A FINITENESS THEOREM FOR GALOIS REPRESENTATIONS OF FUNCTION FIELDS OVER FINITE FIELDS (AFTER DELIGNE)

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ABSTRACT. We give a detailed account of Deligne's letter [13] to Drinfeld dated June 18, 2011, in which he shows that there are finitely many irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves with bounded ramification, up to isomorphism and up to twist, on a smooth variety defined over a finite field. The proof relies on Lafforgue's Langlands correspondence over curves [27]. In addition, Deligne shows the existence of affine moduli of finite type over $\mathbb{Q}$. A corollary of Deligne's finiteness theorem is the existence of a number field which contains all traces of the Frobenii at closed points, which was the main result of [12] and which answers positively his own conjecture [9, Conj. 1.2.10 (ii)].

1. INTRODUCTION

In Weil II [9, Conj. 1.2.10] Deligne conjectured that if X is a normal connected scheme of finite type over a finite field of characteristic p, and V is an irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaf of rank r, with finite determinant, then

- (i) V has weight 0,
- (ii) there is a number field $E(V) \subset \bar{\mathbb{Q}}_\ell$ containing all the coefficients of the local characteristic polynomials $\det(1 - tF_x|_{V_x})$, where x runs through the closed points of X and F_x is the geometric Frobenius at the point x,
- (iii) V admits ℓ'-companions for all prime numbers $\ell' \neq p$.

As an application of his Langlands correspondence for GL_r, Lafforgue [27] proved (i), (ii), (iii) for X a smooth curve, out of which one deduces (i) in general. Using Lafforgue's results, Deligne showed (ii) in [12]. Using (ii) and ideas of Wiesend, Drinfeld [15] showed (iii) assuming

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in addition X to be smooth. A slightly more elementary variant of Deligne's argument for (ii) was given in [18].

Those conjectures were formulated with the hope that a more motivic statement could be true, which would say that those lisse sheaves come from geometry. On the other hand, over smooth varieties over the field of complex numbers, Deligne in [11] showed finiteness of $\mathbb{Q}$-summands of polarized variations of pure Hodge structures over $\mathbb{Z}$ of bounded rank, a theorem which, in weight one, is due to Faltings [19]. Those are always regular singular, while lisse $\bar{\mathbb{Q}}_\ell$-sheaves are not necessarily tame. However, any lisse sheaf has bounded ramification (see Proposition 3.9 for details). Furthermore, one may twist a lisse $\bar{\mathbb{Q}}_\ell$-sheaf by a character coming from the ground field. Thus it is natural to expect:

Theorem 1.1 (Deligne). *There are only finitely many irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves of given rank up to twist on X with suitably bounded ramification at infinity.*

Deligne shows this theorem in [13] by extending his arguments from [12]. A precise formulation is given in Theorem 2.1 based on the ramification theory explained in Section 3.3.

Our aim in this note is to give a detailed account of Deligne's proof of this finiteness theorem for lisse $\bar{\mathbb{Q}}_\ell$-sheaves and consequently of his proof of (ii). For some remarks on the difference between our method and Deligne's original argument for proving (ii) in [12] see Section 8.2.

In fact Deligne shows a stronger finiteness theorem which comprises finiteness of the number of what we call *2-skeleton sheaves* [1] on X. A 2-skeleton sheaf consists of an isomorphism class of a semi-simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf on every smooth curve mapping to X, which are assumed to be compatible in a suitable sense. These 2-skeleton sheaves were first studied by Drinfeld [15]. His main theorem roughly says that if a 2-skeleton sheaf is tame at infinity along each curve then it comes from a lisse sheaf on X, extending the rank one case treated in [35], [36]. Deligne suggests that a more general statement should be true:

Question 1.2. Does any 2-skeleton sheaf with bounded ramification come from a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X?

For a precise formulation of the question see Question 2.3. The answer to this question is not even known for rank one sheaves, in which case the problem has been suggested already earlier in higher dimensional class field theory. On the other hand Deligne's finiteness

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[1]we thank Lars Kindler for suggesting this terminology

for 2-skeleton sheaves has interesting consequences for relative Chow groups of 0-cycles over finite fields, see Section 8.1.

Some comments on the proof of the finiteness theorem: Deligne uses in a crucial way his key theorem [12, Prop. 2.5] on curves asserting that a semi-simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf is uniquely determined by its characteristic polynomials of the Frobenii at all closed points of some explicitly bounded degree, see Theorem 5.1. This enables him to construct a coarse moduli space of 2-skeleton sheaves $L_r(X, D)$ as an affine scheme of finite type over $\mathbb{Q}$, such that its $\bar{\mathbb{Q}}_\ell$-points correspond to the 2-skeleton sheaves of rank r and bounded ramification by the given divisor D at infinity.

We simplify Deligne's construction of the moduli space slightly. Our method yields less information on the resulting moduli, yet it is enough to deduce the finiteness theorem. In fact finiteness is seen by showing that irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves up to twist are in bijection with (some of) the one-dimensional irreducible components of the moduli space (Corollary 7.2).

We give some applications of Deligne's finiteness theorem in Section 8.

Firstly, it implies the existence of a number field $E(V)$ as in (ii) above, see Theorem 8.2. This number field is in fact stable by an ample hyperplane section if X is projective, see Proposition 7.4.

Secondly, as mentioned above the degree zero part of the relative Chow group of 0-cycles with bounded modulus is finite (Theorem 8.1).

Deligne addresses the question of the number of irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaves with bounded ramification. In [14] some concrete examples on the projective line minus a divisor of degree ≤ 4 are computed. In Section 9 we formulate Deligne's qualitative conjecture. This formulation rests on emails he sent us and on his lecture in June 2012 in Orsay on the occasion of the Laumon conference.

Acknowledgment: Our note gives an account of the 9 dense pages written by Deligne to Drinfeld [13]. They rely on [12] and [15] and contain a completely new idea of great beauty, to the effect of showing finiteness by constructing moduli of finite type and equating the classes of the sheaves one wants to count with some of the irreducible components. We thank Pierre Deligne for his willingness to read our note and for his many enlightening comments.

Parts of the present note are taken from our seminar note [18]. They grew out of discussions at the Forschungsseminar at Essen during summer 2011. We thank all participants of the seminar.

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2. The finiteness theorem and some consequences

2.1. **Deligne's finiteness theorem for sheaves.** We begin by formulating a version of Deligne's finiteness theorem for ℓ-adic Galois representations of functions fields. Later in this section we introduce the notion of a 2-skeleton ℓ-adic representation, which is necessary in order to state a stronger form of Deligne's finiteness result.

Let $\mathrm{Sm}_{\mathbb{F}_q}$ be the category of smooth separated schemes $X/\mathbb{F}_q$ of finite type over the finite field $\mathbb{F}_q$. We fix once for all an algebraic closure $\mathbb{F} \supset \mathbb{F}_q$. To $X \in \mathrm{Sm}_{\mathbb{F}_q}$ connected one associates functorially the *Weil group* $W(X)$ [9, 1.1.7], a topological group, well-defined up to an inner automorphism by $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F})$ when X is geometrically connected over $\mathbb{F}_q$. If so, then it sits in an exact sequence

$$0 \to \pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}) \to W(X) \to W(\mathbb{F}_q) \to 0.$$

There is a canonical identification $W(\mathbb{F}_q) = \mathbb{Z}$.

We fix a prime number ℓ with $(\ell, q) = 1$. Let $\mathcal{R}_r(X)$ be the set of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves on X of dimension r up to isomorphism and up to semi-simplification. For X connected, a lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaf on X is the same as a continuous representations $W(X) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$. As we do not want to talk about a topology on $\bar{\mathbb{Q}}_\ell$ we define the latter continuous representations ad hoc as the homomorphisms which factor through a continuous homomorphism $W(X) \to \mathrm{GL}_r(E)$ for some finite extension E of $\mathbb{Q}_\ell$, see [9, (1.1.6)].

The weaker form of the finiteness theorem says that the number of classes of irreducible sheaves in $\mathcal{R}_r(X)$ with bounded wild ramification is finite up to twist. Let us give some more details. Let $X \subset \bar{X}$ be a normal compactification of the connected scheme X such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. In Section 3.3 we will define a subset $\mathcal{R}_r(X, D)$ of representations whose Swan conductor along any smooth curve mapping to $\bar{X}$ is bounded by the pullback of D to the completed curve. We show that for any $V \in \mathcal{R}_r(X)$ there is a divisor D with $V \in \mathcal{R}_r(X, D)$, see Proposition 3.9.

For $V \in \mathcal{R}_r(X, D)$ we have the notion of twist $\chi \cdot V$ by an element $\chi \in \mathcal{R}_1(\mathbb{F}_q)$.

Theorem 2.1 (Deligne). *Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be connected and $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. The set of irreducible sheaves $V \in \mathcal{R}_r(X, D)$ is finite up to twist by elements of $\mathcal{R}_1(\mathbb{F}_q)$.*

In particular the theorem implies that for any integer $N > 0$ there are only finitely many irreducible $V \in \mathcal{R}_r(X, D)$ with $\det(V)^{\otimes N} = 1$. Theorem 2.1 is a consequence of the stronger Finiteness Theorem 2.4.

Remark 2.2. Any irreducible lisse Weil sheaf on X is a twist of an étale sheaf, Proposition 4.3. So the theorem could also be stated with étale sheaves instead of Weil sheaves.

2.2. **Existence problem and finiteness theorem for 2-skeleton sheaves.** By $\mathrm{Cu}(X)$ we denote the set of normalizations of closed integral subschemes of X of dimension one.

We say that a family $(V_C)_{C \in \mathrm{Cu}(X)}$ with $V_C \in \mathcal{R}_r(C)$ is *compatible* if for all pairs (C, C') we have

$$V_C|_{(C \times_X C')_{\mathrm{red}}} = V_{C'}|_{(C \times_X C')_{\mathrm{red}}} \in \mathcal{R}_r((C \times_X C')_{\mathrm{red}}).$$

We write $\mathcal{V}_r(X)$ for the set of compatible families – also called *2-skeleton sheaves*.

It is not difficult to see that the canonical map $\mathcal{R}_r(X) \to \mathcal{V}_r(X)$ is injective, Proposition 4.1. One might ask, what the image of $\mathcal{R}_r(X)$ in $\mathcal{V}_r(X)$ is.

With the notation as above we can also define the set $\mathcal{V}_r(X, D)$ of 2-skeleton sheaves with bounded wild ramification, see Definition 3.6. Deligne expresses the hope that the following question about existence of ℓ-adic sheaves might have a positive answer.

Question 2.3. Is the map $\mathcal{R}_r(X, D) \to \mathcal{V}_r(X, D)$ bijective for any Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$?

To motivate the question one should think of the set of curves $\mathrm{Cu}(X)$ together with the systems of intersections of curves as the 2-skeleton of X. To be more precise, the analogy is as follows: For a CW-complex S let $S_{\leq d}$ be the union of i-cells of S ($i \leq d$), i.e. its d-skeleton. Assume that $S_{\leq 0}$ consists of just one point.

CW-complex S (with $S_{\leq 0} = \{*\}$)	Variety $X/\mathbb{F}_q$
1-sphere S^1 with topological fundamental group $\pi_1(S^1) = \mathbb{Z}$	Finite field $\mathbb{F}_q$ with Weil group $W(\mathbb{F}_q) = \mathbb{Z}$
S^1-bouquet $S_{\leq 1}$	Set of closed points $|X|$
2-cell in S	Curve in $\mathrm{Cu}(X)$
Relation in $\pi_1(S)$ coming from 2-cell	Reciprocity law on curve
2-skeleton $S_{\leq 2}$	System of curves $\mathrm{Cu}(X)$
Local system on S	Lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaf on X

In the sense of this analogy, Deligne's Question 2.3 is the analog of the fact that the fundamental groups of S and $S_{\leq 2}$ are the same [23, Thm. 4.23], except that we consider only the information contained in ℓ-adic representations, in addition only modulo semi-simplification, and that there is no analog of wild ramification over CW-complexes.

For $D = 0$ a positive answer to Deligne's question is given by Drinfeld [15, Thm 2.5]. His proof uses a method developed by Wiesend [36] to reduce the problem to Lafforgue's theorem. For $r = 1$ and $D = 0$ it was first shown by Schmidt–Spiess [35] using motivic cohomology, and later by Wiesend [37] using more elementary methods.

The stronger form of Deligne's finiteness theorem says that Theorem 2.1 remains true for 2-skeleton sheaves. We say that a *2-skeleton sheaf* $V \in \mathcal{V}_r(X)$ *on a connected scheme* X is *irreducible* if it cannot be written in the from $V_1 \oplus V_2$ with $V_i \in \mathcal{V}_{r_i}(X)$ and $r_1, r_2 > 0$. In Appendix B, Proposition B.1, we give a proof of the well known fact that a sheaf $V \in \mathcal{R}_r(X)$ is irreducible if and only if its image in $\mathcal{V}_r(X)$ is irreducible.

The main result of this note now says:

Theorem 2.4 (Deligne). *Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be connected and $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor supported in $\bar{X} \setminus X$. The set of irreducible 2-skeleton sheaves $V \in \mathcal{V}_r(X, D)$ is finite up to twist by elements from $\mathcal{R}_1(\mathbb{F}_q)$. Its cardinality does not depend on $\ell \neq p$.*

The theorem implies in particular that for a given integer $N > 0$ there are only finitely many $V \in \mathcal{V}_r(X, D)$ with $\det(V)^{\otimes N} = 1$. Following Deligne we will reduce the theorem to the one-dimensional case,

where it is a well known consequence of the Langlands correspondence of Drinfeld–Lafforgue. Some hints how the one-dimensional case is related to the theory of automorphic forms are given in Section 4.3. The proof of Theorem 2.4 is completed in Section 7.

Idea of proof. The central idea of Deligne is to define an algebraic moduli space structure on the set $\mathcal{V}_r(X, D)$, such that it becomes an affine scheme of finite type over $\mathbb{Q}$. In fact $\mathcal{V}_r(X, D)$ will be the $\bar{\mathbb{Q}}_\ell$-points of this moduli space. One shows that the irreducible components of the moduli space over $\bar{\mathbb{Q}}_\ell$ are 'generated' by certain twists of 2-skeleton sheaves, which implies the finiteness theorem, because there are only finitely many irreducible components.

Firstly, one constructs the moduli space structure of finite type over $\mathbb{Q}$ for $\dim(X) = 1$. Then one immediately gets an algebraic structure on $\mathcal{V}_r(X, D)$ in the higher dimensional case and the central point is to show that $\mathcal{V}_r(X, D)$ is of finite type over $\mathbb{Q}$ for higher dimensional X too.

The main method to show the finite type property is a result of Deligne (Theorem 5.1), relying on Weil II and the Langlands correspondence, which says that for one-dimensional X there is a natural number N depending logarithmically on the genus of $\bar{X}$ and the degree of D such that $V \in \mathcal{V}_r(X, D)$ is determined by the polynomials $f_V(x)$ with $\deg(x) \leq N$. Here for $V \in \mathcal{V}_r(X, D)$ we denote by $f_V(x)$ the characteristic polynomial of the Frobenius at the closed point $x \in |X|$, see Section 4.1 for a precise definition.

3. Ramification theory

In this section we review some facts from ramification theory. We work over the finite field $\mathbb{F}_q$. In fact all results remain true over a perfect base field of positive characteristic and for lisse étale ℓ-adic sheaves.

3.1. **Local ramification.** We follow [28, Sec. 2.2]. Let K be a complete discretely valued field with perfect residue field of characteristic $p > 0$. Let $G = \mathrm{Gal}(\bar{K}/K)$, where $\bar{K}$ is a separable closure of K. There is a descending filtration $(I^{(\lambda)})_{0 \leq \lambda \in \mathbb{R}}$ by closed normal subgroups of G with the following properties:

- $\bigcap_{\lambda' < \lambda} I^{(\lambda')} = I^{(\lambda)}$,
- $\bigcap_{\lambda \in \mathbb{R}} I^{(\lambda)} = 0$,
- $I^{(0+)}$ is the unique maximal pro-p subgroup of the inertia group $I^{(0)}$, where $I^{(\lambda+)}$ is defined as $\overline{\bigcup_{\lambda' > \lambda} I^{(\lambda')}}$.

Let $G \to \mathrm{GL}(V)$ be a continuous representation on a finite dimensional $\bar{\mathbb{Q}}_\ell$-vector space V with $\ell \neq p$.

Definition 3.1. The *Swan conductor* of V is defined as

$$\text{Sw}(V) = \sum_{\lambda > 0} \lambda \dim(V^{I^{(\lambda+)}}/V^{I^{(\lambda)}}).$$

The Swan conductor is additive with respect to extensions of ℓ-adic Galois representations, it does not change if we replace V by its semi-simplification.

For later reference we recall the behavior of the Swan conductor with respect to direct sum and tensor product. If V, V' are two $\bar{\mathbb{Q}}_\ell$-G-modules as above and $V^\vee$ denotes the dual representation, then

$$(3.1) \qquad\qquad \text{Sw}(V \oplus V') = \text{Sw}(V) + \text{Sw}(V')$$

$$(3.2) \qquad\qquad \frac{\text{Sw}(V \otimes V')}{\text{rank}(V)\text{rank}(V')} \leq \frac{\text{Sw}(V)}{\text{rank}(V)} + \frac{\text{Sw}(V')}{\text{rank}(V')}$$

$$(3.3) \qquad\qquad \text{Sw}(V^\vee) = \text{Sw}(V)$$

3.2. **Global ramification** (dim = 1). Let $X/\mathbb{F}_q$ be a smooth connected curve with smooth compactification $X \subset \bar{X}$. Let V be in $\mathcal{R}_r(X)$.

The *Swan conductor* $\text{Sw}(V)$ is defined to be the effective Cartier divisor

$$\sum_{x \in |\bar{X}|} \text{Sw}_x(V) \cdot [x] \in \text{Div}^+(\bar{X}).$$

Here $\text{Sw}_x(V)$ is the Swan conductor of the restriction of the representation class V to the complete local field $\text{frac}(\widehat{\mathcal{O}_{\bar{X},x}})$. We say that V is *tame* if $\text{Sw}(V) = 0$.

Clearly the Swan conductor of V is the same as the Swan conductor of any twist $\chi \cdot V$, $\chi \in \mathcal{R}_1(\mathbb{F}_q)$.

Let $\phi : X' \to X$ be an étale covering of smooth curves with compactification $\bar{\phi} : \bar{X}' \to \bar{X}$. By $\text{D}_{\bar{X}'/\bar{X}} \in \text{Div}^+(\bar{X})$ we denote the discriminant [32] of $\bar{X}'$ over $\bar{X}$, cf. Section 3.3.

Lemma 3.2 (Conductor-discriminant-formula). *For $V \in \mathcal{R}_r(X)$ with $\phi^*(V)$ tame the inequality of divisors*

$$\text{Sw}(V) \leq \text{rank}(V)\,\text{D}_{\bar{X}'/\bar{X}}$$

holds on $\bar{X}$.

Proof. By abuse of notation we write V also for a sheaf representing V. There is an injective map of sheaves on X

$$V \to \phi_* \circ \phi^*(V)$$

For any $x \in |X|$

$$\mathrm{Sw}_x(V) \leq \mathrm{Sw}_x(\phi_* \circ \phi^*(V)) \leq \mathrm{rank}(V)\,\mathrm{mult}_x(\mathrm{D}_{\bar{X}'/\bar{X}}).$$

The second inequality follows from [30, Prop. 1(c)]. $\square$

Definition 3.3. Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor. The subset $\mathcal{R}_r(X, D) \subset \mathcal{R}_r(X)$ is defined by the condition $\mathrm{Sw}(V) \leq D$. If $V \in \mathcal{R}(X)$ lies in $\mathcal{R}_r(X, D)$, we say that its *ramification is bounded by* D.

Let $\mathbb{F}_{q^n}$ be the algebraic closure of $\mathbb{F}_q$ in $k(X)$.

Definition 3.4. For a divisor $D \in \mathrm{Div}^+(\bar{X})$ we define the *complexity* of D to be

$$\mathcal{C}_D = 2g(\bar{X}) + 2\deg_{\mathbb{F}_{q^n}}(D) + 1,$$

where $g(\bar{X})$ is the genus of $\bar{X}$ over $\mathbb{F}_{q^n}$ and $\deg_{\mathbb{F}_{q^n}}$ is the degree over $\mathbb{F}_{q^n}$.

Proposition 3.5. *Assume $X/\mathbb{F}_q$ is geometrically connected. For $D \in \mathrm{Div}^+(\bar{X})$ with $\mathrm{supp}(D) = \bar{X} \setminus X$ and for $V \in \mathcal{R}_r(X, rD)$, the inequality*

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^0(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) + \dim_{\bar{\mathbb{Q}}_\ell} H_c^1(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) \leq \mathrm{rank}(V)\,\mathcal{C}_D$$

holds.

Proof. Grothendieck-Ogg-Shafarevich theorem says that

$$\chi_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) = (2 - 2g(\bar{X}))\,\mathrm{rank}(V) - \sum_{x \in \bar{X} \setminus X} (\mathrm{rank}(V) + s_x(V)),$$

see [28, Théorème 2.2.1.2]. Furthermore

$$\dim H_c^0(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) \leq r \text{ and}$$
$$\dim H_c^2(X \otimes_{\mathbb{F}_q} \mathbb{F}, V) = \dim H^0(X \otimes_{\mathbb{F}_q} \mathbb{F}, V^\vee) \leq r.$$

$\square$

3.3. **Global ramification** $(\dim \geq 1)$. We follow an idea of Alexander Schmidt for the definition of the discriminant for higher dimensional schemes.

Let X be a connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$. Let $X \subset \bar{X}$ be a normal compactification of X over k such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. Clearly, such a compactification always exists.

Let $\mathrm{Cu}(X)$ be the set of normalizations of closed integral subschemes of X of dimension one. For C in $\mathrm{Cu}(X)$ denote by $\phi : C \to X$ the

natural morphism. By $\bar{C}$ we denote the smooth compactification of C over $\mathbb{F}_q$ and by $\bar{\phi} : \bar{C} \to \bar{X}$ we denote the canonical extension.

Recall that in Section 2 we introduced the set of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves $\mathcal{R}_r(X)$ and of 2-skeleton sheaves $\mathcal{V}_r(X)$ on X of rank r.

Definition 3.6. For $V \in \mathcal{R}_r(X)$ or $V \in \mathcal{V}_r(X)$ and $D \in \mathrm{Div}^+(\bar{X})$ an effective Cartier with support in $\bar{X} \setminus X$ we (formally) write $\mathrm{Sw}(V) \leq D$ and say that the *ramification of V is bounded by D* if for every curve $C \subset \mathrm{Cu}(X)$ we have
$$\mathrm{Sw}(\phi^*(V)) \leq \bar{\phi}^*(D)$$
in the sense of Section 3.2. The subsets $\mathcal{R}_r(X, D) \subset \mathcal{R}_r(X)$ and $\mathcal{V}_r(X, D) \subset \mathcal{V}_r(X)$ are defined by the condition $\mathrm{Sw}(V) \leq D$.

In the rest of this section we show that for any $V \in \mathcal{R}_r(X)$ there is an effective divisor D with $\mathrm{Sw}(V) \leq D$.

Let $\psi : X' \to X$ be an étale covering (thus finite) and let $\bar{\psi} : \bar{X}' \to \bar{X}$ be the finite, normal extension of X' over $\bar{X}$.

Definition 3.7 (A. Schmidt). The *discriminant* $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})$ is the coherent ideal sheaf in $\mathcal{O}_{\bar{X}}$ locally generated by all elements
$$\det(\mathrm{Tr}_{K'/K}(x_i\,x_j))_{i,j}$$
where $x_1, \ldots, x_n \in \psi_*(\mathcal{O}_{\bar{X}'})$ are local sections restricting to a basis of K' over K. Here $K = k(X)$ and $K' = k(X')$.

Clearly, $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})|_X = \mathcal{O}_X$. This definition extends the classical definition for curves [32], in which case $\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}}) = \mathcal{O}_{\bar{X}}(-\mathrm{D}_{\bar{X}'/\bar{X}})$, where $X \subset \bar{X}$ and $X' \subset \bar{X}'$ are the smooth compactifications.

The following lemma is easy to show.

Lemma 3.8 (Semi-continuity). *In the situation of Definition 3.7 let $\bar{\phi} : \bar{C} \to \bar{X}$ be a smooth curve mapping to $\bar{X}$ with $C = \bar{\phi}^{-1}(X)$ nonempty. Let C' be a connected component of $C \times_X X'$ and let $C' \hookrightarrow \bar{C}'$ be the smooth compactification. Then*
$$\bar{\phi}^{-1}(\mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})) \subset \mathcal{O}_{\bar{C}}(-\mathrm{D}_{\bar{C}'/\bar{C}}).$$

Proposition 3.9. *For $V \in \mathcal{R}_r(X)$ there is an effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ such that $\mathrm{Sw}(V) \leq D$.*

Proof. By Remark 2.2 we can assume that V is an étale sheaf on X. Then there is a local field $E \subset \bar{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_E$ such that V comes form an ℓ-adic $\mathcal{O}_E$-sheaf V_1. Let $\hat{E}$ be the finite residue field of $\mathcal{O}_E$. There is a connected étale covering $\psi : X' \to X$

such that $\psi^*(V_1 \otimes_{\mathcal{O}_E} \hat{E})$ is trivial. This implies that $\psi^*(V)$ is tame. Let $D_1 \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$ such that $\mathcal{O}_{\bar{X}}(-D_1) \subset \mathcal{I}(\mathrm{D}_{\bar{X}'/\bar{X}})$ and set $D = \mathrm{rank}(V)D_1$. With the notation of Lemma 3.8 we obtain

$$\bar{\phi}^*(D_1) \geq \mathrm{D}_{\bar{C}'/\bar{C}}$$

As the pullback of V to C' is tame we obtain from Lemma 3.2 the first inequality in

$$\mathrm{Sw}(\phi^*(V)) \leq \mathrm{rank}(V)\mathrm{D}_{\bar{C}'/\bar{C}} \leq \bar{\phi}^*(D).$$

$\square$

Remark 3.10. We do not know any example for a $V \in \mathcal{V}_r(X)$ for which there does *not* exist a divisor D with $\mathrm{Sw}(V) \leq D$. If such an example existed, it would in particular show, in view of Proposition 3.9, that not all 2-skeleton sheaves are actual sheaves.

We conclude this section by a remark on the relation of our ramification theory with the theory of Abbes-Saito [4]. We expect that for $V \in \mathcal{R}_r(X)$, $\mathrm{Sw}(V) \leq D$ is equivalent to the following: For every open immersion $X \subset X_1$ over $\mathbb{F}_q$ with the property that $X_1 \setminus X$ is a simple normal crossing divisor and for any morphism $h : X_1 \to \bar{X}$, the Abbes-Saito log-ramification Swan conductor of $h^*(V)$ at a maximal point of $X_1 \setminus X$ is $\leq$ the multiplicity of $h^*(D)$ at the maximal point.

For $D = 0$ this equivalence is shown in [26] relying on [36]. For $r = 1$ it is known modulo resolution of singularities by work of I. Barrientos (forthcoming Ph.D. thesis, Universität Regensburg).

4. ℓ-ADIC SHEAVES

4.1. **Basics.** For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ we defined in Section 2 the set $\mathcal{R}_r(X)$ of lisse $\bar{\mathbb{Q}}_\ell$-Weil sheaves on X of rank r up to isomorphism and up to semi-simplification and the set $\mathcal{V}_r(X)$ of 2-skeleton sheaves. Clearly, $\mathcal{R}_r$ and $\mathcal{V}_r$ form contravariant functors from $\mathrm{Sm}_{\mathbb{F}_q}$ to the category of sets.

For $V \in \mathcal{R}_r(X)$, taking the characteristic polynomials of Frobenius defines a function

$$f_V : |X| \to \bar{\mathbb{Q}}_\ell[t], \quad f_V(x) = \det(1 - t\, F_x, V_{\bar{x}}).$$

For $V \in \mathcal{V}_r(X)$ we can still define $f_V(x)$ by choosing a curve $C \in \mathrm{Cu}(X)$ such that $C \to X$ is a closed immersion in a neighborhood of x and we set $f_V(x) = f_{V_C}(x)$. It follows from the definition that $f_V(x)$ does not depend on the choice of C.

We define the trace

$$t_V^n : X(\mathbb{F}_{q^n}) \to \bar{\mathbb{Q}}_\ell, \quad t_V^n(x) = \mathrm{tr}(F_x, V_{\bar{x}})$$

for $V \in \mathcal{R}_r(X)$ and similarly for $V \in \mathcal{V}_r(X)$.

We define $\mathcal{P}_r$ to be the affine scheme over $\mathbb{Q}$ whose points $\mathcal{P}_r(A)$ with values in a $\mathbb{Q}$-algebra A consist of the set of polynomials $1 + a_1 t + \cdots + a_r t^r \in A[t]$ with $a_r \in A^\times$. Mapping $(\alpha_i)_{1 \le i \le r}$ with $\alpha_i \in A^\times$ to

$$(1 - \alpha_1 t) \cdots (1 - \alpha_r t) \in A[t]$$

defines a scheme isomorphism

$$(4.1) \qquad\qquad \mathbb{G}_m^r / S_r \xrightarrow{\simeq} \mathcal{P}_r,$$

where S_r is the permutation group of r elements.

For $d \ge 1$ the finite morphism $\mathbb{G}_m^r \to \mathbb{G}_m^r$ which sends $(\alpha_1, \ldots, \alpha_r)$ to $(\alpha_1^d, \ldots, \alpha_r^d)$ descends to $\mathcal{P}_r$ to define the finite scheme homomorphism $\psi_d : \mathcal{P}_r \to \mathcal{P}_r$.

Let $\mathcal{L}_r(X)$ be the product $\prod_{|X|} \mathcal{P}_r$ with one copy of $\mathcal{P}_r$ for every closed point of X. It is an affine scheme over $\mathbb{Q}$ which if $\dim(X) \ge 1$ is not of finite type over $\mathbb{Q}$. Denote by $\pi_x : \mathcal{L}_r(X) \to \mathcal{P}_r$ the projection onto the factor corresponding to $x \in |X|$. We make $\mathcal{L}_r$ into a contravariant functor from $\mathrm{Sm}_{\mathbb{F}_q}$ to the category of affine schemes over $\mathbb{Q}$ as follows: Let $f : Y \to X$ be a morphism of schemes in $\mathrm{Sm}_{\mathbb{F}_q}$. The image of $(P_x)_{x \in |X|} \in \mathcal{L}_r(X)$ under pullback by f is defined to be

$$\left(\psi_{[k(y):k(f(y))]} P_{f(y)} \right)_{y \in |Y|} \in \mathcal{L}_r(Y).$$

For $N > 0$ we similarly define $\mathcal{L}_r^{\le N}(X)$ to be the product over all $x \in |X|$ with $\deg(x) \le N$ over $\mathbb{F}_q$, with the corresponding forgetful morphism $\mathcal{L}_r(X) \to \mathcal{L}_r^{\le N}(X)$.

Putting things together we get morphisms of contravariant functors

$$(4.2) \qquad\qquad \mathcal{R}_r(X) \longrightarrow \mathcal{V}_r(X) \xrightarrow{\kappa : V \mapsto f_V} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell).$$

Proposition 4.1. *For $X \in \mathrm{Sm}_{\mathbb{F}_r}$ the maps $\mathcal{R}_r(X) \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ and $\mathcal{V}_r(X) \xrightarrow{\kappa} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ are injective.*

Proof. We only have to show the injectivity for $\mathcal{R}_r(X)$, since the curve case for $\mathcal{R}_r(X)$ implies already the general case for $\mathcal{V}_r(X)$. We can easily recover the trace functions t_V^n from the characteristic polynomials f_V. The Chebotarev density theorem [20, Ch. 6] implies that the traces of Frobenius determine semi-simple sheaves, see [28, Thm. 1.1.2]. $\qquad\square$

In Section 5 we will prove a much stronger result, saying that a finite number of characteristic polynomials $f_V(x)$ are sufficient to recover V up to twist, as long as V runs over ℓ-adic sheaves with some fixed bounded ramification and fixed rank.

For later reference we recall the relation between Weil sheaves and étale sheaves from Weil II [9, Prop. 1.3.4]. We say that $V \in \mathcal{R}_r(X)$ is *étale* if it comes from a lisse étale $\bar{\mathbb{Q}}_\ell$-sheaf on X.

Proposition 4.2. *For X connected and $V \in \mathcal{R}_1(X)$, which we consider as a continuous homomorphism $V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$, the geometric monodromy group $\mathrm{im}(\pi_1(X_{\bar{\mathbb{F}}})) \subset W(X)/\ker(V)$ is finite, in particular the monodromy group $W(X)/\ker(V)$ is discrete. The sheaf V extends to a continuous homomorphism $\pi_1(X) \to \bar{\mathbb{Q}}_\ell^\times$, i.e. V is étale, if and only if $\mathrm{im}(V) \subset \bar{\mathbb{Z}}_\ell^\times$.*

Proposition 4.3. *For X connected an irreducible $V \in \mathcal{R}_r(X)$ is étale if and only if its determinant $\det(V)$ is étale. In particular there is always a twist $\chi \cdot V$ with $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ which is étale.*

4.2. Implications of Langlands.

In this section we recall some consequences of the Langlands correspondence of Drinfeld and Lafforgue [27] for the theory of ℓ-adic sheaves.

The following theorem is shown in [27, Théorème VII.6].

Theorem 4.4. *For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ connected of dimension one and for $V \in \mathcal{R}_r(X)$ irreducible with determinant of finite order the following holds:*

   (i) *For an arbitrary, not necessarily continuous, automorphism $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$, there is a $V_\sigma \in \mathcal{R}_r(X)$, called σ-companion, such that*

$$f_{V_\sigma} = \sigma(f_V),$$

     where σ acts on the polynomial ring $\bar{\mathbb{Q}}_\ell[t]$ by σ on $\bar{\mathbb{Q}}_\ell$ and by $\sigma(t) = t$.
  (ii) *V is pure of weight 0.*

Later, we deduce from the theorem that σ-companions exist for arbitrary $V \in \mathcal{R}_r(X)$ in dimension one, not necessarily of finite determinant, see Corollary 4.7.

For $\dim(X)$ arbitrary and $V \in \mathcal{R}_1(X)$, which we consider as a continuous homomorphism $V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$, the σ-companion V_σ simply corresponds to the continuous map $\sigma \circ V : W(X) \to \bar{\mathbb{Q}}_\ell^\times$. In fact $\sigma \circ V$ is continuous, because $W(X)/\ker(V)$ is discrete by Proposition 4.2.

From Lafforgue's theorem one can deduce certain results on higher dimensional schemes.

Corollary 4.5. *Let X be a connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$ of arbitrary dimension. For an irreducible $V \in \mathcal{R}_r(X)$ the following are equivalent:*

    (i) *V is pure of weight 0,*
    (ii) *there is a closed point $x \in X$ such that $V_{\bar{x}}$ is pure of weight 0,*
    (iii) *there is $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ pure of weight 0 such that the determinant $\det(\chi \cdot V)$ is of finite order.*

Proof. (iii) $\Rightarrow$ (i):
For a closed point $x \in X$ choose a curve C/k and a morphism $\phi : C \to X$ such that x is in the set theoretic image of ϕ and such that ϕ^*V is irreducible. A proof of the existence of such a curve is given in an appendix, Proposition B.1. Then by Theorem 4.4 the sheaf ϕ^*V is pure of weight 0 on C, so $V_{\bar{x}}$ is also pure of weight 0.

(i) $\Rightarrow$ (ii): Trivially.

(ii) $\Rightarrow$ (iii):
Choose $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ such that $(\chi|_{k(x)})^{\otimes r} = \det(V_{\bar{x}})^{\vee}$. By Proposition 4.2 it follows that the determinant $\det(\chi \cdot V)$ has finite order. $\square$

Let $\mathcal{W}$ be the quotient of $\bar{\mathbb{Q}}_\ell^{\times}$ modulo the numbers of weight 0 in the sense of [9, Def. 1.2.1] (algebraic numbers all complex conjugates of which have absolute value 1).

Corollary 4.6. *A sheaf $V \in \mathcal{R}_r(X)$, resp. a 2-skeleton sheaf $V \in \mathcal{V}_r(X)$, can be decomposed uniquely as a sum*

$$V = \bigoplus_{w \in \mathcal{W}} V_w$$

with the property that $V_w \in \mathcal{R}(X)$, resp. $V_w \in \mathcal{V}(X)$, such that for each point $x \in |X|$, all eigenvalues of the Frobenius F_x on V_w lie in the class w.

Corollary 4.7. *Assume $\dim(X) = 1$. For $V \in \mathcal{R}_r(X)$ and an automorphism $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$, there is a σ-companion to V, i.e. $V_\sigma \in \mathcal{R}_r(X)$ such that*

$$f_{V_\sigma} = \sigma(f_V).$$

Proof. Without loss of generality we may assume that V is irreducible. In the same way as in the proof of Corollary 4.5 we find $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ such that $\chi \cdot V$ has determinant of finite order. A σ-companion of $\chi \cdot V$ exists by Theorem 4.4 and a σ-companion of χ exists by the remarks

below Theorem 4.4. As the formation of σ-companions is compatible with tensor products, $V_\sigma = (V \cdot \chi)_\sigma \cdot (\chi_\sigma)^\vee$ is a σ-companion of V. $\square$

Deligne showed a compatibility result [10, Thm. 9.8] for the Swan conductor of σ-companions.

Proposition 4.8. *Let V and V_σ be σ-companions on a one-dimensional $X \in \mathrm{Sm}_{\mathbb{F}_q}$ as in as in Corollary 4.7. Then $\mathrm{Sw}(V) = \mathrm{Sw}(V_\sigma)$.*

Recall from (4.2) that there is a canonical injective map of sets $\mathcal{V}_r(X) \xrightarrow{\kappa} \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$. In the following corollary we use the notation of Section 3.3.

Corollary 4.9. *For $X \in \mathrm{Sm}_{\mathbb{F}_q}$ and an effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ the action of $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ on $\mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$ stabilizes $\alpha(\mathcal{V}_r(X))$ and $\alpha(\mathcal{V}_r(X, D))$.*

Remark 4.10. Drinfeld has shown [15] that Corollary 4.7 remains true for higher dimensional $X \in \mathrm{Sm}_{\mathbb{F}_q}$. His argument relies on Deligne's Theorem 8.2.

4.3. **Proof of Thm. 2.1** (dim = 1). Theorem 2.1 for one-dimensional schemes is a well-known consequence of Lafforgue's Langlands correspondence for GL_r [27]. Let $X \in \mathrm{Sm}_{\mathbb{F}_q}$ be of dimension one with smooth compactification $\bar{X}$, $L = k(X)$. The Langlands correspondence says that there is a natural bijective equivalence between cuspidal automorphic irreducible representations π of $\mathrm{GL}_r(\mathbb{A}_L)$ (with values in $\bar{\mathbb{Q}}_\ell$) and continuous irreducible representations of the Weil group $\sigma_\pi : W(L) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$, which are unramified almost everywhere. For such an automorphic π one defines an (Artin) conductor $\mathrm{Ar}(\pi) \in \mathrm{Div}^+(X)$ and one constructs an open compact subgroup $K \subset \mathrm{GL}_r(\mathbb{A}_L)$ depending only on $\mathrm{Ar}(\pi)$ such that the space of K invariant vectors of π has dimension one, see [22].

The divisor $\mathrm{Ar}(\pi)$ has support in $\bar{X} \setminus X$ if and only if σ_π is unramified over X. Moreover

$$\mathrm{Sw}_x(\sigma_\pi) + r \geq \mathrm{Ar}_x(\pi)$$

for $x \in |\bar{X}|$.

For an arbitrary compact open subgroup $K \subset \mathrm{GL}_r(\mathbb{A}_L)$ the number of cuspidal automorphic irreducible representations π with fixed central character and which have a non-trivial K-invariant vector is finite by work of Harder, Gelfand and Piatetski-Shapiro, see [29, Thm. 9.2.14].

Via the Langlands correspondence this implies that for given $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ and for given $W \in \mathcal{R}_1(X)$ the number

of irreducible $V \in \mathcal{R}_r(X)$ with $\det(V) = W$ and with $\mathrm{Sw}(V) \leq D$ is finite. Recall that the determinant of σ_π corresponds to the central character of π via class field theory.

As written by Deligne in an email to us dated July 30, 2012, one can also use the quasi-orthogonality relations from Claim 5.4 and a sphere packing argument to conclude, but Claim 5.4 relies on purity, which again comes from the Langlands correspondence on curves, so we do not make the argument explicit.

4.4. Structure of a lisse $\bar{\mathbb{Q}}_\ell$-sheaf over a scheme over a finite field.
Let the notation be as above. The following proposition is shown in [5, Prop. 5.3.9].

Proposition 4.11. *Let V be irreducible in $\mathcal{R}_r(X)$.*

 (i) *Let m be the number of irreducible constituents of $V_{\mathbb{F}}$. There is a unique irreducible $V^\flat \in \mathcal{R}_{r/m}(X_{\mathbb{F}_{q^m}})$ such that*
  - *the pullback of $V^\flat$ to $X \otimes_{\mathbb{F}_q} \mathbb{F}$ is irreducible,*
  - *$V = b_{m,*}V^\flat$, where b_m is the natural map $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} \to X$.*
 (ii) *V is pure of weight 0 if and only if $V^\flat$ is pure of weight 0.*
 (iii) *If $V' \in \mathcal{R}_r(X)$ is another sheaf on X with $V'_{\mathbb{F}} = V_{\mathbb{F}}$, then there is a unique sheaf $W \in \mathcal{R}_1(\mathbb{F}_{q^m})$ with*

$$V' = b_{m,*}(V^\flat \otimes W).$$

A special case of the Grothendieck trace formula [28, (1.1.1.3)] says:

Proposition 4.12. *Let V and m be as in Proposition 4.11. For $n \geq 1$ and $x \in X(\mathbb{F}_{q^n})$*

$$t_V^n(x) = \sum_{\substack{y \in X_{\mathbb{F}_{q^m}}(\mathbb{F}_{q^n}) \\ y \mapsto x}} t_{V^\flat}^n(y).$$

Concretely, $t_V^n(x) = 0$ if m does not divide n.

5. Frobenius on curves

We now present Deligne's key technical method for proving his finiteness theorems. It strengthens Proposition 4.1 on curves by allowing us to recover an ℓ-adic sheaf from an effectively determined finite number of characteristic polynomials of Frobenius.

Our notation is explained in Section 2 and Section 4.1. Throughout this section X is a geometrically connected scheme in $\mathrm{Sm}_{\mathbb{F}_q}$ with $\dim(X) = 1$.

Theorem 5.1 (Deligne). *The natural map*

$$\mathcal{R}_r(X, D) \xrightarrow{\kappa_N} \mathcal{L}_r^{\leq N}(X)(\bar{\mathbb{Q}}_\ell)$$

is injective if

(5.1) $$N \geq 4r^2 \lceil \log_q(2r^2\mathcal{C}_D) \rceil$$

Here for a real number w we let $\lceil w \rceil$ be the smallest integer larger or equal to w. Theorem 5.1 relies on the Langlands correspondence and weight arguments form Weil II. The Langlands correspondence enters via Corollary 4.6.

We deduce Theorem 5.1 from the following trace version:

Proposition 5.2. *If $V, V' \in \mathcal{R}_r(X, D)$ are pure of weight 0 and satisfy $t_V^n = t_{V'}^n$, for all*

(5.2) $$n \leq 4r^2 \lceil \log_q(2r^2\,\mathcal{C}_D) \rceil,$$

then $V = V'$.

Prop. 5.2 $\Rightarrow$ Thm. 5.1. Let $V, V' \in \mathcal{R}_r(X, D)$. We write

$$V = \bigoplus_{w \in \mathcal{W}} V_w \quad \text{and} \quad V' = \bigoplus_{w \in \mathcal{W}} V'_w$$

as in Corollary 4.6. The condition $\alpha_N(V) = \alpha_N(V')$ implies $\alpha_N(V_w) = \alpha_N(V'_w)$, thus $t_{V_w}^n = t_{V'_w}^n$ for all $w \in \mathcal{W}$ and all n as in (5.2). By Proposition 5.2, applied to some twist of weight 0 of V_w and V'_w by the same χ, this implies $V_w = V'_w$ for all $w \in \mathcal{W}$. $\square$

5.1. **Proof of Proposition 5.2.** Let J be the set of irreducible $W \in \mathcal{R}_s(X)$, $1 \leq s \leq r$, which are twists of direct summands of $V \oplus V'$. Set $I = J/\text{twist}$. Choose representative sheaves $S_i \in \mathcal{R}(X)$ which are pure of weight 0 ($i \in I$). In particular this implies that $\mathrm{Hom}_{X \otimes_{\mathbb{F}_q}\mathbb{F}}(S_{i_1}, S_{i_2}) = 0$ for $i_1 \neq i_2 \in I$ by Proposition 4.11. Also for each $i \in I$ we have

$$S_i = b_{m_i,*}S_i^\flat$$

for positive integers m_i and geometrically irreducible $S_i^\flat \in \mathcal{R}(X_{\mathbb{F}_{q^{m_i}}})$ with the notation of Proposition 4.11.

It follows from Proposition 4.11 that there are $W_i, W'_i \in \mathcal{R}(\mathbb{F}_{q^{m_i}})$ pure of weight 0 such that

$$V = \bigoplus_{i \in I} b_{m_i,*}(S_i^\flat \otimes_{\bar{\mathbb{Q}}_\ell} W_i)$$

and

$$V' = \bigoplus_{i \in I} b_{m_i,*}(S_i^\flat \otimes_{\bar{\mathbb{Q}}_\ell} W'_i).$$

For $n > 0$ set
$$I_n = \{i \in I, \ m_i | n\}.$$

Lemma 5.3. *Let S_i be in $\mathcal{R}(X)$ pairwise distinct, geometrically irreducible, pure of weight 0. Then the functions*
$$t^n_{S_i} : X(\mathbb{F}_{q^n}) \to \bar{\mathbb{Q}}_\ell \qquad (i \in I_n)$$
are linearly independent over $\bar{\mathbb{Q}}_\ell$ for $n \geq 2 \log_q(2r^2 \mathcal{C}_D)$.

Proof. Fix an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Assume we have a linear relation

(5.3)
$$\sum_{i \in I_n} \lambda_i \, t^n_{S_i} = 0, \ \lambda_i \in \bar{\mathbb{Q}}_\ell,$$

such that not all λ_i are 0. Multiplying by a constant in $\bar{\mathbb{Q}}_\ell^\times$, we may assume that $|\iota(\lambda_{i_\circ})| = 1$ for one $i_\circ \in I_n$ and $|\iota(\lambda_i)| \leq 1$ for all $i \in I_n$. Set
$$\langle S_{i_1}, S_{i_2} \rangle_n = \sum_{x \in X(\mathbb{F}_{q^n})} t^n_{Hom(S_{i_1}, S_{i_2})}(x)$$

for $i_1, i_2 \in I_n$. Observe that
$$t^n_{Hom(S_{i_1}, S_{i_2})} = t^n_{S_{i_1}^\vee} \cdot t^n_{S_{i_2}}.$$

Multiplying (5.3) by $t^n_{S_{i_\circ}^\vee}$ and summing over all $x \in X(\mathbb{F}_{q^n})$ one obtains

(5.4)
$$\sum_{i \in I_n} \lambda_i \, \langle S_{i_\circ}, S_i \rangle_n = 0.$$

Claim 5.4. One has

(i)
$$|\iota \langle S_{i_\circ}, S_i \rangle_n| \leq \text{rank}(S_{i_\circ}) \text{rank}(S_i) \, \mathcal{C}_D \, q^{n/2}$$
for $i \neq i_\circ$,

(ii)
$$|m_{i_\circ} \, q^n - \iota \langle S_{i_\circ}, S_{i_\circ} \rangle_n| \leq \text{rank}(S_{i_\circ})^2 \, \mathcal{C}_D \, q^{n/2}.$$

Proof of (i):
By [9, Théorème 3.3.1] the eigenvalues α of F^n on $H^k_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_{i_\circ}))$ for $k \leq 1$ fulfill
$$|\iota \alpha| \leq q^{n/2}.$$
On the other hand
$$\dim_{\bar{\mathbb{Q}}_\ell}(H^0_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i))) + \dim_{\bar{\mathbb{Q}}_\ell}(H^1_c(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i))) \ \leq$$
$$\text{rank}(S_{i_\circ}) \text{rank}(S_i) \, \mathcal{C}_D$$

by Proposition 3.5. In fact the we have

$$\mathrm{Sw}(Hom(S_{i_\circ}, S_i)) \le \mathrm{rank}(S_{i_\circ})\mathrm{rank}(S_i)D$$

by (3.1) - (3.3). Under the assumption $i \ne i_\circ$ one has

$$H_c^2(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i)) = \mathrm{Hom}_{X \otimes_{\mathbb{F}_q} \mathbb{F}}(S_i, S_{i_\circ}) \otimes \bar{\mathbb{Q}}_\ell(-1) = 0$$

by Poincaré duality. Putting this together and using Grothendieck's trace formula [28, 1.1.1.3] one obtains (i).

Proof of (ii):
It is similar to (i) but this time we have

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^2(X \otimes_{\mathbb{F}_q} F, Hom(S_{i_\circ}, S_i)) = m_{i_\circ}$$

and for an eigenvalue α of F^n on

$$H_c^2(X \otimes_{\mathbb{F}_q} \mathbb{F}, Hom(S_{i_\circ}, S_i)) = \mathrm{Hom}_{X \otimes_{\mathbb{F}_q} \mathbb{F}}(S_i, S_{i_\circ}) \otimes \bar{\mathbb{Q}}_\ell(-1)$$

we have $\alpha = q^n$. This finishes the proof of the claim.

Since under the assumption on n from Lemma 5.3

$$\mathcal{C}_D \mathrm{rank}(S_{i_\circ}) \sum_{i \in I_n} \mathrm{rank}(S_i) < q^{n/2},$$

we get a contradiction to the linear dependence (5.3).

$\square$

By Proposition 4.12 for any $n \ge 0$ we have

$$t_V^n = \sum_{i \in I_n} t_{W_i}^n t_{S_i}^n$$

and

$$t_{V'}^n = \sum_{i \in I_n} t_{W_i'}^n t_{S_i}^n.$$

Under the assumption of equality of traces from Theorem 5.2 and using Lemma 5.3 we get

(5.5) $$\mathrm{Tr}(F^n, W_i) = \mathrm{Tr}(F^n, W_i') \qquad i \in I_n$$

for

$$2 \log_q(2r^2 \mathcal{C}_D) \le n \le 4r^2 \lceil \log_q(2r^2 \mathcal{C}_D) \rceil.$$

In particular this means that equality (5.5) holds for

$$n \in \{m_i A, m_i(A+1), \ldots, m_i(A+2r-1)\},$$

where $A = \lceil 2 \log_q(2r^2 \mathcal{C}_D) \rceil$. So Lemma 5.5 applied to the set $\{b_1, \ldots, b_w\}$ of eigenvalues of F^{m_i} of W_i and W_i' (so $w \le 2r$) shows that $W_i = W_i'$ for all $i \in I$.

Lemma 5.5. *Let k be a field and consider elements $a_1, \ldots, a_w \in k$, $b_1, \cdots b_w \in k^\times$ such that*

$$F(n) := \sum_{1 \le j \le w} a_j \, b_j^n = 0$$

for $1 \le n \le w$. Then $F(n) = 0$ for all $n \in \mathbb{Z}$.

Proof. Without loss of generality we can assume that the b_j are pairwise different for $1 \le j \le w$. Then the Vandermonde matrix

$$(b_j^n)_{1 \le j, n \le w}$$

has non-vanishing determinant, which implies that $a_j = 0$ for all j. $\square$

6. MODULI SPACE OF ℓ-ADIC SHEAVES

In Section 4.1 we introduced an injective map

$$\kappa : \mathcal{V}_r(X) \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell)$$

from the set of 2-skeleton ℓ-adic sheaves to the $\bar{\mathbb{Q}}_\ell$-points of an affine scheme $\mathcal{L}_r(X)$ defined over $\mathbb{Q}$, which is not of finite type over $\mathbb{Q}$ if $\dim(X) \ge 1$. Assume that there is a connected normal projective compactification $X \subset \bar{X}$ such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor on $\bar{X}$. We use the notation of Section 4.1.

The existence of the moduli space of ℓ-adic sheaves on X is shown in the following theorem of Deligne.

Theorem 6.1. *For any effective Cartier divisor $D \in \mathrm{Div}^+(\bar{X})$ with support in $\bar{X} \setminus X$ there is a unique reduced closed subscheme $L_r(X, D)$ of $\mathcal{L}_r(X)$ which is of finite type over $\mathbb{Q}$ and such that*

$$L_r(X, D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X, D)).$$

Uniqueness is immediate from Proposition A.1. In Section 6.2 we construct $L_r(X, D)$ for $\dim(X) = 1$. In Section 6.3 we construct $L_r(X, D)$ for general X. Before we begin the proof we introduce some elementary constructions on $\mathcal{L}_r(X)$.

6.1. **Direct sum and twist as scheme morphisms.** For $r = r_1 + r_2$ the isomorphism

$$\mathbb{G}_m^{r_1} \times_{\mathbb{Q}} \mathbb{G}_m^{r_2} \xrightarrow{\cong} \mathbb{G}_m^{r}$$

together with the embedding of groups $S_{r_1} \times S_{r_2} \subset S_{r_1+r_2}$ induces a finite surjective map

(6.1) $- \oplus - : \mathcal{P}_{r_1} \times_{\mathbb{Q}} \mathcal{P}_{r_2} \to \mathcal{P}_r, \quad (P, Q) \mapsto PQ$

via the isomorphism (4.1). We call it the direct sum.

There is a twisting action by $\mathbb{G}_m$

$$\mathbb{G}_m \times_{\mathbb{Q}} \mathcal{P}_r \to \mathcal{P}_r, \quad (\alpha, P) \mapsto \alpha \cdot P$$

defined by the diagonal action of $\mathbb{G}_m$ on $\mathbb{G}_m^r$

$$(\alpha, (\alpha_1, \ldots, \alpha_r)) \mapsto (\alpha \cdot \alpha_1, \ldots, \alpha \cdot \alpha_r)$$

and the isomorphism (4.1).

We now extend the direct sum and twist morphisms to $\mathcal{L}(X)$.

By taking direct sum on any factor of $\mathcal{L}(X)$ we get for $r_1 + r_2 = r$ a morphism of schemes over $\mathbb{Q}$

(6.2) $$- \oplus - : \mathcal{L}_{r_1}(X) \times \mathcal{L}_{r_2}(X) \to \mathcal{L}_r(X)$$

Note that the direct sum is not a finite morphism in general, since we have an infinite product over closed points of X.

The twist is an action of $\mathbb{G}_m$

(6.3) $$\mathbb{G}_m \times_{\mathbb{Q}} \mathcal{L}_r(X) \to \mathcal{L}_r(X)$$

given by

$$(\alpha, (P_x)_{x \in |X|}) \mapsto \alpha \cdot (P_x)_{x \in |X|} = (\alpha^{\deg(x)} \cdot P_x)_{x \in |X|}$$

where we take the degree of a point x over $\mathbb{F}_q$.

Let k be a field containing $\mathbb{Q}$ and $P_i \in \mathcal{L}_{r_i}(k)$, $i = 1, \ldots, n$. Assume $r_i > 0$ for all i and set $r = r_1 + \cdots + r_n$.

Lemma 6.2. *The morphism of schemes over the field k*

$$\rho : \mathbb{G}_m^n \to \mathcal{L}_r(X), \quad (\alpha_i)_{i=1,\ldots,n} \mapsto \alpha_1 \cdot P_1 \oplus \cdots \oplus \alpha_n \cdot P_n$$

is finite.

Proof. In fact already the composition of ρ with the projection to one factor $\mathcal{P}_r$ of $\mathcal{L}_r(X)$, corresponding to a point $x \in |X|$, is finite. To see this write this morphism as the composition of finite morphisms over k

$$\mathbb{G}_m^n \xrightarrow{\psi_{\deg(x)}} \mathbb{G}_m^n \xrightarrow{\cdot(P_1,\ldots,P_n)} \mathcal{P}_{r_1} \times \cdots \times \mathcal{P}_{r_n} \xrightarrow{\oplus} \mathcal{P}_r.$$

$\square$

6.2. **Moduli over curves.** In this section we prove Theorem 6.1 for $\dim(X) = 1$. The dimension one case of Theorem 2.1 was shown in Section 4.3. In particular we get:

Lemma 6.3. *There are up to twist only* finitely many *irreducible direct summands of the sheaves $V \in \mathcal{R}_r(X, D) = \mathcal{V}_r(X, D)$.*

Step 1:
Consider $V_1 \oplus \cdots \oplus V_n \in \mathcal{R}_r(X, D)$ and the map
(6.4)
$$(\mathcal{R}_1(\mathbb{F}_q))^n \to \mathcal{L}_r(X)(\bar{\mathbb{Q}}_\ell), \quad (\chi_1, \ldots, \chi_n) \mapsto \kappa(\chi_1 \cdot V_1 \oplus \cdots \oplus \chi_n \cdot .V_n)$$

This map is just the induced map on $\bar{\mathbb{Q}}_\ell$-points of the finite scheme morphism over $k = \bar{\mathbb{Q}}_\ell$ from Lemma 6.2, where we take $P_i = \kappa(V_i)$. By Proposition A.3 there is a unique reduced closed subscheme $L(V_i)$ of $\mathcal{L}_r(X) \otimes \bar{\mathbb{Q}}_\ell$ of finite type over $\bar{\mathbb{Q}}_\ell$ such that $L(V_i)(\bar{\mathbb{Q}}_\ell)$ is the image of the map (6.4).

Step 2:
By Lemma 6.3 there are only finitely many direct sums

(6.5) $$V_1 \oplus \cdots \oplus V_n \in \mathcal{R}_r(X, D)$$

with V_i irreducible up to twists $\chi_i \mapsto \chi_i \cdot V_i$ with $\chi_i \in \mathcal{R}_1(\mathbb{F}_q)$. Let

$$L_r(X, D)_{\bar{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$$

be the reduced scheme, which is the union of the finitely many closed subschemes $L(V_i) \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ corresponding to representatives of the finitely many twisting classes of direct sums (6.5). Clearly $L_r(X, D)_{\bar{\mathbb{Q}}_\ell}(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{R}_r(X, D))$ and $L_r(X, D)_{\bar{\mathbb{Q}}_\ell}$ is of finite type over $\bar{\mathbb{Q}}_\ell$.

Step 3:
By Corollary 4.9 the automorphism group $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ acting on $\mathcal{L}_r(X)$ stabilizes $\kappa(\mathcal{R}_r(X, D))$. Therefore by the descent Proposition A.2 the scheme $L_r(X, D)_{\bar{\mathbb{Q}}_\ell} \hookrightarrow \mathcal{L}_r(X) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ over $\bar{\mathbb{Q}}_\ell$ descends to a closed subscheme $L_r(X, D) \hookrightarrow \mathcal{L}_r(X)$. This is the *moduli space of ℓ-adic sheaves* on curves, the existence of which was claimed in Theorem 6.1.

From the proof of Lemma 6.2 and the above construction we deduce:

Lemma 6.4. *For any $x \in |X|$ the composite map*

$$L_r(X, D) \to \mathcal{L}_r(X) \xrightarrow{\pi_x} \mathcal{P}_r$$

is a finite morphism of schemes.

6.3. **Higher dimension.** Now the dimension $d = \dim(X)$ of $X \in \mathrm{Sm}_{\mathbb{F}_q}$ is allowed to be arbitrary. In order to prove Theorem 6.1 in general we first construct a closed subscheme $L_r(X, D) \hookrightarrow \mathcal{L}_r(X)$ such that

$$L_r(X, D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X, D))$$

relying on Theorem 6.1 for curves. However from this construction it is not clear that $L_r(X, D)$ is of finite type over $\mathbb{Q}$. The main step is to show that it is of finite type using Theorem 5.1.

Step 1:
We define the reduced closed subscheme $L_r(X, D) \hookrightarrow \mathcal{L}_r(X)$ by the Cartesian square (in the category of reduced schemes)

$$
\begin{array}{ccc}
L_r(X, D) & \longrightarrow & \mathcal{L}_r(X) \\
\downarrow & & \downarrow \\
\displaystyle\prod_{C \in \mathrm{Cu}(X)} L_r(C, \bar{\phi}^*(D)) & \longrightarrow & \displaystyle\prod_{C \in \mathrm{Cu}(X)} \mathcal{L}_r(C)
\end{array}
$$

where $\mathrm{Cu}(X)$ is defined in Section 2.2. Clearly, from the curve case of Theorem 6.1 and the definition of $\mathcal{V}_r(X, D)$ we get

$$L_r(X, D)(\bar{\mathbb{Q}}_\ell) = \kappa(\mathcal{V}_r(X, D)).$$

In addition, as $\mathcal{L}(X) \to \displaystyle\prod_{C \in \mathrm{Cu}(X)} \mathcal{L}_r(C)$ is a closed immersion, so is $\mathcal{L}(X, D) \to \displaystyle\prod_{C \in \mathrm{Cu}(X)} L_r(C, \bar{\phi}^*(D))$.

Step 2:
Let C be a purely one-dimensional scheme which is separated and of finite type over $\mathbb{F}_q$. Let $\phi_i : E_i \to C$ ($i = 1, \dots, m$) be the normalizations of the irreducible components of C and let $\phi : E = \coprod_i E_i \to C$ be the disjoint union. Let $D \in \mathrm{Div}^+(\bar{E})$ be an effective divisor with supports in $\bar{E} \setminus E$. Here $\bar{E}$ is the canonical smooth compactification of E. Define the reduced scheme $L_r(C, D)$ by the Cartesian square (in the category of reduced schemes)

$$
\begin{array}{ccc}
L_r(C, D) & \longrightarrow & \prod_{j=1,\dots,m} L_r(E_j, D_j) \\
\downarrow & & \downarrow \\
\prod_{i=1,\dots,m} L_r(E_i, D_i) & \longrightarrow & \prod_{(i,j)} L_r((E_i \times_C E_j)_{\mathrm{red}})
\end{array}
$$

Step 3:
By an *exhaustive system of curves* on X we mean a sequence $(C_n)_{n \geq 0}$ of purely one-dimensional closed subschemes $C_n \hookrightarrow X$ with the properties

(a) – (d) listed below. We write $\phi : E_n \to X$ for the normalization of C_n. For a divisor $D' \in \text{Div}^+(\bar{E}_n)$ we let $\mathcal{C}_{D'}$ be the maximum of the complexities of the irreducible components of $E_n \otimes \mathbb{F}$, see Definition 3.4.

(a) $C_n \hookrightarrow C_{n+1}$ for $n \geq 0$,
(b) $E_n(\mathbb{F}_{q^n}) \to X(\mathbb{F}_{q^n})$ is surjective,
(c) the fields of constants of the irreducible components of E_n ($n \geq 0$) are bounded,
(d) the complexity $\mathcal{C}_{\bar{\phi}_n^*(D)}$ of E_n satisfies

$$\mathcal{C}_{\bar{\phi}_n^*(D)} = O(n).$$

Lemma 6.5. *Any $X \in \text{Sm}_{\mathbb{F}_q}$ admits an exhaustive system of curves.*

The proof of the lemma is given below.

In order to show that $L_r(X, D)$ is of finite type, one can assume that the support of D is $\bar{X} \setminus X$. Let now (C_n) be an exhaustive system of curves on X. Set $D_n = \bar{\phi}_n^*(D) \in \text{Div}^+(\bar{E}_n)$. An immediate consequence of (a)–(d) and the Riemann hypothesis for curves is that for $n \gg 0$ any irreducible component of C_{n+1} meets C_n. This implies by Lemma 6.4 that the tower of affine schemes of finite type over $\mathbb{Q}$

$$\cdots \to L_r(C_{n+1}, D_{n+1}) \xrightarrow{\tau} L_r(C_n, D_n) \to \cdots$$

has finite transition morphisms. Clearly, $L_r(X, D)$ maps to this tower. Since the complexities of the irreducible curves grow linearly in n and the fields of constants are bounded, Theorem 5.1 implies that there is $N \geq 0$ such that the map

$$L_r(C_{n+1}, D_{n+1})(\bar{\mathbb{Q}}_\ell) \to \mathcal{L}_r^{\leq n}(E_{n+1})$$

is injective for $n \geq N$. As this map factors through

$$\tau : L_r(C_{n+1}, D_{n+1})(\bar{\mathbb{Q}}_\ell) \to L_r(C_n, D_n)(\bar{\mathbb{Q}}_\ell)$$

by (b), we get injectivity of τ on $\bar{\mathbb{Q}}_\ell$-points for $n \geq N$. Consider the intersection of the images

$$I_n = \bigcap_{i \geq 0} \tau^i(L_r(C_{n+i}, D_{n+i})) \hookrightarrow L_r(C_n, D_n),$$

endowed with the reduced closed subscheme structure. Then the transition maps in the tower

$$\cdots \to I_{n+1} \to I_n \to \cdots$$

are finite and induce bijections on $\bar{\mathbb{Q}}_\ell$-points for $n \geq N$. By Proposition A.4 we get an $N' \geq 0$ such that $I_{n+1} \to I_n$ is an isomorphism of schemes for $n \geq N'$. The closed immersion $L_r(X, D) \to \mathcal{L}_r(X)$ factors

through $L_r(X, D) \to \varprojlim_n L_r(C_n, D_n)$, which is therefore itself a closed immersion. Thus we obtain a closed immersion

$$L_r(X, D) \to \varprojlim_n L_r(C_n, D_n) \cong \varprojlim_n I_n \overset{\simeq}{\to} I_{N'},$$

and therefore $L_r(X, D)$ is of finite type over $\mathbb{Q}$.

Proof of Lemma 6.5. Using Noether normalization we find a finite number of finite surjective morphisms

$$\bar{\eta}_s : \bar{X} \to \mathbb{P}^d, \quad s = 1, \ldots, w$$

with the property that every point $x \in |X|$ is in the étale locus of one of the $\eta_s = \bar{\eta}_s|_X$. See [25, Theorem 1] for more details.

Claim 6.6. For a point $y \in \mathbb{P}^d(\mathbb{F}_{q^n})$ there is a morphism $\phi_y : \mathbb{P}^1 \to \mathbb{P}^d$ of degree $< n$ with $y \in \phi_y(\mathbb{P}^1(\mathbb{F}_{q^n}))$.

Proof of Claim. The closed point y lies in an affine chart

$$\mathbb{A}^d_{\mathbb{F}_q} = \operatorname{Spec}(\mathbb{F}_q[T_1, \ldots, T_d]) \hookrightarrow \mathbb{P}^d_{\mathbb{F}_q}$$

and gives rise to a non-constant homomorphism $\mathbb{F}_q[T_1, \ldots, T_d] \to \mathbb{F}_{q^n}$. We choose an embedding $\operatorname{Spec} \mathbb{F}_{q^n} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q} = \operatorname{Spec}(\mathbb{F}_q[T])$ and a lifting

$$\mathbb{F}_q[T_1, \ldots, T_d] \to \mathbb{F}_q[T]$$

with $\deg(\phi(T_i)) < n$ $(1 \leq i \leq d)$. By projective completion we obtain a morphism $\phi_y : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{P}^d_{\mathbb{F}_q}$ of degree less than n factoring the morphism $y \to \mathbb{P}^d$. $\square$

For $x \in |X|$ of degree n choose a lift $x \in X(\mathbb{F}_{q^n})$ and an s such that x is in the étale locus of η_s. Furthermore choose $\phi_y : \mathbb{P}^1 \to \mathbb{P}^d$ as in the claim with $y = \eta_s(x)$. Clearly x lifts to a smooth point of $(\mathbb{P}^1 \times_{\mathbb{P}^d} X)(\mathbb{F}_{q^n})$ contained in an irreducible component which we call Z. Let $\phi_x : C_x \to X$ be the normalization of the image of Z in X. Then $x \in \phi_x(C_x(\mathbb{F}_{q^n}))$.

We assume now that we have made the choice of the curve $\phi_x : C_x \to X$ above for any point $x \in |X|$. As usual $\bar{\phi}_x : \bar{C}_x \to \bar{X}$ denotes the smooth compactification of C_x. From the Riemann-Hurwitz formula [21, IV,Cor. 2.4] we deduce the growth property

$$\mathcal{C}_{\bar{\phi}_x^*(D)} = \mathcal{O}(\deg(x))$$

for the complexity of $\bar{C}_x$. Furthermore it is clear that the fields of constants of the curves C_x are bounded. Therefore the subschemes

$$C_n = \bigcup_{\deg(x) \leq n} \phi_x(C_x) \hookrightarrow X$$

satisfy the conditions (a)–(d) above.   □

7. Irreducible components and proof of finiteness theorems

Recall that we defined irreducible 2-skeleton sheaves in Section 2 and that in Section 6 we constructed an affine scheme $L_r(X, D)$ of finite type over $\mathbb{Q}$, the $\bar{\mathbb{Q}}_\ell$-points of which are in bijection with 2-skeleton sheaves of rank r with ramification bounded by D. For this we had to assume that $\bar{X}$ is a normal projective variety defined over $\mathbb{F}_q$ and D is an effective Cartier divisor supported in $\bar{X} \setminus X$.

The following theorem describes the irreducible components of $L_r(X, D)$ over $\bar{\mathbb{Q}}$ or, what is the same, over $\bar{\mathbb{Q}}_\ell$.

Theorem 7.1.   1) *Given $V_1, \ldots, V_m$ irreducible in $\mathcal{V}(X)$ such that $V_1 \oplus \ldots \oplus V_m \in \mathcal{V}_r(X, D)$, there is a unique irreducible component $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}$ such that*

$$(7.1) \qquad Z(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi_1 \cdot V_1 \oplus \ldots \oplus \chi_m \cdot V_m) \,|\, \chi_i \in \mathcal{R}_1(\mathbb{F}_q)\}$$

2) *If $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}$ is an irreducible component, then there are $V_1, \ldots, V_m$ irreducible in $\mathcal{V}(X, D)$ such that (7.1) holds true.*

Proof. We first prove 2). Let d be the dimension of Z, so $\bar{\mathbb{Q}}(Z)$ has transcendence degree d over $\bar{\mathbb{Q}}$. Let $\kappa(V) \in Z(\bar{\mathbb{Q}}_\ell)$ be a geometric generic point, corresponding to $\iota : \bar{\mathbb{Q}}(Z) \hookrightarrow \bar{\mathbb{Q}}_\ell$.

By definition, the coefficients of the local polynomials $f_V(x)$, $x \in |X|$ span $\iota(\bar{\mathbb{Q}}(Z))$. The subfield K of $\bar{\mathbb{Q}}_\ell$ spanned by the (inverse) roots of the $f_V(x)$ is algebraic over $\iota(\bar{\mathbb{Q}}(Z))$, and thus has transcendence degree d over $\bar{\mathbb{Q}}$ as well.

Writing

$$(7.2) \qquad\qquad\qquad V = \oplus_{w \in \mathcal{W}} V_w$$

thanks to Corollary 4.6, the number m of such w with $V_w \neq 0$ is $\geq d$. Indeed those w have the property that they span K.

On the other hand, the map (6.4) corresponding to the decomposition (7.2) is the $\bar{\mathbb{Q}}_\ell$-points of a finite map with source $\mathbb{G}_m^m$, which is irreducible, and has image contained in Z. So we conclude $m = d$ and that the morphism $\mathbb{G}_m^m \to Z$ is finite surjective.

We prove 1). By Corollary 4.6, the V_i have the property that there is a $w_i \in \mathcal{W}$ such that all the inverse eigenvalues of the Frobenius F_x on V_i lie in the class of w_i. Replacing V_i by $\chi_i \cdot V_i$ for adequately chosen $\chi_i \in \mathcal{R}_1(\mathbb{F}_q)$, we may assume that $w_i \neq w_j$ in $\mathcal{W}$ if $i \neq j$. We consider the irreducible reduced closed subscheme $Z \hookrightarrow L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$ defined

by its $\bar{\mathbb{Q}}_\ell$-points $\{\kappa(\chi_1 \cdot V_1 \oplus \ldots \oplus \chi_m \cdot V_m) \mid \chi_i \in \mathcal{R}_1(\mathbb{F}_q)\}$. Let Z' be an irreducible component of $L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$ containing Z. Thus by B),

$$Z'(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi'_1 \cdot V'_1 \oplus \ldots \chi'_{m'} \cdot V'_{m'}) \mid \chi'_i \in \mathcal{R}_1(\mathbb{F}_q)\}.$$

So there are χ'_i such that

(7.3) $$V_1 \oplus \ldots \oplus V_m = \chi'_1 V'_1 \oplus \ldots \oplus \chi'_{m'} V'_{m'}$$

As V'_j is irreducible for any $j \in \{1, \ldots, m'\}$, it is of class w for some $w \in \mathcal{W}$ in the sense of Corollary 4.6. So for each $j \in \{1, \ldots, m'\}$, there is a $i \in \{1, \ldots, m\}$ with $\chi'_j \cdot V'_j \subset V_i$, and thus $\chi'_j \cdot V'_j = V_i$ as V_i is irreducible. This implies $m = m'$ and the decompositions (7.3) are the same, up to ordering. So $Z = Z'$. $\square$

Corollary 7.2. *A 2-skeleton sheaf $V \in \mathcal{V}_r(X, D)$ is irreducible if and only if $\kappa(V)$ lies on a one-dimensional irreducible component of $L_r(X, D) \otimes \bar{\mathbb{Q}}_\ell$. In this case $\kappa(V)$ lies on a unique irreducible component $Z/\bar{\mathbb{Q}}_\ell$. The component Z has the form*

$$Z(\bar{\mathbb{Q}}_\ell) = \{\kappa(\chi \cdot V) \mid \chi \in \mathcal{R}_1(\mathbb{F}_q)\}$$

and it does not meet any other irreducible component.

Remark 7.3. If Question 2.3 had a positive answer and using a more refined analysis of Deligne [13] one could deduce that the moduli space $L_r(X, D)$ is smooth and any irreducible component is of the from $\mathbb{G}_m^{s_1} \times \mathbb{A}^{s_2}$ ($s_1, s_2 \geq 0$).

Proof of Theorem 2.4. Using the Chow lemma [1, Sec. 5.6] we can assume without loss of generality that $\bar{X}$ is projective. By Corollary 7.2, the set of one-dimensional irreducible components of $L_r(X, D) \otimes \bar{\mathbb{Q}}$ is in bijection with the set of irreducible 2-skeleton sheaves on X up to twist by $\mathcal{R}_1(\mathbb{F}_q)$. Since $L_r(X, D)$ is of finite type, there are only finitely many irreducible components. $\square$

Proof of Theorem 8.2. By Corollary 4.9 there is a natural action of $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ on $\mathcal{V}_r(X, D)$ compatible via f_V with the action on $\bar{\mathbb{Q}}_\ell[t]$ which fixes t. Let $N > 0$ be an integer such that $\det(V)^{\otimes N} = 1$. For $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ we then have

$$1 = \sigma(\det(V)^{\otimes N}) = \det(\sigma(V))^{\otimes N}.$$

Then Theorem 2.4, (see also the remark following the theorem), implies that the orbit of V under $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ is finite. Let $H \subset \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ be the stabilizer group of V. As $[\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}) : H] < \infty$ we get that $E(V) = \bar{\mathbb{Q}}_\ell^H$ is a number field. $\square$

In order to effectively determine the field $E(V)$ for $V \in \mathcal{R}_r(X)$ with $X \in \mathrm{Sm}_{\mathbb{F}_q}$ projective one can use the following simple consequence of a theorem of Drinfeld [15], which itself relies on Deligne's Theorem 8.2.

Proposition 7.4. *For $X/\mathbb{F}_q$ a smooth projective geometrically connected scheme and $H \hookrightarrow X$ a smooth hypersurface section with $\dim(H) > 0$ consider $V \in \mathcal{R}_r(X)$. Then $E(V) = E(V|_H)$.*

Proof. Observe that the Weil group of H surjects onto the Weil group of X, so we get an injection $\mathcal{R}_r(X) \to \mathcal{R}_r(H)$. By [15] Corollary 4.7 remains true for higher dimensional smooth schemes $X/\mathbb{F}_q$, i.e. for any $\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ there exists a σ-companion V_σ to V. By the above injectivity, the sheaves V and $V|_H$ have the same stabilizer G in $\mathrm{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$. We get

$$E(V) = \bar{\mathbb{Q}}_\ell^G = E(V|_H).$$

$\square$

8. Applications

We now explain applications of the finiteness theorem for 2-skeleton sheaves to a conjecture from Weil II [9, Conj. 1.2.10 (ii)] and to Chow groups of 0-cycles.

8.1. **Finiteness of relative Chow group of 0-cycles.** It was shown by Colliot-Thélène–Sansuc–Soulé [8] and by Kato–Saito [24] that over a finite field, the Chow group of 0-cycles of degree 0 of a proper variety is finite.

Assume now that $X \subset \bar{X}$ is a compactification as above and let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. For a curve $C \in \mathrm{Cu}(X)$ and an effective divisor $E \in \mathrm{Div}^+(\bar{C})$ with support in $\bar{C} \setminus C$, where $\bar{C}$ is the smooth compactification of C, let

$$P_{k(C)}(E) = \{g \in k(C)^\times | \mathrm{ord}_x(1-g) \geq \mathrm{mult}_x(E) + 1 \text{ for } x \in \bar{C} \setminus C\}$$

be the unit group with modulus well known from the ideal theoretic version of global class field theory. Set

$$\mathrm{CH}_0(X, D) = Z_0(X)/\mathrm{im}[\bigoplus_{C \in \mathrm{Cu}(X)} P_{k(C)}(\bar{\phi}^* D)].$$

Here $\bar{\phi} : \bar{C} \to \bar{X}$ is the extension of the natural morphism $\phi : C \to X$. A similar Chow group of 0-cycles is used in [17], [31] to define 2-skeleton Albanese varieties. For $D = 0$ and $\bar{X} \setminus X$ a simple normal crossing divisor it is isomorphic to the Suslin homology group $H_0(X)$ [34]. For $\dim(X) = 1$ it is the classical ideal class group with modulus $D + E$, where E is the reduced divisor with support $\bar{X} \setminus X$.

From Deligne's finiteness Theorem 2.4 and class field theory one immediately obtains a finiteness result which was expected to hold in higher dimensional class field theory.

Theorem 8.1. *For any $D \in \mathrm{Div}^+(\bar{X})$ as above the kernel of the degree map from $\mathrm{CH}_0(X, D)$ to $\mathbb{Z}$ is finite.*

8.2. **Coefficients of characteristic polynomial of the Frobenii at closed points.** In [9, Conjecture 1.2.10] Deligne conjectured that sheaves $V \in \mathcal{R}_r(X)$ with certain obviously necessary properties should behave as if they all came from geometry, i.e. as if they were ℓ-adic realizations of pure motives over X. In particular they should not only be 'defined over' $\bar{\mathbb{Q}}_\ell$, but over $\bar{\mathbb{Q}}$. In this section we explain how this latter conjecture of Deligne (for the precise formulation see Corollary 8.3 below), follows from Theorem 2.4.

In fact Corollary 8.3 is the main result of Deligne's article [12]. His proof uses Bombieri's upper estimates for the ℓ-adic Euler characteristic of an affine variety defined over a finite field, (and Katz' improvement for the Betti numbers) in terms of the embedding dimension, the number and the degree of the defining equations, which rests, aside of Weil II, on Dwork's p-adic methods. In [18] it was observed that one could replace the use of p-adic cohomology theory by some more elementary ramification theory. After this Deligne extended his methods in [13] to obtain the Finiteness Theorem 2.4.

For $V \in \mathcal{V}_r(X)$ and $x \in |X|$ one defines the characteristic polynomial of Frobenius $f_V(x) \in \bar{\mathbb{Q}}_\ell[t]$ at the point x, see Section 4.1. Let $E(V)$ be the subfield of $\bar{\mathbb{Q}}_\ell$ generated by all coefficients of all the polynomials $f_V(x)$ where $x \in |X|$ runs through the closed points.

Theorem 8.2. *Let $D \in \mathrm{Div}^+(\bar{X})$ be an effective Cartier divisor with support in $\bar{X} \setminus X$. For $V \in \mathcal{V}_r(X, D)$ irreducible with $\det(V)$ of finite order, the field $E(V)$ is a number field.*

In Section 7 we deduce Theorem 8.2 from Theorem 2.4. By associating to $V \in \mathcal{R}_r(X)$ its 2-skeleton sheaf in $\mathcal{V}_r(X)$, one finally obtains Deligne's conjecture [12, Conj. 1. 2.10(ii)] from Weil II.

Corollary 8.3. *For $V \in \mathcal{R}_r(X)$ irreducible with $\det(V)$ of finite order the field $E(V)$ is a number field.*

In fact by Proposition 3.9 there is a divisor D such that $V \in \mathcal{R}_r(X, D)$. Then apply Theorem 8.2 to the induced 2-skeleton sheaf in $\mathcal{V}_r(X, D)$.

9. Deligne's conjecture on the number of irreducible lisse sheaves of rank r over a smooth curve with prescribed local monodromy at infinity

Let C be a smooth quasi-projective geometrically irreducible curve over $\mathbb{F}_q$, and $C \hookrightarrow \bar{C}$ be a smooth compactification. One fixes an algebraic closure $\mathbb{F} \supset \mathbb{F}_q$ of $\mathbb{F}_q$. For each point $s \in (\bar{C} \setminus C)(\mathbb{F})$, one fixes a $\bar{\mathbb{Q}}_\ell$-representation V_s of the inertia

$$I(s) = \mathrm{Gal}(K_s^{\mathrm{sep}}/K_s)$$

where K_s is the completion of the function field $K = k(C)$ at s. We write

$$I(\bar{s}) = P \rtimes \prod_{\ell' \neq p} \mathbb{Z}_\ell(1),$$

where P is the wild inertia, a pro-p-group. A generator $\xi_{\ell'}$ of $\mathbb{Z}_{\ell'}(1)$, $\ell' \neq p$, acts on V_s for all $s \in (\bar{C} \setminus C)(\mathbb{F})$. Since the open immersion $j : C \hookrightarrow \bar{C}$ is defined over $\mathbb{F}_q$, if $s \in (\bar{C} \setminus C)(\mathbb{F})$ is defined over $\mathbb{F}_{q^n}$, for any conjugate point $s' \in (\bar{C} \setminus C)(\mathbb{F})$, the group $I(s')$ is conjugate to $I(s)$ by $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. One requires the following condition to be fulfilled.

   i) If $s' \in (\bar{C} \setminus C)(\mathbb{F})$ is $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-conjugate to s, the conjugation which identifies $I(s')$ and $I(s)$ identifies $V_{s'}$ and V_s.

Let V be an irreducible lisse $\bar{\mathbb{Q}}_\ell$ sheaf of rank r on $C \otimes_{\mathbb{F}_q} \mathbb{F}$ such that the set of isomorphism classes of restrictions $\{V \otimes K_s\}$ to $\mathrm{Spec}\, K_s$ is the set $\{V_s\}$ defined above with the condition i). Then if for a natural number $n \geq 1$, V is F^n invariant, V descends to a Weil sheaf on $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. By Weil II, (1.3.3), $\det(V)$ is torsion. Thus by the dimension one case of Theorem 2.1 the cardinality of the set of such F^n-invariant sheaves V is finite.

If such a V exists, then the set $\{V_{\bar{s}}\}$ satisfies automatically

   ii) For any $\ell' \neq p$, $\xi_{\ell'}$ acts trivially on $\otimes_{s \in (\bar{C} \setminus C)(\mathbb{F})} \det(V_s)$.

Indeed, as $\det(V)$ is torsion, a p power $\det(V)^{p^N}$ has torsion t prime to p, thus defines a class in $H^1(C \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t)$. The exactness of the localization sequence $H^1(C \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t) \xrightarrow{\mathrm{res}} \oplus_{s \in (\bar{C} \setminus C)(\mathbb{F})} \mathbb{Z}/t \xrightarrow{\mathrm{sum}} H^2(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, \mu_t) = \mathbb{Z}/t$ implies that the sum of the residues is 0. This shows ii).

Furthermore, if such a V exists, then the set $\{V_{\bar{s}}\}$ satisfies automatically

   iii) The action of $\xi_{\ell'}$ on V_s is quasi-unipotent for all $\ell' \neq p$ and all $s \in (\bar{C} \setminus C)(\mathbb{F})$.

Indeed, this is Grothendieck's theorem, see [33, Appendix].

Given a set $\{V_s\}$ for all $s \in (\bar{C} \setminus C)(\mathbb{F})$, satisfying the conditions i), ii), iii), Conjecture 9.1 predicts a qualitative shape for the cardinality of the F^n invariants of the set M of irreducible lisse $\bar{\mathbb{Q}}_\ell$ sheaves on $C \otimes_{\mathbb{F}_q} \mathbb{F}$ of rank r with $V \otimes K_s$ isomorphic to V_s.

If V is an element of M, then $H^0(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd(V)) = \bar{\mathbb{Q}}_\ell$, spanned by the identity. Indeed, a global section is an endomorphism $V \xrightarrow{f} V$ on $C \otimes_{\mathbb{F}_q} \mathbb{F}$. f is defined by an endomorphism of the $\bar{\mathbb{Q}}_\ell$ vector space V_a which commutes with the action of $\pi_1(\bar{C}, a)$, where $a \in C(\mathbb{F})$ is a given closed geometric point. Since this action is irreducible, the endomorphism is a homothety. We write $\mathcal{E}nd(V) = \mathcal{E}nd(V)^0 \oplus \bar{\mathbb{Q}}_\ell$, where $\mathcal{E}nd(V)^0$ is the trace-free part, thus $j_*\mathcal{E}nd(V) = j_*\mathcal{E}nd(V)^0 \oplus \bar{\mathbb{Q}}_\ell$. Thus $H^0(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) = 0$. The cup-product

$$j_*\mathcal{E}nd(V)) \times j_*\mathcal{E}nd(V) \to j_*\bar{\mathbb{Q}}_\ell = \bar{\mathbb{Q}}_\ell$$

obtained by composing endomorphisms and then taking the trace induces the perfect duality

(9.1)
$$H^i(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) \times H^{2-i}(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) \to H^2(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{\mathbb{Q}}_\ell).$$

For $i = 1$, the bilinear form (9.1) is symplectic. We conclude that $H^2(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V)) = 0$ and that $H^1(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd^0(V))$ is even dimensional. But $\dim H^1(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{\mathbb{Q}}_\ell) = 2g$ thus $H^1(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd(V))$ is even dimensional as well. We define

$$2d = \dim H^1(\bar{C} \otimes_{\mathbb{F}_q} \mathbb{F}, j_*\mathcal{E}nd(V)).$$

Conjecture 9.1. (Deligne's conjecture)

    i) There are finitely many Weil numbers a_i, b_j of weight between 0 and $2d$ such that

$$N(n) = \sum_i a_i^n - \sum_j b_j^n$$

    ii) If $M \neq \emptyset$, there is precisely one of the numbers a_i, b_j of weight $2d$ and moreover, it is one of the a_i and is equal to q^d.

An example where $M = \emptyset$ is given by $\bar{C} = \mathbb{P}^1$, C is the complement of 3 rational points $\{0, 1, \infty\}$, the rank r is 2 and the V_s are unipotent, so in particular, the Swan conductor at the 3 points is 0. Indeed, fixing ℓ', the inertia groups $I(s)$ at the 3 points, which depend on the choice of a base point, can be chosen so the product over the 3 points of the $\xi_{\ell'}$ is equal to 1. Thus the set $\{V_s, s = 0, 1, \infty\}$ is defined by 3 unipotent matrices A_0, A_1, A_∞ in $\mathrm{GL}(2, \bar{\mathbb{Q}}_\ell)$ such that $A_0 \cdot A_1 \cdot A_\infty = 1$. Since $A_0 \cdot A_1$ is then unipotent, A_0 and A_1, and thus A_∞, lie in

the same Borel subgroup of $GL(2, \bar{\mathbb{Q}}_\ell)$. Thus the 3 matrices have one common eigenvector. Since the tame fundamental group is spanned by the images of $I(0), I(1), I(\infty)$, a $\bar{\mathbb{Q}}_\ell$-sheaf of rank 2 with $V \otimes K_s$ isomorphic to V_s is not irreducible. Thus $M = \emptyset$.

Two further examples are computed in [14]. For the first case [14, section 7], $C = \mathbb{P}^1 \setminus D$ where D is a reduced degree 4 divisor, with unipotent $V_{\bar{s}}$. The answer is $N(n) = q^n$. For the second case, $C = \mathbb{P}^1 \setminus D$ where D is a reduced non-irreducible degree 3 divisor with unipotent $V_{\bar{s}}$ with only one Jordan block (a condition which could be forced by the irreducibility condition for V). Then $N(n) = q^n$ as well.

Appendix A

In this appendix we gather a few facts on how to recognize through their closed points affine schemes of finite type as subschemes of affine schemes not necessarily of finite type.

Proposition A.1. *Let k be an algebraically closed field, let Y be an affine k-scheme. Then the map*

$$Z \mapsto Z(k)$$

embeds the set of reduced closed subschemes $Z \hookrightarrow Y$ of finite type into the power set $\mathcal{P}(Y(k))$.

Proof. Choose a filtered direct system $B_\alpha \subset B = k(Y)$ of affine k-algbras (of finite type), such that $B = \varinjlim_\alpha B_\alpha$. Set $Y_\alpha = \operatorname{Spec} B_\alpha$. Consider two closed subschemes

(A.1) $\qquad Z_1 = \operatorname{Spec} B/I_1 \hookrightarrow Y, \quad Z_2 = \operatorname{Spec} B/I_2 \hookrightarrow Y$

of finite type over k such that $Z_1(k) = Z_2(k) \subset Y(k)$. After replacing the direct system α by a cofinal subsystem we can assume that $B_\alpha \to B/I_1$ and $B_\alpha \to B/I_2$ are surjective. Hilbert's Nullstellensatz for the closed subschemes $Z_1 \hookrightarrow Y_\alpha$ and $Z_2 \to Y_\alpha$ implies $I_1 \cap B_\alpha = I_2 \cap B_\alpha$. So $I_1 = I_2$ and the closed subschemes (A.1) agree. $\qquad\square$

Proposition A.2. *Let k be a characteristic 0 field, let $K \supset k$ be an algebraically closed field extension. Let Y be an affine scheme over k, and $Z \hookrightarrow Y \otimes_k K$ be a closed embedding of an affine scheme of a finite type. If the subset $Z(K)$ of $Y(K)$ is invariant under the automorphism group of K over k, then there is a reduced closed subscheme $Z_0 \hookrightarrow Y$ of finite type over k such that*

$$(Z \hookrightarrow Y \otimes_k K) = (Z_0 \hookrightarrow Y) \otimes_k K.$$

Proof. Let $G = \mathrm{Aut}(K/k)$, $B = k(Y)$, $Z = \mathrm{Spec}\,((B \otimes_k K)/I)$. The G-stability of $Z(K) \subset Y(K)$ and Proposition A.1 imply that $I \subset B \otimes_k K$ is stable under G. Then [6, Sec. V.10.4] implies that $I_0 = I^G \subset B$ satisfies $I_0 \otimes_k K = I$. Set $Z_0 = \mathrm{Spec}\,B/I_0$.

$\square$

Proposition A.3. *Let k be an algebraically closed field, let $\varphi : Z \to Y$ be an integral k-morphism of affine schemes, with Z of finite type over k. Then there is a uniquely defined reduced closed subscheme $X \hookrightarrow Y$ of finite type over k such that*

$$\varphi(Z(k)) = X(k).$$

Proof. Write $Y = \mathrm{Spec}\,B$, $Z = \mathrm{Spec}\,C$, for commutative k-algebras B, C with C of finite type over k. Without loss of generality assume that B and C are reduced. There are finitely many elements of C which span C as a k-algebra. They are integral over B. This defines finitely many minimal polynomials, thus finitely many coefficients of those polynomials in B. Thus there is an affine k-algebra of finite type $B_0 \subset B$ containing them all. It follows that C is finite over B_0. Choose a filtered inverse system $Y_\alpha = \mathrm{Spec}\,B_\alpha$ of affine k-schemes of finite type, such that $B_\alpha \subset B$ and

$$Y = \mathrm{Spec}\,B = \varprojlim_\alpha Y_\alpha.$$

The morphisms $\varphi_\alpha : Z \xrightarrow{\varphi} Y \to Y_\alpha$ are all finite. Let $X_\alpha = \mathrm{Spec}\,C_\alpha \hookrightarrow Y_\alpha$ be the (reduced) image of φ_α. We obtain finite ring extensions $C_\alpha \subset C$. By Noether's basis theorem the filtered direct system C_α stabilizes at some α_0. Then

$$X = \mathrm{Spec}\,C_{\alpha_0} = \varprojlim_\alpha \mathrm{Spec}\,C_\alpha \hookrightarrow Y$$

is of finite type over k and satisfies $\varphi(Z(k)) = X(k)$. $\square$

Proposition A.4. *Let k be an algebraically closed field of characteristic 0, let Y be an affine k-scheme, such that $Y = \mathrm{Spec}\,B = \varprojlim_n Y_n$, $n \in \mathbb{N}$ is the projective limit of reduced affine schemes Y_n of finite type. If the transition morphisms induce bijections $Y_{n+1}(k) \xrightarrow{\cong} Y_n(k)$ on closed points, then there is a $n_0 \in \mathbb{N}$ such that $Y_n \to Y_{n_0}$ is an isomorphism for all $n \geq n_0$. In particular, $Y \to Y_{n_0}$ is an isomorphism as well.*

Proof. Applying Zariski's Main Theorem [2, Thm.4.4.3], one constructs inductively affine schemes of finite type $\bar{Y}_n$, $\bar{Y}_0 = Y_0$, together with an open embedding $Y_n \hookrightarrow \bar{Y}_n$, such that the transition morphisms $Y_{n+1} \to Y_n$ extend to finite transition morphisms $\bar{Y}_{n+1} \to \bar{Y}_n$. On the

other hand, the assumption implies that the morphisms $Y_{n+1} \to Y_n$ are birational on every irreducible component. So the same property holds true for $\bar{Y}_{n+1} \to \bar{Y}_n$. One thus has a factorization $\widetilde{Y}_0 \to \bar{Y}_n \to Y_0$ for all n, where $\widetilde{Y}_0 \to Y_0$ the normalization morphism. Since $\widetilde{Y}_0$ is of finite type, there is a n_0 such that $\bar{Y}_n \to \bar{Y}_{n_0}$ is an isomorphism for all $n \geq n_0$. Thus the composite morphism $Y_n \to Y_{n_0} \to \bar{Y}_{n_0}$ is an open embedding for all $n \geq n_0$, and thus $Y_{n+1} \to Y_n$ is an open embedding as well. Since it induces a bijection on points, and the Y_n are reduced, the transition morphisms $Y_{n+1} \to Y_n$ are isomorphisms for $n \geq n_0$. $\square$

Remark A.5. If in Proposition A.4, one assumes in addition that the transition morphisms $Y_{n+1} \to Y_n$ are finite, then one does not need Zariski's Main Theorem to conclude.

Appendix B

In the proof of Corollary 4.5 we claim the existence of a curve with certain properties. The Bertini argument given in [27, p. 201] for the construction of such a curve is, as such, not correct. We give a complete proof here relying on Hilbert irreducibility instead of Bertini.

Let X be in $\mathrm{Sm}_{\mathbb{F}_q}$.

Proposition B.1. *For $V \in \mathcal{R}_r(X)$ irreducible and a closed point $x \in X$, there is an irreducible smooth curve $C/\mathbb{F}_q$ and a morphism $\psi : C \to X$ such that*

- *$\psi^*(V)$ is irreducible,*
- *x is in the image of ψ.*

Lemma B.2. *For an irreducible $\bar{\mathbb{Q}}_\ell$-étale sheaf V on X there is a connected étale covering $X' \to X$ with the following property:*
For a smooth irreducible curve $C/\mathbb{F}_q$ and a morphism $\psi : C \to X$ the implication

$$C \times_X X' \text{ irreducible} \implies \psi^*(V) \text{ irreducible}$$

holds.

Proof. Choose a finite normal extension R of $\mathbb{Z}_\ell$ with maximal ideal $\mathfrak{m} \subset R$ such that V is induced by a continuous representation

$$\rho : \pi_1(X) \to GL(R, r).$$

Let H_1 be the kernel of $\pi_1(X) \to GL(R/\mathfrak{m}, r)$ and let G be the image of ρ. The subgroup

$$H_2 = \bigcap_{\nu \in \mathrm{Hom}(H_1, \mathbb{Z}/\ell)} \ker(\nu)$$

is open normal in $\pi_1(X)$ according to [3, Th. Finitude]. Indeed observe that $H_1/H_2 = H_1^{\mathrm{ab}}/\ell$ is Pontryagin dual to $H^1_{\mathrm{\acute{e}t}}(X_{H_1}, \mathbb{Z}/\ell)$, where X_{H_1} is the étale covering of X associated to H_1. Since the image of H_1 in G is pro-ℓ, and therefore pro-nilpotent, any morphism of pro-finite groups $K \to \pi_1(X)$ satisfies:

$$(K \to \pi_1(X)/H_2 \quad \text{surjective }) \implies (K \to G \text{ surjective }).$$

(Use [6, Cor. I.6.3.4].)

Finally, let $X' \to X$ be the Galois covering corresponding to H_2. $\quad\square$

Proof of Proposition B.1. We can assume that X is affine. By Proposition 4.3 we can, after some twist, assume that V is étale. Let X' be as in the lemma. By Noether normalization, e.g. [16, Corollary 16.18], there is a finite generically étale morphism

$$f : X \to \mathbb{A}^d.$$

Let $U \subset \mathbb{A}^d$ be an open dense subscheme such that $f^{-1}(U) \to U$ is finite étale. Let $y \in \mathbb{A}^d$ be the image of x. Choose a linear projection $\pi : \mathbb{A}^d \to \mathbb{A}^1$ and set $z = \pi(y)$ and consider the map $h : U \to \mathbb{A}^1$. By definition, $U_{k(\mathbb{A}^1)} \subset \mathbb{A}^{d-1}_{k(\mathbb{A}^1)}$.

Let $F = k(\Gamma) \supset k(\mathbb{A}^1)$ be a finite extension such that $X' \otimes_{k(\mathbb{A}^1)} F$ is irreducible and the smooth curve $\Gamma \to \mathbb{A}^1$ contains a closed point z' with $k(z') = k(y)$.

It is easy to see that there is an $\hat{F}$-point in $U_{k(\mathbb{A}^1)}$ which specializes to y. By Hilbert irreducibility, see [15, Cor. A.2], we find an F-point $u \in U_{k(\mathbb{A}^1)}$ which specializes to y and such that u does not split in $X' \times_{\mathbb{A}^1} \Gamma$.

Let $v \in X$ be the unique point over u. By the going-down theorem [7, Thm. V.2.4.3] the closure $\overline{\{v\}}$ contains x. Finally, we let C be the normalization of $\overline{\{v\}}$. $\quad\square$

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