write the first coordinate as $x_1 = r_1 \cdot \exp(\sqrt{-1} \cdot \theta_1)$, the local relatively flat sections s of $\dot{\mathcal{M}}$ satisfy a differential equation

$$\partial_{\theta_1} s = -(\sqrt{-1} \cdot \exp(-\sqrt{-1} \cdot \theta_1) \cdot \nabla_0 + \sqrt{-1} \cdot r_1 \cdot \nabla_1) \cdot s$$

for ∇_0 in $p^{-1}(\mathcal{O}_S)$ and ∇_1 in $\mathcal{O}_X$ and $\nabla_{\mathrm{rel}} = \left(\frac{\nabla_0}{x_1} + \nabla_1\right) \cdot dx_1$. The argument of [3], II; théorème 1.17, carries over to the case considered here.

d) By c), T_i defines an endomorphism $\bar{\gamma}_i$ of $\mathcal{M} \otimes \mathcal{O}_{\tilde{D}_i}$, where $\tilde{D}_i$ is the normalization of D_i. Since $\bar{\gamma}_i$ is a limit of conjugates of γ_i, its eigenvalues are constant. Define the residues $\Gamma_i = \mathrm{Res}_i(\nabla)$ of ∇ by $\Gamma_i = \mathrm{res}_i \circ \nabla$, where

$$\mathrm{res}_i\colon \Omega^1_X\langle D\rangle \otimes \mathcal{M} \to \mathcal{O}_{\tilde{D}_i} \otimes \mathcal{M}$$

is the Poincaré residue. Of course, using the notations from c) one has $\nabla_0|_{D_1} = \Gamma_1$. Therefore one has $\exp(-2 \cdot \pi \cdot \sqrt{-1} \cdot \Gamma_i) = \bar{\gamma}_i$, regarding Γ_i as an element of $\mathrm{End}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{D}_i})$. Especially, its eigenvalues are constant.

e) If $B = \sum_1^l b_j \cdot B_j$ is any divisor supported in D, the Leibnitz rule implies that ∇ induces a connection ∇^B with logarithmic poles along D_{red} on

$$\mathcal{M}(B) = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(B),$$

whose residues Γ_i^B are elements of

$$\mathrm{End}(\mathcal{M}(B) \otimes \mathcal{O}_{\tilde{D}_i}) = \mathrm{End}(\mathcal{M} \otimes \mathcal{O}_{\tilde{D}_i}).$$

Therefore one may compare Γ_i and Γ_i^B and one has

$$\Gamma_i^B = \Gamma_i - b_i \cdot \text{identity}.$$

Especially the eigenvalues differ by integers. If Γ_i does not have any positive integer as an eigenvalue, the same is true for Γ_i^B for any effective B.

(1.3) a) On the other hand, if V is a local constant system on U, denote $\mathcal{M}_U = \mathcal{O}_U \otimes_{\mathbb{C}} V$ and ∇_U the unique holomorphic connection whose flat sections are V. Then there exist a locally free sheaf $\mathcal{M}$ and a connection ∇ with logarithmic poles along D extending $(\mathcal{M}_U, \nabla_U)$. As explained in Appendix C, this extension is far from being unique. However, a choice of a logarithm function fixes a unique extension ([3], II; 5.3 and 5.4). For example, there is a unique extension (up to isomorphism) such that the real part of the eigenvalues of Γ_i lies in $[0, 1[$. This extension is called in the sequel the *canonical extension* of V and is denoted by V_{can}. If the monodromies are unipotent, then V_{can} is characterized by the fact that the eigenvalues of the residues are zero.

b) The canonical extension is compatible with subsystems and quotient systems:

For $W \subset V$, one has

$$W_{\mathrm{can}} \subset V_{\mathrm{can}} \quad \text{and} \quad (V/W)_{\mathrm{can}} = V_{\mathrm{can}}/W_{\mathrm{can}}.$$

Moreover it is compatible with covers in the following sense: For any cover $\mu: X' \to X$, ramified only along D, and with X' smooth, one has

$$\mu^*(V_{\mathrm{can}}) \subset (\mu^{-1} V)_{\mathrm{can}}.$$

Moreover, if $(\mathcal{M}, \nabla)$ is any other extension of V such that for some cover μ as before $\mu^* \mathcal{M}$ is contained in $(\mu^{-1} V)_{\mathrm{can}}$, then one has $\mathcal{M} \subset V_{\mathrm{can}}$.

(1.4) The following description of $DR_D \mathcal{M}$ will play a central role: a) ([3], II; 3.13 and 3.14) If $(\mathcal{M}, \nabla)$ is a logarithmic connection along D such that none of the eigenvalues of the residues Γ_i is a strictly positive integer, then $DR_D \mathcal{M}$ is quasi-isomorphic to $Rj_* V$ (for example for $\mathcal{M} = V_{\mathrm{can}}$). By (1.2, e) this condition is preserved if one tensors with $\mathcal{O}_X(B)$ for an effective divisor B supported on D. Hence under this assumption $DR_D \mathcal{M}$ and $DR_D \mathcal{M}(B)$ are quasi-isomorphic.

b) If the monodromies γ_i do not have one as eigenvalue then, independently of the divisor B, we can apply part a) to $\mathcal{M}(B)$. Therefore the condition on the monodromy implies that for all divisors B supported on D the complexes $DR_D \mathcal{M}$, $DR_D \mathcal{M}(B)$ and $Rj_* V$ are quasi-isomorphic.

(1.5) **Corollary.** *If U is an affine manifold of dimension n and V a local constant system on U, then*

$$H^k(U, V) = 0 \quad \textit{for } k > n.$$

If moreover the monodromies γ_i of V around D_i (for $i = 1 \ldots s$) *do not have* 1 *as eigenvalue, then*

$$H^k(U, V) = 0 \quad \textit{for } k \neq n.$$

The *proof* of the first part can be found in [3], II; 6.2: One chooses X to be a projective compactification satisfying the assumptions of (1.1), and $(\mathcal{M}, \nabla)$ to be V_{can}. If one denotes for a moment the algebraic sheaves by $(\ )^{\mathrm{alg}}$ and the analytical ones by $(\ )^{\mathrm{an}}$ one uses the following argument due to Grothendieck.

By (1.2), a) and (1.4), one has

$$\begin{aligned} H^k(U, V) &= \mathbb{H}^k(X, Rj_* V) = \mathbb{H}^k(X, (DR_D(\mathcal{M}))^{\mathrm{an}}) \\ &= \mathbb{H}^k(X, \varinjlim_{B \geqq D \geqq B_{\mathrm{red}}} (DR_D \mathcal{M}(B)^{\mathrm{an}})) = \mathbb{H}^k(X, \varinjlim_{B \geqq D \geqq B_{\mathrm{red}}} (DR_D \mathcal{M}(B)^{\mathrm{alg}})) \\ &= \mathbb{H}^k(U, DR_D \mathcal{M}|_U^{\mathrm{alg}}) = H^k(H^0(U, \Omega^{\bullet}_U \otimes \mathcal{M}^{\mathrm{alg}})) = 0 \quad \text{for } k > n. \end{aligned}$$

The second part of (1.5) follows from the first part, (1.6) and the Poincaré duality since

$$H^k(U, V) = \mathbb{H}^k(X, Rj_* V) = \mathbb{H}^k(X, j_! V) = H^k_c(U, V) = H^{2n-k}(U, V^{\vee}).$$

If one does not want to use (1.6) one can argue using Serre's vanishing theorem and Serre duality:

Assume that the γ_i do not have 1 as eigenvalue. If one chooses B to be a sufficiently high multiple of an ample divisor, we have

$$H^q(X, \Omega^p_X\langle D\rangle \otimes \mathcal{M}^{\vee}(B - D_{\mathrm{red}})) = 0 \quad \text{for } q > 0.$$

Here $\mathcal{M}^{\vee}$ is the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$. By Serre-duality one has

$$H^{q'}(X, \omega_X \otimes (\Omega^p_X\langle D\rangle \otimes \mathcal{M}^{\vee}(B - D_{\mathrm{red}}))^{\vee}) = H^{q'}(X, \Omega^{n-p}_X\langle D\rangle \otimes \mathcal{M}(-B)) = 0$$

for $q'<n$. Looking at the spectral sequence associated to $DR_D\mathcal{M}(-B)$ with the "filtration bête" (see [4], 1.4) and converging to

$$\mathbb{H}^k(X, DR_D\mathcal{M}(-B)) = \mathbb{H}^k(X, Rj_* V) = \mathbb{H}^k(U, V)$$

one obtains $H^k(U, V)=0$ for $k<n$.

(1.6) **Lemma.** *Let V be a local constant system on U such that the monodromies γ_i of V around D_i (for $i=1\ldots s$) do not have 1 as eigenvalue. Then $j_! V$ and $Rj_* V$ are quasi-isomorphic. Especially $j_! V = j_* V$ and*

$$R^q j_* V = 0 \quad \textit{for } q>0.$$

Proof. Since we have a natural morphism $j_! V \to Rj_* V$, it is enough to prove (1.6) locally. We have to show that $R^p j_* V(W) = H^p(U\cap W, V)=0$ for all p and W running in a fundamental system of neighbourhoods of a given point on D. If $0\to V'\to V\to V''\to 0$ is an exact sequence of local systems on U and if V' and V'' have no cohomology on $U\cap W$, the same holds for V.

Choosing W small enough we may assume that $W=\prod_{j=1}^{n} \Delta_j$ and $U\cap W = \prod_{j=1}^{r} \Delta_j^* \times \prod_{j=r+1}^{n} \Delta_j$ where Δ_j is a small disk and Δ_j^* the punctured disk. Since the monodromies γ_j around the components of $D\cap W$ commute we can find a local subsystem V' of V stable by the γ_j such that the cokernel is a local system of lower rank. By induction on the rank of V we are reduced to the case $rk(V) = 1$.

We may write $V = p_1^{-1} V_1 \otimes \ldots \otimes p_r^{-1} V_r$ where $p_j\colon U\cap W \to \Delta_j^*$ is the j-th projection and V_j the local constant system on Δ_j^* corresponding to the representation of γ_j on a one dimensional vector space L.

By the Künneth formula we just have to show that $H^k(\Delta_j^*, V_j)=0$ for $k=0$ and $k=1$. We may replace Δ_j^* by its boundary $S^1 = \partial\Delta_j^*$ and we parametrize S^1 by $e^{2i\pi\cdot t}$, $t\in\mathbb{R}$. Take

$$U_1 = \{e^{2i\pi\cdot t}, t\in]0,1[\} \quad \text{and} \quad U_2 = \{e^{2i\pi\cdot s}, s\in]-\tfrac{1}{2}, \tfrac{1}{2}[\}$$

as cover of S^1. Then U_1, U_2 and the two connected components W^+ and W^- of $U_1\cap U_2$ are simply connected. The coordinate change from U_1 to U_2 is

$$W^+ \sqcup W^- \to W^+ \sqcup W^-$$
$$t \mapsto s = \begin{cases} t & \text{if } t\in W^+ \\ t-1 & \text{if } t\in W^-. \end{cases}$$

The Čech cohomology with values in V_j is computed by the the cohomology of the complex

$$0 \to L_{U_1} \times L_{U_2} \xrightarrow{d} L_{W^+} \times L_{W^-} \to 0$$
$$(l_1, l_2) \to (l_1 - l_2, l_1 - \gamma_j l_2).$$

However, if $\gamma_j \neq 1$, d is an isomorphism.

Lemma (1.6) gives the following improvement of (1.4, b)

(1.7) **Corollary.** *Let* $(\mathcal{M}, \nabla)$ *be a logarithmic connection along* D *and* $V = \mathrm{Ker}(\nabla_{|U})$. *If* $\mathrm{Res}_i(\nabla)$ *has no integer as eigenvalue (for* $i=1\ldots s$) *then* $DR_D(\mathcal{M})$, $j_!(DR_D(\mathcal{M})_{|U})$, $Rj_* V$ *and* $j_! V$ *are quasi-isomorphic.*

(1.8) *Remark.* If $\mathbb{D}$ denotes the Verdier-duality functor, $V^v = \mathcal{H}om_{\mathbb{C}}(V, \mathbb{C})$ and $\mathcal{M}^v = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ with the dual connection, (1.7) implies that

$$\mathbb{D}(DR_D\mathcal{M}) \simeq \mathbb{D}(j_! V) \simeq Rj_* V^v \simeq DR_D\mathcal{M}^v \simeq DR_D\mathcal{M}^v(-B)$$

for all divisors B with support on D, whenever the monodromies do not have one as eigenvalue. We prove without this assumption in Appendix A that $\mathbb{D}(DR_D\mathcal{M}) \simeq DR_D\mathcal{M}^v(-D_{\mathrm{red}})$. This implies using (1.4, a) that $DR_D\mathcal{M} = j_! V$ whenever the eigenvalues of the residues of $(\mathcal{M}, \nabla)$ are not lying in $-\mathbb{N}$. For example this applies to $\mathcal{M} = ((V^v)_{\mathrm{can}})^v \otimes \mathcal{O}_X(-D_{\mathrm{red}})$. Moreover this gives another proof of (1.6).

§ 2. The $E_1(\mathcal{M})$-degeneration, applications to global vanishing theorems and examples

From now on we allow X to be algebraic over $\mathbb{C}$ or – as in § 1 – analytic. We keep the assumptions made in (1.1). Since we only deal with hypercohomology of logarithmic De Rham complexes over compact manifolds we can use GAGA theorems and switch from the algebraic case to the analytic case whenever it is necessary.

(2.1) On the logarithmic De Rham complex $DR_D\mathcal{M}$ one considers the "filtration bête"

$$F^p\colon 0 \to \Omega^p_X\langle D\rangle \otimes \mathcal{M} \to \Omega^{p+1}_X\langle D\rangle \otimes \mathcal{M} \to \ldots \to \Omega^n_X\langle D\rangle \otimes \mathcal{M}$$

and the associated E_1-spectral sequence

$$(E_1^{pq}(\mathcal{M}), d_1) = (H^q(X, \Omega^p_X\langle D\rangle \otimes \mathcal{M}), H^q(\nabla)),$$

which converges to $\mathbb{H}^{p+q}(X, DR_D\mathcal{M})$ (see [4], 1.4).

By definition of a spectral sequence, the following two conditions are equivalent:

A) $\dim \mathbb{H}^k(X, DR_D\mathcal{M}) = \sum_{p+q=k} \dim H^p(X, \Omega^p_X\langle D\rangle \otimes \mathcal{M})$

B) *The spectral sequence* $E_1^{pq}(\mathcal{M})$ *degenerates at* E_1.

If A and B hold, we say that $(\mathcal{M}, \nabla)$ *satisfies "the* $E_1(\mathcal{M})$ *degeneration".*

(2.2) **Main Lemma.** *Let* $(\mathcal{M}, \nabla)$ *be a logarithmic connection along* D *satisfying the* $E_1(\mathcal{M})$ *degeneration. Assume that* U *is affine. Then*

1) *if* $DR_D\mathcal{M}$ *is quasi-isomorphic to* $Rj_* V$, *one has*

$$H^q(X, \Omega^p_X\langle D\rangle \otimes \mathcal{M}) = 0 \quad \text{for } p+q>n.$$

2) *if* $DR_D\mathcal{M}$ *is quasi-isomorphic to* $j_! V$, *one has*

$$H^q(X, \Omega^p_X\langle D\rangle \otimes \mathcal{M}) = 0 \quad \text{for } p+q<n.$$

3) *if for $i=1\dots s$ the monodromy around D_i does not have* 1 *as eigenvalue, one has*

$$H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) = 0 \qquad \textit{for } p+q \neq n.$$

Remark. As we have seen in (1.4) the assumption in 1) is satisfied for the canonical extension or – more generally – if for all i, $\mathrm{Res}_i(\nabla)$ has no strictly positive integer as eigenvalue. Correspondingly the assumption of 2) is satisfied if, for all i, $\mathrm{Res}_i(\nabla)$ has no eigenvalue lying in $-\mathbb{N}$ (see (1.8)).

Proof. One just writes

$$\dim \mathbb{H}^k(X, DR_D\mathcal{M}) = \bigoplus_{p+q=k} \dim H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}).$$

In case 1) or 3) (use (1.4, b)) this is nothing but

$$\dim \mathbb{H}^k(X, Rj_* V) = \dim H^k(U, V)$$

and in case 2) this is

$$\dim \mathbb{H}^k(X, j_! V) = \dim H_c^k(U, V) = \dim H^{2n-k}(U, V^v)$$

and the Main Lemma follows from the "topological vanishing" (1.5).

(2.3) *Remarks.* a) In fact, by a small modification of the arguments given, it is enough to assume that the conditions in 1), 2), 3) are satisfied along enough components of D, such that the complement remains affine. For example 3) could be replaced by

3′) if for $i=1,\dots,r$ the monodromy around D_i does not have 1 as eigenvalue and $X - \bigcup_{i=1}^{r} D_i$ is affine, then

$$H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) = 0 \qquad \text{for } p+q \neq n.$$

b) If $U = X - D$ is not affine, but if there exists a proper surjective morphism g from U to an affine variety W, one still obtains the vanishing of some of the cohomology groups. In fact, if $H^k(U, V') = 0$ for all local constant systems V' and $k > n+r$ the additional assumptions made in (2.2, 3) imply that

$$H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) = 0 \qquad \text{for } q+p < n-r \quad \text{or} \quad q+p > n+r.$$

Using the Leray spectral sequence, the cohomological dimension of affine varieties for constructible sheaves and the fact that

$$\dim\{x; \dim(g^{-1}(x)) \geqq k\} \leqq n-1-k \qquad \text{for } k > n - \dim(W),$$

one can choose

$$r = \mathrm{Max}\{n - \dim(W), (\text{maximal fibre dimension of } g) - 1\}.$$

(2.4) Let Y be a normal manifold and $\pi\colon Y \to X$ be a Galois cover ramified only along the normal crossing divisor D. Let $\sigma\colon Z \to Y$ be a desingularization of Y such that $\sigma^{-1}\pi^{-1}D=\Delta$ is a normal crossing divisor too. Y has rational singularities (see [5]) and $\pi_*\mathcal{O}_Y$ is locally free. By [5], § 1,

$$R(\pi\circ\sigma)_*\Omega_Z^p\langle\Delta\rangle=\Omega_X^p\langle D\rangle\otimes_{\mathcal{O}_X}\pi_*\mathcal{O}_Y.$$

The push down of the Kähler differential $d\colon \mathcal{O}_Z\to\Omega_Z^1\langle\Delta\rangle$ induces a connection ∇' on $\pi_*\mathcal{O}_Y$ and

$$R(\pi\circ\sigma)_*DR_\Delta(\mathcal{O}_Z)=DR_D(\pi_*\mathcal{O}_Y).$$

The Galois group G operates on $\mathcal{O}_Y$ and $\pi_*\mathcal{O}_Y$. Let $\mathcal{M}$ be a direct summand, invariant under G. Then ∇' induces a logarithmic connection ∇ on $\mathcal{M}$ and $DR_D\mathcal{M}$ is a summand of the complex $DR_D(\pi_*\mathcal{O}_Y)$. Hence $(\mathcal{M},\nabla)$ satisfies the $E_1(\mathcal{M})$-degeneration, if $(\mathcal{O}_Z,d)$ satisfies the $E_1(\mathcal{O}_Z)$-degeneration.

By Deligne's mixed Hodge theory for open varieties [4] this is true if X (and hence Z) is algebraic or Kähler or – more generally – if there exists a Kähler manifold X' and a bimeromorphic map $\tau\colon X'\to X$. In the last case we will say that X is *bimeromorphically dominated* by a Kähler manifold.

By definition $(\mathcal{M},\nabla)$ is the canonical extension. The local constant system V of flat (analytic) sections is given by a representation of $\pi_1(U)$ on a vector space L factorizing over G. The assumption made in (2.2, 3) says that

(*) The induced representation of the ramification group of none of the components D_i of D has a trivial summand.

Alltogether we obtain:

(2.5) **Corollary.** *Let $(\mathcal{M},\nabla)$ be the logarithmic connection constructed above. Assume that X is a proper algebraic (or compact Moisežon) manifold and U is affine, then $H^q(X,\Omega_X^p\langle D\rangle\otimes\mathcal{M})=0$ for $q+p>n$. Moreover, if $(\nabla,\mathcal{M})$ satisfies (*), then*

$$H^q(X,\Omega_X^p\langle D\rangle\otimes\mathcal{M})=0 \quad \textit{for } q+p\neq n.$$

(2.6) *Remark.* The Main Lemma (2.2) applies as well in the following situation: Let V be a local system on U given by a unitary representation of $\pi_1(U)$ and $\mathcal{M}=V_{\mathrm{can}}$, the canonical extension. If the monodromies γ_i do not have 1 as eigenvalue, one can use S. Zucker's methods from [12] to prove that the $E_1(\mathcal{M})$ degeneration holds, as was pointed out to us by P. Deligne. The complete proof is given by K. Timmerscheidt in Appendix D at the end of this article. Using this we obtain:

Corollary. *Under the assumptions made in* (2.6) *one has*

$$H^q(X,\Omega_X^p\langle D\rangle\otimes\mathcal{M})=0 \quad \textit{for } q+p\neq n.$$

In fact, S. Zucker studied in [12] arbitrary variations of Hodge structures, but he had to assume that X is a curve. A "good" extension of variations of Hodge structures together with the degeneration of the corresponding spectral sequence might imply vanishing theorems for certain subquotients of the variation of Hodge structures. Some more precise questions can also be found in J. Kollár's paper [10], § 5.

(2.7) The simplest case of the covering construction given in (2.4) is that of a cyclic cover.

Let $\mathscr{L}$ be an invertible sheaf on X and $D=\sum_{i=1}^{s} v_i D_i$ be an effective normal crossing divisor, such that for some $N>1$ one has $\mathscr{L}^N=\mathcal{O}_X(D)$. Define for $0\leqq j\leqq N-1$ the sheaves $\mathscr{L}^{(j)}=\mathscr{L}^j\left(-\left[\frac{j\cdot D}{N}\right]\right)$ where [ ] denotes the integral part of the $\mathbb{Q}$-divisor $\frac{j\cdot D}{N}$ (see [5] or [11]). Let $L\to X$ and $L^N\to X$ be the line bundles corresponding to $\mathscr{L}$ and $\mathscr{L}^N$ and $\eta\colon L\to L^N$ the map obtained by taking the N-th power. Let $s\colon X\to L^N$ be the section corresponding to D and Y the normalization of $\eta^{-1}((s(X))$. The cover $\pi\colon Y\to X$ obtained is a cyclic cover, ramified over D. It is the same cover constructed in [5] or [11] as normalization of $\mathscr{S}pec\left(\bigoplus_{j=0}^{N-1}\mathscr{L}^{-j}\right)$. One has $\pi_*\mathcal{O}_Y=\bigoplus_{j=0}^{N-1}\mathscr{L}^{(j)^{-1}}$ and the sheaves $\mathscr{L}^{(j)^{-1}}$ correspond to the different sheaves of eigenspaces.

By the construction of (2.4) the sheaves $\mathscr{L}^{(j)^{-1}}$ are endowed with a natural logarithmic connection along D. It can locally be described in the following way:

If t^{-1} is a local generator of $\mathscr{L}$ and $f=x_1^{v_1}\cdot\ldots\cdot x_r^{v_r}$ a local equation for D, one has $t^N=f$. A local generator of $\mathscr{L}^{(j)^{-1}}$ is given by

$$\sigma_j=t^j\cdot x_1^{-\left[\frac{j\cdot v_1}{N}\right]}\cdot\ldots\cdot x_r^{-\left[\frac{j\cdot v_r}{N}\right]}.$$

One has

$$\nabla(\sigma_j)=\sigma_j\cdot\left(j\frac{dt}{t}-\sum_{i=1}^{r}\left[\frac{j\cdot v_i}{N}\right]\frac{dx_i}{x_i}\right)=\sigma_j\cdot\left(\sum_{i=1}^{r}\left(\frac{j\cdot v_i}{N}-\left[\frac{j\cdot v_i}{N}\right]\right)\frac{dx_i}{x_i}\right).$$

The condition (*) of (2.4), saying that the monodromy of $(\mathscr{L}^{(j)},\nabla)$ does not have 1 as eigenvalue means exactly that $\frac{j\cdot v_i}{N}\notin\mathbb{Z}$ for $i=1\ldots s$.

Rewriting (2.5) in this case one obtains:

(2.8) **Global vanishing theorem for integral parts of $\mathbb{Q}$-divisors.** *Let X be a proper algebraic (or compact Moišezon) manifold and U affine. Let $\mathscr{L}$ be an invertible sheaf and $\mathscr{L}^N=\mathcal{O}_X(D)$. Then*

1) *for $0\leqq j\leqq N-1$ and $p+q>n$ one has $H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{L}^{(j)^{-1}})=0$.*

2) *if moreover, for some j, $1\leqq j\leqq N-1$, and for all i, one has $\frac{j\cdot v_i}{N}\notin\mathbb{Z}$, then $H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{L}^{(j)^{-1}})=0$ for $p+q\neq n$.*

(2.9) *Remarks.* 1) Let $D'=\Sigma D_i$, where the sum is taken over all components D_i with $\frac{j\cdot v_i}{N}\in\mathbb{Z}$. Then $\mathscr{L}^{(j)}(-D_{\mathrm{red}})=\mathscr{L}^{(N-j)^{-1}}(-D')$. Using Serre duality one obtains in (2.8.1) the vanishing of

$$H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{L}^{(N-j)^{-1}}(-D'))\quad\text{for } p+q<n.$$

2) Using (2.3, a) it is in (2.8, 2) again sufficient to ask for the condition "$\frac{j \cdot \nu_i}{N} \notin \mathbb{Z}$" for "enough" components of D. Moreover – as remarked in (2.3, 6) – one can weaken the condition "U affine" and obtains still the vanishing of some cohomology groups.

3) Replacing $\mathscr{L}$ by $\mathscr{L}^j$ and N by $j \cdot N$ we may always assume that the sheaf considered is of the form $\mathscr{L}^{(1)}$. Moreover, since $\mathscr{L}^{(1)}$ does not change if we replace D by $D - N \cdot D_i$ and $\mathscr{L}$ by $\mathscr{L}(-D_i)$ for some i with $\nu_i \geqq N$, we can as well assume that all $0 < \nu_i < N$. In this case the assumption of (2.8, 2)) is satisfied for the new divisor D. However, if from the beginning $\frac{j \cdot \nu_i}{N} \notin \mathbb{Z}$, D_{red} does not change.

At the end of this section we want to show how to obtain from (2.8) several of the classical vanishing theorems.

(2.10) **Kodaira-Nakano-vanishing theorem** (see for example [7]). *Let X be a projective manifold and $\mathscr{L}$ be an invertible ample sheaf. Then $H^q(X, \Omega_X^p \otimes \mathscr{L}^{-1}) = 0$ for $p + q < n$.*

Proof. For some $N > 1$ we can find a smooth very ample divisor D such that $\mathscr{L}^N = \mathcal{O}_X(D)$. One has an exact sequence $0 \to \Omega_X^p \to \Omega_X^p \langle D \rangle \to \Omega_D^{p-1} \to 0$ and a long exact sequence

$$\ldots \to H^{q-1}(X, \Omega_X^p \langle D \rangle \otimes \mathscr{L}^{-1}) \to H^{q-1}(D, \Omega_D^{p-1} \otimes \mathscr{L}^{-1}) \to H^q(X, \Omega_X^p \otimes \mathscr{L}^{-1})$$
$$\to H^q(X, \Omega_X^p \langle D \rangle \otimes \mathscr{L}^{-1}) \to \ldots$$

By construction $U = X - D_{\text{red}}$ is affine and (2.8.2) implies

$$H^{q-1}(D, \Omega_D^{p-1} \otimes \mathscr{L}^{-1}) \cong H^q(X, \Omega_X^p \otimes \mathscr{L}^{-1})$$

for $q + p < n$ (or $q + p > n + 1$).

The sheaf $\mathscr{L}_{|D}$ is ample and – by induction on the dimension – we may assume that $H^{q-1}(D, \Omega_D^{p-1} \otimes \mathscr{L}^{-1}) = 0$ for $p + q \leqq n$.

(2.11) **Bogomolov-Sommese-Vanishing theorem** (see for example [11]). *Let X be a proper algebraic (or compact Moišezon) manifold. $\mathscr{L}$ an invertible sheaf with $\kappa(\mathscr{L}) = n$. Then $H^0(X, \Omega_X^p \otimes \mathscr{L}^{-1}) = 0$ for $p < n$.*

Proof. The statement is compatible with blowing up. Therefore we may assume X to be projective. As well known, $\kappa(\mathscr{L}) = n$ if and only if one can find $N > 1$, a very ample sheaf $\mathscr{H}$ and an effective divisor B such that $\mathscr{L}^N = \mathscr{H}(B)$ (see for example [11], p. 17). Let $\sigma: X' \to X$ be an embedded desingularization of B and $-E$ a relative ample divisor, E supported in the exceptional locus of σ. Replacing N by $\nu \cdot N$ and $\mathscr{H}$ by $\mathscr{H}^\nu$ we may assume that $\mathscr{H}' = \sigma^* \mathscr{H}(-E)$ is very ample and for $\mathscr{L}' = \sigma^* \mathscr{L}$ we have an effective normal crossing divisor $B' = \sigma^*(B) + E$ with $\mathscr{L}'^N = \mathscr{H}'(B')$. Hence we may assume that from the beginning B was a normal crossing divisor. Of course, in order to prove (2.11) we may replace $\mathscr{L}$ be a smaller sheaf and hence we can also assume that the multiplicities of all components of B are strictly smaller than N. Let H be a general divisor of $\mathscr{H}$. Then $D=H+B$ is a normal crossing divisor. As in (2.9) we have $\mathscr{L}^{(1)}=\mathscr{L}$. Since $\mathscr{H}$ is very ample $U=X-D_{\mathrm{red}}$ is affine and (2.11) follows from (2.8.2).

(2.12) **The vanishing theorem for numerically effective sheaves** (see [8] or [11]). *Let X be a proper algebraic (or compact Moišezon) manifold, $\mathscr{L}$ a numerically effective invertible sheaf (i.e.* $\deg(\mathscr{L}_{|C})\geqq 0$ *for all curves $C\subset X$) and $c_1(\mathscr{L})^n>0$. Then $H^q(X,\mathscr{L}^{-1})=0$ for $q<n$.*

Proof. Again (2.12) is compatible with blowing up and we may assume X to be projective. For numerically effective sheaves the condition $c_1(\mathscr{L})^n>0$ is equivalent to $\kappa(\mathscr{L})=n$ (the proof is quite simple, see for example [11]). As in the proof of (2.11) we can find – after blowing up again – an ample sheaf $\mathscr{H}$ and a normal crossing divisor B such that $\mathscr{L}^N=\mathscr{H}(B)$. Since $\mathscr{L}$ is numerically effective $\mathscr{H}\otimes\mathscr{L}^{\nu}$ is ample for all $\nu\geqq 0$. Replacing N by $N+\nu$, we may assume that N is larger than the multiplicities of the components of B and – replacing N, $\mathscr{L}$, $\mathscr{H}$, B by $\mu\cdot N$, $\mathscr{L}^{\mu}$, $\mathscr{H}^{\mu}$, $\mu\cdot B$ – that $\mathscr{H}$ is a very ample. Let H be a general divisor of $\mathscr{H}$ and $D=B+H$. Then $\mathscr{L}^{(1)}=\mathscr{L}$,

$$U=X-D_{\mathrm{red}}=(X-H_{\mathrm{red}})-B_{\mathrm{red}}$$

is affine and (2.12) follows from (2.8.2).

(2.12) can be generalized to $\mathbb{Q}$-divisors. The most general form is

(2.13) **Theorem** (see [8], [11] or [5]). *Let X be a proper algebraic (or compact Moišezon) manifold, $\mathscr{L}$ an invertible sheaf and C an effective normal crossing divisor such that for some $N>1$ $\mathscr{L}^N(-C)$ is numerically effective. If for some $j<N$ the "$\mathscr{L}$-dimension"*

$$\kappa\left(\mathscr{L}^j\left(-\left[\frac{j\cdot C}{N}\right]\right)\right)=n, \quad \textit{then} \quad H^q\left(X,\mathscr{L}^{-j}\left(\left[\frac{j\cdot C}{N}\right]\right)\right)=0 \quad \textit{for } q<n.$$

The *proof* is similar to (2.12): If $\sigma\colon X'\to X$ is a blowing up, such that $\sigma^*C=C'$ is again a normal crossing divisor, then $R\sigma_*\mathcal{O}_{X'}\left(\left[\frac{j\cdot C'}{N}\right]\right)=\mathcal{O}_X\left(\left[\frac{j\cdot C}{N}\right]\right)$. This follows from the fact that the cover Y of X constructed in (2.7) has at most rational singularities, or from elementary local calculations (see [11]). Hence the statement of (2.13) is compatible with blowing up.

If we allow "fractional powers of sheaves", one has

$$\mathscr{L}^j\left(-\left[\frac{j\cdot C}{N}\right]\right)=(\mathscr{L}^N(-C))^{\frac{j}{N}}\otimes\mathcal{O}\left(\frac{j}{N}\cdot C-\left[\frac{j\cdot C}{N}\right]\right).$$

Hence the assumption says that we can find (after replacing N by some high multiple) a subdivisor C' of C such that $\left[\frac{j\cdot C}{N}\right]=\left[\frac{j\cdot(C-C')}{N}\right]$ and such that $\mathscr{L}^N(-C+C')$ contains an ample sheaf $\mathscr{H}$. After blowing up we may assume that $\mathscr{L}^N(-(C-C'))=\mathscr{H}(B)$ where $B+C$ is a normal crossing divisor. Replacing $\mathscr{H}$ by $\mathscr{H}\otimes\mathscr{L}^{\nu\cdot N}(-C)$ we can increase N without changing the multiplicity of the components of B. Alltogether we are reduced to the case that $\mathscr{L}^N=\mathcal{O}(D)$ where $D=H+B+(C-C')$ is a normal crossing divisor, H is ample and $\left[\frac{j\cdot D}{N}\right]=\left[\frac{j\cdot C}{N}\right]$. Now (2.13) follows from (2.8, 2).

(2.14) *Remark.* a) If $\mathscr{L}$ is an invertible sheaf on X such that some power of $\mathscr{L}$ is generated by its global sections, $\mathscr{L}$ is of course numerically effective. Therefore (2.12) implies the global Grauert-Riemenschneider vanishing theorem. The relative version for modifications of algebraic varieties as well as its generalisation for integral parts of $\mathbb{Q}$-divisors are easy corollaries of (2.12) and (2.13) (see for example [11]).

b) It seems surprising that the vanishing theorems (2.11) for $q=0$ and (2.12) for $p=0$ are more general than (2.10). However, it is well known that (2.10) is no longer true, if one replaces the condition "$\mathscr{L}$ ample" by "$\kappa(\mathscr{L})=n$ and $\mathscr{L}^\mu$ generated by global sections for some $\mu>0$". In this case one could still choose a normal crossing divisor D with small multiplicities, such that $\mathscr{L}^N=\mathcal{O}(D)$ and such that $U=X-D_{\mathrm{red}}$ is affine. One obtains the vanishing of $H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{L}^{-1})$ for $q+p\neq n$, but the induction used in the proof of (2.10) breaks down, since for some components D_i, $\kappa(\mathscr{L}_{|D_i})$ might be too small.

c) The proof of (2.12) and (2.13) in [11] used Hodge duality to reduce the vanishing of cohomology of invertible sheaves to the Bogomolov-Sommese vanishing theorem. In the approach described here, both follow from the same statement, the E_1-degeneration of the spectral sequence associated to the Hodge filtration, and one does not use the Hodge duality.

§ 3. Applications to the vanishing of the cohomology of morphisms

We keep the notations and assumptions made in (1.1) and (2.1). Whereas in § 2 we just considered the dimension of $E_1^{pq}(\mathscr{M})$ for a logarithmic connection $\mathscr{M}$, we will now regard the differentials d_p of the spectral sequence.

(3.1) As usual $[i]$ denotes the shift operator for complexes. Hence $F^p[p]$ is the complex starting with $\Omega_X^p\langle D\rangle\otimes\mathscr{M}$ in degree zero and - if $\mathscr{F}$ is any complex - one has $\mathbb{H}^k(\mathscr{F})=\mathbb{H}^{k+i}(\mathscr{F}[-i])$.

The differential

$$d_1\colon H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{M})\to H^q(X,\Omega_X^{p+1}\langle D\rangle\otimes\mathscr{M})=\mathbb{H}^{q+1}(X,F^{p+1}/F^{p+2}[p]),$$

is the connecting morphism of

$$0\to F^{p+1}/F^{p+2}[p]\to F^p/F^{p+2}[p]\to\Omega_X^p\langle D\rangle\otimes\mathscr{M}\to 0.$$

Hence $d_1=0$ implies that $\mathbb{H}^q(X,F^p/F^{p+2}[p])\to H^q(X,\Omega_X^p\langle D\rangle\otimes\mathscr{M})$ is surjective and in this case d_2 is the connecting morphism of

$$0\to F^{p+2}/F^{p+3}[p]\to F^p/F^{p+3}[p]\to F^p/F^{p+2}[p]\to 0.$$

If $d_2=0$ one gets a surjection

$$\mathbb{H}^q(X, F^p/F^{p+3}[p]) \to H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M})$$

and repeating this construction long enough one finds the wellknown equivalence of the following two conditions:

A) *For all p, q the connecting morphisms*

$$\delta_p\colon H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) \to \mathbb{H}^q(X, F^{p+1}[p+1]) = \mathbb{H}^{q+1}(X, F^{p+1}[p])$$

of

$$0 \to F^{p+1}[p] \to F^p[p] \to \Omega_X^p\langle D\rangle \otimes \mathcal{M} \to 0$$

are zero.

B) $(\mathcal{M}, \nabla)$ *satisfies* $E_1(\mathcal{M})$ *degeneration.*

Of course δ_p is induced by $\Omega_X^p\langle D\rangle \otimes \mathcal{M} \xrightarrow{\nabla} F^{p+1}[p+1]$.

Under the additional condition that $DR_D\mathcal{M}$ is quasi-isomorphic to $j_! V$, where V denotes as usual the flat (analytic) sections of $\mathcal{M}$, the $E_1(\mathcal{M})$ degeneration can be interpretated in a more geometric way. In the Lemma below part 1) and 3) use the whole vanishing of δ, whereas 2) follows from the vanishing of d_1.

(3.2) **Main Lemma.** *Let* $(\mathcal{M}, \nabla)$ *be a logarithmic connection satisfying* $E_1(\mathcal{M})$ *degeneration. Assume that the monodromies of* $(\mathcal{M}, \nabla)$ *around the components* D_i *of* D *do not have* 1 *as an eigenvalue.*

1) *Then for any effective divisor* B *with* $B_{\mathrm{red}} \leqq D_{\mathrm{red}}$, *and all* $q \geqq 0$, *the morphism, induced by restriction of* $\mathcal{M}$ *to* B,

$$H^q(R^0)\colon H^q(X, \mathcal{M}) \to H^q(B, \mathcal{M}_{|B}),$$

is zero.

2) *Let* C *be a smooth subdivisor of* D_{red} *and* $D' = D_{\mathrm{red}} - C$. *Then for all* $q \geqq 0$ *and* $p \geqq 0$ *the morphism, induced by restriction of differentials,*

$$H^q(R^p)\colon H^q(X, \Omega_X^p\langle D'\rangle \otimes \mathcal{M}) \to H^q(C, \Omega_C^p\langle D' \cap C\rangle \otimes \mathcal{M}),$$

is zero. Especially, if D *is smooth, the map*

$$H^q(X, \Omega_X^p \otimes \mathcal{M}) \to H^q(D, \Omega_D^p \otimes \mathcal{M})$$

is zero.

3) *Then for all* $q \geqq 0$ *and* $p \geqq 0$ *the morphism, induced by the connection* ∇,

$$H^q(\nabla)\colon H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) \to H^q(X, \nabla(\Omega_X^p\langle D\rangle \otimes \mathcal{M}))$$

is zero.

Proof. 1) By (1.4, b) $DR_D\mathcal{M}$ and $DR_D\mathcal{M}(-B)$ are quasi-isomorphic. By (3.1, A) the morphism $\delta_0\colon H^q(X, \mathcal{M}) \to \mathbb{H}^q(X, F^1[1])$ is zero. Hence in the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^q(X, DR_D\mathcal{M}(-B)) & \longrightarrow & H^q(X, \mathcal{M}(-B)) \\ \downarrow \beta & & \downarrow \gamma \\ \mathbb{H}^q(X, DR_D\mathcal{M}) & \xrightarrow{\alpha} & H^q(X, \mathcal{M}). \end{array}$$

β is an isomorphism and α surjective. Therefore γ is also surjective.

2) By assumption $\mathrm{Res}_C(\nabla)$ can not have zero as eigenvalue and this just means ([3], p. 78) that the composition

$$\mathrm{Res}_C(\nabla)\colon \mathcal{M} \xrightarrow{\nabla} \Omega^1_X\langle D\rangle \otimes \mathcal{M} \xrightarrow{\mathrm{res}} \mathcal{O}_C \otimes \mathcal{M}$$

is surjective. Hence one has

$$\begin{array}{ccc} \Omega^p_X\langle D'\rangle\otimes\mathcal{M} & \xrightarrow{\nabla} & \Omega^{p+1}_X\langle D\rangle\otimes\mathcal{M} \\ {\scriptstyle R^p}\downarrow & \searrow{\scriptstyle\gamma} & \downarrow{\scriptstyle \mathrm{res}} \\ \Omega^p_C\langle D'\cap C\rangle\otimes\mathcal{M} & \xrightarrow{\cong} & \Omega^p_C\langle D'\cap C\rangle\otimes\mathcal{M}\end{array}$$

$H^q(\nabla)=0$ implies that $H^q(\gamma)$ and $H^q(R^p)$ are both zero.

3) We have a quasi-isomorphism (1.7) $j_! V \to F^0 = DR_D\mathcal{M}$ and therefore $\nabla(\Omega^p_X\langle D\rangle\otimes\mathcal{M}) \to F^{p+1}[p+1]$ is a quasi-isomorphism for $p \geqq 0$. Hence 3) is just saying that δ_p in (3.1, A) is zero.

Applying (3.2, 1 and 2) to invertible sheaves arising from cyclic covers of X (2.7) we obtain:

(3.3) **Relative vanishing theorem for integral parts of $\mathbb{Q}$-divisors.** *Let X be a proper algebraic manifold or a compact analytic manifold which is bimeromorphically dominated by a Kähler manifold. Let $\mathscr{L}$ be an invertible sheaf on X, D be an effective normal crossing divisor and $\mathscr{L}^N = \mathcal{O}_X(D)$ for some $N>1$. Let $1 \leqq j \leqq N-1$.*

1) *Let B be an effective divisor supported in* $\mathrm{supp}\left(j\cdot D - N\cdot\left[\frac{j\cdot D}{N}\right]\right)$. *Then the maps*

$$H^0(R^0)\colon H^q(X, \mathscr{L}^{(j)^{-1}}) \to H^q(B, \mathscr{L}^{(j)^{-1}}|_B)$$

are zero for all $q \geqq 0$.

2) *Let C be a smooth subdivisor of*

$$D_{\mathrm{red}} \cap \mathrm{supp}\left(j\cdot D - N\cdot\left[\frac{j\cdot D}{N}\right]\right) \quad \text{and} \quad D' = D_{\mathrm{red}} - C.$$

Then the maps

$$H^q(R^p)\colon H^q(X, \Omega^p_X\langle D'\rangle \otimes \mathscr{L}^{(j)^{-1}}) \to H^q(C, \Omega^p_C\langle D'\cap C\rangle \otimes \mathscr{L}^{(j)^{-1}})$$

are zero for all $p \geqq 0$ and $q \geqq 0$.

(3.4) *Remark.* As described in (2.9, 3) one may rephrase (3.3) in the following way.

Assume that for an effective normal crossing divisor D one has, $\mathscr{L}^N = \mathcal{O}(D)$, where N is larger than the multiplicities of the components of D, and let B be any divisor supported in D_{red}. Then the maps $H^q(X, \mathscr{L}^{-1}) \to H^q(B, \mathscr{L}^{-1}|_B)$ are zero for all $q \geqq 0$.

If C is a smooth subdivisor of D_{red}, then the maps

$$H^q(X, \Omega^p_X\langle D'\rangle \otimes \mathscr{L}^{-1}) \to H^q(C, \Omega^p_C\langle D'\cap C\rangle \otimes \mathscr{L}^{-1})$$

are zero for all $p, q \geqq 0$.

(3.5) **Corollary** (Kollár, [9], 2.2). *Let X be as in* (3.3), *$\mathscr{L}$ an invertible sheaf, such that some power of $\mathscr{L}$ is generated by its global sections, and B an effective divisor, such that $\mathcal{O}_X(B)$ is contained in a power of $\mathscr{L}$. Then the restriction maps $H^q(X, \mathscr{L}^{-1}) \to H^q(B, \mathscr{L}^{-1}|_B)$ are zero for all $q \geqq 0$.*

Proof. We choose D' such that $\mathcal{O}_X(D'+B) = \mathscr{L}^\mu$. In order to show that $H^q(X, \mathscr{L}^{-1}(-B)) \to H^q(X, \mathscr{L}^{-1})$ is surjective, we may replace X be a blowing up and thereby we may assume $B+D'$ to be a normal crossing divisor. By assumption $\mathscr{L}^\nu$ is generated by its global sections for some $\nu \gg 0$ and one finds a smooth divisor D'' such that $D = B + D' + D''$ is a normal crossing divisor. Choosing ν large enough one may assume that the multiplicities of the components of D are smaller than $N = \mu + \nu$ and obtains (3.5) from (3.3, 1) or (3.4, 1).

In (3.2, 2) and correspondingly in (3.2, 2) and correspondingly in (3.3, 2) one can weaken the hypothesis "C smooth" to "C reduced with non singular components. However, in this case we just get that the natural map

$$H^q(\tilde{R})\colon H^q(X, \Omega^p_X\langle D'\rangle \otimes \mathscr{M}) \to H^q(C, \Omega^p_{\tilde{C}}\langle \tilde{D}\rangle \otimes \mathscr{M})$$

is zero, where $\tilde{C}$ is the normalization of C and $\tilde{D}$ the pullback of the one by one intersections of D to $\tilde{C}$.

Of course the map we are really interested in is

$$H^q(R)\colon H^q(X, \Omega^p_X\langle D'\rangle \otimes \mathscr{M}) \to H^q(C, \Omega^p_C\langle D'\rangle \otimes \mathscr{M}).$$

The only cases where we know that $H^q(\tilde{R}) = 0$ implies $H^q(R) = 0$ are the trivial one, $q = 0$, or the case $p = 0$, handled in (3.2, 1) by different methods.

In [6], 1.1, we proved (3.3, 2) for $q = 0$ by direct calculation, and – similarly to the global case (see (2.14, c)) – we used Hodge duality to obtain the $p = 0$ case. Finally we had to use the strict compatibility of the restriction map with the Hodge and the weight filtration ([4], 8.2.7) to show that for $p = 0$, $H^q(\tilde{R}) = 0$ implies $H^1(R) = 0$ (see [6], 1.6).

If one tries to consider more complicated restriction maps, the picture is even worse and the interpretation of the morphisms nearly impossible. Nevertheless, we will try in the last part of this chapter to use (3.2, 3) to obtain some generalizations of (3.2, 1) and (3.2, 2).

We assume in the sequel that D is a reduced normal crossing divisor with non singular components.

The idea of the constructions following is quite simple. We try to find $\mathcal{O}_X$-modules (or complexes) $\mathscr{N}^p$ and $\mathscr{K}^p$ and an $\mathcal{O}_X$-linear map $\gamma\colon \mathscr{N}^p \to \mathscr{K}^p$ which fits into a commutative diagram

$$\begin{array}{ccc} \Omega^p_X\langle D\rangle \otimes \mathscr{M} & \xrightarrow{\nabla} & \nabla(\Omega^p_X\langle D\rangle \otimes \mathscr{M}) \hookrightarrow \Omega^{p+1}_X\langle D\rangle \otimes \mathscr{M} \\ {\scriptstyle\alpha}\uparrow & & \downarrow{\scriptstyle\beta} \\ \mathscr{N}^p & \xrightarrow{\gamma} & \mathscr{K}^p \end{array}$$

of $\mathbb{C}_X$ sheaves. Then $H^q(\nabla) = 0$ implies $H^q(\gamma) = 0$

(3.6) The sheaves $\mathcal{N}^p$ will be given by the weight filtration (see [4]) W_k of $\Omega_X^p\langle D\rangle$ where $W_k(\Omega_X^p\langle D\rangle)=\Omega_X^k\langle D\rangle\wedge\Omega_X^{p-k}$.

We denote by $\mathcal{C}_k^p$ the quotient sheaf $\Omega_X^p\langle D\rangle\otimes\mathcal{M}/W_k(\Omega_X^p\langle D\rangle)\otimes\mathcal{M}$ and by $\mathcal{K}_k^p$ the image of $\nabla(\Omega_X^{p-1}\langle D\rangle\otimes\mathcal{M})$ in $\mathcal{C}_k^p$.

By the Leibnitz rule one has

$$\nabla(W_k(\Omega_X^p\langle D\rangle\otimes\mathcal{M})\subset W_{k+1}(\Omega_X^{p+1}\langle D\rangle)\otimes\mathcal{M},$$

and ∇ induces a map

$$\nabla'\colon \mathcal{C}_k^p\to\mathcal{C}_{k+1}^{p+1} \quad \text{such that } \mathcal{K}_k^p\subseteq\mathcal{K}er(\nabla').$$

In general ∇' is not $\mathcal{O}_X$-linear and $\mathcal{K}_k^p$ is not an $\mathcal{O}_X$-module. Applying again the Leibnitz rule we obtain an $\mathcal{O}_X$-linear map

$$\operatorname{Res}_k^{p-1}(\nabla)\colon W_k(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M}\xrightarrow{\nabla}\Omega_X^p\langle D\rangle\otimes\mathcal{M}\to\mathcal{C}_k^p$$

and $\mathcal{I}m(\operatorname{Res}_k^{p-1}(\nabla))\subseteq\mathcal{K}_k^p$.

(3.7) Denote by $D^{[s]}$ the normalization of the s by s intersections of the components D_i of D and by D^{s+1} the normal crossing divisor on $D^{[s]}$ obtained by pulling back the $(s+1)$ by $(s+1)$ intersections of the components of D. One has an inclusion

$$\mathcal{C}_k^p\hookrightarrow\Omega_{D^{[k+1]}}^{p-k-1}\langle D^{k+2}\rangle\otimes\mathcal{M}$$

given locally at a point on $D=$ zero set of $x_1\cdot\ldots\cdot x_r=0$ by

$$\frac{dx_{i_1}}{x_{i_1}}\wedge\ldots\wedge\frac{dx_{i_p}}{x_{i_p}}\otimes m\mapsto\bigoplus\pm\bigwedge_{i\in J}\frac{dx_{i_j}}{x_{i_j}}\otimes m\Big|_{\substack{\{x_{i_l}=0,\, l\notin J\}\\ l=1\ldots r}}$$

where $1\leqq i_1<\ldots<i_p\leqq r$, and where the direct sum is taken over all subsets $J\subseteq\{1,\ldots,r\}$ of $r-k-1$ elements, and the signs are given by the usual rule.

If $\Gamma_i\colon \mathcal{M}\xrightarrow{\nabla}\Omega_X^1\langle D\rangle\otimes\mathcal{M}\xrightarrow{\operatorname{res}_i}\mathcal{M}|_{D_i}$ denotes the residue of ∇ along D_i then, for example, $\operatorname{Res}_k^{p-1}(\nabla)$ maps $\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_k}{x_k}\wedge\varphi\otimes m$ to $\bigoplus_{i>k}\pm\varphi\wedge\Gamma_i(m)$. Especially, if the monodromies of $(\mathcal{M},\nabla)$ around the components D_i of D do not have one as eigenvalue, then Γ_i is surjective as well as $\operatorname{Res}_k^{p-1}(\nabla)$ at the general points of the components of $D^{[k+1]}$. Moreover $\operatorname{Res}_k^{p-1}(\nabla)$ factors in the following way

$$\begin{aligned}W_k(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M}&\to W_k(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M}/W_{k-1}(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M}\\ &\cong\Omega_{D^{[k]}}^{p-k-1}\otimes\mathcal{M}\to\Omega_{\tilde{D}^{k+1}}^{p-k-1}\otimes\mathcal{M}\xrightarrow{\rho}\Omega_{D^{[k+1]}}^{p-k-1}\otimes\mathcal{M}\hookrightarrow\mathcal{C}_k^p.\end{aligned}$$

Here $\tilde{D}^{k+1}$ is the normalization of D^{k+1} and ρ is mapping $\varphi\otimes m$ to the alternating sum of the possible restrictions $\varphi\otimes\Gamma_i(m)$.

By the $E_1(\mathcal{M})$ degeneration we obtain.

(3.8) *Claim.*

$$H^q(\mathrm{Res}_k^{p-1}(\nabla)): H^q(X, W_k(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M})\to H^q(X, \Omega_{D[k+1]}^{p-k-1}\langle D^{k+2}\rangle\otimes\mathcal{M})$$

is the zero map.

Of course, the $\mathcal{O}_X$-linear map $\mathrm{Res}_k^{p-1}(\nabla)$ depends on the residues of the connection and the only case where one can find an isomorphism α of $\Omega_{D[k+1]}^{p-k-1}\langle D^{k+2}\rangle\otimes\mathcal{M}$ such that $\alpha\cdot\mathrm{Res}_k^{p-1}(\nabla)$ does not, is for $k=0$. In fact (3.2.3) implies a stronger statement:

(3.9) $$H^q(\mathrm{Res}_k^{p-1}(\nabla)): H^q(X, W_k(\Omega_X^{p-1}\langle D\rangle)\otimes\mathcal{M})\to H^q(X, \mathcal{K}_k^p)$$

is the zero map, where $\mathcal{K}_k^p=\mathcal{K}er(\nabla' : \mathcal{C}_k^p\to\mathcal{C}_{k+1}^{p+1})$.

However, both sheaves, $\mathcal{K}_k^p$ and $\mathcal{C}_k^p$ are quite difficult to describe.

For $\mathcal{C}_k^p$, at least, we have a reasonable filtration. If $\mathcal{W}_l$ denotes the image of $W_l(\Omega_X^p\langle D\rangle)\otimes\mathcal{M}$ in $\mathcal{C}_k^p$, one obtains a filtration $0=\mathcal{W}_k\subset\mathcal{W}_{k+1}\subset\ldots\subset\mathcal{W}_p=\mathcal{C}_k^p$ such that $\mathcal{W}_l/\mathcal{W}_{l-1}=\Omega_{D[l]}^{p-l}\otimes\mathcal{M}$.

For $k=p-1$, one obtains $\mathcal{C}_{p-1}^p=\mathcal{O}_{D[p]}\otimes\mathcal{M}$. However,

$$\nabla' : \mathcal{M}|_{D[p]}\to\mathcal{M}|_{D[p+1]}=\mathcal{C}_p^{p+1}$$

is given by the alternating sum of the Γ_i, considered as an isomorphism of $\mathcal{M}|_{D_i}$.

Define $\gamma^{[p]}$: $\mathcal{M}|_{D[p]}\to\mathcal{M}|_{D[p]}$ to be the automorphism given by

$$\Gamma_{i_1}\circ\Gamma_{i_2}\circ\ldots\circ\Gamma_{i_p}$$

on $\mathcal{M}|_{D_{i_1}\cap\ldots\cap D_{i_p}}$. Since ∇ is integrable $\gamma^{[p]}$ is independent of the numbering of the components. One obtains a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}|_{D[p]} & \xrightarrow{\nabla'} & \mathcal{M}|_{D[p+1]} \\
\downarrow{\scriptstyle \gamma^{[p]}} & & \downarrow{\scriptstyle \gamma^{[p+1]}} \\
\mathcal{M}|_{D[p]} & \xrightarrow[\varepsilon^p\otimes id_\mu]{} & \mathcal{M}|_{D[p+1]}
\end{array}$$

where ε^p is the usual map $\mathcal{O}_{D[p]}\to\mathcal{O}_{D[p+1]}$. Hence $\gamma^{[p]}$ maps $\mathcal{K}_{p-1}^p$ to $\mathcal{K}er(\varepsilon^p)\otimes\mathcal{M}$ $=\mathcal{I}m(\varepsilon^{p-1})\otimes\mathcal{M}$. Locally, if D is the zero set of $x_1\cdot\ldots\cdot x_r$, $\gamma^{[p]^{-1}}\circ\mathrm{Res}_{p-1}^p(\nabla)$ maps $\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{p-1}}{x_{p-1}}\otimes m$ to

$$\bigoplus_{i\geqq p}\pm\gamma^{[p]^{-1}}\circ\Gamma_i(m)|_{D_1\cap\ldots\cap D_{p-1}\cap D_i}$$
$$=\bigoplus_{i\geqq p}\pm\Gamma_1^{-1}\circ\ldots\circ\Gamma_{p-1}^{-1}(m)|_{D_1\cap\ldots\cap D_{p-1}\cap D_i}.$$

Hence we obtain

(3.10) **Claim.** *Keeping the assumptions made in* (3.2) *and the notations introduced above the map*

$$H^q(\gamma^{[p]^{-1}} \circ \mathrm{Res}^p_{p-1}(\nabla)) \colon H^q(X, \Omega^{p-1}_X\langle D\rangle \otimes \mathcal{M}) \to H^q(X, \mathcal{K}er(\varepsilon^p) \otimes \mathcal{M})$$

is zero.

For $p=1$ and $D=B$ this is the same as (3.2, 1). For $p>1$ the map $\gamma^{[p]^{-1}} \circ \mathrm{Res}^p_{p-1}(\nabla)$ depends on ∇. Of course, we can apply (3.10) to the situation of invertible sheaves coming from cyclic covers (as in (3.3)). In this case, one can give a more explicit description of the morphism considered, but we don't.

Appendix A: Duality for logarithmic De Rham complexes

We keep the notations and assumptions introduced in (1.1), except that X is a (not necessarily compact) analytic manifold and – to simplify the notation – that D is reduced.

(A.1) Let $D^b_c(X)$ be the derived category of bounded complexes of $\mathbb{C}$-sheaves with constructible cohomology. The Verdier dual is given by the functor

$$\mathbb{D} \colon D^b_c(X) \to D^b_c(X)$$
$$\mathcal{F}^\bullet \mapsto \mathbb{D}(\mathcal{F}^\bullet) = R\,\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^\bullet, \mathbb{C}_X).$$

For an $\mathcal{O}_X$-module $\mathcal{M}$ we write $\mathcal{M}^v = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$ and, if $(\mathcal{M}, \nabla)$ is a logarithmic connection along D, ∇^v denotes the dual connection. The main result of this appendix is:

(A.2) **Proposition.** *In* $D^b_c(X)$ *one has*

$$DR_D\mathcal{M} \cong \mathbb{D}(DR_D\mathcal{M}^v(-D)).$$

The arguments needed to prove (A.2) are quite similar to a proof of the corresponding statement for $\mathcal{D}_X$-modules, due to J. Bernstein ([2], §5). We recall some notations from the theory of $\mathcal{D}_X$-modules. Details can be found in [2].

(A.3) $\mathcal{D}_X$ denotes the sheaf of holomorphic differential operators on X and $\mathcal{D}_X\langle -D\rangle$ the subalgebra of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and $T_X\langle -D\rangle = (\Omega^1_X\langle D\rangle)^v$ the sheaf of vectorfields preserving $\mathcal{O}_X(-D)$.

Locally we choose a parameter system of X such that D is given by $x_1 \cdot \ldots \cdot x_r = 0$.

Let $\partial_1, \ldots, \partial_n$ be the vectorfields orthogonal to $x_1, \ldots, x_n$ and define

$$\delta_i = \begin{cases} x_i \cdot \partial_i & \text{for } 1 \leqq i \leqq r \\ \partial_i & \text{for } r+1 \leqq i \leqq n. \end{cases}$$

δ_i is dual to $\dfrac{dx_i}{x_i}$ $(1 \leqq i \leqq r)$ or dx_i $(r+1 \leqq i \leqq n)$, and $T_X\langle -D\rangle$ is generated by $\delta_1, \ldots, \delta_n$.

The logarithmic connection ∇ on $\mathcal{M}$ gives $\mathcal{M}$ the structure of a left $\mathcal{D}_X\langle -D\rangle$ module and for $m\in\mathcal{M}$,

$$\nabla m=\sum_{i=1}^{r}\delta_i\cdot m\frac{dx_i}{x_i}+\sum_{i=r+1}^{r}\delta_i\cdot m\,dx_i.$$

(A.4) **Claim.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two left $\mathcal{D}_X\langle -D\rangle$-modules. Then*

a) $\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A},\mathcal{B})\cong\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A}(D),\mathcal{B}(D))$

b) *One has an isomorphism*

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{B}))\xrightarrow{\sim}\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A},\mathcal{B})$$

given by $\varphi\mapsto\varphi(1)$.

Proof. a) If $\mathcal{M}$ is a $\mathcal{D}_X\langle -D\rangle$ module, and $\varphi\in\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A},\mathcal{B})$, then the induced morphism $\varphi\otimes 1$ from $\mathcal{A}\otimes\mathcal{M}$ to $\mathcal{B}\otimes\mathcal{M}$ is also defined over $\mathcal{D}_X\langle -D\rangle$. Therefore, taking $\mathcal{M}$ to be $\mathcal{O}_X(D)$, we obtain an equivalence of the category of $\mathcal{D}_X\langle -D\rangle$ modules to itself, whose inverse is defined by $\otimes_{\mathcal{O}_X}\mathcal{O}_X(-D)$.

b) As for connections the $\mathcal{D}_X\langle -D\rangle$-module structure on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{B})$ is given by $(\delta_i\Psi)(a)=\delta_i(\Psi(a))-\Psi(\delta_i a)$ for $\Psi\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{B})$. The morphism $\varphi(1)$ is $\mathcal{O}_X$-linear and

$$\delta_i(\varphi(1)(a))-\varphi(1)(\delta_i a)=(\delta_i(\varphi(1))(a)=(\varphi(\delta_i 1))(a)=0.$$

Hence $\varphi(1)$ is $\mathcal{D}_X\langle -D\rangle$ linear. On the other hand, if $\eta\in\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A},\mathcal{B})$ we define $\varphi\in\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A},\mathcal{B}))$ by $\varphi(\lambda)=\lambda\cdot\eta$. In fact, φ is $\mathcal{D}_X\langle -D\rangle$ linear since

$$\begin{aligned}\varphi(\delta_i\lambda)(a)&=(\delta_i\lambda)\cdot\eta(a)=\delta_i(\lambda\cdot\eta(a))-\lambda\cdot\delta_i(\eta(a))\\&=\delta_i(\varphi(\lambda)(a))-\varphi(\delta_i a)\end{aligned}$$

and $\varphi(1)=\eta$.

$\mathcal{O}_X$ has a locally free resolution as $\mathcal{D}_X\langle -D\rangle$-module by the Koszul complex

$$\begin{aligned}0\to\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\Lambda^n T_X\langle -D\rangle&\xrightarrow{d^{n-1}}\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\Lambda^{n-1}T_X\langle -D\rangle\xrightarrow{d^{n-2}}\\\dots&\xrightarrow{d^1}\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}T\langle -D\rangle\xrightarrow{d^0}\mathcal{D}_X\langle -D\rangle\to 0\end{aligned}$$

where, for local sections $v_{i_1},\dots,v_{i_{p+1}}$ of $T^1\langle -D\rangle$, and ρ a local section of $\mathcal{D}_X\langle -D\rangle$, one has

$$\begin{aligned}&d^p(\rho\otimes(v_{i_1}\wedge\dots\wedge v_{i_{p+1}}))\\&\quad=\sum_{1}^{p+1}(-1)^{j-1}\rho\cdot v_{i_j}\otimes(v_{i_1}\wedge\dots\wedge\hat{v}_{i_j}\wedge\dots\wedge v_{i_{p+1}})\\&\quad+\sum_{1\leq k<l\leq p+1}(-1)^{k+l}\rho\cdot[v_{i_k},v_{i_l}]\wedge v_{i_1}\dots\wedge\hat{v}_{i_k}\wedge\dots\wedge\hat{v}_{i_l}\wedge\dots\wedge v_{i_{p+1}}.\end{aligned}$$

With the generators δ_i, this gives

$$d^P(\rho\otimes(\delta_{i_1}\wedge\ldots\wedge\delta_{i_{p+1}}))=\sum_{j=1}^{p+1}(-1)^{j-1}\rho\cdot\delta_{i_j}\otimes(\delta_{i_1}\wedge\ldots\wedge\hat{\delta}_{i_j}\wedge\ldots\wedge\delta_{i_{p+1}}).$$

(A.5) **Claim.**

$$\begin{aligned} DR_D\mathcal{M} &\cong R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{M})\cong \mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle,\mathcal{M})\\ &\cong \mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D),\mathcal{M}(D))\end{aligned}$$

Proof. The last quasi-isomorphism follows from (A.4, a)). The Koszul complex is a locally free resolution and therefore one obtains the second quasi-isomorphism. Since

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\Lambda^pT_X\langle -D\rangle,\mathcal{M})\cong\Omega_X^p\langle D\rangle\otimes_{\mathcal{O}_X}\mathcal{M}$$

we just have to verify that the differentials d_p of the third complex are the same as ∇_p. For simplicity we assume $p=0$.

Let $m=\varphi(1)$ for $\varphi\in\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle,\mathcal{M})$.

One has $d_0m=\sum_{i=1}^{r}n_i\cdot\frac{dx_i}{x_i}+\sum_{i=r+1}^{n}n_i\cdot dx_i$ for

$$n_i=(\varphi\circ d)(\delta_i)=\varphi(\delta_i)=\delta_i\varphi(1)=\delta_im.$$

By definition of the $\mathcal{D}_X\langle -D\rangle$-module structure on $\mathcal{M}$ we have $d_0m=\nabla m$.

(A.6) **Claim.** $DR_D\mathcal{M}^v\cong R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{M},\mathcal{O}_X)$.

Proof. By taking $I^\cdot$ to be an injective resolution of $\mathcal{O}_X$ over $\mathcal{D}_X\langle -D\rangle$ we obtain $\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{M},I^\cdot)=R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{M},\mathcal{O}_X)$ and by (A.4, b) this is quasi-isomorphic to

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},I^\cdot)).$$

Since $\mathcal{M}$ is locally free and $\mathcal{O}_X$ quasi-isomorphic to $I^\cdot$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},I^\cdot)\cong\mathcal{M}^v\otimes_{\mathcal{O}_X}I^\cdot$ is an injective resolution of $\mathcal{M}^v$. In fact, $\mathcal{M}^v\otimes_{\mathcal{O}_X}I^\cdot$ is locally a direct sum of copies of $I^\cdot$ and

$$\mathcal{M}^v=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)\cong R\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)\cong\mathcal{H}om_{\mathcal{O}X}(\mathcal{M},I^\cdot).$$

Therefore

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},I^\cdot))&=R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},I^\cdot))\\&=R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{M}^v)\end{aligned}$$

and using (A.5) we are done.

(A.7) *There is a natural pairing, non degenerate over U*

$$DR_D\mathcal{M}\overset{\mathbb{L}}{\otimes}_{\mathbb{C}}DR_D\mathcal{M}^v(-D)\to\mathbb{C}.$$

Proof. Again, let $I^\cdot$ be an injective resolution of $\mathcal{O}_X$ as $\mathcal{D}_X\langle -D\rangle$-module. Using (A.5) and (A.6) we obtain the pairing

$$\begin{array}{c}
DR_D\mathcal{M}\overset{\mathbb{L}}{\otimes}_{\mathbb{C}}DR_D\mathcal{M}^v(-D)\\
\wr|\\
\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D),\mathcal{M}(D))\otimes_{\mathbb{C}}\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{M}(D),I^\cdot)\\
\downarrow\\
\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D),I^\cdot)\\
\wr|\\
\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D),\mathcal{O}_X)
\end{array}$$

The last sheaf is by scalar extension isomorphic to

$$\mathcal{H}om_{\mathcal{D}_X}((\mathcal{D}_X\otimes_{\mathcal{D}_X\langle -D\rangle}\mathcal{D}_X\langle -D\rangle)\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}(D),\mathcal{O}_X)$$

In fact, if φ is a $\mathcal{D}_X\langle -D\rangle$ linear morphism

$$\varphi\colon \mathcal{D}_X\langle -D\rangle\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D)\to\mathcal{O}_X$$

one can extend the operation of $\mathcal{D}_X\langle -D\rangle$ to $\mathcal{D}_X$ using the $\mathcal{O}_X$-linearity and writing $\partial_i=\frac{\delta_i}{x_i}$ for $i\leqq r$.

The inclusion $\dot{\Lambda}T_X\to\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}(D)$ gives a morphism

$$\begin{array}{c}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X\langle -D\rangle\otimes_{\mathcal{O}_X}\mathcal{O}_X(D),\mathcal{O}_X)\\
\downarrow\\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X\otimes_{\mathcal{O}_X}\dot{\Lambda}T_X,\mathcal{O}_X)=R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{O}_X)=\mathbb{C}.
\end{array}$$

As a corollary we obtain:

(A.8) *There is a natural morphism, isomorphic over U:*

$$DR_D\mathcal{M}\xrightarrow{\Phi}\mathbb{D}\,DR_D\mathcal{M}^v(-D).$$

Proof of (*A*.2). If $0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{M}''\to 0$ is an exact sequence of logarithmic connections along D and the morphism in (A.8) an isomorphism for $\mathcal{M}'$ and $\mathcal{M}''$, it is an isomorphism for $\mathcal{M}$ as well.

Moreover the question whether Φ is an isomorphism can be answered locally.

So we may assume X to be a polydisk and D to be the coordinate axes. By (1.2, c), the action of the loops T_i on V extend to $\mathcal{M}$. Its eigenvalues are constant. Therefore, there is a subconnection $\mathcal{N}$ of $\mathcal{M}$ of rank 1, with constant T_i-action, such that the quotient $\mathcal{M}/\mathcal{N}$ is a connection of lower rank. Arguing by induction on the rank, we may assume that $\operatorname{rank}(\mathcal{M})=1$.

Choosing the neighbourhood small enough we may write $(X,D)\cong(X_1,D_1)\times(X_2,D_2)$ and $\mathcal{M}$ as $\mathcal{M}=p_1^*\mathcal{M}_1\otimes p_2^*\mathcal{M}_2$ where $\mathcal{M}_i$ is a rank one connection on X_i, logarithmic along D_i (see [3], p. 81). Then $DR_D\mathcal{M}=p_1^*DR_{D_1}\mathcal{M}_1\otimes p_2^*DR_{D_2}\mathcal{M}_2$ and since the Verdier duality is also compatible with products we are reduced to the case of curves:

Let X be a curve, $D\in X$ a point, given by $x=0$, and $\mathcal{M}$ a rank one bundle whose connection has constant coefficients. If the residue $\operatorname{Res}\nabla$ of $\mathcal{M}$ at D is given by multiplication with a, $\operatorname{Res}(\nabla^{v,-D})$ of $\mathcal{M}^v(-D)$ is given by multiplication with $(1-a)$. Hence changing the role of $\mathcal{M}$ and $\mathcal{M}^v(-D)$ if necessary, we may assume that $1-a\notin\mathbb{N}-\{0\}$. By [3], II, 3.14, $DR_D\mathcal{M}^v(-D)\cong Rj_*V^v$ and $\mathbb{D}DR_D\mathcal{M}^v(-D)\cong j_!V$ where $V=\operatorname{Ker}(\nabla_{|U})$. Therefore we just have to show that $-a\notin\mathbb{N}$ implies that $DR_D\mathcal{M}=(0\to\mathcal{M}\to\Omega_X^1\langle D\rangle\otimes\mathcal{M}\to 0)$ is quasi-isomorphic to $j_!V$.

Since $\operatorname{Res}\nabla\colon\mathcal{M}\to\mathcal{M}_{|D}$ is given by multiplication with $a\neq 0$, $\operatorname{Ker}\nabla\subset\mathcal{M}(-D)$. Similarly, since $\operatorname{Res}(\nabla_{|\mathcal{M}(-l\cdot D)})$ is given by multiplication with $a+l$ and is nontrivial for $l\geqq 0$, one obtains

$$\operatorname{Ker}\nabla\subset\bigcap_{l\geqq 0}\mathcal{M}(-l\cdot D)\cap j_*V=j_!V.$$

On the other hand, if e is a generating section of $\mathcal{M}$ such that

$$\nabla(f\cdot e)=(x\cdot\partial+a)f\cdot e\frac{dx}{x}$$

and

$$g\cdot e\cdot\frac{dx}{x}=\Big(\sum_l\lambda_l\cdot x^l\Big)\cdot e\frac{dx}{x}\in\Omega_X^1\langle D\rangle\otimes\mathcal{M},\quad\text{then }\sum_l\frac{\lambda_l}{a+l}\cdot x^l$$

converges as well and

$$\nabla\left(\left(\sum_l\frac{\lambda_l}{a+l}\cdot x^l\right)\cdot e\right)=\left(\sum_l\frac{\lambda_l}{a+l}(l\cdot x^l+a\cdot x^l)\right)\cdot e\frac{dx}{x}=g\cdot e\cdot\frac{dx}{x}.$$

Hence ∇ is surjective and the quasi-isomorphism is established.

Appendix B: Chern classes and logarithmic connections

Let $(\mathcal{M},\nabla)$ be a connection on a proper algebraic or compact analytic manifold X with logarithmic poles along a normal crossing divisor D. As we have seen in §2 the classical positivity conditions on a C^∞ curvature matrix of a differentiable connection on $\mathcal{M}$ can be replaced by conditions on the residues of ∇ along the components of D, if one is interested in vanishing theorems of Kodaira-Nakano type. In this appendix we want to show how to define the Chern classes of $\mathcal{M}$ using the logarithmic connection ∇. This is a second example indicating that both, the theory of $\mathscr{C}^\infty$-connections without singularities but with nontrivial curvature matrix and the theory of holomorphic integrable connections with logarithmic singularities can be applied in a quite similar way in algebraic geometry.

The computation of the Chern classes and the Atiyah class described here was done independently by J.L. Verdier and the first author about one year ago.

Let $\tilde{D}$ be the normalization of D and Res: $\Omega_X^1\langle D\rangle\otimes\mathcal{M}\to\mathcal{O}_{\tilde{D}}\otimes\mathcal{M}$ be the Poincaré residue. The element $\Gamma=\mathrm{res}\circ\nabla\in\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{M}_{|\tilde{D}})$ is mapped under the connecting morphism of the exact sequence

$$0\to\Omega_X^1\otimes\mathcal{M}\to\Omega_X^1\langle D\rangle\otimes\mathcal{M}\to\mathcal{O}_{\tilde{D}}\otimes\mathcal{M}\to 0$$

to an element $\gamma\in\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{M},\Omega_X^1\otimes\mathcal{M})$.

(B.1) **Proposition.** *$-\gamma$ is the Atiyah class of $\mathcal{M}$.*

Proof. We consider here holomorphic, not necessarily integrable connections. Let $\{U_i\}$ be an open covering such that $\mathcal{M}$ has a connection ∇_i^0 on U_i. Then each connection on U_i is given by $\nabla_i=\nabla_i^0+\alpha_i$ for some $\alpha_i\in\Omega_{U_i}^1\otimes\mathcal{E}nd\,\mathcal{M}$. Therefore the connections on $\mathcal{M}$ form a $\Omega_X^1\otimes\mathcal{E}nd\,\mathcal{M}$ principal space (torseur), whose class $\{\nabla_i-\nabla_j\}\in H^1(X,\Omega_X^1\otimes\mathcal{E}nd\,\mathcal{M})$ is by definition [1] the Atiyah class of $\mathcal{M}$.

Assume now that $\mathcal{M}$ has a global connection ∇ with logarithmic poles along D and with residue $\Gamma\in H^0(\mathcal{O}_{\tilde{D}}\otimes_{\mathcal{O}_X}\mathcal{E}nd\,\mathcal{M})$. On U_i the holomorphic connections on $\mathcal{M}$ are described by $\nabla_i=\nabla+\alpha_i$, where $\alpha_i\in\Omega_{U_i}^1\langle D\rangle\otimes\mathcal{E}nd\,\mathcal{M}$ is of Poincaré residue $-\Gamma$. The class of $\alpha_i-\alpha_j=\nabla_i-\nabla_j$ in $H^1(X,\Omega_X^1\otimes\mathcal{E}nd\,\mathcal{M})$ is by definition the image of $-\Gamma$ under the connecting morphism of the Poincaré residue sequence tensorized by $\mathcal{E}nd\,\mathcal{M}$ over $\mathcal{O}_X$.

(B.2) *Remarks.* 1) Atiyah described his class as the extension class

$$0\to\Omega_X^1\otimes\mathcal{M}\to P^1(\mathcal{M})\to\mathcal{M}\to 0$$

in $\mathrm{Ext}^1(\mathcal{M},\Omega_X^1\otimes\mathcal{M})=H^1(X,\Omega_X^1\otimes\mathcal{E}nd\,\mathcal{M})$, where $P^1(\mathcal{M})$ is the first order jet bundle of $\mathcal{M}$. A connection ∇ on $\mathcal{M}$ is a $\mathcal{O}_X$ splitting s: $\mathcal{M}\to P^1(\mathcal{M})$ by $s(m)=(1\otimes 1)m-\nabla(m)$ ([3], p. 7). For the embedding of the holomorphic forms in the holomorphic forms with logarithmic poles, one obtains an induced extension $P_D^1(\mathcal{M})$ in $\mathrm{Ext}^1(\mathcal{M},\Omega_X^1\langle D\rangle\otimes\mathcal{M})$. A logarithmic connection is a $\mathcal{O}_X$ splitting s: $\mathcal{M}\to P_D^1(\mathcal{M})$ by $s(m)=(1\otimes 1)m-\nabla(m)$. The residue of ∇ is by definition the morphism from $\mathcal{M}$ to $P_D^1(\mathcal{M})/P^1(\mathcal{M})=\Omega_X^1\langle D\rangle/\Omega_X^1\otimes\mathcal{M}$ induced by $-s$, whereas the class of s in $\mathrm{Ext}^1(\mathcal{M},\Omega_X^1\otimes\mathcal{M})$ is by definition $P^1(\mathcal{M})$. This gives another way of understanding the proposition.

2) In both proofs, it does not change anything to replace the one forms with logarithmic poles by forms with poles of order k along D, and the residues of the logarithmic connection by the residue $\Gamma\in\mathrm{Hom}(\mathcal{M},\Omega_X^1(k\cdot D)/\Omega_X^1\otimes\mathcal{M})$. The image of $-\Gamma$ is again the Atiyah class of $\mathcal{M}$.

Atiyah himself explained how to use the Atiyah class to compute the Chern classes (i.e.: the symmetric functions of the Chern roots). Usually one gives the formula for the Newton classes N_p (i.e. the sum over the p-th powers of the Chern roots) and obtains the Chern classes by the interchange formulas.

(B.3) **Corollary** ([1], Prop. 13). *Let* $\Gamma_i = \mathrm{Res}_i \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}_{|D_i})$ *and* $[D_i]$ *the class of* D_i *in* $H^1(X, \Omega^1_X)$. *Then*

$$N_p(\mathcal{M}) = (-1)^p \sum_{\alpha_1 + \ldots + \alpha_s = p} \binom{p}{\alpha} \mathrm{Tr}(\Gamma_1^{\alpha_1} \circ \ldots \circ \Gamma_s^{\alpha_s}) \cdot [D_1]^{\alpha_1} \cdot \ldots \cdot [D_s]^{\alpha_s}.$$

Especially $C_1(\mathcal{M}) = N_1(\mathcal{M}) = -\sum_{i=1}^{s} \mathrm{Tr}(\Gamma_i) \cdot [D_i]$.

Appendix C: Local structure of logarithmic connections

In this section we reproduce a classification of free logarithmic connections $(\mathcal{M}, \nabla)$ on $X = \Delta^n$ with poles along $D \subset \bigcup_1^l pr_i^{-1}(0)$, due to P. Deligne.

(C.1) Let $p: X \to S = \Delta^l$ be the projection on the first l coordinates and $D' = p(D)$. If s is the zero section of p, $\mathcal{N} = s^*(\mathcal{M})$ carries a connection ∇', induced by ∇, with logarithmic singularities along D'.

Lemma. *$(\mathcal{M}, \nabla)$ is the pull-back of $(\mathcal{N}, \nabla')$.*

Proof. By induction we may assume $l = n-1$. ∇ induces the relative connection

$$\nabla_{\mathrm{rel}}: \mathcal{M} \to \Omega^1_{X/S} \otimes \mathcal{M} = \Omega^1_X\langle D\rangle / p^* \Omega^1_S \langle D' \rangle \otimes \mathcal{M}.$$

The kernel of ∇_{rel} is a relative local constant system V_{rel} and $s^{-1}(V_{\mathrm{rel}}) \simeq \mathcal{N}$. As in [3], p. 17, the existence and uniqueness of a Cauchy problem with parameters implies that a local section of $\mathcal{N}$ is obtained as the pull-back of a unique relative flat section of $\mathcal{M}$. Therefore one can extend the isomorphism $s^{-1}(V_{\mathrm{rel}}) \simeq \mathcal{N}$ to an isomorphism $p^{-1}(\mathcal{N}) \simeq V_{\mathrm{rel}}$.

(C.2) From now on we assume $D = \bigcup_1^n pr_i^{-1}(0)$ because of (C.1).

By (1.2, c) the loop T_i around $D_i = pr_i^{-1}(0)$ acts on $\mathcal{M}$ with constant eigenvalues. As in [3], p. 95, we can decompose $\mathcal{M}$ in generalized eigenspaces (which means especially subspaces stable under the action of the T_i).

Write $\mathcal{M} = \bigoplus_{\lambda \in (\mathbb{C}^*)^n} U_\lambda \otimes \mathcal{M}_\lambda$, where U_λ is the rank one connection on which T_i acts by multiplication with λ_i and where the monodromy on $\mathcal{M}_\lambda$ is unipotent.

Hence, in order to classify the logarithmic connections, one may and one does assume that the monodromy is unipotent, or, equivalently (1.2, d) that the residues Γ_i have integers as eigenvalues.

(C.3) Let x_j be the local coordinates in $X = \Delta^n$ and V be the local constant system on $U = X - D$. For $a \in \mathbb{Z}^n$ we define the logarithmic connection $x^a \cdot V_{\mathrm{can}} = V_{\mathrm{can}} \otimes_{\mathcal{O}_X} \mathcal{O}\left(-\sum_1^n a_j \cdot D_j\right)$. Of course for any torsion free coherent logarithmic connection $\mathcal{M}$ we find $c \in \mathbb{Z}^n$ such that $x^c \cdot V_{\mathrm{can}} \subset \mathcal{M} \subset x^{-c} \cdot V_{\mathrm{can}}$. The connection on $\mathcal{M}$ is the restriction of the connection on $x^{-c} \cdot V_{\mathrm{can}}$ because they are the same on U.

(C.4) *Definition.* a) A *filtration* P_a, $a\in\mathbb{Z}^n$, $a=(a_1,\dots,a_n)$, of a local constant system V consists of subsystems P_a such that

i) $\bigcup_{a\in\mathbb{Z}^n} P_a = V$

ii) $P_b\subset P_a$ for $b\leqq a$ (coordinate-wise).

b) A filtration P_a of V is called *splittable* (scindable) if for some point $p\in U$ there are sub-vectorspaces W_b of V_p verifying $(P_a)_p=\bigoplus_{b\leqq a} W_b$

c) If P_a is a filtration of V one defines the torsion-free module

$$\Phi(P_a)=\sum_{a\in\mathbb{Z}^n} x^a\cdot(P_a)_{\mathrm{can}}$$

where the sum is taken in $j_* V_{\mathrm{can}|U}$.

(C.5) **Lemma.** *If P_a is a splittable filtration, then $\Phi(P_a)$ is free.*

Proof. One has $(P_a)_{\mathrm{can}}=\bigoplus_{b\leqq a} W_b\otimes\mathcal{O}_X\subset V_p\otimes\mathcal{O}_X\simeq V_{\mathrm{can}}$. Hence $x^a\cdot(P_a)_{\mathrm{can}}$ is generated by $x^{a-b}\cdot(x^b\cdot W_b)$, $b\leqq a$. Since the later is already contained in $x^b\cdot(P_b)_{\mathrm{can}}$, $\Phi(P_a)$ is generated by $\{x^b\cdot W_b\}_{b\in\mathbb{Z}^n}$, and one finds the right number of generators.

(C.6) **Theorem.** *Let V be a local constant system on $U=X-D$ with unipotent monodromies. Then Φ defines a one-to-one correspondence between the splittable filtrations P_a, $a\in\mathbb{Z}^n$, of V and the free $\mathcal{O}_X$ modules with a logarithmic connection extending V.*

Proof. Let $(\mathcal{M},\nabla)$ be given. We denote as in (A.3) by δ_i the operation of $x_i\cdot\partial_i$ on $\mathcal{M}$ induced by ∇. Especially $\delta_{i|D_i}$ is the residue Γ_i. In order to find the inverse of Φ, we construct a filtration of $\mathcal{M}$ by free subconnections $\mathcal{P}_a$ generated over $\mathcal{O}_X$ by elements in the kernel of some high power of δ_i-b_i, $b_i\leqq a_i$, in such a way that the residue classes in $L=\mathcal{M}/x\cdot\mathcal{M}$ fulfill the condition (C.4, ii). Since the subconnections of $\mathcal{M}$ are in one-to-one correspondence with the subconnections of $\hat{\mathcal{M}}=\mathcal{M}\otimes_{\mathcal{O}_X}\mathbb{C}[\![x]\!]$, we may construct $\mathcal{P}_a$ in the formal case.

As a $\mathbb{C}$-vector space, $\hat{\mathcal{M}}$ can be decomposed in $\hat{\mathcal{M}}=\prod_{a\in\mathbb{Z}^n} M_a$ where the product is infinite, the M_a are the generalized finite dimensional eigenspaces on which δ_i-a_i is nilpotent, and $x^b\cdot M_a\subset M_{a+b}$. Actually this is obvious for $x^c\cdot V_{\mathrm{can}}$ and, since one can squeeze $\hat{\mathcal{M}}$ between two such modules (C.3), this carries over to $\hat{\mathcal{M}}$.

Write $L=\bigoplus_{b\in\mathbb{Z}^n} L_b$, the decomposition in generalized eigenspaces. Write L'_b for a lifting of L_b as a $\mathbb{C}$-vector space in M_b. Define $\mathcal{W}_b$ as the submodule of $\hat{\mathcal{M}}$ generated over $\mathcal{O}_X$ by L'_b, and $\mathcal{P}_a$ as the one generated by M_b, for all $b\leqq a$. By the Leibnitz rule, $\mathcal{P}_a$ is stable under δ_i for all i, and contains all $\mathcal{W}_b$, for $b\leqq a$. On the other band, $\hat{\mathcal{M}}$ is free, and therefore freely generated by the $\mathcal{W}_b$. Since $M_a=\bigoplus_{b\leqq a} x^{a-b}\cdot L'_b$ one has $\mathcal{P}_a=\bigoplus_{b\leqq a}\mathcal{W}_b$.

If $\mathcal{M}=\Phi(P_a)$ for a splittable filtration P_a, then

$$\mathcal{P}_a=\sum_{b\leqq a} x^b\cdot(P_b)_{\mathrm{can}} \quad\text{and}\quad \mathcal{W}_b=x^b\cdot(W_b\otimes\mathcal{O}_X).$$

(C.7) 1) For $n=1$, the condition splittable is empty. No longer assuming the monodromy to be unipotent, one may directly write that the logarithmic connections are in one-to-one correspondence to the filtration $(P_a)_{a\in\mathbb{Z}+\lambda}$ of V_λ, where V_λ is the local system associated to $U_\lambda\otimes\mathcal{M}_\lambda$ in the decomposition of (C.2).

2) For any n and in the unipotent case, one defines n filtrations $P^{(i)}_{a_i}=P_{\infty,\dots,\infty,a_i,\infty,\dots,\infty}$, where a_i is at the i-th place. If P_a is splittable, one recovers P_a as the intersection of the $P^{(i)}_{a_i}$.

3) For $n=2$, and $P^{(1)}$ and $P^{(2)}$ two filtrations by subsystems of V, the filtration P_a defined as before as the intersection of $P^{(1)}$ and $P^{(2)}$ is always splittable (by a trivial lemma of two filtrations).

(C.8) One knows that $\mathcal{O}$-coherent $\mathcal{D}$-modules are locally free. This no longer true for $\mathcal{D}\langle -D\rangle$ modules. For example, the maximal ideal $\mathfrak{m}$ has a logarithmic connection along the coordinate axis. Taking $P_{e_i}=\mathbb{C}$, for $e_i=(0,\dots 0,1,0,\dots,0)$, where the 1 is at the i-th place, $P_{(0,\dots,0)}=0$ and $P_{(1,\dots,1)}=\mathbb{C}$, one sees that P_a is not a splittable filtration, nor the condition (C.7, 2) is fulfilled. One has $\Phi(P_a)=\mathfrak{m}$.

Remark. In the proof of theorem (C.6) we only used the freeness of $\mathcal{M}$ in order to construct the splitting $\mathcal{W}_b$ of $\mathcal{P}_a$. Hence the same proof gives immediately:

(C.9) **Proposition.** *Φ defines a one-to-one correspondence between the filtrations P_a, $a\in\mathbb{Z}^n$, of V, and the torsion free $\mathcal{O}_X$ coherent logarithmic connections extending V.*

(C.10) **Corollary.** *Φ defines a one-to-one correspondence between the filtrations P_a, $a\in\mathbb{Z}^n$, of V, such that*

$$P_a=\bigcap_1^n P^{(i)}_{a_i}, \quad \text{where } P^{(i)}_{a_i}=P_{\infty,\dots,\infty,a_i,\infty,\infty,\dots,\infty}$$

(a_i is at the i-th place) and the reflexive $\mathcal{O}_X$ coherent logarithmic connections.

Proof. As a torsion free module $\Phi(P_a)$ is locally free at the generic point of D_i. By (C.1), it is determined there by its restriction to a generic curve transverse to D_i. The corresponding filtration is nothing but $P^{(i)}_{a_i}$.

Since the reflexive hull of $\Phi(P_a)$ is generated by local sections of $\Phi(P_a)$ defined outside of codimension 2, it carries also a logarithmic connection, and therefore is of the shape $\Phi(P'_a)$ (C.9). Since both coincide in codimension 1, one has $P^{(i)}_{a_i}=P'^{(i)}_{a_i}$. Therefore, one has $P'_a\subset P_a=\bigcap_1^n P'^{(i)}_{a_i}$ as P'_a is an ascending filtration. This implies $\Phi(P'_a)\subset\Phi(P_a)$, and $\Phi(P_a)$ is reflexive.

Conversely, if $\mathcal{M}$ is reflexive, it is torsion free, and therefore of the shape $\Phi(P_a)$ (C.9). Since $P_a\subset P'_a$, where P'_a is the intersection of the filtrations in codimension 1, one has $\Phi(P_a)\subset\Phi(P'_a)$, and both are equal.

(C.10) The combinatorial properties of the filtrations given in (C.7) have now a well-known interpretation on the side of modules:

1) corresponds to the fact that a torsion free module on a smooth curve is free.

2) corresponds to the fact that a locally free sheaf is the reflexive hull of its restriction to a subset whose complement is of complex codimension 2.

3) corresponds to the fact that a reflexive sheaf on a smooth surface is locally free.

(C.11) In general, for $\mathcal{O}_X$ coherent logarithmic connections, the structure is more complicated. For example even the skyscraper sheaf $\mathcal{O}/\mathfrak{m}$ has a logarithmic connection as a quotient of two compatible logarithmic connections: the zero map! Of course, this is not realizable by the correspondence (C.9), as $\mathcal{O}/\mathfrak{m}$ is a torsion module.

Appendix D: Hodge decomposition for unitary local systems

Klaus Timmerscheidt

We continue to use the notations introduced in (1.1); X is a compact Kähler manifold.

In this appendix we will show that for a unitary local system V on U, the cohomology of $j_* V$ on X has a Hodge decomposition. The idea of proof is taken from parts of S. Zucker's paper [12] where he treats the case of one-dimensional X and V the underlying local system of a variation of Hodge structures.

Let $(\mathcal{M}, \nabla)$ be the canonical extension of V.

Using the inner product on V, we shall first introduce a subcomplex of $DR_D(\mathcal{M})$ whose hypercohomology is the cohomology of $j_* V$:

(D.1) On U, let $\tilde{\Omega}^p_X(\mathcal{M})_{|U} := \Omega^p_U \otimes \mathcal{M}_{|U}$. For $x \in D$, let Δ be a small polycylinder around x such that $\Delta \cap D = \bigcup_{i=1}^{r} D_i^\Delta$ is a union of coordinate hyperplanes. For $i \in \{1, \ldots, r\}$, let γ_i^Δ be the monodromy of $V_{|\Delta \setminus D}$ around D_i^Δ, and let V_i^Δ be the (well-defined!) subsystem $\mathrm{Ker}(\gamma_i^\Delta - id)$. Let W_i^Δ be its orthogonal complement.

Now let $\tilde{\Omega}^1_X(\mathcal{M})_{|\Delta} \subseteq \Omega^1_X\langle D\rangle \otimes \mathcal{M}_{|\Delta}$ be generated over $\mathcal{O}_\Delta$ by $\Omega^1_\Delta \otimes_{\mathcal{O}_\Delta} \mathcal{M}_{|\Delta}$ and $\Omega^1_\Delta\langle D_i^\Delta\rangle \bigotimes_{\mathbb{C}} W_i^\Delta$, $i = 1, \ldots, r$. Let $\tilde{\Omega}^p_X(\mathcal{M})_{|\Delta} := \Lambda^p \tilde{\Omega}^1_X(\mathcal{M})_{|\Delta}$. It is clear that these locally defined sheaves patch together and give a subcomplex $\tilde{D}R_D(\mathcal{M}) := (\tilde{\Omega}^\bullet_X(\mathcal{M}), \nabla)$ of $DR_D(\mathcal{M})$. Note that if $\mathrm{Res}_i(\nabla)$ does not have 1 as eigenvalue, then $\tilde{D}R_D(\mathcal{M}) = DR_D(\mathcal{M})$.

We want to prove the following

(D.2) **Theorem.** a) *$j_* V$ is quasiisomorphic to $\tilde{D}R_D(\mathcal{M})$.*

b) *The spectral sequence*

$$E_1^{p,q} = H^q(X, \tilde{\Omega}^p_X(\mathcal{M})) \Rightarrow H^{p+q}(X, j_* V)$$

degenerates at E_1, i.e. $H^k(X, j_* V) \cong \bigoplus_{p+q=k} H^q(X, \tilde{\Omega}^p_X(\mathcal{M}))$.

c) *There exists a conjugate linear isomorphism*

$$H^q(X, \tilde{\Omega}^p_X(\mathcal{M})) \cong H^p(X, \tilde{\Omega}^q_X(\mathcal{N})),$$

where $\mathcal{N}$ *is the canonical extension of* $V^\vee$.

The *proof* of a) is parallel to (1.6): The statement being local, one restricts to the case $rk(V)=1$, using that locally (with respect to X) V is an orthogonal direct sum of unitary systems of rank 1. But in that case, $\tilde{D}R_D(\mathcal{M})$ is simply $DR_{D'}(\mathcal{M})$ where D' is the union of all components of D across which V does not extend to a local system.

For the proof of b) and c), the idea is to represent the hypercohomology of $\tilde{D}R_D(\mathcal{M})$ by harmonic L^2-forms with respect to an appropriate Kähler metric, and then use the Kähler identities as in the classical case.

The metric is a complete Kähler metric ω on U which locally near a point x of D has the same asymptotic form as the Poincaré metric of $\Delta \cap U$ where Δ is a small polycylinder centered at x. This means that in local coordinates $(z_1, \ldots, z_n)$ on Δ such that $\Delta \cap D_{\mathrm{red}}$ is given by $z_1 \cdot \ldots \cdot z_r = 0$, ω has the asymptotic form

$$\omega \sim \sum_{i=1}^{r} \frac{i dz_i \wedge d\bar{z}_i}{|z_i|^2 \cdot \log^2 |z_i|^2} + \sum_{i=r+1}^{n} i dz_i \wedge d\bar{z}_i.$$

For the construction of such a metric, see [12, §3].

Let $\mathcal{L}^k(V)_{(2)}$ be the sheaf on X of measurable k-forms σ with values in V such that both σ and $(\nabla + \bar{\partial})\sigma$ (understood in the distributional sense) are locally square-integrable. Let $L^k(V)_{(2)}$ be the vectorspace of k-forms σ on U such that σ and $(\nabla + \bar{\partial})\sigma$ are square-integrable, and

$$H^k(U, V)_{(2)} := H^k(L^{\cdot}(V)_{(2)}, \nabla + \bar{\partial}).$$

Similarly we define $\mathcal{L}^{p,q}(V)_{(2)}$, $L^{p,q}(V)_{(2)}$,

$$H^{p,q}(U, V) := H^q(L^{p,\cdot}(V)_{(2)}, \bar{\partial}),$$

insisting on σ and $\bar{\partial}\sigma$ being square-integrable. Note that

$$L^k(V)_{(2)} = \Gamma(X, \mathcal{L}^k(V)_{(2)}) \quad \text{and} \quad L^{p,q}(V)_{(2)} = \Gamma(X, \mathcal{L}^{p,q}(V)_{(2)})$$

since X is compact.

The reason for introducing L^2-cohomology is the following

(D.3) **Theorem.** *Let U be any complex manifold with a complete Kähler metric, V a unitary local system on U with* $\dim H^k(U, V)_{(2)}$ *finite. Then we have*

$$H^k(U, V)_{(2)} \cong \bigoplus_{p+q=k} H^{p,q}(U, V)_{(2)}$$

and $H^{p,q}(U, V)_{(2)}$ *is conjugate-isomorphic to* $H^{q,p}(U, V^\vee)_{(2)}$.

For a *proof*, see [12, §7]. Note that the Kähler identities $\square_{\nabla + \bar{\partial}} = 2\square_{\nabla} = 2\square_{\bar{\partial}}$ (on C^∞-forms with compact support) follow directly from V being unitary. For the last part, use the "conjugation map" from (p,q)-forms in V to (q,p)-forms in $V^\vee$; clearly it maps harmonic forms to harmonic forms.

The following proposition gives the connection between $H^k(X, j_* V)$ and $H^q(X, \tilde{\Omega}^p(\mathscr{M}))$ and L^2-cohomology, thereby finishing the proof of (D.2).

(D.4) **Proposition.** a) *$j_* V$ is quasiisomorphic to $\mathscr{L}^{\bullet}(V)_{(2)}$, hence*

$$H^k(X, j_* V) \cong H^k(U, V)_{(2)}.$$

b) *$\tilde{\Omega}^p_X(\mathscr{M})$ is quasiisomorphic to $\mathscr{L}^{p,\bullet}(V)_{(2)}$, hence*

$$H^q(X, \tilde{\Omega}^p_X(\mathscr{M})) \cong H^{p,q}(U, V)_{(2)}.$$

Proof. The statement being local and well-known for points of U, we restrict to a small polycylinder $\Delta = \Delta_1 \times \ldots \times \Delta_n$ around $x \in D$ such that $\Delta \cap D_{\text{red}}$ is given by $z_1 \cdot \ldots \cdot z_r = 0$ where z_i is a coordinate on Δ_i. As $V_{|\Delta}$ splits into an orthogonal direct sum of unitary local systems of rank 1, we may assume that V itself has rank 1. Then $\mathscr{M}_{|\Delta} = \mathscr{O}_\Delta$, the connection is given by $\nabla(f) = \partial f + f \cdot \sum_{i=1}^{r} a_i \frac{dz_i}{z_i}$, with a_i a real constant, $0 \leqq a_i < 1$. $\prod_{i=1}^{r} z_i^{-a_i}$ is a (multivalued) section of $V_{|\Delta \cap}$, and the fibre metric is given by

$$\|f\|^2 = \prod_{i=1}^{r} |z_i|^{-2a_i} \cdot |f|^2.$$

For $j_* V = H^0(\mathscr{L}^{\bullet}(V)_{(2)})$, we only have to show that sections of $j_* V$ are square-integrable; this follows from the fact that $\Delta \cap U$ has finite volume with respect to ω.

For $\tilde{\Omega}^p(\mathscr{M}) = H^0(\mathscr{L}^{p,\bullet}(V)_{(2)})$, we have to show that the sections of $\tilde{\Omega}^p(\mathscr{M})$ are exactly the square-integrable sections of $j_*(\Omega^p_{\Delta \cap U} \otimes \mathscr{M}_{|\Delta \cap U})$. This follows directly from

$$\|f dz_I\|^2 = \int |f|^2 \cdot \prod_{\substack{i=1 \\ i \notin I}}^{r} |z_i|^{2a_i} \cdot \Big(\prod_{\substack{i=1 \\ i \in I}}^{r} |z_i|^{2-2a_i} \cdot \log^2 |z_i|^2 \Big)^{-1} dV$$

(where dV is the Euclidean volume element on Δ, $I = (i_1, \ldots, i_p)$ with $i_1 < \ldots < i_p$, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$) and the fact that for square-integrable $f dz_I$, $g dz_J$ one has $(f dz_I, g dz_J) = 0$ if $I \neq J$.

Notice in passing that this characterizes the canonical extension as the sheaf of square-integrable sections of $j_*(\mathscr{O}_U \otimes_{\mathbb{C}} V)$.

Let us now turn to the proof of the exactness of $\mathscr{L}^{p,\bullet}(V)_{(2)}$ at a level $q \geqq 1$:

We will always work in the Hilbert space H_k of square-integrable k-forms on $\Delta \setminus D$, viewing $\bar{\partial}$ and $\nabla + \bar{\partial}$ as densely defined operators. C_0^∞ will denote the subspace of C^∞-forms with compact support in $\Delta \setminus D$.

It suffices to construct for any $i \in \{1, \ldots, n\}$ and any $\alpha \in H_k$, not involving $d\bar{z}_j$ for $j < i$, with $\bar{\partial}\alpha = 0$, some $\beta \in H_{k-1}$ such that $\alpha - \bar{\partial}\beta$ does not involve $d\bar{z}_j$ for $j \leqq i$. - Here and in the following the writing of $\bar{\partial}\alpha$ and $\bar{\partial}\beta$ indicates that α and β are in the domain of definition of $\bar{\partial}$. Let us assume $i \leqq r$, the other case being similar but easier.

By the computations made in A. Andreotti, E. Vesentini: Carleman estimates ..., IHES 25, p. 92, there exists a sequence (α_ν), $\alpha_\nu \in C_0^\infty$, with $\alpha_\nu \to \alpha$ and $\bar{\partial}\alpha_\nu \to 0$ in the L^2-norm on a relatively compact polycylinder in Δ, containing 0.

Introduce polar coordinates $z_j = r_j e^{i\theta_j}$. Denote by $\bar{\partial}_j$ the weak closure of the operator $\sigma \to \frac{1}{2} e^{i\theta_j} d\bar{z}_j \wedge \left(\frac{\partial}{\partial r_j} + \frac{i}{r_j} \frac{\partial}{\partial \theta_j}\right) \sigma$ on C_0^∞. Notice that $\bar{\partial} = \sum_{j=1}^{n} \bar{\partial}_j$ on C_0^∞.

We will define a bounded right inverse G_i to $\bar{\partial}_i$ on all forms involving $d\bar{z}_i$:

First, for $d\bar{z}_i \wedge \sigma \in H_k$, σ not involving dz_i, $d\bar{z}_i$, we look for

$$\tau = G_i(d\bar{z}_i \wedge \sigma) \in H_{k-1} \quad \text{with} \quad \bar{\partial}_i \tau = d\bar{z}_i \wedge \sigma.$$

Write σ and τ as "Fourier series"

$$\sigma = \sum_{n=-\infty}^{+\infty} \sigma_n(r_i) e^{in\theta_i}, \qquad \tau = \sum_{n=-\infty}^{+\infty} \tau_n(r_i) e^{in\theta_i},$$

where in the notation we suppressed the dependence on the other parameters.

Now for $\sigma \in C_0^\infty$, $\bar{\partial}_i \tau = d\bar{z}_i \wedge \sigma$ translates to $\frac{1}{2}\left(\tau_n' - \frac{n}{r}\tau_n\right) = \sigma_{n+1}$, which we solve by

$$\tau_n(r_i) = \begin{cases} 2 \int\limits_0^{r_i} \sigma_{n+1}(\rho) \rho^{-n} d\rho \cdot r_i^n, & \text{for } 2n + 2a_i + 1 \leqq 0 \\ -2 \int\limits_{r_i}^{R_i} \sigma_{n+1}(\rho) \rho^{-n} d\rho \cdot r_i^n, & \text{for } 2n + 2a_i + 1 > 0 \end{cases};$$

here $R_i < 1$ is the radius of Δ_i. Note that

$$\|d\bar{z}_i \wedge \sigma\|^2 = 4\pi \sum_{n=-\infty}^{+\infty} \int\limits_0^{R_i} \|\sigma_n(r_i)\|^2 r_i^{2a_i+1} dr_i,$$

$$\|\tau\|^2 = 2\pi \sum_{n=-\infty}^{+\infty} \int\limits_0^{Ri} \|\tau_n(r_i)\|^2 \frac{r_i^{2a_i-1}}{\log^2 r_i^2} dr_i,$$

where $\|\sigma_n(r_i)\|$ is the L^2-norm of $\sigma_{n|\Delta_1 \times \ldots \times \{r_i e^{i\theta_i}\} \times \ldots \times \Delta_n}$. For $2n + 2a_i + 1 < 0$, we estimate

$$\begin{aligned} \int\limits_0^{R_i} \|\tau_n(r_i)\|^2 \frac{r_i^{2a_i-1}}{\log^2 r_i^2} dr_i &\leqq C' \cdot \int\limits_0^{R_i} \left\| \int\limits_0^{r_i} \sigma_{n+1}(\rho) \rho^{-n} d\rho \right\|^2 r_i^{2n+2a_i-1} dr_i \\ &\leqq C' \cdot \int\limits_0^{R_i} \int\limits_0^{r_i} \|\sigma_{n+1}(\rho)\|^2 \rho^{-2n} d\rho \, r_i^{2n+2a_i} dr_i \\ &= C' \int\limits_0^{R_i} \|\sigma_{n+1}(\rho)\|^2 \rho^{-2n} \left(\int\limits_\rho^{R_i} r_i^{2n+2a_i} dr_i \right) d\rho \\ &\leqq \frac{C'}{|2n+2a_i+1|} \int\limits_0^{R_i} \|\sigma_{n+1}(\rho)\|^2 \rho^{2a_i+1} d\rho. \end{aligned}$$

In the other cases one can get similar estimates, giving

$$\|\tau\| \leqq C \|d\bar{z}_i \wedge \sigma\|.$$

Next, for $d\bar{z}_i \wedge dz_i \wedge \sigma \in H_k$, we look for τ such that $dz_i \wedge \tau \in H_{k-1}$ and $\bar{\partial}_i(dz_i \wedge \tau) = d\bar{z}_i \wedge dz_i \wedge \sigma$. In this case, define the Fourier coefficients of τ by

$$\tau_n(r_i) = \begin{Bmatrix} 2\int\limits_0^{r_i} \sigma_{n+1}(\rho)\rho^{-n} d\rho \cdot r_i^n, & \text{for } 2n+2a_i+3 \leqq 0 \\ -2\int\limits_{r_i}^{R_i} \sigma_{n+1}(\rho)\rho^{-n} d\rho \cdot r_i^n, & \text{for } 2n+2a_i+3 > 0 \end{Bmatrix}.$$

As above, we get $\|dz_i \wedge \tau\| \leqq C\|d\bar{z}_i \wedge dz_i \wedge \sigma\|$.

Finally defining $G_i(\sigma)=0$ for σ not involving $d\bar{z}_i$, we get a bounded operator G_i on H_k with $\bar{\partial}_i G_i(d\bar{z}_i \wedge \sigma) = d\bar{z}_i \wedge \sigma$. It is clear that $\bar{\partial}_j G_i = G_i \bar{\partial}_j$ on C_0^∞ for $j \neq i$, whereas $(G_i\bar{\partial}_i - \bar{\partial}_i G_i)\sigma = \sigma$ for $\sigma \in C_0^\infty$ not involving $d\bar{z}_i$; especially, $\bar{\partial}G_i - G_i\bar{\partial}$ is bounded on C_0^∞.

Now we are done: let $\alpha_\nu \in C_0^\infty$, $\alpha_\nu \to \alpha$, $\bar{\partial}\alpha_\nu \to 0$ (on a smaller neighbourhood of 0). Then $G_i\alpha_\nu$ and $\bar{\partial}G_i\alpha_\nu = (\bar{\partial}G_i - G_i\bar{\partial})\alpha_\nu + G_i\bar{\partial}\alpha_\nu$ converge, so $G_i\alpha$ is in the domain of definition of $\bar{\partial}$. Furthermore, $\bar{\partial}_j\alpha_\nu \to 0$ for $j<i$. Hence $G_i\alpha$ is in the domain of definition of $\bar{\partial}_j$ with $\bar{\partial}_j G_i\alpha = 0$, for $j<i$. But then $\alpha - \bar{\partial}G_i\alpha$ does not involve dz_j, $d\bar{z}_j$ for $j \leqq i$.

At last we have to show the exactness of $\mathscr{L}^\cdot(V)_{(2)}$ at a level $k \geqq 1$. We show that for $i \in \{1,\dots,n\}$, $\alpha \in H_k$ not involving dz_j, $d\bar{z}_j$ for $j<i$, with $(\nabla + \bar{\partial})\alpha = 0$, there exists $\beta \in H_{k-1}$ such that $\alpha - (\nabla + \bar{\partial})\beta$ does not involve dz_j, $d\bar{z}_j$ for $j \leqq i$. – Let us again only consider the case $i \leqq r$. Denote by $(\nabla + \bar{\partial})_j$ the weak closure of the operator

$$\sigma \to dz_j \wedge \left(\frac{\partial}{\partial z_j} + \frac{a_j}{z_j}\right)\sigma + d\bar{z}_j \wedge \frac{\partial \sigma}{\partial \bar{z}_j} = dr_j \wedge \left(\frac{\partial}{\partial r_j} + \frac{a_j}{r_j}\right)\sigma + d\theta_j \wedge \left(\frac{\partial}{\partial \theta_j} + ia_j\right)\sigma$$

on C_0^∞. Then $\nabla + \bar{\partial} = \sum_{j=1}^{n} (\nabla + \bar{\partial})_j$ on C_0^∞.

Since $(\nabla + \bar{\partial})_i = \bar{\partial}_i$ on all forms of the type $dz_i \wedge \sigma$, we can use the above analysis to remove all terms involving $dz_i \wedge d\bar{z}_i$ in α. Hence, now using polar coordinates, we proceed to the case $\alpha = dr_i \wedge \alpha' + d\theta_i \wedge \alpha'' + \alpha'''$ with α', α'', α''' not involving dr_j, $d\theta_j$ for $j \leqq i$.

If $a_i > 0$, then for a form $d\theta_i \wedge \sigma \in H_k$, σ not involving dr_i, $d\theta_i$,

$$\sigma = \sum_{n=-\infty}^{+\infty} \sigma_n e^{in\theta_i},$$

we define $G_i'(d\theta_i \wedge \sigma) = \tau$ where τ has Fourier coefficients $\tau_n = \dfrac{\sigma_n}{i(n+a_i)}$. – For $dr_i \wedge d\theta_i \wedge \sigma \in H_k$, we define $G_i'(dr_i \wedge d\theta_i \wedge \sigma) = dr_i \wedge \tau$ where τ has the same Fourier coefficients as above. On forms not involving $d\theta_i$, $G_i' \equiv 0$.

If $a_i = 0$, then we replace τ_n by

$$\tau_n = \begin{Bmatrix} \dfrac{\sigma_n}{in}, & \text{for } n \neq 0 \\ 0, \text{ for } n=0 & \end{Bmatrix}.$$

Furthermore, $G_i'(dr_i \wedge \sigma) = -\int_{r_i}^{R_i} \sigma_0(\rho)\,d\rho$, which makes sense since

$$\int_0^{R_i} \left\| \int_{r_i}^{R_i} \sigma_0(\rho)\,d\rho \right\|^2 \frac{dr_i}{r_i \log^2 r_i^2} \leqq \int_0^{R_i} \int_{r_i}^{R_i} \|\sigma_0(\rho)\|^2 \frac{\rho\,d\rho}{|\log\rho|^{\frac{1}{2}}} \cdot \int_{r_i}^{R_i} \frac{|\log\rho|^{\frac{1}{2}}}{\rho}\,d\rho \cdot \frac{dr_i}{r_i \log^2 r_i^2}$$

$$\leqq \frac{1}{6} \int_0^{R_i} \int_{r_i}^{R_i} \|\sigma_0(\rho)\|^2 \frac{\rho\,d\rho}{|\log\rho|^{\frac{1}{2}}} \frac{dr_i}{r_i |\log r_i|^{\frac{1}{2}}}$$

$$= \frac{1}{6} \int_0^{R_i} \|\sigma_0(\rho)\|^2 \frac{\rho}{|\log\rho|^{\frac{1}{2}}} \cdot \int_0^{\rho} \frac{dr_i}{r_i |\log r_i|^{\frac{1}{2}}}\,d\rho$$

$$= \frac{1}{3} \int_0^{R_i} \|\sigma_0(\rho)\|^2 \rho\,d\rho.$$

On forms not involving dr_i, $d\theta_i$, $G_i' \equiv 0$.

In either case, one easily checks that G_i' is bounded. As before,

$$(\nabla + \bar\partial)_j G_i' = G_i'(\nabla + \bar\partial)_j \quad \text{for } j \neq i, \quad \text{and} \quad (\nabla + \bar\partial)_i G_i' - G_i'(\nabla + \bar\partial)_i$$

is bounded on C_0^∞. Furthermore the definition of G_i' was made in order to get

$$\|(\nabla + \bar\partial)_i G_i' \sigma - \sigma\| \leqq C \|(\nabla + \bar\partial)_i \sigma\|$$

on forms $\sigma \in C_0^\infty$ not involving $dr_i \wedge d\theta_i$.

Now let $\alpha_\nu \in C_0^\infty$, $\alpha_\nu \to \alpha$, $(\nabla + \bar\partial)\alpha_\nu \to 0$ (again one has to shrink Δ to get convergence). We may assume that $\alpha_\nu = dr_i \wedge \alpha_\nu' + d\theta_i \wedge \alpha_\nu'' + \alpha_\nu'''$, with α_ν', α_ν'', α_ν''' not involving dr_i, $d\theta_i$. Then $(\nabla + \bar\partial)_i(dr_i \wedge \alpha_\nu' + d\theta_i \wedge \alpha_\nu'') \to 0$, and so $(\nabla + \bar\partial)_i G_i' \alpha_\nu \to dr_i \wedge \alpha''$. As $(\nabla + \bar\partial) G_i' \alpha_\nu$ converges, $G_i'\alpha$ is in the domain of definition of $\nabla + \bar\partial$ and $\alpha - (\nabla + \bar\partial) G_i' \alpha$ does not involve dr_j, $d\theta_j$ for $j \leqq i$, q.e.d.

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