RATIONAL POINTS OVER FINITE FIELDS FOR REGULAR MODELS OF ALGEBRAIC VARIETIES OF HODGE TYPE ≥ 1

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Abstract. Let R be a discrete valuation ring of mixed characteristics $(0, p)$, with finite residue field k and fraction field K, let k' be a finite extension of k, and let X be a regular, proper and flat R-scheme, with generic fibre X_K and special fibre X_k. Assume that X_K is geometrically connected and of Hodge type ≥ 1 in positive degrees. Then we show that the number of k'-rational points of X satisfies the congruence $|X(k')| \equiv 1 \bmod |k'|$. Thanks to [BBE07], we deduce such congruences from a vanishing theorem for the Witt cohomology groups $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}})$, for $q > 0$. In our proof of this last result, a key step is the construction of a trace morphism between the Witt cohomologies of the special fibres of two flat regular R-schemes X and Y of the same dimension, defined by a surjective projective morphism $f : Y \to X$.

Contents

1. Introduction and first reductions

Let R be a discrete valuation ring of mixed characteristics $(0, p)$, with perfect residue field k, and fraction field K. The main goal of this article is to prove the following theorem.

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Theorem 1.1. *Let X be a proper and flat R-scheme, with generic fibre X_K, such that the following conditions hold:*

a) *X is a regular scheme.*

b) *X_K is geometrically connected.*

c) *$H^q(X_K, \mathcal{O}_{X_K}) = 0$ for all $q \geq 1$.*

If k is finite, then, for any finite extension k' of k, the number of k'-rational points of X satisfies the congruence

$$|X(k')| \equiv 1 \mod |k'|. \tag{1.1.1}$$

Condition c) should be viewed as a Hodge theoretic property of X_K, which can be stated by saying that X_K has Hodge type ≥ 1 in positive degrees. From this point of view, this theorem fits in the general analogy between the vanishing of Hodge numbers for varieties over a field of characteristic 0, and congruences on the number of rational points with values in finite extensions for varieties over a finite field. This analogy came to light with the coincidence between the numerical values in Deligne's theorem on smooth complete intersections in a projective space [SGA 7 II, Exposé XI, Th. 2.5], and in the Ax-Katz theorem on congruences on the number of solutions of systems of algebraic equations [Kz71, Th. 1.0]. It has been made effective by Katz's conjecture [Kz71, Conj. 2.9] relating the Newton and Hodge polygons associated to the cohomology of a proper and smooth variety (and generalizing earlier results of Dwork for hypersurfaces [Dw64]). For varieties in characteristic p, this conjecture was proved by Mazur ([Ma72], [Ma73]) and Ogus [BO78, Th. 8.39]. In the mixed characteristic case, where a stronger form can be given using the Hodge polygon of the generic fibre, it is a consequence of the fundamental results in p-adic Hodge theory. Our proof of Theorem 1.1 makes essential use of the unequality between these two polygons, but the setup of the theorem is actually more general, since the scheme X is not supposed to be semi-stable over R.

Let us also recall that a result similar to Theorem 1.1 has been proved by the second author [Es06, Th. 1.1] by ℓ-adic methods, with condition c) replaced by a coniveau condition: for any $q \geq 1$, any cohomology class in $H^q_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ vanishes in $H^q_{\text{ét}}(U_{\overline{K}}, \mathbb{Q}_\ell)$ for some non empty open subset $U \subset X_K$. It is easy to see, using [De71], that this coniveau condition implies that the Hodge level of X_K is ≥ 1 in degree $q \geq 1$. It would actually follow from Grothendieck's generalized Hodge conjecture [Gr69] that the two conditions are equivalent. In this article, the use of p-adic methods, and in particular of p-adic Hodge theory, allows us to derive congruence (1.1.1) directly from Hodge theoretic hypotheses.

1.2. As explained by Ax [Ax64], congruences such as (1.1.1) can be expressed in terms of the zeta function of the special fibre X_k of X. We recall that the rationality of the zeta function $Z(X_k, t)$ allows to define the slope < 1 part $Z^{<1}(X_k, t)$ of $Z(X_k, t)$ as follows [BBE07, 6.1]. Let $|k| = p^a$, and write

$$Z(X_k, t) = \prod_i (1 - \alpha_i t) / \prod_j (1 - \beta_j t),$$

with $\alpha_i, \beta_j \in \overline{\mathbb{Q}}_p$ and $\alpha_i \neq \beta_j$ for all i, j. Normalizing the p-adic valuation v of $\overline{\mathbb{Q}}_p$ by $v(p^a) = 1$, one sets

$$Z^{<1}(X_k, t) = \prod_{v(\alpha_i)<1} (1 - \alpha_i t) / \prod_{v(\beta_j)<1} (1 - \beta_j t).$$

Then the congruences (1.1.1) are equivalent to

$$Z^{<1}(X_k, t) = \frac{1}{1-t} \tag{1.2.1}$$

[BBE07, Prop. 6.3].

On the other hand, let $W(\mathcal{O}_{X_k})$ be the sheaf of Witt vectors with coefficients in $\mathcal{O}_{X_k}$, and $W\mathcal{O}_{X_k,\mathbb{Q}} = W(\mathcal{O}_{X_k}) \otimes \mathbb{Q}$. Then the identification of the slope < 1 part of rigid cohomology with Witt vector cohomology provides the cohomological interpretation

$$Z^{<1}(X_k, t) = \prod_i \det(1 - tF^a | H^i(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}))^{(-1)^{i+1}}, \tag{1.2.2}$$

where F is induced by the Frobenius endomorphism of $W(\mathcal{O}_{X_k})$ [BBE07, Cor. 1.3]. Therefore, Theorem 1.1 is a consequence of the following theorem, where k is only assumed to be perfect:

Theorem 1.3. *Let X be a regular, proper and flat R-scheme. Assume that $H^q(X_K, \mathcal{O}_{X_K}) = 0$ for some $q \geq 1$. Then:*

$$H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) = 0. \tag{1.3.1}$$

Proof of Theorem 1.1, assuming Theorem 1.3. Let us prove here this implication, which is easy and does not use the regularity assumption on X. Let $W = W(k)$, and $K_0 = \mathrm{Frac}(W)$. Thanks to (1.2.1) and (1.2.2), Theorem 1.3 implies that it suffices to prove that the homomorphism $K_0 \to H^0(X_k, W\mathcal{O}_{X_k,\mathbb{Q}})$ is an isomorphism.

Since X is proper and flat over R, $H^0(X, \mathcal{O}_X)$ is a free finitely generated R-module. As the generic fibre X_K is geometrically connected and geometrically reduced, the rank of $H^0(X, \mathcal{O}_X)$ is 1. The homomorphism $R \to H^0(X, \mathcal{O}_X)$ maps 1 to 1, hence Nakayama's lemma implies that it is an isomorphism. Applying Zariski's connectedness theorem, it follows that X_k is connected, and even geometrically connected, since the same argument can be applied after any base change from R to R', where R' is the ring of integers of a finite extension of K.

On the other hand, let $\bar{k}$ be an algebraic closure of k, and let k' be a finite extension of k such that $X_{\bar{k}\,\mathrm{red}}$ is defined over k'. As k' is separable over k, the homomorphisms $W_n(k) \to W_n(k')$ are finite étale liftings of $k \to k'$, and the homomorphisms $W_n(k') \otimes_{W_n(k)} W_n(\mathcal{O}_{X_k}) \to W_n(\mathcal{O}_{X_{k'}})$ are isomorphisms [Il79, I, Prop. 1.5.8]. It follows that the homomorphism $W(k') \otimes_{W(k)} H^0(X_k, W(\mathcal{O}_{X_k})) \to H^0(X_{k'}, W(\mathcal{O}_{X_{k'}}))$ is an isomorphism, and that it suffices to prove the claim for $X_{k'}$. Using the fact that

$$H^0(X_{k'}, W\mathcal{O}_{X_{k'},\mathbb{Q}}) \xrightarrow{\sim} H^0(X_{k'\,\mathrm{red}}, W\mathcal{O}_{X_{k'\,\mathrm{red}},\mathbb{Q}})$$

by [BBE07, Prop. 2.1 (i)], it suffices to check that, if Z is a proper, geometrically connected and geometrically reduced k-scheme, the homomorphism $W(k) \to H^0(Z, W(\mathcal{O}_Z))$ is an isomorphism.

Under these assumptions, the homomorphism $k \to H^0(Z, \mathcal{O}_Z)$ is an isomorphism. As the homomorphism $R : W_n(\mathcal{O}_Z) \to W_{n-1}(\mathcal{O}_Z)$ is the projection of a product onto one of its factors, the homomorphisms $H^0(Z, W_n(\mathcal{O}_Z)) \to H^0(Z, W_{n-1}(\mathcal{O}_Z))$ are surjective, and one gets by induction that the homomorphism $W_n(k) \to H^0(Z, W_n(\mathcal{O}_Z))$ is an isomorphism for all n. Taking inverse limits, the claim follows. □

1.4. Theorem 1.3 is deeper, and most of our paper is devoted to developing the techniques used in its proof. We may observe though that, in the context of Theorem 1.1, there is a case where (1.3.1) is trivial: namely, if we replace the condition on the Hodge numbers of X_K, which is equivalent to requiring that the modules $H^q(X, \mathcal{O}_X)$ be p-torsion modules, by the stronger condition that $H^q(X, \mathcal{O}_X)$ vanishes for all $q \geq 1$. Indeed, the flatness of X over R allows to apply the derived base change formula for coherent cohomology and to conclude that $H^q(X_k, \mathcal{O}_{X_k}) = 0$ for all $q \geq 1$. By induction on n, one gets that $H^q(X_k, W_n(\mathcal{O}_{X_k})) = 0$ for all $n, q \geq 1$, and (1.3.1) follows for all $q \geq 1$ (even before tensoring with $\mathbb{Q}$).

In the general case, where the $H^q(X, \mathcal{O}_X)$ are p-torsion modules, we do not know any direct argument to derive the vanishing property stated in (1.3.1). Our strategy is then to use the results of p-adic Hodge theory relating the Hodge and Newton polygons of certain filtered F-isocrystals on k, which allow to study separately the cohomology groups for a given q as in Theorem 1.3. In particular, when X is semistable on R, a straightforward argument using the fundamental comparison theorems of p-adic Hodge theory allows to deduce (1.3.1) from the unequality between the two polygons defined by the log crystalline cohomology of X_k. We explain this argument in Theorem 2.1.

In the rest of Section 2, we show that this argument can be modified to prove the vanishing of $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}})$ in the general case. For any finite extension K' of K, with ring of integers R', let $X_{R'}$ be deduced from X by base change from R to R'. After reducing to the case where R is complete, the first step is to apply de Jong's alteration theorem to construct for any m an m-truncated simplicial scheme $Y_\bullet$ over the ring of integers R' of a suitable extension K' of K, endowed with an augmentation morphism $Y_0 \to X_{R'}$, such that the Y_i's are pullbacks of proper semistable schemes, and $Y_\bullet \to X_{R'}$ induces an m-truncated proper hypercovering of $X_{K'}$ (see Lemma 2.2 for a precise statement). Then, using Tsuji's extension of the comparison theorems to truncated simplicial schemes [Ts98], we show that, in this situation, the cohomology group $H^q(Y_{\bullet k}, W\mathcal{O}_{Y_{\bullet k},\mathbb{Q}})$ vanishes. However, due to the possible presence of vertical components in the coskeletons, the special fibre $Y_{\bullet k}$ of the m-truncated simplicial scheme $Y_\bullet$ may not be a proper hypercovering of X_k, and it is unclear how the groups $H^q(Y_{\bullet k}, W\mathcal{O}_{Y_{\bullet k},\mathbb{Q}})$ are related to the groups $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}})$. Therefore another ingredient will be necessary to complete the proof. It will be provided by the following injectivity theorem, the proof of which will be given in section 8.

Theorem 1.5. *Let X, Y be two flat, regular R-schemes of finite type, of the same dimension, and let $f : Y \to X$ be a projective and surjective R-morphism, with* *reduction f_k over* $\operatorname{Spec} k$*. Then, for all $q \geq 0$, the functoriality homomorphism*

$$f_k^* : H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) \longrightarrow H^q(Y_k, W\mathcal{O}_{Y_k,\mathbb{Q}}) \tag{1.5.1}$$

is injective.

1.6. We will deduce Theorem 1.5 from the existence of a trace morphism

$$\tau_{i,\pi} : \mathbb{R}f_*(W\mathcal{O}_{Y_k,\mathbb{Q}}) \longrightarrow W\mathcal{O}_{X_k,\mathbb{Q}}, \tag{1.6.1}$$

defined by means of a factorization $f = \pi \circ i$, where π is the projection of a projective space $\mathbb{P}^d_X$ on X, and i is a closed immersion. The key fact used in the construction of this trace morphism is that, under the assumptions of Theorem 1.5, i is a regular immersion of codimension d, or, said otherwise, that f is a complete intersection morphism of virtual relative dimension 0, in the sense of [SGA 6, Exposé VIII].

Sections 3 to 7 are devoted to the construction of $\tau_{i,\pi}$. In section 3, we state a similar result for $\mathcal{O}_X$, providing a canonical trace morphism

$$\tau_f : \mathbb{R}f_*(\mathcal{O}_Y) \to \mathcal{O}_X,$$

whenever X is a noetherian scheme with a relative dualizing complex, and $f : Y \to X$ is a proper complete intersection morphism of virtual relative dimension 0 (see Theorem 3.1). The existence of τ_f has been observed by El Zein as a particular case of his construction of the relative fundamental class [El78, IV, Prop. 6]. However, there does not seem to be in the literature a complete proof of the properties listed in Theorem 3.1. Due to the many corrections and complements to [Ha66] made by Conrad in [Co00], we have included in an Appendix the details of a proof of Theorem 3.1 based on [Co00]. So we refer to B.7 for the definition of τ_f, and to B.9 for the proof of Theorem 3.1. When Y is finite locally free of rank r over X, the composition of the functoriality morphism $\mathcal{O}_X \to \mathbb{R}f_*(\mathcal{O}_Y)$ with τ_f is multiplication by r on $\mathcal{O}_X$. This has striking consequences for the functoriality maps induced by f on coherent cohomology (see Theorem 3.2). For example, if r is invertible on X, one obtains an injectivity theorem which may be of independent interest. An outline of the construction of τ_f is given in the introduction to the Appendix.

To construct the trace morphism $\tau_{i,\pi}$, we consider more generally a projective complete intersection morphism $f : Y \to X$ of virtual relative dimension 0 between two noetherian $\mathbb{F}_p$-schemes with dualizing complexes. Under these assumptions, we construct a compatible family of morphisms

$$\tau_{i,\pi,n} : \mathbb{R}f_*(W_n(\mathcal{O}_Y)) \to W_n(\mathcal{O}_X)$$

for $n \geq 1$, with $\tau_{i,\pi,1} = \tau_f$. Our main tool here is the theory of the relative de Rham-Witt complex developped by Langer and Zink [LZ04]. In Section 5, we recall some basic facts about their construction, and we extend to the relative case some structure theorems proved by Illusie [Il79] when the base scheme is perfect (see in particular Proposition 5.7 and Theorem 5.13). Then we define $\tau_{i,\pi,n}$ by combining two morphisms. On the one hand, we consider a projective space $P := \mathbb{P}^d_X$ with projection π on X, and we define in Section 6 a trace morphism

$$\mathrm{Trp}_{\pi,n} : \mathbb{R}\pi_*(W_n\Omega^d_{P/X}[d]) \to W_n(\mathcal{O}_X),$$

using the d-th power of the Chern class of the canonical bundle $\mathcal{O}_P(1)$. On the other hand, we consider a regularly embedded closed subscheme Y of a smooth X-scheme P, and we define in Section 7 a relative Hodge-Witt fundamental class for Y in P, which is a section of $\mathcal{H}^d_Y(W_n\Omega^d_{P/X})$ and defines a morphism

$$\gamma_{i,\pi,n} : i_*W_n(\mathcal{O}_Y) \to W_n\Omega^d_{P/X}[d],$$

with $i : Y \hookrightarrow P$ and $d = \mathrm{codim}_P(Y)$. This allows to define the morphism $\tau_{i,\pi,n}$ as being the composition $\mathrm{Trp}_{\pi,n} \circ \mathbb{R}\pi_*(\gamma_{i,\pi,n})$. The proof of Theorem 1.5 is then completed in Section 8 thanks to a theorem relating the morphisms $\tau_{i,\pi,n}$ defined by the reduction mod p of a factorization of the given morphism $f : Y \to X$ over R, and the morphism τ_f defined by f.

It may be worth pointing out here that these results seem to indicate that Grothendieck's relative duality theory for coherent $\mathcal{O}$-modules can be generalized to some extent to the Hodge-Witt sheaves, as was already apparent from [Ek84] when the base scheme is a perfect field. We do not try to develop such a generalization in this article, and we limit ourselves to the properties needed for the proof of Theorem 1.1. For example, it is very likely that the morphisms $\tau_{i,\pi,n}$ only depend on f, and not on the chosen factorization $f = \pi \circ i$, but this is not needed here, and we did not check it. A natural context one might think of for developing our results is the theory of the trace map for projectively embeddable morphisms outlined in [Ha66, III, 10.5 and §11]. Unfortunately, as discussed by Conrad in [Co00, p. 103-104], the foundation work needed for the definition of such a theory has not really been done even for coherent $\mathcal{O}$-modules.

Finally, we conclude in Section 9 by giving a family of examples to which Theorem 1.1 can be applied, but which are not covered by earlier results, nor by cases where Theorem 1.3 can be proved directly, such as the trivial case where $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$, or the semi-stable case. These examples are obtained for $p \geq 7$, and are quotients of an hypersurface of degree p in a projective space $\mathbb{P}^{p-2}_R$ by a free $(\mathbb{Z}/p\mathbb{Z})$-action. Their generic fibre is a smooth variety of general type, and their special fibre has isolated singularities, at least when p is not a Fermat number.

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General conventions

1) All schemes under consideration are supposed to be separated. By a projective morphism $f : Y \to X$, we always mean a morphism which can be factorized as $f = \pi \circ i$, where i is a closed immersion in some projective space $\mathbb{P}^n_X$, and π is the natural projection $\mathbb{P}^n_X \to X$.

2) In this paper, we use the terminology of [SGA 6] for complete intersection morphisms: a morphism of schemes $f : Y \to X$ is said to be a *complete intersection morphism* if, for any $y \in Y$, there exists an open neighbourhood U of y in Y such that the restriction of f to U can be factorized as $f|_U = \pi \circ i$, where π is a smooth morphism and i a regular immersion [SGA 6, VIII, 1.1]. Note that this notion of complete intersection morphism is more general than the notion of "local complete

intersection map" used in [Ha66] and [Co00], where "lci map" is only used for regular immersions.

If d is the codimension of i at y, and n the relative dimension of π at $i(y)$, the integer $m = n - d$ does not depend upon the local factorization $f|_U = \pi \circ i$, and is called the *virtual relative dimension* of f at y [SGA 6, VIII, 1.9]. One says that f has *constant virtual relative dimension* m if the integer m does not depend upon y. We will always assume in this paper that the virtual relative dimension of the morphisms under consideration is constant (however, the dimension of the fibres of such morphisms can vary).

3) Apart from the previous remark, we will use the definitions and sign conventions from Conrad's book [Co00]. In particular, when $i : Y \hookrightarrow P$ is a regular immersion of codimension d defined by an ideal $\mathcal{I} \subset \mathcal{O}_P$, we define $\omega_{Y/P}$ by

$$\omega_{Y/P} = \wedge^d((\mathcal{I}/\mathcal{I}^2)^\vee)$$

rather than $(\wedge^d(\mathcal{I}/\mathcal{I}^2))^\vee$ as in [Ha66, III, p. 141] (see [Co00, p. 7]). The canonical identification between both definitions is given by [Bo70, III, §11, Prop. 7].

4) If R, S are commutative rings, $R \to S$ a ring homomorphism, and X an R-scheme, we denote by X_S the S-scheme $\operatorname{Spec} S \times_{\operatorname{Spec} R} X$.

5) If $\mathcal{E}^\bullet$ is a complex, we denote by $(\sigma_{\geq i}\mathcal{E}^\bullet)_{i\in\mathbb{Z}}$ the naive filtration on $\mathcal{E}$, i.e., the filtration defined by $\sigma_{\geq i}\mathcal{E}^n = 0$ if $n < i$, $\sigma_{\geq i}\mathcal{E}^n = \mathcal{E}^n$ if $n \geq i$.

2. Application of p-adic Hodge theory

We explain in this section how the fundamental results of p-adic Hodge theory can be used to prove Theorem 1.3. We begin with the semi-stable case, where p-adic Hodge theory suffices to conclude, and which will serve as a model for the general case. We use the notations R, K, k as in the introduction.

Theorem 2.1. *Let X be a proper and semi-stable R-scheme, with generic fibre X_K and special fibre X_k, and let $q \geq 0$ be an integer. If $H^q(X_K, \mathcal{O}_{X_K}) = 0$, then $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) = 0$.*

Proof. We may assume that R is a complete discrete valuation ring. Indeed, if $\widehat{R}$ is the completion of R, $\widehat{K} = \operatorname{Frac}(\widehat{R})$ and $\widetilde{X} = X_{\widehat{R}}$, then $\widetilde{X}$ is proper and semi-stable over $\widehat{R}$, $H^q(\widetilde{X}_{\widehat{K}}, \mathcal{O}_{\widetilde{X}_{\widehat{K}}}) = \widehat{K} \otimes_K H^q(X_K, \mathcal{O}_{X_K}) = 0$, and X and $\widetilde{X}$ have isomorphic special fibres. So the theorem for $\widetilde{X}$ implies the theorem for X.

We endow $S = \operatorname{Spec} R$ with the log structure defined by the divisor $\operatorname{Spec} k \subset S$, $S_0 = \operatorname{Spec} k$ with the induced log structure, and we denote by $\underline{S}$, $\underline{S}_0$ the corresponding log schemes. Similarly, we endow X with the log structure defined by the special fibre X_k, X_k with the induced log structure, and we denote by $\underline{X}$, $\underline{X}_k$ the corresponding log schemes. Then $\underline{X}$ is smooth over $\underline{S}$, and $\underline{X}_k$ is smooth of Cartier type [Ka89, (4.8)] over $\underline{S}_0$.

Let $W_n = W_n(k)$ (resp. $W = W(k)$), and let $\underline{\Sigma}_n$ (resp. $\underline{\Sigma}$) be the log scheme obtained by endowing $\Sigma_n = \operatorname{Spec} W_n$ (resp. $\Sigma = \operatorname{Spec} W$) with the log structure associated to the pre-log structure defined by the morphism $M_{S_0} \to \mathcal{O}_{S_0} = \mathcal{O}_{\Sigma_1} \to \mathcal{O}_{\Sigma_n}$ (resp. $\mathcal{O}_\Sigma$) provided by composition with the Teichmüller representative map.

We can then consider the log crystalline cohomology groups $H^q_{\text{crys}}(\underline{X}/\underline{\Sigma}_n)$, which are finitely generated W_n-modules endowed with a Frobenius action φ and a monodromy operator N. The log scheme $\underline{X}_k$ also carries a logarithmic de Rham-Witt complex $W\Omega^\bullet_{\underline{X}_k} = \varprojlim_n W_n\Omega^\bullet_{\underline{X}_k}$, constructed by Hyodo [Hy91] in the semi-stable case, and generalized by Hyodo and Kato [HK94, (4.1)] to the case of smooth $\underline{S}_0$-log schemes of Cartier type. In degree 0, we have

(2.1.1) $$W_n\Omega^0_{\underline{X}_k} = W_n(\mathcal{O}_{X_k}),$$

by [HK94, Prop. (4.6)].

It follows from [HK94, Th. (4.19)] that, for all q, there are canonical isomorphisms

(2.1.2) $$H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}_n) \xrightarrow{\sim} H^q(\underline{X}_k, W_n\Omega^\bullet_{\underline{X}_k}),$$

which are compatible when n varies, and commute with the Frobenius actions. As X_k is proper over S_0, these cohomology groups are artinian W-modules. Therefore one can apply the Mittag-Leffler criterium to get canonical isomorphisms

(2.1.3)
$$H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \xrightarrow{\sim} \varprojlim_n H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}_n) \xrightarrow{\sim} \varprojlim_n H^q(\underline{X}_k, W_n\Omega^\bullet_{\underline{X}_k}) \xleftarrow{\sim} H^q(\underline{X}_k, W\Omega^\bullet_{\underline{X}_k})$$

compatible with the Frobenius actions. Using the naive filtration of $W\Omega^\bullet_{\underline{X}_k}$ and tensoring by K_0, one obtains a spectral sequence

(2.1.4) $$E_1^{i,j} = H^j(\underline{X}_k, W\Omega^i_{\underline{X}_k}) \otimes K_0 \implies H^{i+j}_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0$$

endowed by functoriality with a Frobenius action F^*. The operators d, F and V on the logarithmic de Rham-Witt complex satisfy the same relations than on the usual de Rham-Witt complex [HK94, (4.1)], and the structure theorems of [Il79] remain valid in the logarithmic case [HK94, Th. (4.4) and Cor. (4.5)]. It follows that one can argue as in the proof of [Il79, II, Th. 3.2] to prove that, for all i, j, the K_0-vector space $H^j(\underline{X}_k, W\Omega^i_{\underline{X}_k}) \otimes K_0$ is finite dimensional, and that the action of F^* on this space has slopes in $[i, i+1[$. Therefore, the spectral sequence (2.1.4) degenerates at E_1, and yields in particular an isomorphism $(H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0)^{<1} \xrightarrow{\sim} H^q(\underline{X}_k, W\Omega^0_{\underline{X}_k}) \otimes K_0$, the source being the part of $H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0$ where Frobenius acts with slope < 1. Thanks to (2.1.1), we finally get a canonical isomorphism

(2.1.5) $$(H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0)^{<1} \xrightarrow{\sim} H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}).$$

On the other hand, the choice of an uniformizer π of R determines a Hyodo-Kato isomorphism [HK94, Th. (5.1)]

(2.1.6) $$\rho_\pi : H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes_W K \xrightarrow{\sim} H^q(X_K, \Omega^\bullet_{X_K/K}).$$

This allows to endow $H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes_W K$ with the filtration deduced via ρ_π from the Hodge filtration of $H^q(X_K, \Omega^\bullet_{X_K/K})$. Together with its Frobenius action and monodromy operator, $H^q_{\text{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes_W K$ is then a filtered (φ, N)-module as defined by Fontaine [Fo94, 4.3.2 and 4.4.8]. As such, it has both a Newton polygon, built as usual from the slopes of the Frobenius action, and a Hodge polygon, built as usual from the Hodge numbers of $H^q(X_K, \Omega^\bullet_{X_K/K})$. Now, let $\overline{K}$ be an algebraic closure of

K, and let B_{st}, B_{dR} be the Fontaine p-adic period rings. Then Tsuji's comparison theorem [Ts99, Th. 0.2] provides a B_{st}-linear isomorphism

$$B_{\mathrm{st}} \otimes_{K_0} H^q_{\mathrm{crys}}(\underline{X}_k/\underline{\Sigma}) \xrightarrow{\sim} B_{\mathrm{st}} \otimes_K H^q_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p), \tag{2.1.7}$$

compatible with the natural Galois, Frobenius and monodromy actions on both sides, and with the natural Hodge filtrations defined on both sides after scalar extension from B_{st} to B_{dR}. Thus $H^q_{\mathrm{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0$ is an admissible filtered (φ, N)-module [Fo94, 5.3.3], and therefore is weakly admissible [Fo94, 5.4.2]. This implies that its Newton polygon lies above its Hodge polygon [Fo94, 4.4.6]. In particular, either $H^q_{\mathrm{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0 = 0$, or the smallest slope of its Newton polygon is bigger than the smallest slope of its Hodge polygon. By assumption, the latter is at least 1, which forces the part of slope < 1 of $H^q_{\mathrm{crys}}(\underline{X}_k/\underline{\Sigma}) \otimes K_0$ to vanish. Thanks to (2.1.5), this implies the theorem. □

In the general case, we will use truncated simplicial log schemes satisfying the conditions of the next lemma. We will assume that all the log schemes under consideration are fine log schemes [Ka89, (2.3)], and all constructions involving log schemes will be done in the category of fine log schemes. For any finite extension K' of K, with ring of integers R', we will endow $\operatorname{Spec} R'$ with the log structure defined by its closed point, and pullbacks of log schemes to $\operatorname{Spec} R'$ will mean pullbacks in the category of log schemes. Note that, because of [Ka89, (4.4) (ii) and (4.3.1)], the underlying scheme of such a pullback is the usual pullback in the category of schemes. We will denote log schemes by underlined letters, and drop the underlining to denote the underlying schemes.

Lemma 2.2. *Assume that R is complete, and that X is an integral, flat R-scheme of finite type. Let $m \geq 0$ be an integer. Then there exists a finite extension K' of K, with ring of integers R', a split m-truncated simplicial R'-log scheme $\underline{Y}_\bullet = (Y_\bullet, M_{Y_\bullet})$, and an augmentation morphism $u : Y_0 \to X_{R'}$ over R', such that the following conditions hold:*

a) *Each $\underline{Y}_r$ is projective over $X_{R'}$, and is a disjoint union of pullbacks to R' of semi-stable schemes over the integers of sub-K-extensions of K' endowed with the log structure defined by their special fibre;*

b) *Via the augmentation morphism induced by u, $Y_{\bullet,K'}$ is an m-truncated proper hypercovering of $X_{K'}$;*

c) *There exists a projective R-alteration $f : Y \to X$, where Y is semi-stable over the ring of integers R_1 of a sub-K-extension K_1 of K', and there exists finitely many R-embeddings $\sigma_i : R_1 \hookrightarrow R'$, such that, if $u_1 : Y \to X_{R_1}$ denotes the R_1-morphism defined by f, and if Y_i (resp. $u_i : Y_i \to X_{R'}$) denotes the R'-scheme (resp. R'-morphism) deduced by base change via σ_i from Y (resp. u_1), then $Y_0 = \coprod_i Y_i$ and $u|_{Y_i} = u_i$.*

Proof. This is a well known consequence of de Jong's alteration theorem [dJ96, Th. 6.5]. For the sake of completeness, we briefly recall how to construct such a simplicial log scheme. One proceeds by induction on m.

Assume first that $m = 0$. De Jong's theorem provides a finite extension K_1 of K, an integral semi-stable scheme Y over the ring of integers R_1 of K_1, and an

R-morphism $f : Y \to X$ which is a projective alteration. Let $u_1 : Y \to X_{R_1}$ be the morphism defined by f. Let K' be a finite extension of K_1 such that K'/K is Galois, and R' its ring of integers. For any $g \in \mathrm{Gal}(K'/K)$, let σ_g be the composition $K_1 \to K' \xrightarrow{g} K'$, and let $\underline{Y}_g$ (resp. $u_g : Y_g \to X_{R'}$) be the R'-log scheme (resp. R'-morphism) deduced from $\underline{Y}$ (resp. u_1) by base change via $\sigma_g : R_1 \to R'$. Then one defines $\underline{Y}_0$ and u by setting

$$\underline{Y}_0 = \coprod_{g \in \mathrm{Gal}(K'/K)} \underline{Y}_g, \qquad u|_{Y_g} = u_g.$$

One easily checks by Galois descent that $Y_{0,K'} \to X_{K'}$ is surjective, and conditions a) - c) are then satisfied.

Assume now that the lemma has been proved for $m-1$. Over the ring of integers R'' of some finite extension K'' of K, this provides a split $(m-1)$-truncated simplicial log scheme $\underline{Y}''_\bullet$, together with an augmentation morphism $u'' : Y''_0 \to X_{R''}$, so as to satisfy conditions a) - c). Note that these conditions remain satisfied after a base change to the ring of integers of any finite extension of K''. Let $\mathrm{cosk}_{m-1}(\underline{Y}''_\bullet)$ be the coskeleton of $\underline{Y}''_\bullet$ in the category of simplicial fine R''-log schemes, and $\underline{Z} = \mathrm{cosk}_{m-1}(\underline{Y}''_\bullet)_m$ its component of index m. Denote by $Z_1, \ldots, Z_c$ those irreducible components of Z which are flat over R'', and endow each Z_j with the log structure induced by the log structure of $\underline{Z}$. As a consequence of condition a), this log structure induces the trivial log structure on the generic fibre $\underline{Z}_{j,K''}$. Applying de Jong's theorem to Z_j, one can find a finite extension K'_j of K'', with ring of integers R'_j, an integral semi-stable scheme T_j over R'_j and an alteration $f_j : T_j \to Z_j$. One endows T_j with the log structure defined by its special fibre. Because the log structure of the generic fibre $\underline{Z}_{j,K''}$ is trivial, the morphism f_j extends uniquely to a log morphism $f_j : \underline{T}_j \to \underline{Z}_j$. Let K' be a Galois extension of K'' containing K'_j for all j, $1 \le j \le c$, and let R' be its ring of integers. Arguing as in the case $m = 0$ above, one can deduce from the alterations f_j an R'-morphism

$$\underline{T} \longrightarrow \coprod_{j=1}^{c} \underline{Z}_{j,R'} \longrightarrow \underline{Z}_{R'} \xrightarrow{\sim} \mathrm{cosk}_{m-1}(\underline{Y}''_{\bullet,R'})_m \tag{2.2.1}$$

where $\underline{T}$ satisfies condition a), and $T_{K'} \to \mathrm{cosk}_{m-1}(Y''_{\bullet,K'})_m$ is projective and surjective (note that, since all log structures are trivial on the generic fibres, the generic fibre of the coskeleton computed in the category of fine log schemes is the coskeleton of the generic fibres computed in the category of schemes). One can then follow Deligne's method in [De74, (6.2.5)] to extend $\underline{Y}''_{\bullet,R'}$ as a split m-truncated simplicial log scheme $\underline{Y}_\bullet$ over R'. The R'-log scheme $\underline{Y}_m$ satisfies condition a) because $\underline{T}$ does and $\underline{Y}_\bullet$ is split. Similarly, the morphism $Y_{m,K'} \to \mathrm{cosk}_{m-1}(Y_{\bullet,K'})_m$ is proper and surjective because the morphism $T_{K'} \to \mathrm{cosk}_{m-1}(Y''_{\bullet,K'})_m$ is proper and surjective. Thus the m-truncated simplicial scheme $Y_{\bullet,K'}$ is an m-truncated proper hypercovering of $X_{K'}$. Finally, condition c) is satisfied thanks to the induction hypothesis. □

2.3. We recall how to associate cohomological invariants to simplicial schemes and truncated simplicial schemes (see [De74, 5.2] and [Ts98, (6.2)]).

For $r \geq 0$, we denote by $[r]$ the ordered set $\{0, \dots, r\}$. Let Δ (resp. $\Delta[m]$) be the category which has the sets $[r]$ (resp. with $r \leq m$) as objects, the set of morphisms from $[r]$ to $[s]$ being the set of non-decreasing maps $[r] \to [s]$. If $\mathcal{T}$ is a topos, we denote by $\mathcal{T}^{\Delta}$ (resp. $\mathcal{T}^{\Delta[m]}$) the topos of cosimplicial objects (resp. m-truncated cosimplicial objects) in $\mathcal{T}$. Let $\mathcal{A}$ be a ring in $\mathcal{T}$, and $\mathcal{A}^{\bullet}$ the constant cosimplicial ring defined by $\mathcal{A}$. If $\mathcal{E}^{\bullet}$ is an $\mathcal{A}^{\bullet}$-module of $\mathcal{T}^{\Delta}$ (resp. $\mathcal{T}^{\Delta[m]}$), one associates to $\mathcal{E}^{\bullet}$ the complex

$$\begin{aligned} \varepsilon_* \mathcal{E}^{\bullet} \ &= \ \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^r \xrightarrow{\sum_j (-1)^j \partial^j} \mathcal{E}^{r+1} \to \cdots \\ (\text{resp.} \quad \varepsilon_*^m \mathcal{E}^{\bullet} \ &= \ \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^m \to 0 \to \cdots \quad). \end{aligned}$$

One views $\varepsilon_* \mathcal{E}^{\bullet}$ (resp. $\varepsilon_*^m \mathcal{E}^{\bullet}$) as a filtered complex of $\mathcal{A}$-modules using the naive filtration. The functors ε_* and ε_*^m are exact functors from the category of $\mathcal{A}^{\bullet}$-modules to the category of filtered complexes of $\mathcal{A}$-modules (which means that they transform a short exact sequence of $\mathcal{A}^{\bullet}$-modules into a short exact sequence of filtered complexes, i.e., such that the sequence of Fil^i's is exact for all i). Hence, they factorize so as to define exact functors $\mathbb{R}\varepsilon_*$ and $\mathbb{R}\varepsilon_*^m$ from $D^+(\mathcal{T}^{\Delta}, \mathcal{A}^{\bullet})$ (resp. $D^+(\mathcal{T}^{\Delta[m]}, \mathcal{A}^{\bullet})$) to $D^+F(\mathcal{T}, \mathcal{A})$. For any complex $\mathcal{E}^{\bullet,\bullet} \in D^+(\mathcal{A}^{\bullet})$, they provide functorial spectral sequences

$$E_1^{r,q} = \mathcal{H}^q(\mathcal{E}^{r,\bullet}) \Rightarrow \mathcal{H}^{r+q}(\mathbb{R}\varepsilon_*(\mathcal{E}^{\bullet,\bullet})) \tag{2.3.1}$$

and similarly for $\mathbb{R}\varepsilon_*^m$ with $E_1^{r,q} = 0$ for $r > m$ (we use here the first index to denote the simplicial degree). Note that the truncation functor induces a functorial morphism

$$\mathbb{R}\varepsilon_*(\mathcal{E}^{\bullet,\bullet}) \longrightarrow \mathbb{R}\varepsilon_*^m(\mathrm{sk}_m(\mathcal{E}^{\bullet,\bullet})), \tag{2.3.2}$$

and therefore a morphism between the corresponding spectral sequences (2.3.1). It follows that, if $\mathcal{H}^q(\mathcal{E}^{r,\bullet}) = 0$ for $q < 0$ and all r, then the morphism (2.3.2) is a quasi-isomorphism in degrees $< m$.

Let $Y_{\bullet}$ be a simplicial scheme (resp. m-truncated simplicial scheme), and $\underline{\mathrm{Sets}}$ the topos of sets. If R is a commutative ring, and $\mathcal{E}^{\bullet}$ a sheaf of R-modules on $Y_{\bullet}$, one can associate to $\mathcal{E}^{\bullet}$ a cosimplicial $R^{\bullet}$-module $\Gamma^{\bullet}(Y_{\bullet}, \mathcal{E}^{\bullet}) \in \underline{\mathrm{Sets}}^{\Delta}$ (resp. $\underline{\mathrm{Sets}}^{\Delta[m]}$) by setting for all $r \geq 0$

$$\Gamma^r(Y_{\bullet}, \mathcal{E}^{\bullet}) = \Gamma(Y_r, \mathcal{E}^r).$$

The functor $\Gamma^{\bullet}$ can be derived, and its right derived functor $\mathbb{R}\Gamma^{\bullet}$ can be computed using resolutions by complexes $\mathcal{I}^{\bullet,\bullet}$ such that, for each r, q, the sheaf $\mathcal{I}^{r,q}$ is acyclic on Y_r. The cohomology of $Y_{\bullet}$ with coefficients in a complex $\mathcal{E}^{\bullet,\bullet}$ is then by definition

$$\begin{gathered} \mathbb{R}\Gamma(Y_{\bullet}, \mathcal{E}^{\bullet,\bullet}) = \mathbb{R}\varepsilon_* \mathbb{R}\Gamma^{\bullet}(Y_{\bullet}, \mathcal{E}^{\bullet,\bullet}) \qquad (\text{resp.} \ \ \mathbb{R}\varepsilon_*^m), \\ H^q(Y_{\bullet}, \mathcal{E}^{\bullet,\bullet}) = H^q(\mathbb{R}\Gamma(Y_{\bullet}, \mathcal{E}^{\bullet,\bullet})). \end{gathered}$$

If $Y_{\bullet}$ is a smooth simplicial (resp. m-truncated simplicial) R-scheme, this can be applied to the complex $\Omega^{\bullet}_{Y_{\bullet}/R}$ and to its sub-complexes $\sigma_{\geq i}\Omega^{\bullet}_{Y_{\bullet}/R}$, defining the naive filtration. This provides the definition of the de Rham cohomology of $Y_{\bullet}$, and of its Hodge filtration.

Proposition 2.4. *Let K be a field of charactristic 0, X a proper and smooth K-scheme, $Y_\bullet \to X$ an m-truncated proper hypercovering of X over K such that Y_r is proper and smooth for all r. Then, for all $q < m$, the canonical homomorphism*

$$H^q(X, \Omega^\bullet_{X/K}) \longrightarrow H^q(Y_\bullet, \Omega^\bullet_{Y_\bullet/K}) \tag{2.4.1}$$

is an isomorphism of filtered K-vector spaces for the Hodge filtrations.

Proof. Since algebraic de Rham cohomology (endowed with the Hodge filtration) commutes with base field extensions, standard limit arguments allow to assume that K is of finite type over $\mathbb{Q}$. Choosing an embedding $\iota : K \hookrightarrow \mathbb{C}$, we are reduced to the case where $K = \mathbb{C}$. Using resolution of singularities, we can find a proper and smooth hypercovering $Z_\bullet$ of X such that $\mathrm{sk}_m(Z_\bullet) = Y_\bullet$. As the morphism (2.3.2) for $\sigma_{\geq i}\Omega^\bullet_{Z_\bullet/\mathbb{C}}$ is a quasi-isomorphism in degrees $< m$ for all i, it suffices to prove the proposition with $Y_\bullet$ replaced by $Z_\bullet$. This now follows from [De74, Prop. (8.2.2)]. □

Corollary 2.5. *Under the assumptions a) and b) of Lemma 2.2, assume in addition that X_K is proper and smooth, and that $H^q(X_K, \mathcal{O}_{X_K}) = 0$ for some $q < m$. Then the smallest Hodge slope of $H^q(Y_{\bullet K'}, \Omega^\bullet_{Y_{\bullet K'}})$ is at least 1.*

Proof. Assumption a) and b) imply that the hypotheses of the proposition are satisfied by $Y_{\bullet K'} \to X_{K'}$, and the corollary is then clear. □

2.6. Let $\underline{\Sigma}_n, \underline{\Sigma}$ be as in the proof of Theorem 2.1. We now denote by $\underline{Y}_\bullet = (Y_\bullet, M_{Y_\bullet})$ an m-truncated simplicial log scheme over $\underline{\Sigma}_1$. We assume that each $\underline{Y}_r$ is smooth of Cartier type over $\underline{\Sigma}_1$, so that, for all $n \geq 1$, its de Rham-Witt complex $W_n\Omega^\bullet_{\underline{Y}_r}$ is defined [HK94, (4.1)]. When r varies, the functoriality of the de Rham-Witt complex turns the family of complexes $(W_n\Omega^\bullet_{\underline{Y}_r})_{0\leq r\leq m}$ into a complex $W_n\Omega^\bullet_{\underline{Y}_\bullet}$ on $Y_\bullet$. One defines its cohomology as in 2.3, and one has similar definitions for the de Rham-Witt complex $W\Omega^\bullet_{\underline{Y}_\bullet} = \varprojlim_n W_n\Omega^\bullet_{\underline{Y}_\bullet}$.

For a morphism $\alpha : [r] \to [s]$ in $\Delta[m]$, let $\alpha_{\mathrm{crys}} : (\underline{Y}_s/\underline{\Sigma}_n)_{\mathrm{crys}} \to (\underline{Y}_r/\underline{\Sigma}_n)_{\mathrm{crys}}$ be the morphism between the log crystalline topos induced by the corresponding morphism $\underline{Y}_s \to \underline{Y}_r$. One defines the log crystalline topos $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\mathrm{crys}}$ as being the topos of families of sheaves $(E^r)_{0\leq r\leq m}$, where E^r is a sheaf on the log crystalline site $\mathrm{Crys}(\underline{Y}_r/\underline{\Sigma}_n)$, endowed with a transitive family of morphisms $\alpha^{-1}_{\mathrm{crys}}E^r \to E^s$ for morphisms α in $\Delta[m]$. In particular, the family of sheaves $\mathcal{O}_{\underline{Y}_r/\underline{\Sigma}_n}$ defines the structural sheaf of $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\mathrm{crys}}$, denoted by $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$. There is a canonical morphism $u_{\underline{Y}_\bullet/\underline{\Sigma}_n} : (\underline{Y}_\bullet/\underline{\Sigma}_n)_{\mathrm{crys}} \to Y_{\bullet\mathrm{Zar}}$, such that $u_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(E^\bullet)^r = u_{\underline{Y}_r/\underline{\Sigma}_n*}(E^r)$ for all r. If $E^{\bullet,\bullet}$ is a complex of abelian sheaves in $(\underline{Y}_\bullet/\underline{\Sigma}_n)_{\mathrm{crys}}$, one proceeds as in 2.3 to define its log crystalline cohomology $\mathbb{R}\Gamma_{\mathrm{crys}}(\underline{Y}_\bullet/\underline{\Sigma}_n, E^{\bullet,\bullet})$ and its projection on the Zariski topos $\mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(E^{\bullet,\bullet})$. One gives similar definitions for the log crystalline topos $(\underline{Y}_\bullet/\underline{\Sigma})_{\mathrm{crys}}$ relative to $\underline{\Sigma}$. By construction, there are canonical isomorphisms

$$\mathbb{R}\Gamma(Y_\bullet, \mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}_n*}(E^{\bullet,\bullet})) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{crys}}(\underline{Y}_\bullet/\underline{\Sigma}_n, E^{\bullet,\bullet}), \tag{2.6.1}$$

$$\mathbb{R}\Gamma(Y_\bullet, \mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}*}(E^{\bullet,\bullet})) \xrightarrow{\sim} \mathbb{R}\Gamma_{\mathrm{crys}}(\underline{Y}_\bullet/\underline{\Sigma}, E^{\bullet,\bullet}). \tag{2.6.2}$$

If $\underline{Y}_\bullet \hookrightarrow \underline{P}_\bullet$ is a closed immersion of the m-truncated simplicial log scheme $\underline{Y}_\bullet$ into a smooth m-truncated simplicial $\underline{\Sigma}_n$-log scheme $\underline{P}_\bullet$ (resp. $\underline{\Sigma}$-formal log scheme), the family of PD-envelopes $\mathcal{P}^{\log}_{\underline{Y}_r}(\underline{P}_r)$ (resp. completed PD-envelopes) [Ka89, (5.4)] defines a sheaf $\mathcal{P}^{\log}_{\underline{Y}_\bullet}(\underline{P}_\bullet)$ on $\underline{Y}_\bullet$, and one can form the de Rham complex $\mathcal{P}^{\log}_{\underline{Y}_\bullet}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{P_\bullet}} \Omega^\bullet_{\underline{P}_\bullet/\Sigma_n}$ (resp. $\mathcal{P}^{\log}_{\underline{Y}_\bullet}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{P_\bullet}} \Omega^\bullet_{\underline{P}_\bullet/\underline{\Sigma}}$), which is supported in $\underline{Y}_\bullet$. Because the linearization functor L used in the proof of the comparison theorem between crystalline and de Rham cohomologies [Ka89, (6.9)] makes sense simplicially, this theorem extends to the simplicial case and there is a canonical isomorphism in $D^+(Y_\bullet, W_n)$ (resp. $D^+(Y_\bullet, W)$)

$$\mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}_n *}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} \mathcal{P}^{\log}_{\underline{Y}_\bullet}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{\underline{P}_\bullet}} \Omega^\bullet_{\underline{P}_\bullet/\underline{\Sigma}_n} \tag{2.6.3}$$

$$\text{(resp. } \mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma} *}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}}) \xrightarrow{\sim} \mathcal{P}^{\log}_{\underline{Y}_\bullet}(\underline{P}_\bullet) \otimes_{\mathcal{O}_{P_\bullet}} \Omega^\bullet_{\underline{P}_\bullet/\underline{\Sigma}} \quad). \tag{2.6.4}$$

Proposition 2.7. *With the hypotheses of 2.6, assume that $\underline{Y}_\bullet$ is split. Then there exists in $D^+(Y_\bullet, W_n)$ (resp. $D^+(Y_\bullet, W)$) canonical isomorphisms compatible with the transition morphisms and the Frobenius actions*

$$\mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}_n *}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} W_n\Omega^\bullet_{\underline{Y}_\bullet} \tag{2.7.1}$$

$$(\textit{resp. } \mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma} *}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}}) \xrightarrow{\sim} W\Omega^\bullet_{\underline{Y}_\bullet} \quad). \tag{2.7.2}$$

The proof will use the following lemma, due to Nakkajima [Na09, Lemma 6.1].

Lemma 2.8. *Under the assumptions of 2.7, there exists an m-truncated simplicial log scheme $\underline{Z}_\bullet$ and a morphism of m-truncated simplicial log schemes $\underline{Z}_\bullet \to \underline{Y}_\bullet$ such that, for $0 \leq r \leq m$, Z_r is the disjoint union of affine open subsets of Y_r, and the morphism $Z_r \to Y_r$ induces the natural inclusion on each of these subsets.*

Definition 2.9. Let X be a scheme on which p is locally nilpotent, and $n \geq 1$ an integer. We denote by $|X|$ the topological space underlying X, and by $W_n(X)$ the ringed space $(|X|, W_n(\mathcal{O}_X))$, which is a scheme ([Il79, 0, 1.5] and [LZ04, 1.10]). The ideal $VW_{n-1}(\mathcal{O}_X)$ carries a canonical PD-structure ([Il79, 0, 1.4] and [LZ04, 1.1]), which turns the nilpotent immersion $u : X \hookrightarrow W_n(X)$ into a PD-thickening of X.

If $\underline{X} = (X, M_X)$ is a log scheme, we denote by $W_n(\underline{X}) = (W_n(X), M_{W_n(X)})$ the log scheme obtained by sending M_X to $W_n(\mathcal{O}_X)$ by the Teichmüller representative map, and taking the associated log structure [HK94, Def. (3.1)]. The immersion u is then in a natural way an exact closed immersion $u : \underline{X} \hookrightarrow W_n(\underline{X})$, functorial with respect to $\underline{X}$.

Lemma 2.10. *Under the assumptions of 2.7, there exists a bisimplicial log scheme $\underline{Z}_{\bullet,\bullet}$, m-truncated with respect to the first index and augmented towards $\underline{Y}_\bullet$ with respect to the second index, a bisimplicial formal log scheme $\underline{\mathcal{T}}_{\bullet,\bullet}$ over $\underline{\Sigma}$, m-truncated with respect to the first index, and a closed immersion of bisimplicial formal log schemes $i_{\bullet,\bullet} : \underline{Z}_{\bullet,\bullet} \hookrightarrow \underline{\mathcal{T}}_{\bullet,\bullet}$, such that the following conditions are satisfied:*

a) *For $0 \leq r \leq m$, $Z_{r,0}$ is the disjoint union of affine open subsets of Y_r, the augmentation morphism $Z_{r,0} \to Y_r$ induces the natural inclusion on each of these subsets, and the canonical morphism $\underline{Z}_{r,\bullet} \to \operatorname{cosk}_0^{Y_r}(\operatorname{sk}_0^{Y_r}(\underline{Z}_{r,\bullet}))$ is an isomorphism.*

b) *For $0 \leq r \leq m$ and $t \geq 0$, the formal log scheme $\underline{\mathcal{T}}_{r,t}$ is smooth over $\underline{\Sigma}$ (i.e., its reduction mod p^n is smooth over $\underline{\Sigma}_n$ for all n), and the canonical morphism $\underline{\mathcal{T}}_{r,\bullet} \to \operatorname{cosk}_0^{\Sigma}(\operatorname{sk}_0^{\Sigma}(\underline{\mathcal{T}}_{r,\bullet}))$ is an isomorphism.*

c) *Let* $i_{\bullet,\bullet,n} : \underline{Z}_{\bullet,\bullet} \hookrightarrow \underline{T}_{\bullet,\bullet,n}$ *be the reduction mod* p^n *of* $i_{\bullet,\bullet}$*, and let* $u_{\bullet,\bullet,n} : \underline{Z}_{\bullet,\bullet} \hookrightarrow W_n(\underline{Z}_{\bullet,\bullet})$ *denote the morphism of bisimplicial log schemes defined by the canonical immersions. For variable* n*, there exists a compatible family of* $\underline{\Sigma}_n$*-morphisms of bisimplicial schemes* $h_{\bullet,\bullet,n} : W_n(\underline{Z}_{\bullet,\bullet}) \to \underline{T}_{\bullet,\bullet,n}$ *such that* $h_{\bullet,\bullet,n} \circ u_{\bullet,\bullet,n} = i_{\bullet,\bullet,n}$*.*

Proof. Let $j_\bullet : \underline{Z}_\bullet \to \underline{Y}_\bullet$ be a morphism of m-truncated simplicial log schemes satisfying the conclusions of Lemma 2.8. One chooses a decomposition $\underline{Z}_r = \coprod_\alpha \underline{Z}_r^\alpha$, with $Z_r^\alpha \subset Y_r$ open affine such that $j_r|_{Z_r^\alpha}$ is the natural inclusion.

Let $\underline{Z}_{r,1}^\alpha = \underline{Z}_r^\alpha$. Since $\underline{Z}_r^\alpha$ is affine and smooth over $\underline{\Sigma}_1$, and $\underline{\Sigma}_{n-1} \hookrightarrow \underline{\Sigma}_n$ is a nilpotent exact closed immersion, there exists for each r, α and each $n \geq 2$ a smooth log scheme $\underline{Z}_{r,n}^\alpha$ over $\underline{\Sigma}_n$ endowed with an isomorphism $\underline{Z}_{r,n-1}^\alpha \xrightarrow{\sim} \underline{\Sigma}_{n-1} \times_{\underline{\Sigma}_n} \underline{Z}_{r,n}^\alpha$ [Ka89, Prop. (3.14) (1)]. Taking limits when $n \to \infty$, we obtain a smooth formal log scheme $\underline{\mathcal{Z}}_r^\alpha$ over $\underline{\Sigma}$ and an isomorphism $\underline{Z}_r^\alpha \xrightarrow{\sim} \underline{\Sigma}_1 \times_{\underline{\Sigma}} \underline{\mathcal{Z}}_r^\alpha$. Moreover, the smoothness of $\underline{Z}_{r,n}^\alpha$ over $\underline{\Sigma}_n$ for all n implies that we can find inductively a compatible family of $\underline{\Sigma}_n$-morphisms $g_{r,n}^\alpha : W_n(\underline{Z}_r^\alpha) \to \underline{Z}_{r,n}^\alpha$ such that the composition $\underline{Z}_r^\alpha \hookrightarrow W_n(\underline{Z}_r^\alpha) \to \underline{Z}_{r,n}^\alpha$ is the chosen immersion $\underline{Z}_r^\alpha \hookrightarrow \underline{Z}_{r,n}^\alpha$.

Let $\underline{Z}_{r,n} = \coprod_\alpha \underline{Z}_{r,n}^\alpha$, $\underline{\mathcal{Z}}_r = \coprod_\alpha \underline{\mathcal{Z}}_r^\alpha$, let $v_{r,n} : \underline{Z}_r \hookrightarrow \underline{Z}_{r,n}$, $v_r : \underline{Z}_r \hookrightarrow \underline{\mathcal{Z}}_r$ be defined by the immersions $\underline{Z}_r^\alpha \hookrightarrow \underline{Z}_{r,n}^\alpha$ and $\underline{Z}_r^\alpha \hookrightarrow \underline{\mathcal{Z}}_r^\alpha$, and let $g_{r,n} : W_n(\underline{Z}_r) \to \underline{Z}_{r,n}$ be defined by the morphisms $g_{r,n}^\alpha$. We now use the method of Chiarellotto and Tsuzuki ([CT03, 11.2], [Tz04, 7.3]) to deduce from these data a closed immersion $i_\bullet$ of $\underline{Z}_\bullet$ into an m-truncated simplicial formal log scheme $\underline{\mathcal{T}}_\bullet$, smooth over $\underline{\Sigma}$, with reduction $\underline{T}_{\bullet,n}$ over $\underline{\Sigma}_n$, and a compatible family of $\underline{\Sigma}_n$-morphisms of m-truncated simplicial log schemes $h_{\bullet,n} : W_n(\underline{Z}_\bullet) \to \underline{T}_{\bullet,n}$ such that $h_{\bullet,n} \circ u_{\bullet,n} = i_{\bullet,n}$, where $u_{\bullet,n} : \underline{Z}_{\bullet,n} \hookrightarrow W_n(\underline{Z}_{\bullet,n})$ is the canonical morphism, and $i_{\bullet,n}$ is the reduction mod p^n of $i_\bullet$. First, we set for $0 \leq s \leq m$

$$\Gamma_s(\underline{\mathcal{Z}}_r) = \prod_{\gamma:[r]\to[s]} \underline{\mathcal{Z}}_{r,\gamma},$$

where the product is taken over all morphisms $\gamma : [r] \to [s]$ in $\Delta[m]$, and $\underline{\mathcal{Z}}_{r,\gamma} = \underline{\mathcal{Z}}_r$ for all γ. Then any morphism $\eta : [s'] \to [s]$ in $\Delta[m]$ defines a morphism $\Gamma_s(\underline{\mathcal{Z}}_r) \to \Gamma_{s'}(\underline{\mathcal{Z}}_r)$ having as component of index γ' the projection of $\Gamma_s(\underline{\mathcal{Z}}_r)$ to the factor of index $\eta \circ \gamma'$. One obtains in this way an m-truncated simplicial formal log scheme $\Gamma_\bullet(\underline{\mathcal{Z}}_r)$ over $\underline{\Sigma}$, the terms of which are smooth over $\underline{\Sigma}$.

For each $\gamma : [r] \to [s]$, there is a commutative diagram

$$\begin{array}{ccccccc}
W_n(\underline{Z}_s) & \xrightarrow{W_n(\gamma)} & W_n(\underline{Z}_r) & & & & \\
\uparrow{\scriptstyle u_{s,n}} & & \uparrow{\scriptstyle u_{r,n}} & \overset{g_{r,n}}{\searrow} & & & \\
\underline{Z}_s & \xrightarrow{\gamma} & \underline{Z}_r & \xhookrightarrow{v_{r,n}} & \underline{Z}_{r,n} & \hookrightarrow & \underline{\mathcal{Z}}_r.
\end{array}$$

For fixed r and variable s, the family of morphisms $\underline{Z}_s \to \Gamma_s(\underline{\mathcal{Z}}_r)$ having the composition $\underline{Z}_s \xrightarrow{\gamma} \underline{Z}_r \hookrightarrow \underline{\mathcal{Z}}_r$ as component of index γ defines a morphism of m-truncated simplicial formal log schemes $\underline{Z}_\bullet \to \Gamma_\bullet(\underline{\mathcal{Z}}_r)$. We set

$$\underline{\mathcal{T}}_\bullet = \prod_{0\leq r\leq m} \Gamma_\bullet(\underline{\mathcal{Z}}_r),$$

and we define $i_\bullet : \underline{Z}_\bullet \to \underline{\mathcal{T}}_\bullet$ as having the previous morphism as component of index r, for $0 \le r \le m$. For each r, the morphism $\underline{Z}_r \to \Gamma_r(\underline{\mathcal{Z}}_r)$ has the closed immersion $v_r : \underline{Z}_r \hookrightarrow \underline{\mathcal{Z}}_r$ as component of index $\mathrm{Id}_{[r]}$. It follows that $\underline{Z}_r \to \underline{\mathcal{T}}_r$ is a closed immersion for all r.

Similarly, the family of morphisms $W_n(\underline{Z}_s) \to \Gamma_s(\underline{\mathcal{Z}}_r)$ having the composition $W_n(\underline{Z}_s) \xrightarrow{W_n(\gamma)} W_n(\underline{Z}_r) \xrightarrow{g_{r,n}} \underline{\mathcal{Z}}_{r,n} \hookrightarrow \underline{\mathcal{Z}}_r$ as component of index γ defines a morphism of m-truncated simplicial log schemes $W_n(\underline{Z}_\bullet) \to \Gamma_\bullet(\underline{\mathcal{Z}}_r)$. We define $h_\bullet$: $W_n(\underline{Z}_\bullet) \to \underline{\mathcal{T}}_\bullet$ as having the previous morphism as component of index r for $0 \le r \le m$, and $h_{\bullet,n} : W_n(\underline{Z}_\bullet) \to \underline{\mathcal{T}}_{\bullet,n}$ as being the reduction of $h_\bullet$ mod p^n. It is clear that $h_{\bullet,n} \circ u_{\bullet,n} = i_{\bullet,n}$, and that the morphisms $h_{\bullet,n}$ form a compatible family when n varies.

We now set $\underline{Z}_{\bullet,0} = \underline{Z}_\bullet$, $\underline{\mathcal{T}}_{\bullet,0} = \underline{\mathcal{T}}_\bullet$, and we define

$$\underline{Z}_{\bullet,\bullet} = \mathrm{cosk}_0^{\underline{Y}_\bullet}(\underline{Z}_{\bullet,0}), \qquad \underline{\mathcal{T}}_{\bullet,\bullet} = \mathrm{cosk}_0^{\underline{\Sigma}}(\underline{\mathcal{T}}_{\bullet,0}),$$

the coskeletons being taken respectively in the category of simplicial m-truncated simplicial log schemes over $\underline{Y}_\bullet$ and of simplicial m-truncated simplicial formal log schemes over $\underline{\Sigma}$. The augmentation morphism $\underline{Z}_{\bullet,0} \to \underline{Y}_\bullet$ is given by $j_\bullet$, and the morphism $i_{\bullet,\bullet}$ is defined by setting $i_{\bullet,0} = i_\bullet : \underline{Z}_{\bullet,0} \hookrightarrow \underline{\mathcal{T}}_{\bullet,0}$, and extending $i_{\bullet,0}$ by functoriality to the coskeletons. As seen above, $i_{\bullet,0}$ is a closed immersion, and it follows from the construction of coskeletons that $i_{\bullet,t}$ is a closed immersion for all t. Since $\mathrm{cosk}_0^{\underline{\Sigma}}(\underline{\mathcal{T}}_{r,0})_t = \underline{\mathcal{T}}_r \times_{\underline{\Sigma}} \times \cdots \times_{\underline{\Sigma}} \underline{\mathcal{T}}_r$ ($t+1$ times), $\underline{\mathcal{T}}_{r,t}$ is smooth over $\underline{\Sigma}$ for all r, t. Finally, we define $h_{\bullet,\bullet,n} : W_n(\underline{Z}_{\bullet,\bullet}) \to \underline{\mathcal{T}}_{\bullet,\bullet,n}$ as being the composition

$$W_n(\mathrm{cosk}_0^{\underline{Y}_\bullet}(\underline{Z}_{\bullet,0})) \to \mathrm{cosk}_0^{W_n(\underline{Y}_\bullet)}(W_n(\underline{Z}_{\bullet,0})) \to \mathrm{cosk}_0^{\underline{\Sigma}_n}(\underline{\mathcal{T}}_{\bullet,0,n}) \simeq \underline{\Sigma}_n \times_{\underline{\Sigma}} \mathrm{cosk}_0^{\underline{\Sigma}}(\underline{\mathcal{T}}_{\bullet,0}),$$

where the first map is defined by the universal property of the coskeleton (and is actually an isomorphism), the second one is defined by functoriality by the morphism $h_{\bullet,n} : W_n(\underline{Z}_{\bullet,0}) \to \underline{\mathcal{T}}_{\bullet,n} = \underline{\mathcal{T}}_{\bullet,0,n}$, and the last one is the base change isomorphism for coskeletons. The relations $h_{\bullet,\bullet,n} \circ u_{\bullet,\bullet,n} = i_{\bullet,\bullet,n}$ and the compatibility for variable n follow from the similar properties for the morphisms $h_{\bullet,n}$. Properties a) - c) of the Lemma are then satisfied. □

2.11. *Proof of Proposition 2.7.* Let

$$\begin{array}{ccc} \underline{Z}_{\bullet,\bullet} & \xhookrightarrow{i_{\bullet,\bullet}} & \underline{\mathcal{T}}_{\bullet,\bullet} \\ {\scriptstyle j_{\bullet,\bullet}}\downarrow & & \downarrow \\ \underline{Y}_\bullet & \longrightarrow & \underline{\Sigma} \end{array}$$

be a commutative diagram satisfying the properties of Lemma 2.10. Since, for all $r \le m$, the morphism $j_{r,0}$ is locally an open immersion, the scheme underlying $\underline{Z}_{r,t}$ is the usual fibred product $Z_{r,0} \times_{Y_r} \cdots \times_{Y_r} Z_{r,0}$ ($t+1$ times). It follows that, if $\mathfrak{U}_r = (Z_r^\alpha)_\alpha$ is an affine covering of Y_r such that $Z_{r,0} = \coprod_\alpha Z_r^\alpha$ and $j_{r,0|Z_r^\alpha}$ is the natural inclusion, then, for any abelian sheaf $\mathcal{E}$ on Y_r, the complex

$$\varepsilon_{r\,*}(j_{r,\bullet\,*} j_{r,\bullet}^{-1}\mathcal{E}) = \Big[j_{r,0\,*} j_{r,0}^{-1}\mathcal{E} \to \cdots \to j_{r,t\,*} j_{r,t}^{-1}\mathcal{E} \xrightarrow{\sum_k (-1)^k \partial_k} j_{r,t+1\,*} j_{r,t+1}^{-1}\mathcal{E} \to \cdots \Big]$$

is the Čech resolution of $\mathcal{E}$ defined by the covering $\mathfrak{U}_r$. If $\mathcal{E}^\bullet$ is an abelian sheaf on $Y_\bullet$, the fact that $j_{\bullet,0}$ is an augmentation morphism in the category of m-truncated simplicial schemes implies that the complex $\varepsilon_{r\,*}(j_{r,\bullet\,*}j_{r,\bullet}^{-1}\mathcal{E}^r)$ is functorial with respect to $[r] \in \Delta[m]$, and we obtain a resolution $\varepsilon_{\bullet\,*}(j_{\bullet,\bullet\,*}j_{\bullet,\bullet}^{-1}\mathcal{E}^\bullet)$ of $\mathcal{E}^\bullet$ in the category of abelian sheaves on $Y_\bullet$. In particular, taking into account that each $j_{q,q'}$ is locally an open immersion, we obtain for all n a resolution of the de Rham-Witt complex of $\underline{Y}_\bullet$ given by

$$W_n\Omega^\bullet_{\underline{Y}_\bullet} \xrightarrow{\text{qis}} \varepsilon_{\bullet\,*}(j_{\bullet,\bullet *}W_n\Omega^\bullet_{\underline{Z}_{\bullet,\bullet}}). \tag{2.11.1}$$

On the other hand, one can also define for all r a complex on $\mathrm{Crys}(\underline{Y}_r/\underline{\Sigma}_n)$ by setting

$$\varepsilon_{r\,*}(j_{r,\bullet\,\mathrm{crys}\,*}(\mathcal{O}_{\underline{Z}_{r,\bullet}/\underline{\Sigma}_n})) = \Big[j_{r,0\,\mathrm{crys}\,*}(\mathcal{O}_{\underline{Z}_{r,0}/\underline{\Sigma}_n}) \to \cdots \to j_{r,t\,\mathrm{crys}\,*}(\mathcal{O}_{\underline{Z}_{r,t}/\underline{\Sigma}_n}) \xrightarrow{\sum_k(-1)^k\partial_k} \cdots\Big].$$

Since $Z_{r,\bullet} \to Y_r$ is the Čech simplicial scheme defined by an affine open covering of Y_r, this complex is a resolution of $\mathcal{O}_{\underline{Y}_r/\underline{\Sigma}_n}$ [Be74, III, Prop. 3.1.2 and V, Prop. 3.1.2]. Since $\underline{Z}_{\bullet,\bullet}$ is a bisimplicial scheme, these resolutions are functorial with respect to $[r]$ and yield a resolution $\varepsilon_{\bullet\,*}(j_{\bullet,\bullet\,\mathrm{crys}\,*}(\mathcal{O}_{\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n}))$ of $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$. Let $\underline{T}_{\bullet,\bullet,n}$ be the reduction mod p^n of $\underline{\mathcal{T}}_{\bullet,\bullet}$. The linearization functor L [Ka89, (6.9)] is functorial with respect to embeddings, hence it provides a complex $L(\Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n})$ on $\mathrm{Crys}(\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n)$. This complex is a resolution of $\mathcal{O}_{\underline{Z}_{\bullet,\bullet}/\underline{\Sigma}_n}$ thanks to the log Poincaré lemma which follows from [Ka89, Prop. (6.5)]. For each (r,t) and each i, one checks easily that the term $j_{r,t\,\mathrm{crys}\,*}(L(\Omega^i_{\underline{T}_{r,t,n}/\underline{\Sigma}_n}))$ is acyclic with respect to $u_{\underline{Y}_r/\underline{\Sigma}_n *}$ (use [Be74, V, (2.2.3)] and the equality $u_{\underline{Y}_r/\underline{\Sigma}_n *} \circ j_{r,t\,\mathrm{crys}\,*} = j_{r,t\,*} \circ u_{\underline{Z}_{r,t}/\underline{\Sigma}_n *}$). Hence, the complex $\varepsilon_{\bullet\,*}(j_{\bullet,\bullet\,\mathrm{crys}\,*}(L(\Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n})))$ is an $u_{\underline{Y}_\bullet/\underline{\Sigma}_n *}$-acyclic resolution of $\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}$. Moreover, the closed immersion of bisimplicial schemes $i_{\bullet,\bullet}$ defines a family of PD-envelopes $\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n})$, supported in $Z_{\bullet,\bullet}$. They provide a de Rham complex $\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \otimes \Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n}$, which can be viewed as a complex of abelian sheaves on $Z_{\bullet,\bullet}$, and it follows from [Be74, V, (2.2.3)] that

$$u_{\underline{Y}_\bullet/\underline{\Sigma}_n *}(j_{\bullet,\bullet\,\mathrm{crys}\,*}(L(\Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n}))) = j_{\bullet,\bullet\,*}(\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \otimes \Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n}).$$

So we finally get in $D^+(Z_\bullet, W_n)$ an isomorphism

$$\mathbb{R}u_{\underline{Y}_\bullet/\underline{\Sigma}_n *}(\mathcal{O}_{\underline{Y}_\bullet/\underline{\Sigma}_n}) \xrightarrow{\sim} \varepsilon_{\bullet\,*}(j_{\bullet,\bullet\,*}(\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \otimes \Omega^\bullet_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n})). \tag{2.11.2}$$

To prove Proposition 2.7, it suffices to define a quasi-isomorphism between the right hand sides of (2.11.1) and (2.11.2). Note that, for each r, t, i, the sheaves $W_n\Omega^i_{\underline{Z}_{r,t}}$ and $\mathcal{P}^{\log}_{\underline{Z}_{r,t}}(\underline{T}_{r,t,n}) \otimes \Omega^i_{\underline{T}_{r,t,n}/\underline{\Sigma}_n}$ are $j_{r,t\,*}$-acyclic. Indeed, $Z_{r,t}$ is a disjoint union of affine open subsets of Y_r, and on the one hand $W_n\Omega^i_{\underline{Z}_{r,t}}$ has a finite filtration with subquotients which are quasi-coherent over suitable Frobenius pullbacks of $Z_{r,t}$ [Ka89, Th. (4.4)], on the other hand $\mathcal{P}^{\log}_{\underline{Z}_{r,t}}(\underline{T}_{r,t,n}) \otimes \Omega^i_{\underline{T}_{r,t,n}/\underline{\Sigma}_n}$ is a quasi-coherent $\mathcal{O}_{T_{r,t,n}}$-module with support in $Z_{r,t}$, hence is a direct limit of submodules which have a finite filtration with subquotients which are quasi-coherent over $Z_{r,t}$. Therefore, it suffices to construct a quasi-isomorphism

$$\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \otimes \Omega^{\bullet}_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n} \longrightarrow W_n\Omega^{\bullet}_{\underline{Z}_{\bullet,\bullet}} \tag{2.11.3}$$

in the category of complexes of W_n-modules over $Z_{\bullet,\bullet}$.

We can now argue as in the proof of [HK94, Th. (4.19)]. Since the PD-immersion $u_{r,t,n} : \underline{Z}_{r,t} \hookrightarrow W_n(\underline{Z}_{r,t})$ is an exact closed immersion for all r, t, the morphism $h_{\bullet,\bullet,n} : W_n(\underline{Z}_{\bullet,\bullet}) \to \underline{T}_{\bullet,\bullet,n}$ defines uniquely a PD-morphism $\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \to W_n(\mathcal{O}_{\underline{Z}_{\bullet,\bullet}})$ in the category of sheaves of W-modules on the bisimplicial scheme $T_{\bullet,\bullet,n}$. As $h_{\bullet,\bullet,n}$ is a morphism of bisimplicial log schemes, it defines by functoriality a morphism of complexes $\Omega^{\bullet}_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n} \to \Omega^{\bullet}_{W_n(\underline{Z}_{\bullet,\bullet})/\underline{\Sigma}_n}$ on $T_{\bullet,\bullet,n}$. This morphism extends as a morphism of complexes with support in $Z_{\bullet,\bullet}$

$$\mathcal{P}^{\log}_{\underline{Z}_{\bullet,\bullet}}(\underline{T}_{\bullet,\bullet,n}) \otimes \Omega^{\bullet}_{\underline{T}_{\bullet,\bullet,n}/\underline{\Sigma}_n} \longrightarrow \Omega^{\bullet}_{W_n(\underline{Z}_{\bullet,\bullet})/\underline{\Sigma}_n}/\mathcal{N}^{\bullet}_{\bullet,\bullet},$$

where $\mathcal{N}^{\bullet}_{\bullet,\bullet} \subset \Omega^{\bullet}_{W_n(\underline{Z}_{\bullet,\bullet})/\underline{\Sigma}_n}$ denotes the graded ideal generated by the sections $d(a^{[i]}) - a^{[i-1]}da$ for all sections a of $VW_{n-1}(\mathcal{O}_{Z_{\bullet,\bullet}})$ and all $i \geq 1$. The differential graded algebra $W_n\Omega^{\bullet}_{\underline{Z}_{\bullet,\bullet}}$ is a quotient of $\Omega^{\bullet}_{W_n(\underline{Z}_{\bullet,\bullet})/\underline{\Sigma}_n}$ [HK94, Prop. (4.7)], and the generators of $\mathcal{N}^{\bullet}_{\bullet,\bullet}$ vanish in $W_n\Omega^{\bullet}_{\underline{Z}_{\bullet,\bullet}}$ (because $W\Omega^{\bullet}_{\underline{Z}_{\bullet,\bullet}}$ is p-torsion free), so we finally get the morphism (2.11.3). To check that it is a quasi-isomorphism, it suffice to do so on each $Z_{r,t}$, and this follows from [HK94, Th. (4.19)]. We obtain in this way the isomorphism (2.7.1).

To construct the isomorphism (2.7.2), it suffices to observe that the compatibility of the previous constructions when n varies implies that they make sense in the category of inverse systems indexed by $n \in \mathbb{N}$. Then one can apply the functor $\mathbb{R}\varprojlim_n$ to the isomorphism (2.7.1) viewed an an isomorphism in the derived category of inverse systemes of sheaves of W-modules on $Y_\bullet$, and this provides the isomorphism (2.7.2).

The isomorphisms (2.7.1) and (2.7.2) do not depend upon the choices made in their construction. If $(\underline{Z}_{\bullet,\bullet}, \underline{\mathcal{T}}_{\bullet,\bullet}, j_{\bullet,\bullet}, i_{\bullet,\bullet}, h_{\bullet,\bullet,n})$ and $(\underline{Z}'_{\bullet,\bullet}, \underline{\mathcal{T}}'_{\bullet,\bullet}, j'_{\bullet,\bullet}, i'_{\bullet,\bullet}, h'_{\bullet,\bullet,n})$ are two sets of data provided by Lemma 2.10, one can construct a third set of data $(\underline{Z}''_{\bullet,\bullet}, \underline{\mathcal{T}}''_{\bullet,\bullet}, j''_{\bullet,\bullet}, i''_{\bullet,\bullet}, h''_{\bullet,\bullet,n})$ mapping to the two previous ones by setting

$$\underline{Z}''_{\bullet,\bullet} = \underline{Z}_{\bullet,\bullet} \times_{\underline{Y}_\bullet} \underline{Z}'_{\bullet,\bullet}, \qquad \underline{\mathcal{T}}''_{\bullet,\bullet} = \underline{\mathcal{T}}_{\bullet,\bullet} \times_{\underline{\Sigma}} \underline{\mathcal{T}}'_{\bullet,\bullet},$$

and defining $j''_{\bullet,\bullet}$, $i''_{\bullet,\bullet}$ and $h''_{\bullet,\bullet,n}$ by functoriality. Then the independence property of (2.7.1) and (2.7.2) follows from the functoriality of the canonical isomorphisms used in their construction with respect to the projections from $(\underline{Z}''_{\bullet,\bullet}, \underline{\mathcal{T}}''_{\bullet,\bullet})$ to $(\underline{Z}_{\bullet,\bullet}, \underline{\mathcal{T}}_{\bullet,\bullet})$ and $(\underline{Z}'_{\bullet,\bullet}, \underline{\mathcal{T}}'_{\bullet,\bullet})$. Moreover, one can also prove the functoriality of (2.7.1) and (2.7.2) with respect to $\underline{Y}_\bullet$ by similar arguments using the graph construction: for a morphism $\varphi_\bullet : \underline{Y}'_\bullet \to \underline{Y}_\bullet$ between two m-truncated simplicial log schemes satisfying the assumptions of Lemma 2.7, one can find sets of data $(\underline{Z}_{\bullet,\bullet}, \underline{\mathcal{T}}_{\bullet,\bullet}, j_{\bullet,\bullet}, i_{\bullet,\bullet}, h_{\bullet,\bullet,n})$ and $(\underline{Z}'_{\bullet,\bullet}, \underline{\mathcal{T}}'_{\bullet,\bullet}, j'_{\bullet,\bullet}, i'_{\bullet,\bullet}, h'_{\bullet,\bullet,n})$ satisfying the conditions of Lemma 2.10 relatively to $\underline{Y}_\bullet$ and $\underline{Y}'_\bullet$, and such that there exists morphisms of bisimplicial log schemes $\psi_{\bullet,\bullet} : \underline{Z}'_{\bullet,\bullet} \to \underline{Z}_{\bullet,\bullet}$, $\theta_{\bullet,\bullet} : \underline{\mathcal{T}}'_{\bullet,\bullet} \to \underline{\mathcal{T}}_{\bullet,\bullet}$ satisfying the obvious compatibilities. Then the functoriality of (2.7.1) and (2.7.2) with respect to $\varphi_\bullet$ follows from the functoriality of the canonical isomorphisms used in their construction with respect to $\varphi_\bullet$,

$\psi_{\bullet,\bullet}$ and $\theta_{\bullet,\bullet}$. In particular, one obtains in this way that the isomorphisms (2.7.1) and (2.7.2) are compatible with the Frobenius actions. □

2.12. *Proof of Theorem 1.3, assuming Theorem 1.5.* To conclude this section, we prove that Theorem 1.5 implies Theorem 1.3. We keep the notations of 1.1, and we first observe that if Theorem 1.3 holds when R is complete, then it holds in general. Indeed, let $\widehat{R}$ be the completion of R, and $\widetilde{X} = X_{\widehat{R}}$. Then $\widetilde{X}$ is a regular scheme: on the one hand, its generic fibre is smooth over $\widehat{K} = \mathrm{Frac}(\widehat{R})$; on the other hand, its special fibre is isomorphic to X_k, and the completions of the local rings of X and $\widetilde{X}$ are isomorphic at any corresponding points of their special fibres. It follows that $\widetilde{X}$ satisfies the assumptions of Theorem 1.3 relatively to $\widehat{R}$, and the theorem for $\widetilde{X}$ implies the theorem for X.

Therefore, we assume in the rest of the proof that R is complete. We fix an integer $m > q$. Let K' be a finite extension of K, with ring of integers R' and residue field k', such that there exists an m-truncated simplicial log scheme $\underline{Y}_\bullet$ over R', with an augmentation morphism $u : Y_0 \to X_{R'}$, such that properties a) - c) of Lemma 2.2 are satisfied. Let $W'_n = W_n(k')$, $W' = W(k')$, $K'_0 = \mathrm{Frac}(W')$, and let $\underline{\Sigma}'_n$, $\underline{\Sigma}'$ be the log schemes defined by W'_n, W' as in 2.1.

Thanks to property a) of Lemma 2.2, the log schemes $(\underline{Y}_r)_{k'}$ are smooth of Cartier type over $\underline{\Sigma}'_1$. Therefore, we can consider the log crystalline cohomology of $\underline{Y}_{\bullet\, k'}$

$$\mathbb{R}\Gamma_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) := \mathbb{R}\varepsilon^m_*\mathbb{R}\Gamma^\bullet_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}),$$

as defined in 2.6. Using the naive filtration on the functor $\mathbb{R}\varepsilon^m_*$ (see 2.3), its basic properties follow from those of the log crystalline cohomology of the proper and smooth log schemes $(Y_r)_{k'}$. In particular, since Y_r is proper over $\underline{\Sigma}'_1$ for all r, the complex $\mathbb{R}\Gamma_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'})$ is a perfect complex of W'-modules, and the cohomology space $H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0$ is a finite dimensional K'_0-vector space. By functoriality, it is endowed with the semi-linear Frobenius action defined by the absolute Frobenius endomorphism of $\underline{Y}_{\bullet\, k'}$.

From (2.6.2) and (2.7.2), we deduce an isomorphism

$$H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0 \xrightarrow{\sim} H^q(\underline{Y}_{\bullet\, k'}, W\Omega^\bullet_{\underline{Y}_{\bullet\, k'}}) \otimes K'_0,$$

which is compatible with the Frobenius actions thanks to Proposition 2.7. The filtration of the complex $W\Omega^\bullet_{\underline{Y}_{\bullet\, k'}}$ by the subcomplexes $\sigma_{\geq i}W\Omega^\bullet_{\underline{Y}_{\bullet\, k'}}$ provides a spectral sequence

$$E_1^{i,j} = H^j(\underline{Y}_{\bullet\, k'}, W\Omega^i_{\underline{Y}_{\bullet\, k'}}) \otimes K'_0 \Longrightarrow H^{i+j}_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0,$$

which is endowed with a Frobenius action. Using the naive filtration on $\mathbb{R}\varepsilon^m_*$, we deduce from the case of a single log scheme that each term $E_1^{i,j}$ is a finite dimensional K_0-vector space on which the Frobenius action is bijective with slopes in $[i, i+1[$. Therefore the spectral sequence degenerates at E_1, and, taking (2.1.1) into account, we get in particular an isomorphism

$$(H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0)^{<1} \xrightarrow{\sim} H^q(Y_{\bullet\, k'}, W\mathcal{O}_{Y_{\bullet\, k'},\mathbb{Q}}). \tag{2.12.1}$$

Since $\underline{Y}_\bullet$ satisfies property a) of 2.2, the construction of the monodromy operator N on log crystalline cohomology can be extended to the case of $\underline{Y}_{\bullet\, k'}$ [Ts98, (6.3)].

Moreover, the Hyodo-Kato isomorphism ρ_π can also be extended to the case of $\underline{Y}_{\bullet\, k'}$ [Ts98, (6.3.2)], providing an isomorphism

$$\rho_\pi : H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K' \xrightarrow{\sim} H^q(Y_{\bullet\, K'}, \Omega^\bullet_{Y_{\bullet\, K'}}). \tag{2.12.2}$$

Thus, $H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'$ inherits a filtered (φ, N)-module structure.

It follows from [Ts98, Th. 7.1.1] (generalizing [Ts99, Th. 0.2]) that, endowed with this structure, $H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0$ is an admissible filtered (φ, N)-module, corresponding to the Galois representation $H^q_{\text{ét}}(Y_{\bullet\, \overline{K}}, \mathbb{Q}_p)$. Therefore it is weakly admissible. In particular, either $H^q_{\mathrm{crys}}(\underline{Y}_{\bullet\, k'}/\underline{\Sigma}', \mathcal{O}_{\underline{Y}_{\bullet\, k'}/\underline{\Sigma}'}) \otimes K'_0 = 0$, or its smallest Newton slope is greater or equal to its smallest Hodge slope. Since $H^q(X_K, \mathcal{O}_{X_K}) = 0$, Corollary 2.5 implies that the smallest Hodge slope is at least 1. Therefore the part of Newton slope < 1 vanishes. By (2.12.1), we obtain

$$H^q(Y_{\bullet\, k'}, W\mathcal{O}_{Y_{\bullet\, k'}, \mathbb{Q}}) = 0. \tag{2.12.3}$$

As $Y_\bullet \to X_{R'}$ satisfies property 2.2 c), there exists a sub-K-extension $K_1 \subset K'$, with ring of integers R_1 and residue field k_1, a semi-stable scheme Y over R_1, a projective R-alteration $f : Y \to X$, and finitely many R-embeddings $\sigma_i : R_1 \hookrightarrow K'$ such that, if $u_1 : Y \to X_{R_1}$ denotes the R-morphism defined by f, and if Y_i (resp. $u_i : Y_i \to X_{R'}$) denotes the R'-scheme (resp. R'-morphism) deduced by base change via σ_i from Y (resp. u_1), then $Y_0 = \coprod_i Y_i$, and the augmentation morphism $u : Y_0 \to X_{R'}$ is defined by $u|_{Y_i} = u_i$. This provides a commutative diagram

$$\begin{array}{ccccccc} & & & Y_{\bullet\, k'} & \xrightarrow{u_{\bullet\, k'}} & X_{k'} & \\ & \overset{s_0}{\nearrow} & & & & & \searrow \\ Y_{0,k'} = \coprod_i Y_{i,k'} & & & & & & X_k \\ & \searrow & & & & & \nearrow_{f_k} \\ & & & Y_{k_1} & \hookrightarrow & Y_k & \end{array} \tag{2.12.4}$$

in which we identify schemes with their Zariski topos, $Y_k := \operatorname{Spec} k \times_{\operatorname{Spec} R} Y$, and:

(i) the morphism $u_{\bullet\, k'}$ is such that, for any sheaf E on $X_{k'}$, $u^{-1}_{\bullet\, k'} E$ is the family of sheaves $(u_r)^{-1}_{k'} E$, with $u_r : Y_r \to X_{R'}$ defined by the augmentation morphism,

(ii) the morphism s_0 is such that, for any sheaf $F^\bullet$ on $Y_{\bullet\, k'}$, $s_0^{-1} F^\bullet = F^0$,

(iii) the morphism $Y_{i,k'} \to Y_{k_1}$ is the projection corresponding to σ_i.

By functoriality, we obtain a commutative diagram for the corresponding Witt cohomology spaces

(2.12.5)

$$\begin{array}{ccccc} & H^q(X_{k'}, W\mathcal{O}_{X_{k'}, \mathbb{Q}}) & \longrightarrow & H^q(Y_{\bullet\, k'}, W\mathcal{O}_{Y_{\bullet\, k'}, \mathbb{Q}}) & \\ \nearrow & & & & \searrow \\ H^q(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) & & & & \bigoplus_i H^q(Y_{i,k'}, W\mathcal{O}_{Y_{i,k'}, \mathbb{Q}}). \\ \searrow^{f_k^*} & & & & \nearrow \\ & H^q(Y_k, W\mathcal{O}_{Y_k, \mathbb{Q}}) & \xrightarrow{\sim} & H^q(Y_{k_1}, W\mathcal{O}_{Y_{k_1}, \mathbb{Q}}) & \end{array}$$

In this diagram, the lower horizontal arrow is an isomorphism because $Y_{k_1} \hookrightarrow Y_k$ is a nilpotent immersion [BBE07, Prop. 2.1 (i)]. The lower right arrow is injective on each summand, because each σ_i turns k' into a finite separable extension of k_1, hence it follows from [Il79, 0, Prop. 1.5.8] that

$$W(k') \otimes_{W(k_1)} \Gamma(U, W\mathcal{O}_{Y_{k_1}}) \xrightarrow{\sim} \Gamma(U_i, W\mathcal{O}_{Y_{i,k'}})$$

for any affine open subset $U \subset Y_{k_1}$ with inverse image $U_i \subset Y_{i,k'}$; as one can compute Witt cohomology using Čech cohomology, this implies that

$$W(k') \otimes_{W(k_1)} H^q(Y_{k_1}, W\mathcal{O}_{Y_{k_1}}) \xrightarrow{\sim} H^q(Y_{i,k'}, W\mathcal{O}_{Y_{i,k'}}).$$

Finally, $f : Y \to X$ is a projective alteration between two flat regular schemes of finite type over R, so Theorem 1.5 implies that f_k^* is injective. Therefore, the functoriality map $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) \to \bigoplus_i H^q(Y_{i,k'}, W\mathcal{O}_{Y_{i,k'},\mathbb{Q}})$ is injective. But (2.12.3) implies that the composition of the upper path in the diagram is 0. It follows that $H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) = 0$. □

3. An injectivity theorem for coherent cohomology

We now begin our preliminary work in view of the proof of Theorem 1.5.

One of the key ingredients in this proof is a theorem which bounds the order of elements in the kernel of the functoriality map induced on coherent cohomology by a proper surjective complete intersection morphism $f : Y \to X$ of virtual relative dimension 0. Such a result is a consequence of the existence of a "trace morphism" $\tau_f : \mathbb{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$ which satisfies the properties stated in the following theorem:

Theorem 3.1. *Let X be a noetherian scheme with a dualizing complex, and let $f : Y \to X$ be a proper complete intersection morphism of virtual relative dimension 0. There exists a morphism $\tau_f : \mathbb{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$ which satisfies the following properties:*

(i) *If $g : Z \to Y$ is a second proper complete intersection morphism of virtual relative dimension 0, then the composed morphism*

$$\mathbb{R}(f \circ g)_*\mathcal{O}_Z \cong \mathbb{R}f_*\mathbb{R}g_*\mathcal{O}_Z \xrightarrow{\mathbb{R}f_*(\tau_g)} \mathbb{R}f_*\mathcal{O}_Y \xrightarrow{\tau_f} \mathcal{O}_X \tag{3.1.1}$$

is equal to τ_{fg}.

(ii) *Let X' be another noetherian scheme with a dualizing complex, $u : X' \to X$ a morphism such that X' and Y are Tor-independent over X, and $f' : Y' \to X'$ the pull-back of f by u. If f is projective, or if either f is flat, or u is residually stable* [Co00, p. 132]*, then the morphism*

$$\mathbb{R}f'_*\mathcal{O}_{Y'} \cong \mathbb{L}u^*\mathbb{R}f_*\mathcal{O}_Y \xrightarrow{\mathbb{L}u^*(\tau_f)} \mathcal{O}_{X'}, \tag{3.1.2}$$

defined by the base change isomorphism (A.1.2)*, is equal to $\tau_{f'}$.*

(iii) *If f is finite and flat, then, for any section $b \in f_*\mathcal{O}_Y$,*

$$\tau_f(b) = \mathrm{trace}_{f_*\mathcal{O}_Y/\mathcal{O}_X}(b). \tag{3.1.3}$$

As explained in the introduction, we refer to B.7 for the definition of τ_f, and to B.9 for the proof of the theorem.

It may be worth recalling a few examples of complete intersection morphisms of virtual relative dimension 0 (in short: ci0):

1) If X and Y are two regular schemes with the same Krull dimension, any morphism $f : Y \to X$ which is locally of finite type is ci0. This is the situation where we will use Theorem 3.1 in this article.

2) If X and Y are smooth over a third scheme S, with the same relative dimension, any S-morphism $Y \to X$ is ci0.

3) If X is a scheme, $Z \hookrightarrow X$ a regularly embedded closed subscheme, and $f : Y \to X$ the blowing up of X along Z, then f is ci0 [SGA 6, VII, Proposition 1.8].

The existence of τ_f has a remarkable consequence for the functoriality maps induced on coherent cohomology.

Theorem 3.2. *Let X be a noetherian scheme with a dualizing complex, and $f : Y \to X$ a proper complete intersection morphism of virtual relative dimension 0. Assume that there exists a scheme-theoretically dense open subset $U \subset X$ such that $f^{-1}(U) \to U$ is finite locally free of constant rank $r \geq 1$. Then, for any complex $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_X)$ and any $q \geq 0$, the kernel of the functoriality map*

$$H^q(X, \mathcal{E}) \to H^q(Y, \mathbb{L}f^*\mathcal{E}) \tag{3.2.1}$$

is annihilated by r. In particular, when r is invertible on X, the functoriality maps are injective.

Proof. By 3.1 (iii), the composition $\mathcal{O}_X \to \mathbb{R}f_*\mathcal{O}_Y \xrightarrow{\tau_f} \mathcal{O}_X$ is multiplication by r over U. Since U is scheme-theoratically dense in X, it is multiplication by r over X.

The complete intersection hypothesis implies that f has finite Tor-dimension, hence $\mathbb{L}f^*\mathcal{E}$ belongs to $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_Y)$. Moreover, we can apply the projection formula [SGA 6, III, 3.7] to obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & \mathbb{R}f_*\mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{E} & \xrightarrow{\sim} & \mathbb{R}f_*\mathbb{L}f^*\mathcal{E} \\ & {\scriptstyle \times r}\searrow & \downarrow{\scriptstyle \tau_f \otimes \mathrm{Id}} & \swarrow & \\ & & \mathcal{E} & & \end{array},$$

in which the upper composed morphism is the adjunction morphism. Applying the functors $H^q(X, -)$ to the diagram, the theorem follows. $\square$

4. Koszul resolutions and local description of the trace morphism τ_f

We recall here some well-known explicit constructions based on the Koszul complex which enter in the definition of the trace morphism τ_f. Later on, this will allow us to define generalizations of τ_f for sheaves of Witt vectors. As in the whole article, we follow Conrad's constructions and conventions [Co00].

4.1. Let P be a scheme, and let $\mathbf{t} = (t_1, \ldots, t_d)$ be a regular sequence of sections of $\mathcal{O}_P$, defining an ideal $\mathcal{I} \subset \mathcal{O}_P$. We denote by $Y \subset P$ the closed subscheme defined by $\mathcal{I}$, and by $i : Y \hookrightarrow P$ the corresponding closed immersion. Classically, the Koszul complex $K_\bullet(\mathbf{t})$ defined by the sequence $(t_1, \ldots, t_d)$ is the chain complex concentrated in homological degrees $[0, d]$, such that $\mathcal{E} := K_1(\mathbf{t})$ is a free $\mathcal{O}_P$-module of rank d

with basis $e_1, \dots, e_d$, $K_k(\mathbf{t}) = \bigwedge^k \mathcal{E}$ for all k, and such that the differential is given in degree k by

$$d_k(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j-1} t_{i_j} e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_k}.$$

It is often more convenient to consider $K_\bullet(\mathbf{t})$ as a cochain complex concentrated in cohomological degrees $[-d, 0]$, by setting $(K_\bullet(\mathbf{t}))^k = K_{-k}(\mathbf{t})$ and leaving the differential unchanged [Co00, p. 17].

Since $\mathbf{t}$ is a regular sequence, $K_\bullet(\mathbf{t})$ is a free resolution of $\mathcal{O}_Y$ over $\mathcal{O}_P$. For any $\mathcal{O}_P$-module $\mathcal{M}$, this resolution provides an isomorphism

$$(4.1.1) \qquad \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{M}) := H^d(\mathcal{H}om^\bullet_{\mathcal{O}_P}(K_\bullet(\mathbf{t}), \mathcal{M})) \xrightarrow[\sim]{\psi_{\mathbf{t},\mathcal{M}}} \frac{\mathcal{H}om_{\mathcal{O}_P}(\bigwedge^d \mathcal{E}, \mathcal{M})}{\mathcal{I}\mathcal{H}om_{\mathcal{O}_P}(\bigwedge^d \mathcal{E}, \mathcal{M})},$$

where $\psi_{\mathbf{t},\mathcal{M}}$ is the tautological isomorphism multiplied by $(-1)^{d(d+1)/2}$ (see [Co00, definition of (1.3.28) and (2.5.2)]). For any section m of $\mathcal{M}$, we will denote by

$$(4.1.2) \qquad \begin{bmatrix} m \\ t_1, \dots, t_d \end{bmatrix} \in \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{M})$$

the section corresponding by (4.1.1) to the class of the homomorphism $u_{\mathbf{t},m}$ which sends $e_1 \wedge \dots \wedge e_d$ to m. Note that this section is linear with respect to m, only depends on the class of m mod $\mathcal{I}\mathcal{M}$, and is functorial with respect to $\mathcal{M}$. Its dependence on the regular sequence $\mathbf{t}$ is given by the following lemma.

Lemma 4.2. *Let $\mathbf{t}' = (t'_1, \dots, t'_d)$ be another regular sequence of sections of $\mathcal{O}_P$, generating an ideal $\mathcal{I}'$ such that $\mathcal{I}' \subset \mathcal{I}$. Let $C = (c_{i,j})_{1 \le i,j \le d}$ be a matrix with entries in $\mathcal{O}_P$ such that $t'_i = \sum_{j=1}^d c_{i,j} t_j$ for all i. If $\alpha : \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_P/\mathcal{I}, \mathcal{M}) \to \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_P/\mathcal{I}', \mathcal{M})$ is the functoriality homomorphism, then*

$$(4.2.1) \qquad \alpha(\begin{bmatrix} m \\ t_1, \dots, t_d \end{bmatrix}) = \begin{bmatrix} \det(C) m \\ t'_1, \dots, t'_d \end{bmatrix}.$$

Proof. Let $K_\bullet(\mathbf{t}')$ be the Koszul resolution of $\mathcal{O}_P/\mathcal{I}'$, and $\mathcal{E}' = K_1(\mathbf{t}')$, with basis $e'_1, \dots, e'_d$. One defines a morphism of resolutions $\phi : K_\bullet(\mathbf{t}') \to K_\bullet(\mathbf{t})$ by setting $\phi_1(e'_i) = \sum_j c_{i,j} e_j$, and $\phi_k = \wedge^k \phi_1$ for $0 \le k \le d$. Then ϕ provides a commutative diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}, \mathcal{M}) & \longrightarrow & \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_P/\mathcal{I}, \mathcal{M}) \\ \Big\downarrow{\scriptstyle \phi_d = \det(C)} & & \Big\downarrow{\scriptstyle \alpha} \\ \mathcal{H}om_{\mathcal{O}_P}(\wedge^d \mathcal{E}', \mathcal{M}) & \longrightarrow & \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_P/\mathcal{I}', \mathcal{M}). \end{array}$$

The lemma follows. ☐

4.3. Under the assumptions of 4.1, the morphism $d_1 : \mathcal{E} \twoheadrightarrow \mathcal{I}$ defines an isomorphism $\mathcal{E}/\mathcal{I}\mathcal{E} \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2$. Using the canonical isomorphisms, this provides

$$\begin{aligned}(4.3.1) \qquad \frac{\mathcal{H}om_{\mathcal{O}_P}(\bigwedge^d\mathcal{E}, \mathcal{M})}{\mathcal{I}\mathcal{H}om_{\mathcal{O}_P}(\bigwedge^d\mathcal{E}, \mathcal{M})} &\xrightarrow{\sim} (\textstyle\bigwedge^d\mathcal{E})^\vee/\mathcal{I}(\bigwedge^d\mathcal{E})^\vee \otimes_{\mathcal{O}_Y} \mathcal{M}/\mathcal{I}\mathcal{M} \\ &\xrightarrow{\sim} \textstyle\bigwedge^d((\mathcal{E}/\mathcal{I}\mathcal{E})^\vee) \otimes_{\mathcal{O}_Y} \mathcal{M}/\mathcal{I}\mathcal{M} \\ &\xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^*\mathcal{M}.\end{aligned}$$

Note that, due to the commutation between dual and exterior power, the composition (4.3.1) maps the class of $u_{\mathbf{t},m}$ to $(\bar{t}_d^\vee \wedge \ldots \wedge \bar{t}_1^\vee) \otimes i^*(m)$, where $\bar{t}_k$ denotes the class of t_k mod $\mathcal{I}^2$.

Composing (4.1.1) and (4.3.1), one obtains the *fundamental local isomorphism* [Ha66, III, 7.2] as defined by Conrad [Co00, (2.5.2)] in the local case:

$$(4.3.2) \qquad \eta_{Y/P} : \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{M}) \xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^*\mathcal{M}.$$

Applying Lemma 4.2 to the case of two regular sequences of generators of the ideal $\mathcal{I}$, one sees that the isomorphism $\eta_{Y/P}$ does not depend on the sequence $\mathbf{t}$, so that local constructions can be glued to define $\eta_{Y/P}$ for any regular immersion $i : Y \hookrightarrow P$, without assuming that $\mathcal{I}$ is defined globally by a regular sequence. One obtains in this way the fundamental local isomorphism in the general case [Co00, (2.5.1)].

Lemma 4.4. *Under the assumptions of 4.1, let $\pi : P \to X$ be a smooth morphism of relative dimension d, and $f = \pi \circ i$. Let $\zeta'_{i,\pi} : \omega_{Y/X} \xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^*\omega_{P/X}$ be the canonical identification* (A.2.5)*, and let δ_f be the canonical section of $\omega_{Y/X}$ (defined by* (A.7.2)*). Then*

$$(4.4.1) \qquad \zeta'_{i,\pi}(\delta_f) = \eta_{Y/P}(\begin{bmatrix} dt_1 \wedge \cdots \wedge dt_d \\ t_1, \ldots, t_d \end{bmatrix}).$$

Proof. By definition, $\begin{bmatrix} dt_1 \wedge \cdots \wedge dt_d \\ t_1, \ldots, t_d \end{bmatrix}$ is mapped to $u_{\mathbf{t},dt_1\wedge\ldots\wedge dt_d}$ by (4.1.1), and we observed that $u_{\mathbf{t},dt_1\wedge\ldots\wedge dt_d}$ is mapped to $(\bar{t}_d^\vee \wedge \ldots \wedge \bar{t}_1^\vee) \otimes i^*(dt_1 \wedge \ldots \wedge dt_d)$ by (4.3.1). Since $\zeta'_{i,\pi}(\delta_f) = (\bar{t}_1^\vee \wedge \ldots \wedge \bar{t}_d^\vee) \otimes i^*(dt_d \wedge \ldots \wedge dt_1)$ by construction, relation (4.4.1) follows. □

4.5. Let $\pi : P \to X$ be a smooth morphism of relative dimension d, $i : Y \hookrightarrow P$ a regular immersion of codimension d, and $f = \pi \circ i$. We define the morphism

$$(4.5.1) \qquad \gamma_f : \mathcal{O}_Y \to \omega_{P/X}[d]$$

as being the composition

$$\mathcal{O}_Y \xrightarrow{\varphi_f} \omega_{Y/X} \xrightarrow{\eta^{-1}_{Y/P} \circ \zeta'_{i,\pi}} \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \omega_{P/X}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \omega_{P/X}[d]) \xrightarrow{\text{can}} \omega_{P/X}[d],$$

where φ_f is the morphism sending 1 to δ_f.

Proposition 4.6. *Let X be a separated noetherian scheme with a dualizing complex, $P = \mathbb{P}^d_X$ a projective space over X, $\pi : P \to X$ the structural morphism, $i : Y \hookrightarrow P$*

a regular immersion of codimension d, and $f = \pi \circ i$. Then the trace morphism $\tau_f : \mathbb{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$ of Theorem 3.1 is equal to the composition

$$\mathbb{R}f_*(\mathcal{O}_Y) \xrightarrow{\mathbb{R}\pi_*(\gamma_f)} \mathbb{R}\pi_*(\omega_{P/X}[d]) \xrightarrow{\mathrm{Trp}_\pi} \mathcal{O}_X, \tag{4.6.1}$$

where Trp_π *is the trace morphism for the projective space defined in* [Co00, (2.3.1)-(2.3.5)].

Proof. By construction (see B.7), τ_f is the composition $\mathrm{Tr}_f \circ \mathbb{R}f_*(\lambda_f) \circ \mathbb{R}f_*(\varphi_f)$ in the commutative diagram

$$\begin{array}{ccccc}
\mathbb{R}f_*(\mathcal{O}_Y) & & & & \\
\downarrow{\scriptstyle \mathbb{R}f_*(\varphi_f)} & & & & \\
\mathbb{R}f_*(\omega_{Y/X}) & \xrightarrow[\sim]{\mathbb{R}f_*(\lambda_f)} & \mathbb{R}f_*(f^!(\mathcal{O}_X)) & \xrightarrow{\mathrm{Tr}_f} & \mathcal{O}_X \\
{\scriptstyle \mathbb{R}f_*(\zeta'_{i,\pi})}\downarrow\wr & & {\scriptstyle \mathbb{R}f_*(c^{-1}_{i,\pi})}\uparrow\wr & & {\scriptstyle \mathrm{Tr}_\pi}\uparrow \\
\mathbb{R}f_*(\omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}i^*(\omega_{P/X}[d])) & & \mathbb{R}f_*(i^!\pi^!(\mathcal{O}_X)) & \xrightarrow{\mathbb{R}\pi_*(\mathrm{Tr}_i)} & \mathbb{R}\pi_*(\pi^!(\mathcal{O}_X)) \\
{\scriptstyle \mathbb{R}f_*(\eta_i^{-1})}\downarrow\wr & & {\scriptstyle \mathbb{R}f_*(i^!(e_\pi))}\uparrow\wr & & {\scriptstyle \mathbb{R}\pi_*(e_\pi)}\uparrow\wr \\
\mathbb{R}f_*(\mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \omega_{P/X}[d])) & \xrightarrow[\sim]{\mathbb{R}f_*(d_i)} & \mathbb{R}f_*(i^!(\omega_{P/X}[d])) & \xrightarrow{\mathbb{R}\pi_*(\mathrm{Tr}_i)} & \mathbb{R}\pi_*(\omega_{P/X}[d]),
\end{array}$$

in which the isomorphism λ_f is defined by the commutativity of the left rectangle before applying $\mathbb{R}f_*$ (cf. B.1), and the other arrows are defined as follows:

a) $\zeta'_{i,\pi}$ is the derived category version of the isomorphism used in Lemma 4.4, defined by (A.2.6);

b) η_i is the extension to the derived category of the fundamental local isomorphism $\eta_{Y/P}$, defined by [Co00, (2.5.3)];

c) d_i is the canonical isomorphism of functors $i^\flat := \mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, -) \xrightarrow{\sim} i^!$, defined by [Co00, (3.3.19)];

d) e_π is the canonical isomorphism of functors $\pi^\sharp := \omega_{P/X}[d] \otimes^{\mathbb{L}} \pi^*(-) \xrightarrow{\sim} \pi^!$, defined by [Co00, (3.3.21)].

e) $c_{i,\pi}$ is the transitivity isomorphism $f^! \xrightarrow{\sim} i^!\pi^!$, defined by [Co00, (3.3.14)].

Moreover, the upper right square commutes because of the transitivity of the trace morphism [Co00, 3.4.3, (TRA1)], and the lower right square commutes by functoriality of the trace morphism Tr_i with respect to e_π.

In this diagram, the composition of the right vertical arrows is the projective trace morphism Trp_π [Co00, 3.4.3, (TRA3)], and the isomorphism d_i on the bottom row identifies Tr_i with the trace morphism Trf_i for finite morphisms [Co00, 3.4.3, (TRA2)]. As the latter is the canonical morphism $i_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, -) \to \mathrm{Id}$ defined by $\mathcal{O}_P \twoheadrightarrow \mathcal{O}_Y$, it follows that the composition of the left column and the bottom row of the diagram is equal to $\mathbb{R}\pi_*(\gamma_f)$, which proves the proposition. □

5. Preliminaries on the relative de Rham-Witt complex

We extend here to the relative de Rham-Witt complex constructed by Langer and Zink [LZ04] structure theorems which are classical when the base is a perfect scheme of characteristic p ([Il79], [IR83]). We begin by recalling some basic facts from their construction.

From now on, we fix a prime number p. We denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the prime ideal (p). Although many results of [LZ04] are valid for $\mathbb{Z}_{(p)}$-schemes, we limit our exposition to the case of schemes on which p is locally nilpotent, which will suffice for our applications.

5.1. Let S be a scheme on which p is locally nilpotent, and let $f : X \to S$ be a morphism of schemes. An *F-V-pro-complex of X/S* as defined in [LZ04] is a pro-complex $\{R : E_{n+1} \to E_n\}_{n\geq 1}$ of sheaves on X, where E_n is a differential graded $W_n(\mathcal{O}_X)/f^{-1}W_n(\mathcal{O}_S)$-algebra (i.e., E_n is a commutative graded $W_n(\mathcal{O}_X)$-algebra together with an $f^{-1}W_n(\mathcal{O}_S)$-linear map $d : E_n \to E_n(1)$, satisfying $d(\alpha\beta) = (d\alpha)\beta+(-1)^{\deg\alpha}\alpha d\beta$ and $d^2 = 0$), which is equipped with a map of graded pro-rings

$$F : E_{\bullet+1} \to E_\bullet,$$

called the Frobenius morphism, and with a map of graded abelian groups

$$V : E_\bullet \to E_{\bullet+1},$$

called the Verschiebung morphism, such that the following properties hold:

(i) The structure map $W_\bullet(\mathcal{O}_X) \to E^0_\bullet$ is compatible with F and V.

(ii) The following relations hold:

$$FV = p, \quad FdV = d, \tag{5.1.1}$$

$$V(\omega F(\eta)) = V(\omega)\eta, \quad \text{for all } \omega \in E_n, \eta \in E_{n+1}, n \geq 1, \tag{5.1.2}$$

$$F(d[a]) = [a]^{p-1}d[a], \quad \text{for all } a \in \mathcal{O}_X, \tag{5.1.3}$$

where $[a]$ denotes the Teichmüller lift of a to $W_n(\mathcal{O}_X)$, for any n.

A morphism between two F-V-pro-complexes of X/S is a map of pro-differential graded $W_\bullet(\mathcal{O}_X)/f^{-1}W_\bullet(\mathcal{O}_S)$-algebras compatible with F and V. By [LZ04, Prop. 1.6, Rem. 1.10] there exists an initial object in the category of F-V-pro-complexes of X/S, which is called the *relative de Rham-Witt complex of X/S* and is denoted by $\{R : W_{n+1}\Omega^\bullet_{X/S} \to W_n\Omega^\bullet_{X/S}\}_{n\geq 1}$. Each sheaf $W_n\Omega^q_{X/S}$ is a quasi-coherent sheaf on the scheme $W_n(X) := (|X|, W_n(\mathcal{O}_X))$ defined in 2.9, and the transition morphisms R are epimorphisms. When S is a perfect scheme of characteristic p, the relative de Rham-Witt complex coincides with the one defined in [Il79]. Notice that we have the following properties:

$$W_n\Omega^0_{X/S} = W_n(\mathcal{O}_X), \quad W_1\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S},$$

and that, by [LZ04, (1.17) and (1.19)], relations (5.1.1) and (5.1.2) imply that

$$Vd = pdV, \quad dF = pFd. \tag{5.1.4}$$

In addition, when S is an $\mathbb{F}_p$-scheme, the operators F and V satisfy the relation $VF = p$.

We also recall the behaviour of the de Rham-Witt complex with respect to étale pull-backs. Let

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative diagram in which h is étale and g unramified. Then, for all $q \geq 0$ and $r \geq n \geq 1$, $W_n(X')$ is étale over $W_n(X)$ and we have the $W_r(\mathcal{O}_{X'})$-linear isomorphisms

$$W_r(\mathcal{O}_{X'}) \otimes_{h^{-1}W_r(\mathcal{O}_X)} h^{-1}W_n\Omega^q_{X/S} \xrightarrow{\sim} W_n\Omega^q_{X'/S'}, \tag{5.1.5}$$

(5.1.6)

$$W_r(\mathcal{O}_{X'}) \otimes_{h^{-1}W_r(\mathcal{O}_X)} h^{-1}(F^{r-n}_* W_n\Omega^q_{X/S}) \xrightarrow{\sim} F^{r-n}_* W_n\Omega^q_{X'/S'}, \quad a \otimes \omega \mapsto F^{r-n}(a)\omega,$$

where, for any W_n-module M, $F^{r-n}_* M$ denotes M viewed as a W_r-module via $F^{r-n} : W_r \to W_n$ [LZ04, Prop. 1.11, Prop. A.8 and Cor. A.11].

Finally, the completed relative de Rham-Witt complex is defined by $W\Omega^\bullet_{X/S} := \varprojlim_n W_n\Omega^\bullet_{X/S}$; the canonical morphisms $W\Omega^\bullet_{X/S} \to W_n\Omega^\bullet_{X/S}$ are still epimorphisms.

5.2. Let $S = \operatorname{Spec} A$ be affine. We want to recall the calculation of $W\Omega^q_{A[x_1,\dots,x_d]/A} := \Gamma(\mathbb{A}^d_S, W\Omega^q_{\mathbb{A}^d_S/S})$. We need some notations for this.

A *weight* is a function $k : [1,d] = \{1,2,\dots,d\} \to \mathbb{Z}[\frac{1}{p}]_{\geq 0}$. We write $k_i := k(i)$, for $i \in [1,d]$. The support of k, $\operatorname{supp} k$, consists of those $i \in [1,d]$ with $k_i \neq 0$. For any weight k we choose once and for all a total ordering on the elements of the support of k,

$$\operatorname{supp} k = \{i_1, \dots, i_r\}, \tag{5.2.1}$$

such that:

(i) $\operatorname{ord}_p k_{i_1} \leq \operatorname{ord}_p k_{i_2} \leq \cdots \leq \operatorname{ord}_p k_{i_r}$.
(ii) The ordering on $\operatorname{supp} k$ and on $\operatorname{supp} p^a k$ agree, for any $a \in \mathbb{Z}$.

We say k is *integral* if $k_i \in \mathbb{Z}$, for all $i \in [1,d]$. We say k is *primitive* if it is integral and not all k_i are divisible by p. We set

$$t(k_i) := -\operatorname{ord}_p k_i \quad \text{and} \quad t(k) := \begin{cases} \max\{\, t(k_i) \mid i \in \operatorname{supp} k \,\} & \text{if } \operatorname{supp} k \neq \emptyset, \\ 0 & \text{if } k = 0. \end{cases} \tag{5.2.2}$$

If $k \neq 0$, $t(k)$ is the smallest integer such that $p^{t(k)}k$ is primitive, and we have

$$t(k) = t(k_{i_1}) \geq t(k_{i_2}) \geq \cdots \geq t(k_{i_r}).$$

We denote by $u(k)$ the smallest non-negative integer such that $p^{u(k)}k$ is integral, i.e., $u(k) = \max\{0, t(k)\}$. Notice that k is integral iff $u(k) = 0$ iff $t(k) \leq 0$, and k is primitive iff $t(k) = 0$. An *interval of the support of* k is by definition a subset $I \subset \operatorname{supp} k$ of the form

$$I = \{i_s, i_{s+1}, \dots, i_{s+m}\}.$$

We denote by k_I the weight which equals k on I and is zero on $[1,d] \setminus I$. If k is fixed and I is an interval of the support of k, we write $u(I) := u(k_I)$ and $t(I) := t(k_I)$.

An *admissible partition* $\mathcal{P}$ *of length* q *of* $\operatorname{supp} k$ (or just *of* k) is a tuple of intervals of $\operatorname{supp} k$, $\mathcal{P} = (I_0, I_1, \ldots, I_q)$, such that:

(i) $\operatorname{supp} k = I_0 \sqcup I_1 \sqcup \ldots \sqcup I_q$.

(ii) The elements in I_j are smaller than the elements in I_{j+1} (with respect to the ordering (5.2.1)) for all $j = 0, \ldots, q-1$.

(iii) The intervals $I_1, \ldots, I_q$ are non-empty (but I_0 may be).

Notice that $u(k) = u(I_0)$ if $I_0 \neq \emptyset$ and $u(k) = u(I_1)$ if $I_0 = \emptyset$.

For any $n \leq \infty$, we write $X_i := [x_i] \in W_n(A[x_1, \ldots, x_d])$. If k is an integral weight as above, we write $X^k = X_{i_1}^{k_{i_1}} \cdots X_{i_r}^{k_{i_r}} \in W_n(A[x_1, \ldots, x_d])$.

Let k be any weight and $\eta \in W(A)$. We define

$$e^0(\eta, k) := V^{u(k)}(\eta X^{p^{u(k)}k}) \in W(A[x_1, \ldots x_d]) \tag{5.2.3}$$

and

$$e^1(\eta, k) := \begin{cases} dV^{u(k)}(\eta X^{p^{u(k)}k}) & \text{if } k \text{ is not integral} \\ \eta F^{-t(k)} dX^{p^{t(k)}k} & \text{if } k \text{ is integral} \end{cases} \in W\Omega^1_{A[x_1,\ldots,x_d]/A}. \tag{5.2.4}$$

Definition 5.3 (Basic Witt differentials [LZ04, 2.2])**.** Let k be a weight, $\mathcal{P} = (I_0, I_1, \ldots, I_q)$ an admissible partition of k, and $\xi = V^{u(k)}(\eta) \in W(A)$. The basic Witt-differential $e(\xi, k, \mathcal{P}) \in W\Omega^q_{A[x_1,\ldots,x_d]/A}$ is defined as follows:

$$e(\xi, k, \mathcal{P}) := \begin{cases} e^0(\eta, k_{I_0}) e^1(1, k_{I_1}) \cdots e^1(1, k_{I_q}) & \text{if } I_0 \neq \emptyset, \\ e^1(\eta, k_{I_1}) e^1(1, k_{I_2}) \cdots e^1(1, k_{I_q}) & \text{if } I_0 = \emptyset. \end{cases}$$

Rules 5.4 ([LZ04, Prop. 2.5, Prop. 2.6])**.** Let k be a weight, $\mathcal{P} = (I_0, I_1, \ldots, I_q)$ a partition of k and $\xi = V^{u(k)}(\eta) \in W(A)$. Then:

(i) $\rho e(\xi, k, \mathcal{P}) = e(\rho\xi, k, \mathcal{P})$ for all $\rho \in W(A)$.

(ii) $Fe(\xi, k, \mathcal{P}) = \begin{cases} e(F\xi, pk, \mathcal{P}) & \text{if } I_0 \neq \emptyset \text{ or } k \text{ integral}, \\ e(V^{-1}\xi, pk, \mathcal{P}) & \text{if } I_0 = \emptyset \text{ and } k \text{ not integral}. \end{cases}$

(iii) $Ve(\xi, k, \mathcal{P}) = \begin{cases} e(V\xi, \frac{1}{p}k, \mathcal{P}) & \text{if } I_0 \neq \emptyset \text{ or } \frac{1}{p}k \text{ integral}, \\ e(pV\xi, \frac{1}{p}k, \mathcal{P}) & \text{if } I_0 = \emptyset \text{ and } \frac{1}{p}k \text{ not integral}. \end{cases}$

(iv) $de(\xi, k, \mathcal{P}) = \begin{cases} 0 & \text{if } I_0 = \emptyset, \\ e(\xi, k, (\emptyset, \mathcal{P})) & \text{if } I_0 \neq \emptyset \text{ and } k \text{ not integral}, \\ p^{-t(k)} e(\xi, k, (\emptyset, \mathcal{P})) & \text{if } I_0 \neq \emptyset \text{ and } k \text{ integral}. \end{cases}$

Theorem 5.5 ([LZ04, Thm. 2.8])**.** *Every* $\omega \in W\Omega^q_{A[x_1,\ldots,x_d]/A}$ *can uniquely be written as*

$$\omega = \sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}),$$

where the sum is over all weights k *with* $|\operatorname{supp} k| \geq q$ *and over all admissible partitions of length* q *of* k*, and the sum converges in the sense that, for any* $m \geq 0$*, we have* $\xi_{k,\mathcal{P}} \in V^m W(A)$ *for all but finitely many* $\xi_{k,\mathcal{P}}$*.*

For a weight k, $n \geq 1$ and $\eta \in W_{n-u(k)}(A)$ we define $e_n^0(\eta, k) \in W_n(A[x_1, \dots, x_d])$ and $e_n^1(\eta, k) \in W_n\Omega^1_{A[x_1,\dots,x_d]/A}$ by the same formulas as in (5.2.3) and (5.2.4). For $\mathcal{P}$ an admissible partition of length q of k and $\xi = V^{u(k)}(\eta) \in W_n(A)$, we then define $e_n(\xi, k, \mathcal{P}) \in W_n\Omega^q_{A[x_1,\dots,x_d]/A}$ by the same formula as in Definition 5.3 but with e^i replaced by e_n^i, $i = 0, 1$.

Corollary 5.6 ([LZ04, Prop. 2.17]). *Every $\omega \in W_n\Omega^q_{A[x_1,\dots,x_d]/A}$ may uniquely be written as a finite sum*

$$\omega = \sum_{k,\mathcal{P}} e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P}), \quad \xi_{k,\mathcal{P}} \in V^{u(k)} W_{n-u(k)}(A),$$

where the sum is over all weights k with $|\operatorname{supp} k| \geq q$ and such that $p^{n-1}k$ is integral and over all admissible partitions $\mathcal{P}$ of k of length q.

We now assume that S is an $\mathbb{F}_p$-scheme. The following proposition is known if S is perfect (see [IR83, II, (1.2.2)]).

Proposition 5.7. *Let S be a locally noetherian $\mathbb{F}_p$-scheme and X a smooth S-scheme. Then the sequence*

$$F^n_*\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega^{q-1}_{X/S} \xrightarrow{(1\otimes F^n, -1\otimes F^n d)} F^n_*\Omega^{q-1}_{X/S} \oplus F^n_*\Omega^q_{X/S} \xrightarrow{dV^n + V^n} W_{n+1}\Omega^q_{X/S} \longrightarrow R_* W_n\Omega^q_{X/S} \longrightarrow 0$$

is an exact sequence of $W_{n+1}(\mathcal{O}_S)$-modules.

Proof. The question is local, we thus assume $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$ and B is étale over $B_1 = A[x_1, \dots, x_d]$. As $W\Omega^\bullet_{X/S} \to W_{n+1}\Omega^\bullet_{X/S}$ is an epimorphism, [LZ04, Prop. 2.19] provides the exactness of the second line, and we only have to show that

$$(*_{B/A}): \quad F^n_*A \otimes W_{n+1}\Omega^{q-1}_{B/A} \xrightarrow{(1\otimes F^n, -1\otimes F^n d)} F^n_*\Omega^{q-1}_{B/A} \oplus F^n_*\Omega^q_{B/A} \xrightarrow{dV^n + V^n} W_{n+1}\Omega^q_{B/A}$$

is exact. Notice that it is a complex, as for $a \in A$ and $\omega \in W_{n+1}\Omega^{q-1}_{B/A}$ we have

$$dV^n(aF^n\omega) - V^n(aF^n d\omega) = 0.$$

Notice also that, if we let $W_{2n+2}(B)$ act through $F^{n+1} : W_{2n+2}(B) \to W_{n+1}(B)$, the differentials of this complex are $W_{2n+2}(B)$-linear, since $dF^{n+1} = p^{n+1}F^{n+1}d = 0$ in W_{n+1}. We claim

$$(*_{B/A}) = F^{n+1}_*(*_{B_1/A}) \otimes_{W_{2n+2}(B_1)} W_{2n+2}(B). \tag{5.7.1}$$

Indeed we have the following diagrams:

$$\begin{array}{ccc} F^{n+1}_*(F^n_*A \otimes W_{n+1}\Omega^{q-1}_{B/A}) & \xrightarrow{1\otimes F^n} & F^{2n+1}_*\Omega^{q-1}_{B/A} \\ \uparrow & & \uparrow \\ F^{n+1}_*(F^n_*A \otimes W_{n+1}\Omega^{q-1}_{B_1/A}) \otimes W_{2n+2}(B) & \xrightarrow{(1\otimes F^n)\otimes 1} & F^{n+1}_*(F^n_*\Omega^{q-1}_{B_1/A}) \otimes W_{2n+2}(B), \end{array}$$

$$\begin{array}{ccc} F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega^{q-1}_{B/A}) & \xrightarrow{-1\otimes F^n d} & F_*^{2n+1}\Omega^q_{B/A} \\ \uparrow & & \uparrow \\ F_*^{n+1}(F_*^n A \otimes W_{n+1}\Omega^{q-1}_{B_1/A}) \otimes W_{2n+2}(B) & \xrightarrow{(-1\otimes F^n d)\otimes 1} & F_*^{n+1}(F_*^n \Omega^q_{B_1/A}) \otimes W_{2n+2}(B), \end{array}$$

both with vertical maps

$$(a \otimes \omega) \otimes b \mapsto a \otimes F^{n+1}(b)\omega, \qquad \eta \otimes b \mapsto F^{2n+1}(b)\eta,$$

and

$$\begin{array}{ccc} F_*^{2n+1}\Omega^{q-1}_{B/A} \oplus F_*^{2n+1}\Omega^q_{B/A} & \xrightarrow{dV^n + V^n} & F_*^{n+1} W_{n+1}\Omega^q_{B/A} \\ \uparrow & & \uparrow \\ F_*^{n+1}(F_*^n \Omega^{q-1}_{B_1/A} \oplus F_*^n \Omega^q_{B_1/A}) \otimes W_{2n+2}(B) & \xrightarrow{(dV^n+V^n)\otimes 1} & F_*^{n+1} W_{n+1}\Omega^q_{B_1/A} \otimes W_{2n+2}(B), \end{array}$$

with vertical maps

$$(\eta, \omega) \otimes b \mapsto (F^{2n+1}(b)\eta, F^{2n+1}(b)\omega), \qquad \omega \otimes b \mapsto F^{n+1}(b)\omega.$$

Using again the relation $dF^{n+1} = p^{n+1}F^{n+1}d = 0$ in W_{n+1}, one checks immediately that all three diagrams commute. Now the claim (5.7.1) follows, since the vertical maps are isomorphisms by (5.1.6). As $W_{2n+2}(B_1) \to W_{2n+2}(B)$ is étale [LZ04, Prop. A.8], we are thus reduced to the case $B = B_1 = A[x_1, \dots, x_d]$.

Now take $\alpha \in \Omega^q_{B/A}$ and $\beta \in \Omega^{q-1}_{B/A}$ with $V^n(\alpha) = -dV^n(\beta)$. We have to show that there exists an element $\gamma \in F_*^n A \otimes W_{n+1}\Omega^{q-1}_{B/A}$ with

$$-(1 \otimes F^n d)(\gamma) = \alpha \quad \text{and} \quad (1 \otimes F^n)(\gamma) = \beta. \tag{5.7.2}$$

By Corollary 5.6 (and keeping the notation used there), we can write α and β uniquely as finite sums

$$\alpha = \sum_{k,\mathcal{P}} e_1(\xi_{k,\mathcal{P}}, k, \mathcal{P}), \quad \beta = \sum_{k,\mathcal{Q}} e_1(\eta_{k,\mathcal{Q}}, k, \mathcal{Q}), \quad \text{with } \xi_{k,\mathcal{P}}, \eta_{k,\mathcal{Q}} \in A, \tag{5.7.3}$$

where the sums are over all integral weights k and all admissible partitions $\mathcal{P} = (I_0, \dots, I_q)$ of length q (resp. over all admissible partitions $\mathcal{Q} = (J_0, \dots, J_{q-1})$ of length $q-1$). Using the rules 5.4 (iii) and (iv), we obtain

$$V^n(\alpha) = \sum_{i=0}^{n-1} \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } I_0 = \emptyset}} e_{n+1}(p^{n-i}V^n(\xi_{k,\mathcal{P}}), \tfrac{k}{p^n}, \mathcal{P}) + \sum_{\substack{\frac{k}{p^n} \text{ integral} \\ \text{or } I_0 \neq \emptyset}} e_{n+1}(V^n(\xi_{k,\mathcal{P}}), \tfrac{k}{p^n}, \mathcal{P}) \tag{5.7.4}$$

and

$$(5.7.5)\quad -dV^n(\beta) = \sum_{\substack{\frac{k}{p^n} \text{ integral} \\ \text{and } J_0\neq\emptyset}} -p^{t(\frac{k}{p^n})} e_{n+1}(V^n(\eta_{k,\mathcal{Q}}), \tfrac{k}{p^n}, (\emptyset, \mathcal{Q})) + \sum_{\substack{\frac{k}{p^n} \text{ not integral} \\ \text{and } J_0\neq\emptyset}} -e_{n+1}(V^n(\eta_{k,\mathcal{Q}}), \tfrac{k}{p^n}, (\emptyset, \mathcal{Q})),$$

where $t(k/p^n)$ is defined as in (5.2.2). By the uniqueness of this presentation, and since $V^n : A \to W_{n+1}(A)$ is injective, the equality $V^n(\alpha) = -dV^n(\beta)$ thus gives the following set of equations:

$$\begin{aligned}
\xi_{k,\mathcal{P}} &= -p^{-t(\frac{k}{p^n})}\eta_{k,\mathcal{Q}}, \quad \text{if } \frac{k}{p^n} \text{ is integral, } \mathcal{P} = (\emptyset, \mathcal{Q}) \text{ and } J_0 \neq \emptyset,\\
\eta_{k,\mathcal{Q}} &= -p^{n-i}\xi_{k,\mathcal{P}}, \quad \text{if } \frac{k}{p^i} \text{ is primitive, } \mathcal{P} = (\emptyset, \mathcal{Q}),\ J_0 \neq \emptyset \text{ and } 0 \leq i \leq n-1,\\
\xi_{k,\mathcal{P}} &= 0, \quad \text{if } I_0 \neq \emptyset.
\end{aligned}$$

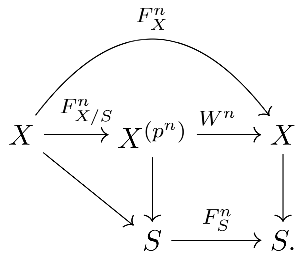
One now easily verifies that (5.7.2) holds for the following choice of $\gamma \in F^n_* A \otimes_{W_{n+1}(A)} W_{n+1}\Omega^{q-1}_{B/A}$:

$$\gamma := \sum_{i=0}^{n-1} \left(\sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0\neq\emptyset}} \Big(-\xi_{k,(\emptyset,\mathcal{Q})} \otimes e_{n+1}(V^{n-i}(1), \tfrac{k}{p^n}, \mathcal{Q})\Big) + \sum_{\substack{\frac{k}{p^i} \text{ primitive} \\ \text{and } J_0=\emptyset}} \Big(\eta_{k,\mathcal{Q}} \otimes e_{n+1}(V^{n-i}(1), \tfrac{k}{p^n}, \mathcal{Q})\Big) \right) + \sum_{\frac{k}{p^n} \text{ integral, } \mathcal{Q}} \eta_{k\mathcal{Q}} \otimes e_{n+1}(1, \tfrac{k}{p^n}, \mathcal{Q}).$$

This proves the proposition. □

5.8. We now recall some facts from [Il79, 0, 2] about the Cartier operator and its iterates.

Let S be an $\mathbb{F}_p$-scheme, and denote by $F_S : S \to S$ its absolute Frobenius endomorphism. Let $X \to S$ be a smooth morphism of $\mathbb{F}_p$-schemes and set $X^{(p^n)} := S \times_{S,F^n_S} X$. We have the usual diagram, which defines the iterates $F^n_{X/S}$ of the relative Frobenius morphism (we write $F_{X/S} = F^1_{X/S}$, $W = W^1$):



Notice that

$$F^n_{X/S} = F_{X^{(p^{n-1})}/S} \circ \ldots \circ F_{X/S}.$$

For an S-morphism $f : X' \to X$ we denote by $f^{(p^n)}$ the base-change morphism $f^{(p^n)} = \mathrm{Id}_S \times f : X'^{(p^n)} \to X^{(p^n)}$.

The inverse Cartier operator is a homomorphism of graded $\mathcal{O}_{X^{(p)}}$-algebras

$$C^{-1}_{X/S} : \Omega^\bullet_{X^{(p)}/S} \to \mathcal{H}^\bullet(\Omega^\bullet_{X/S}),$$

which is uniquely determined by

(5.8.1) $$C^{-1}_{X/S}|_{\mathcal{O}_{X^{(p)}}} = F^*_{X/S} \quad \text{and} \quad C^{-1}_{X/S}(W^*dx) = x^{p-1}dx, \quad \text{for all } x \in \mathcal{O}_X.$$

The inverse Cartier operator is an isomorphism (since X/S is smooth). For $n \geq 0$, one defines abelian subsheaves of $\Omega^q_{X/S}$

(5.8.2) $$B_n\Omega^q_{X/S} \subset Z_n\Omega^q_{X/S} \subset \Omega^q_{X/S}$$

via

$$B_0\Omega^q_{X/S} = 0, \quad Z_0\Omega^q_{X/S} = \Omega^q_{X/S},$$

$$B_1\Omega^q_{X/S} = B\Omega^q_{X/S} = d\Omega^{q-1}_{X/S}, \quad Z_1\Omega^q_{X/S} = Z\Omega^q_{X/S} = \mathrm{Ker}(d : \Omega^q_{X/S} \to \Omega^{q+1}_{X/S}),$$

and, for $n \geq 1$,

$$C^{-1}_{X/S} : B_n\Omega^q_{X^{(p)}/S} \xrightarrow{\simeq} B_{n+1}\Omega^q_{X/S}/B_1\Omega^q_{X/S},$$

$$C^{-1}_{X/S} : Z_n\Omega^q_{X^{(p)}/S} \xrightarrow{\simeq} Z_{n+1}\Omega^q_{X/S}/B_1\Omega^q_{X/S}.$$

We obtain a chain of inclusions

(5.8.3) $$0 \subset B_1\Omega^q_{X/S} \subset \ldots \subset B_n\Omega^q_{X/S} \subset B_{n+1}\Omega^q_{X/S} \subset \ldots$$
$$\subset Z_{n+1}\Omega^q_{X/S} \subset Z_n\Omega^q_{X/S} \subset \ldots \subset Z_1\Omega^q_{X/S} \subset \Omega^q_{X/S}.$$

Proposition 5.9 ([Il79, 0, (2.2.7), Prop. 2.2.8])**.** *Let S be an $\mathbb{F}_p$-scheme and X a smooth S-scheme. Then, for all $q \geq 0$ and $n \geq 1$, the sheaves $Z_n\Omega^q_{X/S}$ and $B_n\Omega^q_{X/S}$ satisfy the following properties.*

(i) *$Z_n\Omega^q_{X/S}$ and $B_n\Omega^q_{X/S}$ are locally free $\mathcal{O}_{X^{(p^n)}}$-modules of finite type, and, for any $h : S' \to S$, we have*

$$h_X^{(p^n)*}Z_n\Omega^q_{X/S} \xrightarrow{\sim} Z_n\Omega^q_{X'/S'}, \quad h_X^{(p^n)*}B_n\Omega^q_{X/S} \xrightarrow{\sim} B_n\Omega^q_{X'/S'},$$

where $h_X : X' := S' \times_S X \to X$ is the base-change map.

(ii) *If $f : X' \to X$ is an étale S-morphism, then there are natural isomorphisms*

$$f^{(p^n)*} Z_n\Omega^q_{X/S} \xrightarrow{\sim} Z_n\Omega^q_{X'/S}, \quad f^{(p^n)*} B_n\Omega^q_{X/S} \xrightarrow{\sim} B_n\Omega^q_{X'/S}.$$

(iii) *$B_n\Omega^q_{X/S}$ is the sub-$\mathcal{O}_S$-module of $\Omega^q_{X/S}$ locally generated by sections of the form $a_1^{p^r-1}\cdots a_q^{p^r-1} da_1 \cdots da_q$, with $a_i \in \mathcal{O}_X$ and $0 \le r \le n-1$.*

(iv) *$Z_n\Omega^q_{X/S}$ is the sub-$\mathcal{O}_S$-module of $\Omega^q_{X/S}$ locally generated by $B_n\Omega^q_{X/S}$ and sections of the form $ba_1^{p^n-1}\cdots a_q^{p^n-1} da_1 \cdots da_q$, with $a_i \in \mathcal{O}_X$ and $b \in \mathcal{O}_{X^{(p^n)}}$.*

Proposition 5.10 (cf. [Il79, I, Prop. 3.3])**.** *For X/S smooth as above, there is a unique map of $W_n(\mathcal{O}_S)$-modules*

$$C_n^{-1} : F_{S*}W_n(\mathcal{O}_S) \otimes_{W_n(\mathcal{O}_S)} W_n\Omega^q_{X/S} \longrightarrow \frac{W_n\Omega^q_{X/S}}{dV^{n-1}\Omega^q_{X/S}},$$

which makes the following diagram commutative

$$\begin{array}{ccc} F_*W_n(\mathcal{O}_S) \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega^q_{X/S} & \xrightarrow{1\otimes F} & W_n\Omega^q_{X/S} \\ \downarrow {\scriptstyle 1\otimes R} & & \downarrow \\ F_*W_n(\mathcal{O}_S) \otimes_{W_n(\mathcal{O}_S)} W_n\Omega^q_{X/S} & \xrightarrow{C_n^{-1}} & \frac{W_n\Omega^q_{X/S}}{dV^{n-1}\Omega^q_{X/S}}. \end{array}$$

For $n = 1$ we have $F_{S*}\mathcal{O}_S \otimes_{\mathcal{O}_S} \Omega^q_{X/S} = \Omega^q_{X^{(p)}/S}$, and $C_n^{-1} = C^{-1} : \Omega^q_{X^{(p)}/S} \longrightarrow \frac{\Omega^q_{X/S}}{d\Omega^q_{X/S}}$ is the inverse Cartier operator.

Proof. Since $1 \otimes R$ is surjective, it is enough to see that the kernel of $1 \otimes R$ is mapped to $dV^{n-1}\Omega^q_{X/S}$ under $1 \otimes F$. But an element in the kernel of $1 \otimes R$ is a sum of elements of the form $a \otimes V^n\omega$ and $a \otimes dV^n\eta$, with $a \in W_n(\mathcal{O}_S)$, $\omega \in \Omega^q_{X/S}$ and $\eta \in \Omega^{q-1}_{X/S}$. We have in $W_n\Omega^q_{X/S}$

$$(1 \otimes F)(a \otimes V^n\omega) = aV^{n-1}(p\omega) = 0, \quad (1 \otimes F)(a \otimes dV^n\eta) = dV^{n-1}(F^{n-1}(a)\eta).$$

This gives the existence and the uniqueness of C_n^{-1}. The second statement follows from the fact that $1 \otimes F$ is compatible with products, and from the formula $1 \otimes F(a \otimes d[x]) = ax^{p-1}dx$, for $a \in \mathcal{O}_S$, $x \in \mathcal{O}_X$. □

Corollary 5.11 (cf. [Il79, I, Prop. 3.11])**.** *Let X/S be as above. Then:*

(i) $\mathrm{Im}(1 \otimes F^n : F^n_*\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega^q_{X/S} \to \Omega^q_{X/S}) = Z_n\Omega^q_{X/S}$.

(ii) $\mathrm{Im}(1 \otimes F^{n-1}d : F^n_*\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} F_*W_n\Omega^{q-1}_{X/S} \to \Omega^q_{X/S}) = B_n\Omega^q_{X/S}$.

Proof. We do induction on n. For $n = 1$, (i) follows from Proposition 5.10 and the relation $d = FdV$, and (ii) holds by definition. Now assume the statements are proven for n. To prove (i) we consider the following commutative diagram:

$$\begin{array}{ccccc}
F_*^{n+1}\mathcal{O}_S \otimes_{W_{n+2}(\mathcal{O}_S)} W_{n+2}\Omega^q_{X/S} & \xrightarrow{1\otimes F^n} & F_*\mathcal{O}_S \otimes_{W_2(\mathcal{O}_S)} W_2\Omega^q_{X/S} & \xrightarrow{1\otimes F} & \Omega^q_{X/S} \\
\downarrow{\scriptstyle 1\otimes R} & & \downarrow{\scriptstyle 1\otimes R} & & \downarrow \\
F_*^{n+1}\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega^q_{X/S} & \xrightarrow{1\otimes F^n} & \Omega^q_{X^{(p)}/S} & \xrightarrow{C^{-1}} & \frac{\Omega^q_{X/S}}{d\Omega^{q-1}_{X/S}}.
\end{array}$$

By induction hypothesis we have

$$\mathrm{Im}\Big((1\otimes R)\circ(1\otimes F^n)\Big) = \mathrm{Im}\Big((1\otimes F^n)\circ(1\otimes R)\Big) = F_{S*}\mathcal{O}_S \otimes_{\mathcal{O}_S} Z_n\Omega^q_{X/S} = Z_n\Omega^q_{X^{(p)}/S},$$

where the last equality follows from the compatibility with base-change. Now, thanks to the relation $d = F^{n+1}dV^{n+1}$, (i) follows from the definition of $Z_{n+1}\Omega^q_{X/S}$. The proof of (ii) is similar. □

Lemma 5.12. *Let X/S be as above. The sheaf $B_n\Omega^q_{X/S}$ is given by*

$$\begin{aligned}
&\mathrm{Im}(1\otimes F^{n-1}d : F_*^n\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} F_*W_n\Omega^{q-1}_{X/S} \to \Omega^q_{X/S}) \\
&\quad = \{(1\otimes F^n d)(\alpha) \,|\, \alpha \in F_*^n\mathcal{O}_S \otimes_{W_{n+1}(\mathcal{O}_S)} W_{n+1}\Omega^{q-1}_{X/S} \text{ with } (1\otimes F^n)(\alpha) = 0\}.
\end{aligned}$$

Proof. We call the left hand side $\mathcal{A}$, and the right hand side $\mathcal{B}$. We know from the previous corollary that $B_n\Omega^q_{X/S} = \mathcal{A}$, and we want now to show that $\mathcal{A} = \mathcal{B}$. In the following, all non-specified tensor products are over $W_{n+1}(\mathcal{O}_S)$. We have the commutative diagram

$$\begin{array}{ccc}
F_*^n\mathcal{O}_S \otimes F_*W_n\Omega^{q-1}_{X/S} & \xrightarrow{1\otimes V} & F_*^n\mathcal{O}_S \otimes W_{n+1}\Omega^{q-1}_{X/S} \\
{\scriptstyle 1\otimes F^{n-1}d}\searrow & & \swarrow{\scriptstyle 1\otimes F^n d} \\
& \Omega^q_{X/S} &
\end{array}.$$

Since we also have $(1\otimes F^n)\circ(1\otimes V) = 0$ it follows that $\mathcal{A} \subset \mathcal{B}$. It remains to show

$$\begin{aligned}
(5.12.1)\quad &\mathrm{Ker}\left(1\otimes F^n : F_*^n\mathcal{O}_S \otimes W_{n+1}\Omega^{q-1}_{X/S} \to F_*^n\Omega^{q-1}_{X/S}\right) \\
&\subset \mathrm{Im}\left(F_*^n\mathcal{O}_S \otimes (F_*W_n\Omega^{q-1}_{X/S} \oplus F_*W_n\Omega^{q-2}_{X/S}) \xrightarrow{1\otimes(V+dV)} F_*^n\mathcal{O}_S \otimes W_{n+1}\Omega^{q-1}_{X/S}\right).
\end{aligned}$$

Indeed, if we take an element α in the kernel on the left hand side and we write it as an element in the right hand side $\alpha = (1\otimes V)(\beta) + (1\otimes dV)(\gamma)$, then $(1\otimes F^n d)(\alpha) = (1\otimes F^{n-1}d)(\beta)$, i.e., $\mathcal{B} \subset \mathcal{A}$. The question is local in X, we may thus assume X is étale over $\mathbb{A}^d_S$. For a $W_n(\mathcal{O}_X)$-module $\mathcal{M}$ we write $F_*^r\mathcal{M}_*F^s$ for $\mathcal{M}$ viewed as a left $W_{n+r}(\mathcal{O}_S)$-module and as a right $W_{n+s}(\mathcal{O}_X)$-module. Then we have the following commutative diagram, in which the most right tensor product in the upper line is over $W_{2n+2}(\mathcal{O}_{\mathbb{A}^d_S})$:

$$\begin{array}{ccc}\Big(F^n_*\mathcal{O}_S\otimes F_*(W_n\Omega^{q-2}_{\mathbb{A}^d_S/S})_*F^{n+2} & \xrightarrow{1\otimes dV} & F^n_*\mathcal{O}_S\otimes (W_{n+1}\Omega^{q-1}_{\mathbb{A}^d_S/S})_*F^{n+1}\Big)\otimes W_{2n+2}(\mathcal{O}_X)\\ \big\downarrow{\scriptstyle (1\otimes\mathrm{can})\otimes 1} & & \big\downarrow{\scriptstyle (1\otimes\mathrm{can})\otimes 1}\\ F^n_*\mathcal{O}_S\otimes F_*(W_n\Omega^{q-2}_{X/S})_*F^{n+2} & \xrightarrow{1\otimes dV} & F^n_*\mathcal{O}_S\otimes (W_{n+1}\Omega^{q-1}_{X/S})_*F^{n+1}.\end{array}$$

If we write V instead of dV and $q-1$ on the left hand side instead of $q-2$, we obtain again a commutative diagram. Since $X/\mathbb{A}^d_S$ is étale, the vertical maps are isomorphisms (in both diagrams). Thus if we denote the image in (5.12.1) by $\mathrm{Im}(X/S)$ we obtain

$$\mathrm{Im}(X/S)\cong \mathrm{Im}(\mathbb{A}^d_S/S)_*F^{n+1}\otimes_{W_{2n+2}(\mathcal{O}_{\mathbb{A}^d_S})} W_{2n+2}(\mathcal{O}_X).$$

Similarly, denoting the kernel in (5.12.1) by $\mathrm{Ker}(X/S)$ one finds

$$\mathrm{Ker}(X/S)\cong \mathrm{Ker}(\mathbb{A}^d_S/S)_*F^{n+1}\otimes_{W_{2n+2}(\mathcal{O}_{\mathbb{A}^d_S})} W_{2n+2}(\mathcal{O}_X).$$

And, since $W_{2n+2}(\mathcal{O}_X)$ is étale over $W_{2n+2}(\mathcal{O}_{\mathbb{A}^d_S})$ [LZ04, Prop. A.8], it is thus enough to prove (5.12.1) in the case $S=\operatorname{Spec} A$, with A an $\mathbb{F}_p$-algebra, and $X=\operatorname{Spec} B$, with $B=A[x_1,\dots,x_d]$.

Now, using the notation of Corollary 5.6, any element $\alpha\in F^n_*A\otimes W_{n+1}\Omega^{q-1}_{B/A}$ can be written as a finite sum

$$\alpha=\sum_i\sum_{\substack{p^nk\text{ integral}\\ \mathcal{P}=(I_0,\dots,I_{q-1})}} a_i\otimes e_{n+1}(V^{u(k)}(\eta_{k,\mathcal{P},i}),k,\mathcal{P}),\qquad \eta_{k,\mathcal{P},i}\in W_{n+1-u(k)}(A). \tag{5.12.2}$$

By the rule 5.4, (ii) we have

$$F^n e_{n+1}(V^{u(k)}(\eta),k,\mathcal{P})=\begin{cases} e_1(F^{n-u(k)}(\eta),p^nk,\mathcal{P}) & \text{if } I_0=\emptyset \text{ or } (I_0\neq\emptyset, k\text{ integral}),\\ 0 & \text{if } I_0\neq\emptyset \text{ and } k \text{ not integral.}\end{cases}$$

It follows that an element α as in (5.12.2) lies in $\mathrm{Ker}(1\otimes F^n)=\mathrm{Ker}(B/A)$ iff it satisfies

$$\sum_i a_iF^{n-u(k)}(\eta_{k,\mathcal{P},i})=0,\quad \text{for } I_0=\emptyset \text{ or } (I_0\neq\emptyset, k\text{ integral}). \tag{5.12.3}$$

We consider the following three cases:

1) *k is integral, i.e., $u(k)=0$.* Then, by Definition 5.3, $e_{n+1}(\eta,k,\mathcal{P})=\eta e_{n+1}(1,k,\mathcal{P})$. By (5.12.3), we get

$$\sum_i a_i\otimes e_{n+1}(\eta_{i,k,\mathcal{P}},k,\mathcal{P})=\left(\sum_i a_iF^n(\eta_{i,k,\mathcal{P}})\right)\otimes e_{n+1}(1,k,\mathcal{P})=0.$$

2) *k is not integral and $I_0=\emptyset$.* In this case $e_{n+1}(\eta,k,\mathcal{P})\in\mathrm{Im}(dV)$ by Definition 5.3. Thus

$$\sum_i a_i\otimes e_{n+1}(\eta_{i,k,\mathcal{P}},k,\mathcal{P})\in\mathrm{Im}(1\otimes dV).$$

3) *k is not integral and $I_0 \neq \emptyset$.* Now $e_{n+1}(\eta, k, \mathcal{P}) \in \mathrm{Im}(V)$ by Definition 5.3. Hence

$$\sum_i a_i \otimes e_{n+1}(\eta_{i,k,\mathcal{P}}, k, \mathcal{P}) \in \mathrm{Im}(1 \otimes V).$$

Putting the three cases together, we see that $\alpha \in \mathrm{Ker}(1 \otimes F^n)$ implies $\alpha \in \mathrm{Im}(1 \otimes V + 1 \otimes dV) = \mathrm{Im}(B/A)$. This gives the statement. □

Theorem 5.13 (cf. [Il79, I, Cor. 3.9])**.** *Let S be an $\mathbb{F}_p$-scheme and let X be a smooth S-scheme. For $n, q \geq 0$, denote by $\mathrm{gr}^n W\Omega^q_{X/S}$ the n-th graded piece of the canonical filtration*

$$\mathrm{Fil}^n W\Omega^q_{X/S} = V^n W\Omega^q_{X/S} + dV^n W\Omega^q_{X/S} = \mathrm{Ker}(W\Omega^q_{X/S} \to W_n\Omega^q_{X/S}).$$

Then we have an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow F^{n+1}_{X*}\frac{\Omega^q_{X/S}}{B_n\Omega^q_{X/S}} \xrightarrow{V^n} \mathrm{gr}^n W\Omega^q_{X/S} \longrightarrow F^{n+1}_{X*}\frac{\Omega^{q-1}_{X/S}}{Z_n\Omega^{q-1}_{X/S}} \longrightarrow 0,$$

where the second map is given by $V^n(\alpha) + dV^n(\beta) \mapsto \beta$ and the $\mathcal{O}_X$-module structure on $\mathrm{gr}^n W\Omega^q_{X/S}$ is given via

$$\mathcal{O}_X = \frac{W_n\mathcal{O}_X}{VW_{n-1}\mathcal{O}_X} \xrightarrow{F} \frac{W_{n+1}\mathcal{O}_X}{pW_n\mathcal{O}_X}.$$

Furthermore $F^n_{X/S*}\frac{\Omega^q_{X/S}}{B_n\Omega^q_{X/S}}$ and $F^n_{X/S*}\frac{\Omega^{q-1}_{X/S}}{Z_n\Omega^{q-1}_{X/S}}$ are locally free $\mathcal{O}_{X^{(p^n)}}$-modules.

Proof. The exactness of the sequence follows from Proposition 5.7, Corollary 5.11 and Lemma 5.12. The second statement is proven as in [Il79, I, Cor. 3.9]. By étale base change (Proposition 5.9, (ii)), we reduce the question of the local freeness of the two extreme $\mathcal{O}_{X^{(p^n)}}$-modules in the exact sequence to the case $X = \mathbb{A}^d_S$. Since everything is compatible with arbitrary base change in the base S (by Proposition 5.9, (i)), we may also assume $S = \mathrm{Spec}\,\mathbb{F}_p$, and even $S = \mathrm{Spec}\,k$ with k algebraically closed. But now the sheaves in question are coherent on $(\mathbb{A}^d_k)^{(p^n)} \cong \mathbb{A}^d_k$, hence locally free in some non-empty open subset, whose translates under certain closed points cover the whole of $(\mathbb{A}^d_k)^{(p^n)}$. As they are invariant under translation, this gives the statement. □

6. The Hodge-Witt trace morphism for projective spaces

Let X be a noetherian $\mathbb{F}_p$-scheme with a dualizing complex, and let $f : Y \to X$ be a projective complete intersection morphism of virtual relative dimension 0. Our goal in the next two sections is to prove that, given a factorization $f = \pi \circ i$, where $\pi : P = \mathbb{P}^d_X \to X$ is the structural morphism of some projective space over X, and $i : Y \hookrightarrow P$ is a closed immersion, one can define for all $n \geq 1$ a morphism

$$\tau_{i,\pi,n} : \mathbb{R}f_* W_n\mathcal{O}_Y \longrightarrow W_n\mathcal{O}_X$$

so as to satisfy the following properties:

(i) For $n = 1$, $\tau_{i,\pi,n}$ is the morphism τ_f of Theorem 3.1;

(ii) For variable n, $\tau_{i,\pi,n}$ commutes with R, F and V.

Our construction of $\tau_{i,\pi,n}$ will be based on a generalization for arbitrary n of the description of τ_f given in Proposition 4.6: we will construct on the one hand a trace morphism $\mathbb{R}\pi_* W_n\Omega^d_{P/X}[d] \to W_n\mathcal{O}_X$, which will be a generalization of the trace morphism Trp_π for the projective space, and on the other hand a morphism $i_* W_n\mathcal{O}_Y \to W_n\Omega^d_{P/X}[d]$ which will be a generalization of the morphism $\gamma_f : \mathcal{O}_Y \to \omega_{P/X}[d]$ defined in (4.5.1).

We begin with the trace morphism for projective spaces.

6.1. We recall first from [Il90, Déf. 1.1] that a smooth proper $\mathbb{F}_p$-morphism $f : X \to S$ is called *ordinary*, if it satisfies

$$R^i f_* B\Omega^q_{X/S} = 0, \quad \text{for all } i, q \geq 0.$$

This notion is compatible with arbitrary base-change in the base S, and $\mathbb{P}^d_{\mathbb{F}_p}$ is ordinary over $\operatorname{Spec} \mathbb{F}_p$ [Il90, Prop. 1.2, Prop. 1.4]. Hence if $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank on some $\mathbb{F}_p$-scheme X, then $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}_{\mathcal{O}_X}\mathcal{E})$ is ordinary over X.

Lemma 6.2. *Let $f : X \to S$ be ordinary. Then, for all $n \geq 1$ and $q \geq 0$,*

$$V^n : F^{n+1}_{S*}\mathbb{R}f_*\Omega^q_{X/S} \xrightarrow{\sim} \mathbb{R}f_*\mathrm{gr}^n W\Omega^q_{X/S}$$

is an isomorphism in the derived category of quasi-coherent $\mathcal{O}_S$-modules (where the $\mathcal{O}_S$-module structure on the right hand side comes from the $\mathcal{O}_X$-module structure defined in Theorem 5.13).

Proof. This follows immediately from Theorem 5.13 and the following claim:

$$(6.2.1)\quad R^i f_* Z_n\Omega^q_{X/S} \xrightarrow{\sim} R^i f_*\Omega^q_{X/S}, \quad R^i f_* B_n\Omega^q_{X/S} = 0, \quad \text{for all } i, q \geq 0, n \geq 1.$$

We prove this by induction on n. The statement for B_1 holds by definition of ordinarity and for Z_1 follows from the exact sequence

$$0 \longrightarrow Z\Omega^q_{X/S} \longrightarrow \Omega^q_{X/S} \xrightarrow{d} B\Omega^{q+1}_{X/S} \longrightarrow 0.$$

Now for the general case consider the following commutative diagram (in which f_* is viewed as a functor on the category of abelian sheaves for the Zariski topology on $|X| = |X^{(p)}|$)

$$\begin{array}{ccccc}
R^i f_* Z_n\Omega^q_{X^{(p)}/S} & \xrightarrow{C^{-1}_{X/S}} & R^i f_* \frac{Z_{n+1}\Omega^q_{X/S}}{B_1\Omega^q_{X/S}} & \longleftarrow & R^i f_* Z_{n+1}\Omega^q_{X/S} \\
\downarrow & & \downarrow & & \downarrow \\
R^i f_*\Omega^q_{X^{(p)}/S} & \xrightarrow{C^{-1}_{X/S}} & R^i f_* \frac{Z_1\Omega^q_{X/S}}{B_1\Omega^q_{X/S}} & \longleftarrow & R^i f_* Z_1\Omega^q_{X/S}
\end{array} .$$

The horizontal maps are isomorphisms as is the vertical map on the left by induction (notice that $X^{(p)}/S$ is also ordinary). Hence all maps in the diagram are isomorphisms, which yields the claim for Z_{n+1}. To prove the statement for B_{n+1} it is enough to consider the upper line in the diagram, with Z replaced by B, and one immediately obtains the statement. □

6.3. Let S be a scheme on which p is locally nilpotent, and X an S-scheme. As in the classical case [Il79, I, 3.23], we define for any $n \geq 1$ the log derivation dlog_n to be the morphism of abelian sheaves

$$\mathrm{dlog}_n : \mathcal{O}_X^\times \longrightarrow W_n\Omega^1_{X/S}, \quad a \mapsto \mathrm{dlog}_n(a) := \frac{d[a]}{[a]}.$$

We may write simply dlog if n is fixed.

For variable n, the maps dlog_n satisfy the following relations:

(6.3.1) $$R(\mathrm{dlog}_n(a)) = \mathrm{dlog}_{n-1}(a), \qquad F(\mathrm{dlog}_n(a)) = \mathrm{dlog}_{n-1}(a).$$

The maps dlog_n allow to define Chern classes for line bundles, and to prove for relative Hodge-Witt cohomology the analog of the classical theorem on the cohomology of projective bundles (cf. [SGA 7 II, XI, Thm. 1.1]).

Theorem 6.4. *Let X be an $\mathbb{F}_p$-scheme, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank $d+1$, $P = \mathbb{P}(\mathcal{E})$, and let $\pi : P \to X$ be the canonical projection. Denote by $\eta_n \in H^0(X, R^1\pi_*W_n\Omega^1_{P/X})$ the image under dlog_n of the class of $\mathcal{O}_P(1)$ in $R^1\pi_*\mathcal{O}_P^\times$, and by $\eta_n^q \in H^0(X, R^q\pi_*W_n\Omega^q_{P/X})$ its q-fold cup product. Then, for all $n \geq 1$ and all q such that $0 \leq q \leq d$, multiplication with η_n^q induces an isomorphism in the derived category of $W_n(\mathcal{O}_X)$-modules*

(6.4.1) $$W_n(\mathcal{O}_X)[-q] \xrightarrow{\sim} \mathbb{R}\pi_*W_n\Omega^q_{P/X}.$$

Furthermore these isomorphisms are compatible with restriction, Frobenius and Verschiebung on both sides.

Proof. We first observe that

$$R^j\pi_*W_n\Omega^q_{P/X} = 0 \quad \text{for } j \neq q.$$

Indeed, we can argue by induction using the exact sequences

$$0 \longrightarrow \mathrm{gr}^n W_{n+1}\Omega^q_{P/X} \longrightarrow W_{n+1}\Omega^q_{P/X} \longrightarrow W_n\Omega^q_{P/X} \longrightarrow 0.$$

For $n = 1$, the claim follows from [SGA 7 II, XI, Thm. 1.1], and, since $\mathbb{P}(\mathcal{E})$ is ordinary over X, Lemma 6.2 implies similarly the claim for all n.

Therefore, we obtain a canonical isomorphism

(6.4.2) $$\mathbb{R}\pi_*W_n\Omega^q_{P/X} \xrightarrow{\sim} R^q\pi_*W_n\Omega^q_{P/X}[-q],$$

and we can define the morphism (6.4.1) as corresponding via (6.4.2) and translation to the morphism

(6.4.3) $$W_n(\mathcal{O}_X) \longrightarrow R^q\pi_*W_n\Omega^q_{P/X}, \quad w \mapsto w\eta_n^q.$$

This reduces the proof of the theorem to proving that (6.4.3) is an isomorphism, compatible with R, F and V.

From (6.3.1), we get for all $w \in W_{n+1}(\mathcal{O}_X)$ the relations

(6.4.4) $$R(w\eta_{n+1}^q) = R(w)\eta_n^q, \qquad F(w\eta_{n+1}^q) = F(w)\eta_n^q$$

in $R^q\pi_*W_n\Omega^q_{P/X}$. From the second relation, we also get

(6.4.5) $$V(w\eta_{n-1}^q) = V(wF(\eta_n^q)) = V(w)\eta_n^q$$

for all $w \in W_{n-1}(\mathcal{O}_X)$. So the homomorphisms (6.4.3) satisfy the required compatibilities.

To prove that the homomorphisms (6.4.3) are isomorphisms, we may now again argue by induction on n, using the compatibility with R and V. Then Lemma 6.2 reduces the proof to the case $n = 1$, which is known by [SGA 7 II, Exp. XI, Thm. 1.1]. □

Definition 6.5. Under the assumptions of Theorem 6.4, we define the Hodge-Witt trace morphism for the projective space $\mathbb{P}(\mathcal{E})$ to be the $W_n\mathcal{O}_X$-linear map

$$\mathrm{Trp}_{\pi,n} : \mathbb{R}\pi_* W_n\Omega^d_{\mathbb{P}(\mathcal{E})/X}[d] \xrightarrow{\sim} W_n\mathcal{O}_X \tag{6.5.1}$$

obtained by inverting the isomorphism (6.4.1), shifting by $-d$ and multiplying by $(-1)^{d(d+1)/2}$. Theorem 6.4 implies that $\mathrm{Trp}_{\pi,n}$ is compatible with restriction, Frobenius and Verschiebung. For $n = 1$, we obtain

$$\mathrm{Trp}_{\pi,1} = \mathrm{Trp}_\pi : \mathbb{R}\pi_* \Omega^d_{\mathbb{P}(\mathcal{E})/X}[d] \xrightarrow{\sim} \mathcal{O}_X. \tag{6.5.2}$$

Indeed, it follows from (6.4.1) that this is a local property on X. So we may assume that $\mathbb{P}(\mathcal{E}) = \mathbb{P}^d_X$. Let $X_0, \dots, X_d$ be homogenous coordinates, and $x_i = X_i/X_0$. Then Trp_π is defined as the isomorphism which maps the Čech cohomology class $dx_1 \wedge \cdots \wedge dx_d/x_1 \cdots x_d$ to $(-1)^{d(d+1)/2}$ [Co00, (2.3.1)-(2.3.3)], and this Čech cohomology class is equal to η_1^d.

7. The Hodge-Witt fundamental class of a regularly embedded subscheme

In this section, we assume that X is a locally noetherian scheme of characteristic p, and we consider a regular immersion $i : Y \hookrightarrow P$ of codimension d, where P is a smooth X-scheme. Under these assumptions, we want to associate to Y a canonical class $\gamma_Y \in \Gamma(P, \mathcal{H}^d_Y(W_n\Omega^d_{P/X}))$, for each $n \geq 1$.

Proposition 7.1. *Under the previous assumptions:*

(i) *If $t_1, \dots, t_d$ is a regular sequence of sections of $\mathcal{O}_P$, then, for all $n \geq 1$ and all $r \geq 1$, $[t_1]^r, \dots, [t_d]^r$ is a regular sequence of sections of $W_n(\mathcal{O}_P)$.*

(ii) *For all $n \geq 1$ and all q, $\mathcal{H}^j_Y(W_n\Omega^q_{P/X}) = 0$ for $j \neq d$.*

Proof. We proceed by induction on n. In the exact sequence of $W_{n+1}(\mathcal{O}_P)$-modules

$$0 \longrightarrow F^n_*\mathcal{O}_P \xrightarrow{V^n} W_{n+1}(\mathcal{O}_P) \xrightarrow{R} W_n(\mathcal{O}_P) \longrightarrow 0,$$

the action of $[t_i]^r$ on $F^n_*\mathcal{O}_P$ is given by multiplication by $t_i^{rp^n}$ on $\mathcal{O}_P$. As P is a locally noetherian scheme, the sequence $t_1^{rp^n}, \dots, t_d^{rp^n}$ is regular in $\mathcal{O}_P$, and the first claim follows easily.

For $n = 1$, the second one is a well known consequence of the regularity of the sequence $t_1, \dots, t_d$. As $\mathcal{O}_P$ is locally free of finite rank over $\mathcal{O}_{P^{(p^n)}}$, we also have $\mathcal{H}^j_Y(\mathcal{O}_{P^{(p^n)}}) = 0$ for $j \neq d$. In the exact sequence

$$0 \longrightarrow \mathrm{gr}^n W_{n+1}\Omega^q_{P/X} \longrightarrow W_{n+1}\Omega^q_{P/X} \xrightarrow{R} W_n\Omega^q_{P/X} \longrightarrow 0,$$

Theorem 5.13 allows to endow the kernel $\mathrm{gr}^n W_{n+1}\Omega^q_{P/X}$ with an $\mathcal{O}_P$-module structure for which it is an extension of two $\mathcal{O}_P$-modules which are locally free over $\mathcal{O}_{P^{(p^n)}}$. Therefore, $\mathcal{H}^j_Y(\mathrm{gr}^n W_{n+1}\Omega^q_{P/X}) = 0$ for $j \neq d$. The second claim follows by induction. □

Theorem 7.2. *Under the assumptions of this section, let* $\mathbf{t} = (t_1, \ldots, t_d)$ *and* $\mathbf{t}' = (t'_1, \ldots, t'_d)$ *be two regular sequences of sections of* $\mathcal{O}_P$ *generating the ideal* $\mathcal{I}$ *of* Y *in* P. *Let* $n \geq 1$ *be an integer, and let* $\mathcal{J} = ([t_1], \ldots, [t_d])$, $\mathcal{J}' = ([t'_1], \ldots, [t'_d])$ *be the ideals of* $W_n(\mathcal{O}_P)$ *generated by the Teichmüller representatives of these generators. If*

$$\beta_{\mathcal{J}} : \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}, W_n\Omega^d_{P/X}) \longrightarrow \mathcal{H}^d_Y(W_n\Omega^d_{P/X})$$

is the canonical homomorphism (and similarly for $\beta_{\mathcal{J}'}$*), then, with the notations of 4.1,*

$$\beta_{\mathcal{J}}(\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1], \ldots, [t_d] \end{bmatrix}) = \beta_{\mathcal{J}'}(\begin{bmatrix} d[t'_1]\cdots d[t'_d] \\ [t'_1], \ldots, [t'_d] \end{bmatrix}). \tag{7.2.1}$$

Proof. It suffices to prove (7.2.1) in a neighbourhood of each point $y \in Y$. Localizing, one can reduce the proof of Theorem 7.2 to the case of a very simple change of generators in $\mathcal{I}$, thanks to the following remarks.

a) If the sequence $(t'_1, \ldots, t'_d)$ is deduced from $(t_1, \ldots, t_d)$ by permutation, then $\mathcal{J} = \mathcal{J}'$, and formula (4.2.1) implies the theorem.

b) If there exists invertible sections $a_1, \ldots, a_d \in \mathcal{O}_P^\times$ such that $t'_i = a_i t_i$ for all i, then $[t'_i] = [a_i][t_i]$ for all i. So $\mathcal{J} = \mathcal{J}'$, we can apply Lemma 4.2, and we can choose the matrix C to be the diagonal matrix with entries $[a_i]$. Then the theorem follows from formula (4.2.1), because an element such as (4.1.2) only depends upon the class of m mod $(t_1, \ldots, t_d)M$, and here we have the congruence

$$d[t'_1]\cdots d[t'_d] \equiv (\prod_{i=1}^d [a_i])\, d[t_1]\cdots d[t_d] \mod \mathcal{J} W_n\Omega^d_{P/X}.$$

c) Given $y \in Y$, there exists a permutation $\sigma \in \mathfrak{S}_d$ such that, for any i, $1 \leq i \leq d$, the sequence $\mathbf{t}^{(i)} = (t'_{\sigma(1)}, \ldots, t'_{\sigma(i)}, t_{i+1}, \ldots, t_d)$ is a regular sequence of generators of $\mathcal{I}$ around y. Indeed, a sequence of elements of $\mathcal{I}_y$ is a regular sequence of generators if and only if it gives a basis of $\mathcal{I}_y/\mathfrak{m}_y\mathcal{I}_y$, and this reduces the claim to an elementary result in linear algebra over a field. If we set $\mathbf{t}^{(0)} = (t_1, \ldots, t_d)$, then $\mathbf{t}^{(0)} = \mathbf{t}$, and $\mathbf{t}^{(d)}$ is deduced from $\mathbf{t}'$ by permutation. So, using remark a), it suffices to prove the theorem for the couple of sequences $\mathbf{t}^{(i-1)}$ and $\mathbf{t}^{(i)}$, for all i, $1 \leq i \leq d$.

This reduces the proof to the case where there exists an integer $i_0 \in \{1, \ldots, d\}$ such that

$$t'_i = t_i \quad \text{for } i \neq i_0, \qquad t'_{i_0} = \sum_{j=1}^d c_{i_0,j} t_j.$$

Using remark a), we may assume that $i_0 = 1$. Moreover, the fact that $\mathbf{t}$ and $\mathbf{t}'$ induce bases of the vector space $\mathcal{I}_y/\mathfrak{m}_y\mathcal{I}_y$ implies that the coefficient $c_{1,1}$ is invertible around y.

d) In this last case, we define inductively elements $t_1^{(j)}$ for $0 \leq j \leq d$ by setting

$$t_1^{(0)} = t_1, \quad t_1^{(1)} = c_{1,1} t_1^{(0)}, \quad t_1^{(j)} = t_1^{(j-1)} + c_{1,j} t_j \quad \text{for } 1 < j.$$

If, for $0 \leq j \leq d$, we define $\mathbf{t}^{(j)} = (t_1^{(j)}, t_2, \dots, t_d)$, then $\mathbf{t}^{(0)} = \mathbf{t}$, $\mathbf{t}^{(d)} = \mathbf{t}'$, and it suffices to prove the theorem for each of the couples $\mathbf{t}^{(j-1)}$, $\mathbf{t}^{(j)}$, for $1 \leq j \leq d$. The theorem is true for $\mathbf{t}^{(0)}$, $\mathbf{t}^{(1)}$, thanks to remark b), and, applying again remark a), we can write all the remaining couples as changes of generators of the form

$$t_1' = t_1 + c t_2, \quad \text{for some } c \in A, \qquad t_i' = t_i \quad \text{for } i \geq 2. \tag{7.2.2}$$

Thus it suffices to prove the theorem for the change of generators of I given by (7.2.2). The generators $t_3, \dots, t_d$ play no role and go unchanged along the computation, so we will drop them to simplify notations, and assume that $d = 2$.

Let $h \in VW_{n-1}(\mathcal{O}_P)$ be defined by setting

$$[t_1] + [c][t_2] = [t_1 + c t_2] + h = [t_1'] + h \tag{7.2.3}$$

in $W_n(\mathcal{O}_P)$. Since $[t_2'] = [t_2]$, this can be rewritten as

$$[t_1] = [t_1'] - [c][t_2'] + h. \tag{7.2.4}$$

The binomial formula gives

$$[t_1]^{p^{n-1}} = ([t_1'] - [c][t_2'])^{p^{n-1}} + \sum_{i=1}^{p^{n-1}} \frac{p^{n-1}!}{(p^{n-1} - i)!i!} h^i ([t_1'] - [c][t_2'])^{p^{n-1} - i}. \tag{7.2.5}$$

Because the ideal $VW_{n-1}(\mathcal{O}_P) \subset W_n(\mathcal{O}_P)$ is a PD-ideal, we can write $h^i = i! h^{[i]}$, with $h^{[i]} \in VW_{n-1}(\mathcal{O}_P)$ when $i \geq 1$. Therefore the numerical coefficient of $h^{[i]}$ in the i-th term of the sum is divisible by p^{n-1} for all $i \geq 1$. Since p^{n-1} kills $VW_{n-1}(\mathcal{O}_P)$, equation (7.2.5) reduces to

$$[t_1]^{p^{n-1}} = ([t_1'] - [c][t_2'])^{p^{n-1}}. \tag{7.2.6}$$

If, for all $k \geq 1$, we denote by $\mathcal{J}^{(k)}$ the ideal $([t_1]^k, [t_2]^k)$, this shows that $\mathcal{J}^{(p^{n-1})} \subset \mathcal{J}'$. So we can apply Lemma 4.2 to the sequences $([t_1'], [t_2'])$ and $([t_1]^{p^{n-1}}, [t_2]^{p^{n-1}})$, which are regular by Lemma 7.1. Moreover, we can write equation (7.2.6) as

$$[t_1]^{p^{n-1}} = [t_1']^{p^{n-1}-1} \cdot [t_1'] + c_{1,2} \cdot [t_2'],$$

so that we can use as matrix C in Lemma 4.2 an upper triangular matrix with diagonal entries $[t_1']^{p^{n-1}-1}$, $[t_2']^{p^{n-1}-1}$ (since $[t_2]^{p^{n-1}} = [t_2']^{p^{n-1}-1} \cdot [t_2']$). In particular, $\det(C) = [t_1']^{p^{n-1}-1} [t_2']^{p^{n-1}-1}$. Thus, formula (4.2.1) provides the equality

$$\alpha'\left(\left[\begin{array}{c} d[t_1']\, d[t_2'] \\ [t_1'][t_2'] \end{array} \right]\right) = \left[\begin{array}{c} [t_1']^{p^{n-1}-1} [t_2']^{p^{n-1}-1}\, d[t_1']\, d[t_2'] \\ [t_1]^{p^{n-1}} [t_2]^{p^{n-1}} \end{array} \right], \tag{7.2.7}$$

where α' is the canonical homomorphism

$$\mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}', W_n\Omega^d_{P/X}) \longrightarrow \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega^d_{P/X}).$$

On the other hand, we also have $\mathcal{J}^{(p^{n-1})} \subset \mathcal{J}$. So we can also apply Lemma 4.2 to the regular sequences $([t_1],[t_2])$ and $([t_1]^{p^{n-1}},[t_2]^{p^{n-1}})$, using now for C the diagonal matrix with entries $[t_1]^{p^{n-1}-1}$, $[t_2]^{p^{n-1}-1}$. If we denote by

$$\alpha : \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}, W_n\Omega^d_{P/X}) \longrightarrow \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega^d_{P/X})$$

the canonical homomorphism, formula (4.2.1) provides the second equality

$$(7.2.8)\qquad \alpha(\begin{bmatrix} d[t_1]\,d[t_2] \\ [t_1][t_2] \end{bmatrix}) = \begin{bmatrix} [t_1]^{p^{n-1}-1}[t_2]^{p^{n-1}-1}\,d[t_1]\,d[t_2] \\ [t_1]^{p^{n-1}}[t_2]^{p^{n-1}} \end{bmatrix}.$$

As $\beta_{\mathcal{J}} = \beta_{\mathcal{J}^{(p^{n-1})}} \circ \alpha$ and $\beta_{\mathcal{J}'} = \beta_{\mathcal{J}^{(p^{n-1})}} \circ \alpha'$, relation (7.2.1) will follow if we prove the equality

$$(7.2.9)\qquad \begin{bmatrix} [t'_1]^{p^{n-1}-1}[t'_2]^{p^{n-1}-1}\,d[t'_1]\,d[t'_2] \\ [t_1]^{p^{n-1}}[t_2]^{p^{n-1}} \end{bmatrix} = \begin{bmatrix} [t_1]^{p^{n-1}-1}[t_2]^{p^{n-1}-1}\,d[t_1]\,d[t_2] \\ [t_1]^{p^{n-1}}[t_2]^{p^{n-1}} \end{bmatrix}$$

in $\mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}^{(p^{n-1})}, W_n\Omega^d_{P/X})$. To prove it, it suffices to prove in $W_n\Omega^d_{P/X}$ the congruence

(7.2.10)

$$[t'_1]^{p^{n-1}-1}[t'_2]^{p^{n-1}-1}\,d[t'_1]\,d[t'_2] \equiv [t_1]^{p^{n-1}-1}[t_2]^{p^{n-1}-1}\,d[t_1]\,d[t_2] \mod ([t_1]^{p^{n-1}},[t_2]^{p^{n-1}}),$$

and, thanks to (5.1.3), the latter will follow by applying F^{n-1} if we prove the congruence

$$(7.2.11)\qquad d[t'_1]\,d[t'_2] \equiv d[t_1]\,d[t_2] \mod ([t_1],[t_2])W_{2n-1}\Omega^d_{P/X}.$$

So let us prove (7.2.11). We still denote by $h \in VW_{2n-2}\mathcal{O}_P$ the difference $h = [t_1] + [ct_2] - [t_1 + ct_2] = [t_1] + [ct_2] - [t'_1]$ computed in $W_{2n-1}\mathcal{O}_P$. Since $t'_2 = t_2$, it suffices to prove the congruence

$$(7.2.12)\qquad dh\,d[t_2] \equiv 0 \mod ([t_1],[t_2])W_{2n-1}\Omega^d_{P/X}.$$

For all i, let

$$S_i(X_0,\dots,X_i,Y_0,\dots,Y_i) \in \mathbb{Z}[X_0,\dots,X_i,Y_0,\dots,Y_i]$$

be the universal polynomial defining the i-th component of the sum of two Witt vectors, and

$$(7.2.13)\qquad s_i(X_0,Y_0) = S_i(X_0,0,\dots,0,Y_0,0,\dots 0) \in \mathbb{Z}[X_0,Y_0].$$

Note that, for $i \geq 1$, the polynomial $s_i(X_0,Y_0)$ is divisible by X_0Y_0, since $(0,\dots,0)$ is the zero element in a Witt vector ring. By definition, we have

$$[t_1] + [ct_2] = (t_1 + ct_2, s_1(t_1,ct_2),\dots,s_{2n-2}(t_1,ct_2)),$$

and

$$h = (0, s_1(t_1,ct_2),\dots,s_{2n-2}(t_1,ct_2)).$$

Since $s_i(X_0,Y_0)$ is divisible by Y_0, we can write $s_i(t_1,ct_2) = z_it_2$ for some section $z_i \in \mathcal{O}_P$. We obtain

$$h = (0, z_1t_2,\dots,z_{2n-2}t_2),$$

which we can write as

$$h = \sum_{i=1}^{2n-2} V^i([z_i][t_2]).$$

For each i, $1 \leq i \leq 2n-2$, we now obtain

$$\begin{aligned} dV^i([z_i][t_2])\, d[t_2] &= dV^i([z_i][t_2]\, F^i(d[t_2])) = dV^i([z_i][t_2]^{p^i} d[t_2]) \\ &= dV^i([z_i]F^i([t_2])d[t_2]) = d([t_2]V^i([z_i]d[t_2])) \equiv d[t_2]\, V^i([z_i]d[t_2]) \\ &= V^i(F^i(d[t_2])[z_i]d[t_2]) = V^i([t_2]^{p^i-1}d[t_2][z_i]d[t_2]) = 0 \end{aligned}$$

mod $[t_2]W_{2n-1}\Omega^d_{P/X}$, which proves (7.2.12). □

Definition 7.3. Under the assumptions of this section, we define the *n-th Hodge-Witt fundamental class* $\gamma_{Y,n}$ of Y in P relatively to X as being the section of $\mathcal{H}^d_Y(W_n\Omega^d_{P/X})$ obtained by glueing the sections $\beta_{\mathcal{J}}(\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1],\ldots,[t_d] \end{bmatrix})$ defined locally by regular sequences of generators of the ideal $\mathcal{I}$ of Y in P.

Proposition 7.4. *For $n \geq 1$, let*

$$\begin{aligned} R &: \mathcal{H}^d_Y(W_{n+1}\Omega^d_{P/X}) \longrightarrow \mathcal{H}^d_Y(W_n\Omega^d_{P/X}), \\ F &: \mathcal{H}^d_Y(W_{n+1}\Omega^d_{P/X}) \longrightarrow \mathcal{H}^d_Y(W_n\Omega^d_{P/X}), \\ V &: \mathcal{H}^d_Y(W_n\Omega^d_{P/X}) \longrightarrow \mathcal{H}^d_Y(W_{n+1}\Omega^d_{P/X}) \end{aligned}$$

be the homomorphisms defined by functoriality. Then

$$R(\gamma_{Y,n+1}) = \gamma_{Y,n}, \qquad F(\gamma_{Y,n+1}) = \gamma_{Y,n}, \qquad V(\gamma_{Y,n}) = p\gamma_{Y,n+1}. \tag{7.4.1}$$

Proof. We may assume that there exists a regular sequence $t_1, \ldots, t_d$ such that $\mathcal{I} = (t_1, \ldots, t_d)$. For each $n \geq 1$, let $\mathcal{J}_n$ be the ideal of $W_n(\mathcal{O}_P)$ generated by the Teichmüller representatives $[t_i]$ of the t_i's, and let $K_\bullet([\mathbf{t}]_n)$ be the Koszul complex defined by the $[t_i]$'s over $W_n(\mathcal{O}_P)$. Since $R([t_i]) = [t_i]$, scalar extension through R yields an isomorphism

$$W_n(\mathcal{O}_P) \otimes_{W_{n+1}(\mathcal{O}_P)} K_\bullet([\mathbf{t}]_{n+1}) \xrightarrow{\sim} K_\bullet([\mathbf{t}]_n).$$

Using the fact that the $[t_i]$'s form a regular sequence both in $W_{n+1}(\mathcal{O}_P)$ and in $W_n(\mathcal{O}_P)$, it can be seen in the derived category of $W_n(\mathcal{O}_P)$-modules as an isomorphism

$$W_n(\mathcal{O}_P) \overset{\mathbb{L}}{\otimes}_{W_{n+1}(\mathcal{O}_P)} W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1} \xrightarrow{\sim} W_n(\mathcal{O}_P)/\mathcal{J}_n. \tag{7.4.2}$$

By adjunction, we obtain a diagram
(7.4.3)

$$\begin{array}{ccccc}
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_{n+1}(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]_{n+1}), W_{n+1}\Omega^d_{P/X})) & \xrightarrow{\sim} & \mathcal{E}xt^d_{W_{n+1}(\mathcal{O}_P)}(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_{n+1}\Omega^d_{P/X}) & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & \mathcal{H}^d_Y(W_{n+1}\Omega^d_{P/X}) \\
\downarrow R & & & & \\
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_{n+1}(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]_{n+1}), W_n\Omega^d_{P/X})) & & \downarrow \rho & & \downarrow R \\
\wr\uparrow & & & & \\
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_n(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]_n), W_n\Omega^d_{P/X})) & \xrightarrow{\sim} & \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}_n, W_n\Omega^d_{P/X}) & \xrightarrow{\beta_{\mathcal{J}_n}} & \mathcal{H}^d_Y(W_n\Omega^d_{P/X}),
\end{array}$$

in which ρ is defined so that the left part of the diagram commutes. On the other hand, (7.4.2) implies that injective $W_n(\mathcal{O}_P)$-modules are acyclic for the functor $\mathcal{H}om_{W_{n+1}(\mathcal{O}_P)}(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, -)$. Replacing $W_{n+1}\Omega^d_{P/X}$ and $W_n\Omega^d_{P/X}$ by injective resolutions, it is then easy to check that the right part of the diagram commutes. As $R(d[t_1]\cdots d[t_d]) = d[t_1]\cdots d[t_d]$, the first relation of (7.4.1) follows.

One proceeds similarly to prove the second one. Since $F([t_i]) = [t_i^p] = [t_i]^p$, and the sequence $[t_i]^p, \dots, [t_d]^p$ is a regular sequence in $W_n(\mathcal{O}_P)$, we obtain isomorphisms

$$W_n(\mathcal{O}_P) \otimes_{W_{n+1}(\mathcal{O}_P)} K_\bullet([\mathbf{t}]_{n+1}) \xrightarrow{\sim} K_\bullet([\mathbf{t}]^p_n),$$

$$W_n(\mathcal{O}_P) \overset{\mathbb{L}}{\otimes}_{W_{n+1}(\mathcal{O}_P)} W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1} \xrightarrow{\sim} W_n(\mathcal{O}_P)/\mathcal{J}^{(p)}_n, \tag{7.4.4}$$

where the tensor products are now taken via $F : W_{n+1}(\mathcal{O}_P) \to W_n(\mathcal{O}_P)$. They provide a commutative diagram similar to (7.4.3)
(7.4.5)

$$\begin{array}{ccccc}
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_{n+1}(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]_{n+1}), W_{n+1}\Omega^d_{P/X})) & \xrightarrow{\sim} & \mathcal{E}xt^d_{W_{n+1}(\mathcal{O}_P)}(W_{n+1}(\mathcal{O}_P)/\mathcal{J}_{n+1}, W_{n+1}\Omega^d_{P/X}) & \xrightarrow{\beta_{\mathcal{J}_{n+1}}} & \mathcal{H}^d_Y(W_{n+1}\Omega^d_{P/X}) \\
\downarrow F & & & & \\
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_{n+1}(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]_{n+1}), W_n\Omega^d_{P/X})) & & \downarrow \phi & & \downarrow F \\
\wr\uparrow & & & & \\
\mathcal{H}^d(\mathcal{H}om^\bullet_{W_n(\mathcal{O}_P)}(K_\bullet([\mathbf{t}]^p_n), W_n\Omega^d_{P/X})) & \xrightarrow{\sim} & \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}^{(p)}_n, W_n\Omega^d_{P/X}) & \xrightarrow{\beta_{\mathcal{J}^{(p)}_n}} & \mathcal{H}^d_Y(W_n\Omega^d_{P/X}).
\end{array}$$

Since $F(d[t_1]\cdots d[t_d]) = [t_1]^{p-1}\cdots[t_d]^{p-1}d[t_1]\cdots d[t_d]$, it follows that

$$F(\beta_{\mathcal{J}_{n+1}}(\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1],\dots,[t_d] \end{bmatrix})) = \beta_{\mathcal{J}^{(p)}_n}(\begin{bmatrix} [t_1]^{p-1}\cdots[t_d]^{p-1}\,d[t_1]\cdots d[t_d] \\ [t_1]^p,\dots,[t_d]^p \end{bmatrix}).$$

On the other hand, if α denotes the canonical homomorphism

$$\alpha : \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}_n, W_n\Omega^d_{P/X}) \longrightarrow \mathcal{E}xt^d_{W_n(\mathcal{O}_P)}(W_n(\mathcal{O}_P)/\mathcal{J}_n^{(p)}, W_n\Omega^d_{P/X}),$$

we have by (4.2.1)

$$\alpha(\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1],\ldots,[t_d] \end{bmatrix}) = \begin{bmatrix} [t_1]^{p-1}\cdots[t_d]^{p-1}\, d[t_1]\cdots d[t_d] \\ [t_1]^p,\ldots,[t_d]^p \end{bmatrix}.$$

As $\beta_{\mathcal{J}_n^{(p)}} \circ \alpha = \beta_{\mathcal{J}_n}$, it follows that $F(\gamma_{Y,n+1}) = \gamma_{Y,n}$.

The last relation of (7.4.1) follows formally, because $V(\gamma_{Y,n}) = V(F(\gamma_{Y,n+1})) = p\gamma_{Y,n+1}$. □

Proposition 7.5. *Let $n \geq 1$ be an integer, and let $\gamma_{Y,n} \in \mathcal{H}^d_Y(W_n\Omega^d_{P/X})$ be the Hodge-Witt fundamental class of Y in P relatively to X, as defined in 7.3.*

(i) *The linear homomorphism $W_n(\mathcal{O}_P) \to \mathcal{H}^d_Y(W_n\Omega^d_{P/X})$ sending 1 to $\gamma_{Y,n}$ vanishes on $W_n(\mathcal{I}) := \mathrm{Ker}(W_n(\mathcal{O}_P) \twoheadrightarrow i_*W_n(\mathcal{O}_Y))$.*

(ii) *Let $\gamma_{i,\pi,n}$ be the composition*

(7.5.1)

$$\gamma_{i,\pi,n} : i_*W_n(\mathcal{O}_Y) \longrightarrow \mathcal{H}^d_Y(W_n\Omega^d_{P/X}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}_Y(W_n\Omega^d_{P/X}[d]) \longrightarrow W_n\Omega^d_{P/X}[d],$$

where the first morphism is defined thanks to the previous assertion. Then $\gamma_{i,\pi,n}$ commutes with R, F and V.

(iii) *For $n = 1$, we have $\gamma_{i,\pi,1} = \gamma_f$, where γ_f is the morphism defined by* (4.5.1).

Proof. To prove assertion (i), we may again assume that $\mathcal{I}$ is generated by a regular sequence $t_1, \ldots, t_d$. Any section w of $W_n(\mathcal{I})$ can then be written as a sum

$$w = \sum_{i=0}^{n-1} V^i([a_{i,1}][t_1] + \cdots + [a_{i,d}][t_d]),$$

with $a_{i,j} \in \mathcal{I}$ and $[a_{i,j}], [t_j] \in W_{n-i}(\mathcal{O}_P)$. By functoriality, we have $V(a)\omega = V(aF(\omega))$ for any $a \in W_i(\mathcal{O}_P)$, $\omega \in \mathcal{H}^d_Y(W_{i+1}\Omega^d_{P/X})$, $i \geq 1$. Using (7.4.1), we obtain

$$V^i([a_{i,j}][t_j])\gamma_{Y,n} = V^i([a_{i,j}][t_j]F^i(\gamma_{Y,n})) = V^i([a_{i,j}][t_j]\gamma_{Y,n-i}).$$

The symbol (4.1.2) is linear with respect to m, therefore we have

$$[a_{i,j}][t_j]\gamma_{Y,n-i} = \beta_{\mathcal{J}}(\begin{bmatrix} [a_{i,j}][t_j]\, d[t_1]\cdots d[t_d] \\ [t_1],\ldots,[t_d] \end{bmatrix}) = 0$$

since the upper entry in the symbol belongs to $([t_1], \ldots, [t_d])W_{n-i}\Omega^d_{P/X}$.

In the definition of $\gamma_{i,\pi,n}$, the last two arrows commute with R, F and V by functoriality. Relations (7.4.1) imply that the first one also commutes with R, F and V, since $R(1) = F(1) = 1$, and $V(1) = p$.

Let us assume that $n = 1$, and check assertion (iii). By construction, $\gamma_{i,\pi,1}$ is the composition of the morphism $i_*\mathcal{O}_Y \to \mathcal{H}^d_Y(\Omega^d_{P/X})$ sending 1 to $\gamma_{Y,1}$ with the canonical morphism

$$\mathcal{H}^d_Y(\Omega^d_{P/X}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}_Y(\Omega^d_{P/X}[d]) \longrightarrow \Omega^d_{P/X}[d].$$

Comparing with the definition of γ_f in 4.5, and using the same notations, it suffices to show that the composed morphism

$$\mathcal{O}_Y \xrightarrow{\varphi_f} \omega_{Y/X} \xrightarrow{\eta_{Y/P}^{-1} \circ \zeta'_{i,\pi}} \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \Omega^d_{P/X}) \xrightarrow{\beta_{\mathcal{I}}} \mathcal{H}^d_Y(\Omega^d_{P/X})$$

sends 1 to $\gamma_{Y,1}$. Since this is a morphism of sheaves (rather than complexes in the derived category), it is a local verification, which is provided by Lemma 4.4. □

Definition 7.6. Let X be a noetherian $\mathbb{F}_p$-scheme with a dualizing complex, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank $d+1$, $P = \mathbb{P}(\mathcal{E})$, $\pi : P \to X$ the canonical projection, $i : Y \hookrightarrow P$ a regular closed immersion of codimension d. For each integer $n \geq 1$, we define a trace morphism $\tau_{i,\pi,n}$ by

(7.6.1) $$\tau_{i,\pi,n} : \mathbb{R}f_*(W_n(\mathcal{O}_Y)) \xrightarrow{\mathbb{R}\pi_*(\gamma_{i,\pi,n})} \mathbb{R}\pi_*(W_n\Omega^d_{P/X}[d]) \xrightarrow{\mathrm{Trp}_{\pi,n}} W_n(\mathcal{O}_X),$$

where $\gamma_{i,\pi,n}$ is the morphism (7.5.1), and $\mathrm{Trp}_{\pi,n}$ is the Hodge-Witt trace morphism defined in (6.5.1).

Proposition 7.7. *Under the assumptions of 7.6, the morphisms $\tau_{i,\pi,n}$ satisfy the following properties.*

(i) *For variable n, $\tau_{i,\pi,n}$ commutes with R, F and V.*

(ii) *For $n = 1$, $\tau_{i,\pi,1} = \tau_f$.*

Proof. Taking into account Proposition 4.6, both assertions follow from the similar properties of $\gamma_{i,\pi,n}$ and $\mathrm{Trp}_{\pi,n}$ proved in 7.5 and 6.5. □

Definition 7.8. Under the assumptions of 7.6, we can use the previous constructions to define a morphism $\tau_{i,\pi} : \mathbb{R}f_*(W(\mathcal{O}_Y)) \longrightarrow W(\mathcal{O}_X)$ which commutes with F and V, and is such that $R_n \circ \tau_{i,\pi} = \tau_{i,\pi,n} \circ R_n$ for all n, R_n denoting both restriction maps $W(\mathcal{O}_X) \to W_n(\mathcal{O}_X)$ and $W(\mathcal{O}_Y) \to W_n(\mathcal{O}_Y)$.

To construct $\tau_{i,\pi}$, we first recall that, for any scheme X, the inverse system $(W_n(\mathcal{O}_X))_{n\geq 0}$ is $\varprojlim$-acyclic, as the cohomology of each term vanishes on affine open subsets, and the inverse system of sections on such a subset has surjective transition maps. So, if $f_{\bullet *}$ denotes the obvious extension of the direct image functor to the category of inverse systems, it suffices to define a morphism

(7.8.1) $$\tau_{i,\pi,\bullet} : \mathbb{R}f_{\bullet *}(W_\bullet(\mathcal{O}_Y)) \longrightarrow W_\bullet(\mathcal{O}_X)$$

in the derived category of inverse systems on X, and to apply the functor $\mathbb{R}\varprojlim$ and the canonical isomorphism $\mathbb{R}f_* \circ \mathbb{R}\varprojlim \simeq \mathbb{R}\varprojlim \circ \mathbb{R}f_{\bullet *}$. On the one hand, the relations $R(\gamma_{Y,n+1}) = \gamma_{Y,n}$ imply that, for variable n, the fundamental classes define a morphism of inverse systems $i_{\bullet *}(W_\bullet(\mathcal{O}_Y)) \to \mathcal{H}^d_Y(W_\bullet\Omega^d_{P/X})$. As the canonical morphisms

$$\mathcal{H}^d_Y(W_\bullet\Omega^d_{P/X}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}_Y(W_\bullet\Omega^d_{P/X}[d]) \longrightarrow W_\bullet\Omega^d_{P/X}[d]$$

make sense in the derived category of inverse systems, we can define in this derived category a morphism $\gamma_{i,\pi,\bullet} : i_{\bullet *}(W_\bullet(\mathcal{O}_Y)) \to W_\bullet\Omega^d_{P/X}[d]$ which has the morphisms $\gamma_{i,\pi,n}$ defined in (7.5.1) as components. On the other hand, the homomorphisms dlog_n used to define Chern classes for invertible bundles form an inverse system of homomorphisms, hence, for variable n, the powers of the Chern classes of $\mathcal{O}_P(1)$

define a morphism $W_\bullet(\mathcal{O}_P)[-d] \to \mathbb{R}\pi_{\bullet *}(W_\bullet\Omega^d_{P/X})$, which is an isomorphism of the derived category of inverse systems. Composing its inverse with the projection by $\mathbb{R}\pi_{\bullet *}$ of $\gamma_{i,\pi,\bullet}$ provides $\tau_{i,\pi,\bullet}$. It is clear that $\tau_{i,\pi,\bullet}$ has the morphisms $\tau_{i,\pi,n}$ as components, and commutes with F and V. Then the morphism

$$\tau_{i,\pi} : \mathbb{R}f_*(W(\mathcal{O}_Y)) \xrightarrow{\sim} \mathbb{R}\varprojlim \mathbb{R}f_{\bullet *}(W_\bullet(\mathcal{O}_Y)) \xrightarrow{\mathbb{R}\varprojlim(\tau_{i,\pi,\bullet})} W(\mathcal{O}_X) \tag{7.8.2}$$

has the required properties.

Finally, as f is a morphism of noetherian schemes, f_* and $\mathbb{R}f_*$ commute with tensorisation with $\mathbb{Q}$. So we can define a morphism again denoted $\tau_{i,\pi} : \mathbb{R}f_*(W\mathcal{O}_{Y,\mathbb{Q}}) \longrightarrow W\mathcal{O}_{X,\mathbb{Q}}$ by

$$\tau_{i,\pi} : \mathbb{R}f_*(W\mathcal{O}_{Y,\mathbb{Q}}) \xrightarrow{\sim} \mathbb{R}f_*(W\mathcal{O}_Y) \otimes \mathbb{Q} \xrightarrow{\tau_{i,\pi}\otimes\mathbb{Q}} W\mathcal{O}_{X,\mathbb{Q}}. \tag{7.8.3}$$

This morphism also commutes with F and V.

8. Proof of the injectivity theorem for Witt vector cohomology

The main result of this section is Theorem 8.1 below, which gives an injectivity property for the functoriality morphisms induced on Witt vector cohomology by some complete intersection morphisms of virtual relative dimension 0. As explained in Remark 8.2, Theorem 1.5 is a particular case of this result.

Theorem 8.1. *Let $f : Y \to X$ be a projective morphism between two flat noetherian $\mathbb{Z}_{(p)}$-schemes with dualizing complexes, which is complete intersection of virtual relative dimension* 0. *We assume that there exists a scheme-theoretically dense open subscheme $U \subset X$ such that $f^{-1}(U) \to U$ is finite locally free of constant rank $r \geq 1$. Let $f_n : Y_n \to X_n$ be the reduction of f mod p^{n+1}.*

(i) *For all $q \geq 0$, the kernels of the functoriality homomorphisms*

$$f^* : H^q(X, \mathcal{O}_X) \longrightarrow H^q(Y, \mathcal{O}_Y), \tag{8.1.1}$$

$$f_n^* : H^q(X_n, \mathcal{O}_{X_n}) \longrightarrow H^q(Y_n, \mathcal{O}_{Y_n}), \tag{8.1.2}$$

$$f_0^* : H^q(X_0, W_n(\mathcal{O}_{X_0})) \longrightarrow H^q(Y_0, W_n(\mathcal{O}_{Y_0})), \tag{8.1.3}$$

$$f_0^* : H^q(X_0, W(\mathcal{O}_{X_0})) \longrightarrow H^q(Y_0, W(\mathcal{O}_{Y_0})), \tag{8.1.4}$$

are annihilated by r.

(ii) *For all $q \geq 0$, the functoriality homomorphism*

$$f_0^* : H^q(X_0, W\mathcal{O}_{X_0,\mathbb{Q}}) \longrightarrow H^q(Y_0, W\mathcal{O}_{Y_0,\mathbb{Q}}) \tag{8.1.5}$$

is injective.

Remark 8.2. Theorem 8.1 implies Theorem 1.5. Indeed, let $f : Y \to X$ be as in 1.5. The morphisms $X_k \hookrightarrow X_0$ and $Y_k \hookrightarrow Y_0$ are nilpotent immersions, hence the canonical homomorphisms

$$H^q(X_0, W\mathcal{O}_{X_0,\mathbb{Q}}) \longrightarrow H^q(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}), \quad H^q(Y_0, W\mathcal{O}_{Y_0,\mathbb{Q}}) \longrightarrow H^q(Y_k, W\mathcal{O}_{Y_k,\mathbb{Q}})$$

are isomorphisms [BBE07, Prop. 2.1]. Therefore it suffices to check that f satisfies the hypotheses of Theorem 8.1. We may assume that X is connected, and replace Y by one of its connected components mapping surjectively to X, so that X and Y are integral schemes. At any closed point $y \in Y$, with image $x = f(y)$, we may choose a closed immersion $Y \hookrightarrow P$ around y, with P smooth over X. If $\dim \mathcal{O}_{X,x} = n$, then $\mathcal{O}_{P,y}$ is a regular local ring of dimension $n + d$ for $d = \dim(P/X)$, and $\mathcal{O}_{Y,y}$ is a regular quotient of $\mathcal{O}_{P,y}$ of dimension n. Therefore, the ideal $\mathcal{I}$ of Y in P is regular of codimension d around y, and it follows that f is complete intersection of virtual relative dimension 0. Moreover, the function field extension $K(X) \hookrightarrow K(Y)$ is finite, hence f is finite and locally free of constant rank ≥ 1 above a non empty open subset U. As X is integral, U is scheme-theoretically dense and the hypotheses of Theorem 8.1 are satisfied.

In order to prove Theorem 8.1, we will choose a factorization $f = \pi \circ i$, where $i : Y \hookrightarrow P = \mathbb{P}^d_X$ is a closed immersion, and $\pi : P \to X$ the structural morphism. Let i_0, π_0 be the reductions mod p of i, π. The key point will be to relate the trace morphisms $\tau_{i_0,\pi_0,n}$ constructed in 7.6 to the trace morphism τ_f given by Theorem 3.1, and this is made possible by the following constructions.

Lemma 8.3. *Let X be a scheme on which p is locally nilpotent, P a smooth X-scheme, $\mathfrak{a} \subset \mathcal{O}_X$ a quasi-coherent ideal, $X' \hookrightarrow X$ the closed subscheme defined by $\mathfrak{a}$, $P' = X' \times_X P$. For each $n \geq 1$, let $\mathcal{N}^\bullet_n \subset W_n\Omega^\bullet_{P/X}$ be the additive subgroup generated by sections of the form*

$$(8.3.1) \qquad V^r([a]\omega), \quad dV^r([a]\omega), \quad \text{with } a \in \mathfrak{a}, \ \ \omega \in W_{n-r}\Omega^\bullet_{P/X}, \quad 0 \leq r \leq n-1.$$

Then, for variable n, the canonical homomorphisms $W_n\Omega^\bullet_{P/X} \to W_n\Omega^\bullet_{P'/X'}$ induce a transitive family of isomorphisms

$$(8.3.2) \qquad W_n\Omega^\bullet_{P/X}/\mathcal{N}^\bullet_n \xrightarrow{\ \sim\ } W_n\Omega^\bullet_{P'/X'}.$$

Proof. Thanks to (5.1.2), one first notices that $\mathcal{N}^\bullet_n$ is a differential graded ideal of $W_n\Omega^\bullet_{P/X}$. Using (5.1.4), one sees that, for all $n \geq 1$, $V(\mathcal{N}^\bullet_n) \subset \mathcal{N}^\bullet_{n+1}$. Using (5.1.1) (and a direct computation for $r = 0$), one sees that $F(\mathcal{N}^\bullet_{n+1}) \subset \mathcal{N}^\bullet_n$. Therefore, the projective system $\{W_n\Omega^\bullet_{P/X}/\mathcal{N}^\bullet_n\}$ is an F-V-procomplex over P/X. In degree 0, it is easy to see by induction on n that the ideal $\mathcal{N}^0_n \subset W_n(\mathcal{O}_P)$ is the kernel of $W_n(\mathcal{O}_P) \to W_n(\mathcal{O}_{P'})$. It follows that $\{W_n\Omega^\bullet_{P/X}/\mathcal{N}^\bullet_n\}$ is actually an F-V-procomplex over P'/X'. It is then clear that it satisfies the universal property which defines $\{W_n\Omega^\bullet_{P'/X'}\}$, which implies that (8.3.2) is an isomorphism of F-V-procomplexes. □

Proposition 8.4. *Let X be a $\mathbb{Z}_{(p)}$-scheme and denote $X_n = X \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/p^{n+1}$.*

(i) *For all $n \geq 1$, there exists a unique homomorphism of sheaves of rings*

$$\widetilde{F}^n : W_n(\mathcal{O}_{X_0}) \longrightarrow \mathcal{O}_{X_{n-1}}$$

making the following diagram commute

$$\begin{array}{ccc} W_{n+1}(\mathcal{O}_{X_{n-1}}) & \xrightarrow{F^n} & \mathcal{O}_{X_{n-1}} \\ \downarrow & \nearrow_{\widetilde{F}^n} & \\ W_n(\mathcal{O}_{X_0}) & & \end{array},$$

where the vertical map is the natural reduction map. Furthermore, if we assume X to be flat over $\mathbb{Z}_{(p)}$ and denote by $R_n : W(\mathcal{O}_{X_0}) \to W_n(\mathcal{O}_{X_0})$ the natural reduction map, then

$$\text{(8.4.1)} \qquad \operatorname{Ker}(F - \operatorname{Id} : W(\mathcal{O}_{X_0}) \to W(\mathcal{O}_{X_0})) \cap \bigcap_{n \geq 1} \operatorname{Ker}(\widetilde{F}^n \circ R_n) \ = \ 0.$$

(ii) *Let P be a smooth X-scheme and denote $P_n = P \times_X X_n$. For all $n \geq 1$, there exists a unique homomorphism of sheaves of graded algebras*

$$\widetilde{F}^n : W_n\Omega^\bullet_{P_0/X_0} \longrightarrow \mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}})$$

making the following diagram commute

$$\begin{array}{ccc} W_{n+1}\Omega^\bullet_{P_{n-1}/X_{n-1}} & \xrightarrow{F^n} & Z\Omega^\bullet_{P_{n-1}/X_{n-1}} \\ \downarrow & & \downarrow \\ W_n\Omega^\bullet_{P_0/X_0} & \xrightarrow{\widetilde{F}^n} & \mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}}). \end{array}$$

Furthermore, for all $a \in \mathcal{O}^\times_{P_0}$ and all $\tilde{a} \in \mathcal{O}^\times_{P_{n-1}}$ lifting a, we have

$$\text{(8.4.2)} \qquad \widetilde{F}^n(\operatorname{dlog}([a])) = \operatorname{cl}(d\tilde{a}/\tilde{a}).$$

When X_0 is a perfect scheme and $X_{n-1} = W_n(X_0)$, $\widetilde{F}^n$ is the isomorphism

$$\text{(8.4.3)} \qquad \theta_n : W_n\Omega^\bullet_{P_0/X_0} \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}})$$

defined by Illusie-Raynaud [IR83, III, (1.5)].

Note that, in formula (8.4.2), the class of $d\tilde{a}/\tilde{a}$ does not depend upon the choice of the liftng $\tilde{a}$: if $\tilde{b} = \tilde{a} + pw$, then

$$d\tilde{b}/\tilde{b} = d\tilde{a}/\tilde{a} + d(\log(1 + p\frac{w}{\tilde{a}})),$$

where $\log(1 + pw/\tilde{a})$ is defined thanks to the canonical divided powers of p.

Proof. (i) We may assume X is affine. The kernel of the vertical map in the diagram is locally generated (as an abelian group) by elements of the form $V^n([a])$ and $V^r([pb])$ for some $a, b \in \mathcal{O}_{P_{n-1}}$ and $0 \leq r \leq n$. As these elements are clearly mapped to 0 under F^n, this gives the unique existence of $\widetilde{F}^n$.

To prove (8.4.1), let $w \in \operatorname{Ker}(F - \operatorname{Id}) \cap \bigcap_n \operatorname{Ker}(\widetilde{F}^n \circ R_n)$. If $w \neq 0$, we can write

$$w = \sum_{i \geq s}^{\infty} V^i([a_i]), \quad \text{with } a_i \in \mathcal{O}_{X_0} \text{ and } a_s \neq 0.$$

Then $R_{s+1}(w) = V^s([a_s]) \in \operatorname{Ker}(\widetilde{F}^{s+1}) \subset W_{s+1}(\mathcal{O}_{X_0})$, which is equivalent to

$$p^s\tilde{a}_s^p \in p^{s+1}\mathcal{O}_X, \quad \text{with } \tilde{a}_s \in \mathcal{O}_X \text{ any lifting of } a_s.$$

Since X is $\mathbb{Z}_{(p)}$-flat, we obtain $\tilde{a}_s^p \in p\mathcal{O}_X$, in particular $a_s^p = 0 \in \mathcal{O}_{X_0}$. But by assumption we have

$$F(w) = \sum_{i \geq s} V^i([a_i^p]) = \sum_{i \geq s} V^i([a_i]) = w.$$

Hence $a_s = a_s^p = 0$, a contradiction.

(ii) First of all, since $dF^n = p^n F^n d$, the image of $F^n : W_{n+1}\Omega^\bullet_{P_{n-1}/X_{n-1}} \to \Omega^\bullet_{P_{n-1}/X_{n-1}}$ is clearly contained in $Z\Omega^\bullet_{P_{n-1}/X_{n-1}}$. Thus the diagram makes sense. Now, Lemma 8.3 and [LZ04, Prop. 2.19] imply that, in degree q, the kernel of the vertical map on the left hand side is locally generated (as an abelian group) by sections of the following form

$$V^n(\alpha), \quad dV^n(\beta), \quad V^r([p]\omega), \quad dV^r([p]\eta), \tag{8.4.4}$$

with $\alpha \in \Omega^q_{P_{n-1}/X_{n-1}}$, $\beta \in \Omega^{q-1}_{P_{n-1}/X_{n-1}}$, $0 \le r \le n$, $\omega \in W_{n+1-r}\Omega^q_{P_{n-1}/X_{n-1}}$ and $\eta \in W_{n+1-r}\Omega^{q-1}_{P_{n-1}/X_{n-1}}$. One immediately sees that F^n maps the first three sections to 0. For the last one, we observe that

$$F^n dV^r([p]\eta) = F^{n-r} d([p]\eta) = F^{n-r}([p]d\eta) = p^{p^{n-r}-(n-r)} d(F^{n-r}(\eta)) = 0$$

in $\mathcal{H}^q(\Omega^\bullet_{P_{n-1}/X_{n-1}})$. Thus F^n maps all elements in the kernel of the vertical map to 0 in $\mathcal{H}^q(\Omega^\bullet_{P_{n-1}/X_{n-1}})$. Since the vertical map is surjective, this yields the statement.

If $\tilde{a} \in \mathcal{O}^\times_{P_{n-1}}$ lifts a, we get by construction

$$\widetilde{F}^n(d[a]/[a]) = \mathrm{cl}(F^n(d[\tilde{a}]/[\tilde{a}])) = \mathrm{cl}([\tilde{a}]^{p^n-1} d[\tilde{a}]/[\tilde{a}]^{p^n}),$$

which gives (8.4.2).

Finally, let us assume that X_0 is perfect and $X_{n-1} = W_n(X_0)$. By [IR83, III, (1.5)], $\mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}})$ has the structure of a differential graded algebra (dga) with the differential $d : \mathcal{H}^i(\Omega^\bullet_{P_{n-1}/X_{n-1}}) \to \mathcal{H}^{i+1}(\Omega^\bullet_{P_{n-1}/X_{n-1}})$ given by the boundary of the long exact cohomology sequence coming from the short exact sequence

$$0 \longrightarrow \Omega^\bullet_{P_{n-1}/X_{n-1}} \xrightarrow{p^n} \Omega^\bullet_{P_{2n-1}/X_{2n-1}} \longrightarrow \Omega^\bullet_{P_{n-1}/X_{n-1}} \longrightarrow 0.$$

The isomorphism θ_n is compatible with the differential and the product, and induces thus an isomorphism of dga's $\theta_n : W_n\Omega^\bullet_{P_0/X_0} \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}})$. On the other hand, it follows from the relation $F^n d = p^n dF^n$ that the morphism $\widetilde{F}^n$ is compatible with the differentials. Therefore $\widetilde{F}^n$ also induces a morphism of dga's $\widetilde{F}^n : W_n\Omega^\bullet_{P_0/X_0} \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega^\bullet_{P_{n-1}/X_{n-1}})$. In degree 0, θ_n is defined by

$$\theta_n(a_0, \dots, a_{n-1}) = \tilde{a}_0^{p^n} + p\tilde{a}_1^{p^{n-1}} + \cdots + p^{n-1}\tilde{a}_{n-1}^p,$$

where $\tilde{a}_0, \dots, \tilde{a}_{n-1}$ are liftings to $\mathcal{O}_{P_{n-1}}$ of $a_0, \dots, a_{n-1}$ [IR83, p. 142, l. 8]. This definition shows that, in degree 0, θ_n is the factorization of the n-th ghost component $W_{n+1}(\mathcal{O}_{P_{n-1}}) \to \mathcal{O}_{P_{n-1}}$. From the definition of the morphism of functors $F^n : W_{n+1} \to W_1$, we also get that, in degree 0, $\widetilde{F}^n$ is the factorization of the n-th ghost component. Since $\widetilde{F}^n = \theta_n$ in degree 0 and $W_n\Omega^\bullet_{P_{n-1}/X_{n-1}}$ is generated as dga by its sections in degree 0, $\widetilde{F}^n$ and θ_n have to be equal. □

Lemma 8.5. *Let S be $\operatorname{Spec}\mathbb{Z}_{(p)}$, X an S-scheme, $\pi : P := \mathbb{P}^d_X \to X$ the structural morphism of a projective space over X. For $n \ge 0$, denote by S_n, X_n, P_n, π_n the reductions modulo p^{n+1}, and let $B\Omega^d_{P_n/X_n} \subset \Omega^d_{P_n/X_n}$ be the subsheaf of exact differential forms.*

(i) *For all $n \ge 0$, the canonical homomorphism*

$$b^d_n : R^d\pi_{n\,*}(\Omega^d_{P_n/X_n}) \longrightarrow R^d\pi_{n\,*}(\Omega^d_{P_n/X_n}/B\Omega^d_{P_n/X_n}) \tag{8.5.1}$$

is an isomorphism.

(ii) *Assume that X is flat over S, and let $Y_0 \hookrightarrow P_0$ be a regular closed immersion of codimension m. Then,*

$$\forall\ j \neq m,\ \forall\ n \geq 0, \quad \mathcal{H}^j_{Y_0}(\Omega^d_{P_n/X_n}/B\Omega^d_{P_n/X_n}) = 0. \tag{8.5.2}$$

Proof. Let $Q = \mathbb{P}^d_S$, and let $T_0, \dots, T_d$ be homogenous coordinates on Q. We define an S-endomorphism $\phi : Q \to Q$ by sending T_i to T_i^p, $0 \leq i \leq d$. By base change by $u : X \to S$, we obtain an X-endomorphism of P, for which we will keep the notation ϕ, as well as for its reduction mod p^{n+1}.

Let us fix $n \geq 0$. We can use the morphism ϕ^{n+1} and view $\phi^{n+1}_*\Omega^\bullet_{P_n/X_n}$ as a complex of quasi-coherent $\mathcal{O}_{P_n}$-modules, the differential of which is then $\mathcal{O}_{P_n}$-linear. But P_n has an open covering by $d+1$ open subsets which are relatively affine with respect to X_n, and therefore $R^d\pi_{n\,*}$ is a right exact functor on the category of quasi-coherent $\mathcal{O}_{P_n}$-modules. As $R^d\pi_{n\,*}(\Omega^{d-1}_{P_n/X_n}) = 0$, the first assertion follows.

To prove the second one, we use ϕ^{n+2} to view $\phi^{n+2}_*\Omega^\bullet_{P_n/X_n}$ as a complex of quasi-coherent $\mathcal{O}_{P_n}$-modules with an $\mathcal{O}_{P_n}$-linear differential, and we claim that the sheaf of $\mathcal{O}_{P_n}$-modules

$$\mathcal{H}^d(\phi^{n+2}_*\Omega^\bullet_{P_n/X_n}) = \phi_*(\phi^{n+1}_*\Omega^d_{P_n/X_n}/B\phi^{n+1}_*\Omega^d_{P_n/X_n})$$

has a filtration by sub-$\mathcal{O}_{P_n}$-modules, the graded of which is locally free over $\mathcal{O}_{P_0}$. As Y_0 is locally defined in P_0 by a regular sequence of m sections, the claim clearly implies assertion (ii).

To prove the existence of this filtration, we may replace X, P by S, Q, because the projection $v : P \to Q$ is flat, and

$$v^*(\phi^{n+2}_*\Omega^\bullet_{Q_n/S_n}) \xrightarrow{\sim} \phi^{n+2}_*\Omega^\bullet_{P_n/X_n}.$$

Now S_0 is a perfect scheme, and $S_n = W_{n+1}(S_0)$. Thanks to the last assertion of Proposition 8.4 (ii), F^{n+1} defines an isomorphism of graded algebras

$$\widetilde{F}^{n+1} : W_{n+1}\Omega^\bullet_{Q_0/S_0} \xrightarrow{\sim} \mathcal{H}^\bullet(\Omega^\bullet_{Q_n/S_n}).$$

We may view $\widetilde{F}^{n+1}$ as an $\mathcal{O}_{Q_n}$-linear isomorphism by endowing $\mathcal{H}^\bullet(\Omega^\bullet_{Q_n/S_n})$ with the $\mathcal{O}_{Q_n}$-module structure provided by the homomorphism $\mathcal{O}_{Q_n} \to \mathcal{H}^0(\Omega^\bullet_{Q_n/S_n})$ defined by ϕ^{n+2}, and $W_{n+1}\Omega^\bullet_{Q_0/S_0}$ with the structure corresponding to the previous one via $(\widetilde{F}^{n+1})^{-1} : \mathcal{H}^0(\Omega^\bullet_{Q_n/S_n}) \xrightarrow{\sim} W_{n+1}(\mathcal{O}_{Q_0})$. The canonical filtration of $W_{n+1}\Omega^d_{Q_0/S_0}$ is then a filtration by sub-$\mathcal{O}_{Q_n}$-modules, which can be transported to $\mathcal{H}^d(\Omega^\bullet_{Q_n/S_n})$ via $\widetilde{F}^{n+1}$. As we know by [Il79, I, Cor. 3.9] that the corresponding graded pieces are locally free $\mathcal{O}_{Q_0}$-modules for the structure defined by the homomorphism

$$\overline{F} : \mathcal{O}_{Q_0} \longrightarrow W_{n+1}(\mathcal{O}_{Q_0})/pW_{n+1}(\mathcal{O}_{Q_0}) \tag{8.5.3}$$

factorizing $F : W_{n+1}(\mathcal{O}_{Q_0}) \to W_{n+1}(\mathcal{O}_{Q_0})$, the proof will be complete if we check the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{O}_{Q_n} & \xrightarrow{\phi^{n+2\,*}} & \mathcal{H}^0(\Omega^\bullet_{Q_n/S_n}) & \xrightarrow[\sim]{(\widetilde{F}^{n+1})^{-1}} & W_{n+1}(\mathcal{O}_{Q_0}) \\ \downarrow & & & & \downarrow \\ \mathcal{O}_{Q_0} & & \xrightarrow{\overline{F}} & & W_{n+1}(\mathcal{O}_{Q_0})/pW_{n+1}(\mathcal{O}_{Q_0}). \end{array} \tag{8.5.4}$$

It is enough to check that the diagram induced on sections over $D_+(T_i) \subset Q_n$ commutes, for $0 \leq i \leq d$. So we may replace $\mathcal{O}_{Q_n}$ by $A = (\mathbb{Z}/p^{n+1}\mathbb{Z})[\underline{x}]$, with $\underline{x} = (x_1, \ldots, x_d)$ and $\phi^*(x_j) = x_j^p$, $1 \leq j \leq d$. Take $f = \sum_I a_I \underline{x}^I \in A$, with $a_I \in \mathbb{Z}/p^{n+1}\mathbb{Z}$. Then

$$(\widetilde{F}^{n+1})^{-1} \circ \phi^{n+2\,*}(f) \;=\; \sum_I a_I (\widetilde{F}^{n+1})^{-1}(\underline{x}^{p^{n+2}I}).$$

As $\widetilde{F}^{n+1}$ is the factorization of the $(n+2)$-th ghost component $w_{n+1} : W_{n+2}(A) \to A$, we see that $(\widetilde{F}^{n+1})^{-1}(x_j^{p^{n+1}}) = [x_j]$, $1 \leq j \leq d$. Therefore, we obtain

$$(\widetilde{F}^{n+1})^{-1} \circ \phi^{n+2\,*}(f) \;=\; \sum_I a_I [\underline{x}]^{pI}.$$

Since $\overline{F}$ is given by lifting an element of A_0 to $W_{n+1}(A_0)$, applying Frobenius and reducing modulo p, this gives the commutativity of (8.5.4). □

Proposition 8.6. *Under the assumptions of Theorem 8.1, let $f = \pi \circ i$ be a factorization of f as the composition of a regular closed immersion $i : Y \hookrightarrow P = \mathbb{P}^d_X$ of Y into a projective space on X, followed by the canonical projection $\pi : P \to X$. For all $n \geq 1$, let f_n, i_n, π_n be the reductions of f, i, π modulo p^{n+1}. Then the compositions*

$$\mathcal{O}_X \xrightarrow{f^*} \mathbb{R}f_*(\mathcal{O}_Y) \xrightarrow{\tau_f} \mathcal{O}_X, \tag{8.6.1}$$

$$\mathcal{O}_{X_n} \xrightarrow{f_n^*} \mathbb{R}f_{n\,*}(\mathcal{O}_{Y_n}) \xrightarrow{\tau_{f_n}} \mathcal{O}_{X_n}, \tag{8.6.2}$$

$$W_n(\mathcal{O}_{X_0}) \xrightarrow{f_0^*} \mathbb{R}f_{0\,*}(W_n(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i_0,\pi_0,n}} W_n(\mathcal{O}_{X_0}), \tag{8.6.3}$$

$$W(\mathcal{O}_{X_0}) \xrightarrow{f_0^*} \mathbb{R}f_{0\,*}(W(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i_0,\pi_0}} W(\mathcal{O}_{X_0}), \tag{8.6.4}$$

$$W\mathcal{O}_{X_0,\mathbb{Q}} \xrightarrow{f_0^*} \mathbb{R}f_{0\,*}(W\mathcal{O}_{Y_0,\mathbb{Q}}) \xrightarrow{\tau_{i_0,\pi_0}} W\mathcal{O}_{X_0,\mathbb{Q}}, \tag{8.6.5}$$

are given by multiplication by r.

Proof. Since the restriction of f above U is finite locally free of rank r, it follows from (3.1.3) that the endomorphism of $\mathcal{O}_U$ induced by $\tau_f \circ f^*$ is mutiplication by r. But U is scheme-theoretically dense in X, therefore the same relation holds on X itself. So (8.6.1) is multiplication by r.

Thanks to the flatness of X and Y over $\mathbb{Z}_{(p)}$, the spectral sequence for the composition of Tor's implies that, for all $n \geq 1$, X_n and Y are Tor-independent over X. Therefore, by Theorem 3.1, (ii), the morphism $\tau_{f_n} \circ f_n^*$ is deduced from $\tau_f \circ f^*$ by base change from X to X_n, and (8.6.2) is also multiplication by r.

We want to deduce from this result that (8.6.3) is also multiplication by r. We observe first that the homomorphisms $\widetilde{F}^n$ defined by Lemma 8.4 provide morphisms

$$\widetilde{F}^n_X : W_n(\mathcal{O}_{X_0}) \longrightarrow \mathcal{O}_{X_{n-1}},$$
$$f_*(\widetilde{F}^n_Y) : f_{0*}(W_n(\mathcal{O}_{Y_0})) \longrightarrow f_{n-1*}(\mathcal{O}_{Y_{n-1}}),$$
$$R^d\pi_*(\widetilde{F}^n_P) : R^d\pi_{0*}(W_n\Omega^d_{P_0/X_0}) \longrightarrow R^d\pi_{n-1*}(\Omega^d_{P_{n-1}/X_{n-1}}/B\Omega^d_{P_{n-1}/X_{n-1}}).$$

Moreover, we can use the isomorphism (8.5.1) and define

$$\widetilde{G}^n_P := (b^d_n)^{-1} \circ R^d\pi_*(\widetilde{F}^n_P) : R^d\pi_{0*}(W_n\Omega^d_{P_0/X_0}) \longrightarrow R^d\pi_{n-1*}(\Omega^d_{P_{n-1}/X_{n-1}}).$$

We consider the diagram

(8.6.6)

$$\begin{array}{ccccccc}
W_n(\mathcal{O}_{X_0}) & \xrightarrow{f_0^*} & f_{0*}(W_n(\mathcal{O}_{Y_0})) & \xrightarrow{\pi_{0*}(\gamma_{i_0,\pi_0,n})} & R^d\pi_{0*}(W_n\Omega^d_{P_0/X_0}) & \xrightarrow[\sim]{\mathrm{Trp}_{\pi_0,n}} & W_n(\mathcal{O}_{X_0}) \\
\downarrow{\scriptstyle \widetilde{F}^n_X} & & \downarrow{\scriptstyle f_*(\widetilde{F}^n_Y)} & & \downarrow{\scriptstyle \widetilde{G}^n_P} & & \downarrow{\scriptstyle \widetilde{F}^n_X} \\
\mathcal{O}_{X_{n-1}} & \xrightarrow{f^*_{n-1}} & f_{n-1*}(\mathcal{O}_{Y_{n-1}}) & \xrightarrow{\pi_{n-1*}(\gamma_{f_{n-1}})} & R^d\pi_{n-1*}(\Omega^d_{P_{n-1}/X_{n-1}}) & \xrightarrow[\sim]{\mathrm{Trp}_{\pi_{n-1}}} & \mathcal{O}_{X_{n-1}},
\end{array}$$

where the compositions of the upper and lower rows are respectively the maps induced by (8.6.3) and (8.6.2) on degree 0 cohomology. Let us prove that this diagram is commutative. The left square commutes because the morphism $\widetilde{F}^n_X$ is functorial with respect to X. To prove that the right square commutes, it suffices to show that, if ξ_{dRW} and ξ_{dR} are the de Rham-Witt and de Rham Chern classes of $\mathcal{O}_P(1)$, then ξ^d_{dRW} and ξ^d_{dR} have same image in $R^d\pi_{n-1*}(\Omega^d_{P_{n-1}/X_{n-1}}/B\Omega^d_{P_{n-1}/X_{n-1}})$. As $R^\bullet\pi_{n-1*}(\widetilde{F}^n_P)$ and $b^\bullet_n$ are compatible with cup-products, it suffices to show that the diagram

$$\begin{array}{ccc}
R^1\pi_{0*}(\mathcal{O}^\times_{P_0}) & \xrightarrow{\mathrm{dlog}} & R^1\pi_{0*}(W_n\Omega^1_{P_0/X_0}) \\
\uparrow & & \downarrow{\scriptstyle R^1\pi_*(\widetilde{F}^n_P)} \\
R^1\pi_{n-1*}(\mathcal{O}^\times_{P_{n-1}}) & \xrightarrow{\mathrm{dlog}} & R^1\pi_{n-1*}(\mathcal{H}^1(\Omega^\bullet_{P_{n-1}/X_{n-1}}))
\end{array}$$

is commutative, which follows from (8.4.2).

To simplify notations, we drop the base scheme from the indices, and denote $C^d_{P_{n-1}} = \Omega^d_{P_{n-1}}/B\Omega^d_{P_{n-1}}$. To prove the commutativity of the central square of (8.6.6), it suffices to prove the commutativity of the diagram

$$\begin{array}{ccccccc}
i_{0*}(W_n(\mathcal{O}_{Y_0})) & \longrightarrow & \mathcal{H}^d_{Y_0}(W_n\Omega^d_{P_0}) & \xleftarrow{\sim} & \mathbb{R}\underline{\Gamma}_{Y_0}(W_n\Omega^d_{P_0})[d] & \longrightarrow & W_n\Omega^d_{P_0}[d] \\
\Big| & & \downarrow{\scriptstyle \mathcal{H}^d_Y(\widetilde{F}^n_P)} & & \downarrow{\scriptstyle \mathbb{R}\underline{\Gamma}_Y(\widetilde{F}^n_P)[d]} & & \downarrow{\scriptstyle \widetilde{F}^n_P[d]} \\
{\scriptstyle i_*(\widetilde{F}^n_Y)} & & \mathcal{H}^d_{Y_{n-1}}(C^d_{P_{n-1}}) & \xleftarrow{\sim} & \mathbb{R}\underline{\Gamma}_{Y_{n-1}}(C^d_{P_{n-1}})[d] & \longrightarrow & C^d_{P_{n-1}}[d] \\
\downarrow & & \uparrow & & \uparrow & & \uparrow \\
i_{n-1*}(\mathcal{O}_{Y_{n-1}}) & \longrightarrow & \mathcal{H}^d_{Y_{n-1}}(\Omega^d_{P_{n-1}}) & \xleftarrow{\sim} & \mathbb{R}\underline{\Gamma}_{Y_{n-1}}(\Omega^d_{Y_{n-1}})[d] & \longrightarrow & \Omega^d_{Y_{n-1}}[d]\,,
\end{array}$$

to apply the functor $\mathbb{R}\pi_{n-1*}$, and to pass to cohomology sheaves in degree 0. In this diagram, the upper left (resp. lower left) horizontal arrow maps 1 to $\gamma_{Y_0,n}$ (resp.

$\gamma_{Y_{n-1},1}$), and the middle horizontal arrow is an isomorphism thanks to Lemma 8.5 (ii). The middle and right squares commute by functoriality, and it suffices to prove that the left rectangle commutes. This part of the diagram comes from a diagram of morphisms of sheaves, therefore the verification is local on P. Thus we may assume that Y is defined by a regular sequence $t_1, \dots, t_d$ in P. Then, since Y and P are flat over $\mathbb{Z}_{(p)}$, the images of this sequence in $\mathcal{O}_{P_{n-1}}$ and $\mathcal{O}_{P_0}$ (still denoted $t_1, \dots, t_d$) are regular sequences defining Y_{n-1} and Y_0. It is enough to show that the symbols

$$\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1],\dots,[t_d] \end{bmatrix} \in \mathcal{E}xt^d_{W_n(\mathcal{O}_{P_0})}(W_n(\mathcal{O}_{Y_0}), W_n\Omega^d_{P_0})$$

$$\text{and} \quad \begin{bmatrix} dt_1\cdots dt_d \\ t_1,\dots,t_d \end{bmatrix} \in \mathcal{E}xt^d_{\mathcal{O}_{P_{n-1}}}(\mathcal{O}_{Y_{n-1}}, \Omega^d_{P_{n-1}})$$

have same image in $\mathcal{H}^d_Y(C^d_{P_{n-1}})$. By functoriality, the image of $\begin{bmatrix} dt_1\cdots dt_d \\ t_1,\dots,t_d \end{bmatrix}$ in $\mathcal{E}xt^d_{\mathcal{O}_{P_{n-1}}}(\mathcal{O}_{Y_{n-1}}, C^d_{P_{n-1}})$ is $\begin{bmatrix} \mathrm{cl}(dt_1\cdots dt_d) \\ t_1,\dots,t_d \end{bmatrix}$. On the other hand, it follows from the construction of $\widetilde{F}^n$ in Proposition 8.4 that $\widetilde{F}^n_P([t_i]) = t_i^{p^n} \in \mathcal{O}_{P_{n-1}}$, and $\widetilde{F}^n_P(d[t_i]) = \mathrm{cl}(t_i^{p^n-1}dt_i) \in \mathcal{H}^1(\Omega^\bullet_{P_{n-1}})$. Since the $t_i^{p^n}$'s form a regular sequence in $\mathcal{O}_{P_{n-1}}$, we may argue as in the proof of Proposition 7.4 to show that the symbols $\begin{bmatrix} d[t_1]\cdots d[t_d] \\ [t_1],\dots,[t_d] \end{bmatrix}$ and $\begin{bmatrix} \mathrm{cl}(t_1^{p^n-1}\cdots t_d^{p^n-1}dt_1\cdots dt_d) \\ t_1^{p^n},\dots,t_d^{p^n} \end{bmatrix}$ have same image in $\mathcal{H}^d_{Y_{n-1}}(C^d_{P_{n-1}})$. The wanted equality is then a consequence of Lemma 4.2, and the commutativity of (8.6.6) follows.

Returning to the homomorphism (8.6.3), we observe that it is defined by multiplication by a section κ_n of $W_n(\mathcal{O}_{X_0})$. Proposition 7.7 (i) implies that, for variable n, the sections κ_n form a compatible family under restriction, and satisfy $F(\kappa_n) = \kappa_{n-1}$. If $\kappa = \varprojlim_n \kappa_n \in \Gamma(X_0, W(\mathcal{O}_{X_0}))$, then $F(\kappa - r) = \kappa - r$. On the other hand, the commutativity of (8.6.6) implies that $\widetilde{F}^n_X(\kappa_n - r) = 0$. So, if $R_n : W(\mathcal{O}_{X_0}) \to W_n(\mathcal{O}_{X_0})$ is the restriction homomorphism, we obtain that

$$\kappa - r \in \mathrm{Ker}(F - \mathrm{Id}) \cap \bigcap_{n\geq 1} \mathrm{Ker}(\widetilde{F}^n_X \circ R_n),$$

which is zero by (8.4.1). Thus $\kappa = r$, hence $\kappa_n = r$ for all n.

If we now consider in the derived category of inverse systems the composition

$$W_\bullet(\mathcal{O}_{X_0}) \xrightarrow{f_{0\bullet}^*} \mathbb{R}f_{0\bullet *}(W_\bullet(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i,\pi,\bullet}} W_\bullet(\mathcal{O}_{X_0}),$$

we obtain a morphism which has (8.6.3) as component of degree n. Therefore, this composition is multiplication by r on the inverse system $W_\bullet(\mathcal{O}_{X_0})$. It follows that the composition

$$W(\mathcal{O}_{X_0}) \xrightarrow{\mathbb{R}\varprojlim f_{0\bullet}^*} \mathbb{R}\varprojlim \mathbb{R}f_{0\bullet *}(W_\bullet(\mathcal{O}_{Y_0})) \xrightarrow{\mathbb{R}\varprojlim \tau_{i,\pi,\bullet}} W(\mathcal{O}_{X_0})$$

is multiplication by r. Using the isomorphism $\mathbb{R}\varprojlim \circ \mathbb{R}f_{0\bullet *} \simeq \mathbb{R}f_{0*} \circ \mathbb{R}\varprojlim$, we obtain that (8.6.4) is multiplication by r. Tensoring by $\mathbb{Q}$ and using the commutation of $\mathbb{R}f_{0*}$ with tensorisation by $\mathbb{Q}$, we obtain that (8.6.5) is multiplication by r. □

8.7. *Proof of Theorem 8.1.* The first assertion is a particular case of Theorem 3.2. To prove the other ones, we choose a factorization $f = \pi \circ i$, where i is a closed immersion of Y into a projective space $P = \mathbb{P}^d_X$ over X, and π is the structural morphism, and we keep the notations of the previous subsections. Applying the functor $H^q(X_n, -)$ (resp. $H^i(X_0, -)$), the morphisms τ_{f_n}, $\tau_{i,\pi,n}$ and $\tau_{i,\pi}$ define homomorphisms

$$H^q(Y_n, \mathcal{O}_{Y_n}) \xrightarrow{\tau_{f_n}} H^q(X_n, \mathcal{O}_{X_n}),$$
$$H^q(Y_0, W_n(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i,\pi,n}} H^q(X_0, W_n(\mathcal{O}_{X_0})),$$
$$H^q(Y_0, W(\mathcal{O}_{Y_0})) \xrightarrow{\tau_{i,\pi}} H^q(X_0, W(\mathcal{O}_{X_0})),$$
$$H^q(Y_0, W\mathcal{O}_{Y_0,\mathbb{Q}}) \xrightarrow{\tau_{i,\pi}} H^q(X_0, W\mathcal{O}_{X_0,\mathbb{Q}}).$$

Proposition 8.6 implies that the composition of these homomorphisms with the functoriality homomorphisms defined by f_n (resp. f_0) is multiplication by r, and this implies Theorem 8.1. □

This also completes the proof of Theorems 1.5, 1.3 and 1.1.

9. An example

Because Theorem 1.1 was previously known in some cases, and can be proved in some other cases without using the most difficult results of this paper, it may be worth giving an example for which we would not know how to prove congruence (1.1.1) without using them. We give here such an example for each $p \geq 7$, except perhaps when p is a Fermat number.

9.1. We begin with a list of conditions that we want our example to satisfy. In these conditions, R, K and k are as in Theorem 1.1, and X is an R-scheme.

(1) X is a regular scheme, projective and flat over R.
(2) $H^0(X_K, \mathcal{O}_{X_K}) = K$, and $H^q(X_K, \mathcal{O}_{X_K}) = 0$ for all $q \geq 1$.
(3) There exists $q \geq 1$ such that $H^q(X_k, \mathcal{O}_{X_k}) \neq 0$.
(4) X is not a semi-stable R-scheme (in particular, not smooth).
(5) $\dim X_K \geq 3$.
(6) X_K is a variety of general type.

Conditions (1) and (2) will ensure that X satisfies the hypotheses of Theorem 1.1. Condition (3) will ensure that we are not in the trivial situation described in the first paragraph of subsection 1.4. Condition (4) will ensure that Theorem 2.1 does not suffice to conclude. Condition (5) will rule out the case of surfaces, for which Theorem 1.1 is already known by [Es06, Th. 1.3]. Condition (6) rules out rationally connected varieties, for which Theorem 1.1 is also known because they satisfy the coniveau condition of [Es06, Th. 1.1]. It also grants that, if X can be embedded as a global complete intersection in some projective space over R, then congruence (1.1.1) cannot be proved by applying Katz's theorem [Kz71, Th. 1.0] to X_k, since a smooth complete intersection in a K-projective space for which Katz's μ invariant is ≥ 1 is a Fano variety.

Remarks 9.2. We begin with a few remarks that make it easier to find an example satisfying the previous conditions.

(i) Examples such that $\dim_k H^1(X_k, \mathcal{O}_{X_k}) > \dim_K H^1(X_K, \mathcal{O}_{X_K}) = 0$ have been known since Serre's construction of a counter-example to Hodge symmetry in characteristic p [Se58, Prop. 16]. The general principle behind such examples is that, by a theorem of Raynaud, the datum of a torsor Y on X under a finite group G defines a morphism $G' \to \underline{\mathrm{Pic}}_{X/R}$, where G' is the Cartier dual of G. Then, under certain conditions, the Lie algebra of G'_k can have a non zero image in the tangent space $H^1(X_k, \mathcal{O}_{X_k})$ to $\underline{\mathrm{Pic}}_{X_k/k}$ (see [Ra70, Prop. 6.2.1] for a precise statement). The simplest case (which was the one considered by Serre) is when G is the étale group $\mathbb{Z}/p\mathbb{Z}$. Then the Artin-Schreier exact sequence shows that, when the torsor Y_k remains non-trivial after extension to an algebraic closure $\overline{k}$ of k, its class gives a non-zero element in $H^1(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$, and therefore $H^1(X_k, \mathcal{O}_{X_k}) \neq 0$. This happens in particular when Y_k is a complete intersection in some projective space, since we then have $\dim_{\overline{k}} H^0(Y_{\overline{k}}, \mathcal{O}_{Y_{\overline{k}}}) = 1$.

To simplify our quest, we will therefore replace condition (3) (and condition (5)) by the more restrictive condition:

(3') X is the quotient of an hypersurface Y in a projective space $\mathbb{P}^n_R$ of relative dimension $n \geq 4$ over R by a free action of the group $\mathbb{Z}/p\mathbb{Z}$.

(ii) Assume that X satisfies condition (3'). Then $H^0(Y_K, \mathcal{O}_{Y_K}) = K$, and $H^q(Y_K, \mathcal{O}_{Y_K}) = 0$ for $q \neq 0, n-1$. Because $\mathrm{char}(K) = 0$, we have $H^q(X_K, \mathcal{O}_{X_K}) = H^q(Y_K, \mathcal{O}_{Y_K})^G$. Hence, $H^0(X_K, \mathcal{O}_{X_K}) = K$, and condition (2) is satisfied if and only if $\chi(\mathcal{O}_{X_K}) = 1$. As Y_K is an étale cover of X_K of degree p, the Riemann-Roch-Hirzebruch formula implies that

$$\chi(\mathcal{O}_{Y_K}) = p\chi(\mathcal{O}_{X_K}). \tag{9.2.1}$$

Then condition (2) is satisfied if and only if $\chi(\mathcal{O}_{Y_K}) = p$. If d is the degree of the hypersurface Y, we obtain

$$\begin{aligned} (-1)^{n-1}(p-1) &= \dim_K H^{n-1}(Y_K, \mathcal{O}_{Y_K}) \\ &= \dim_K H^n(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(-d)) \\ &= \dim_K H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(d-n-1)). \end{aligned}$$

The simplest choice for checking this equation is $d - n - 1 = 1$, so that we get $\dim_K H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(d-n-1)) = n+1$. Then we have to satisfy the conditions

$$p > 2, \qquad n = p - 2, \qquad d = p. \tag{9.2.2}$$

Therefore, we will simplify even further our quest by replacing condition (3') by the following more precise condition, which implies (2), (3) and (5):

(3") X is the quotient of an hypersurface Y of degree p in the projective space $\mathbb{P}^n_R$ of relative dimension $n = p - 2$ over R by a free action of the group $\mathbb{Z}/p\mathbb{Z}$, with $p \geq 7$.

(iii) Assuming that X satisfies conditions (1) and (3”), then condition (6) follows automatically. Indeed, Y_K is smooth over K since $\mathrm{char}(K) = 0$, and its canonical sheaf is then $\mathcal{O}_{Y_K}(-n-1+d) = \mathcal{O}_{Y_K}(1)$. Since Y_K is an étale covering of X_K, it is the inverse image of the canonical sheaf on X, which therefore is ample too.

So it suffices for our purpose to construct an example satisfying conditions (1), (3”) and (4).

9.3. We now begin the construction of our example. Assume that $p \geq 5$, and let E be the free $\mathbb{Z}_{(p)}$-module $(\mathbb{Z}_{(p)})^p$. We denote by $e_0, \ldots, e_{p-1}$ its canonical basis. Let σ be a generator of $G := \mathbb{Z}/p\mathbb{Z}$. We let σ act on E by cyclic permutation of the basis:

$$\sigma : e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{p-1} (\mapsto e_0). \tag{9.3.1}$$

Let $H \subset E$ be the hyperplane consisting of elements for which the sum of coordinates is 0. It is stable under the action of G, and we endow it with the basis $v_1, \ldots, v_{p-1}$ defined by $v_i = e_i - e_{i-1}$. We take as projective space the space $\mathbb{P}(H) \simeq \mathbb{P}^{p-2}_{\mathbb{Z}_{(p)}}$, with the induced G-action, and we denote by $X_1, \ldots, X_{p-1}$ the homogenous coordinates on $\mathbb{P}(H)$ defined by the dual basis to the basis $v_1, \ldots, v_{p-1}$ of H. One checks easily that the orbit of X_1 under the G-action is described by

$$X_1 \mapsto -X_{p-1} \mapsto X_{p-1} - X_{p-2} \mapsto X_{p-2} - X_{p-3} \mapsto \cdots \mapsto X_2 - X_1 \ (\mapsto X_1). \tag{9.3.2}$$

Let $g_0(X_1, \ldots, X_{p-1})$ be the sum of the elements of the orbit of X_1^p, i.e.,

$$g_0(X_1, \ldots, X_{p-1}) = X_1^p + (-X_{p-1})^p + \sum_{i=2}^{p-1} (X_i - X_{i-1})^p. \tag{9.3.3}$$

Then $g_0 \in p\mathbb{Z}[X_1, \ldots, Z_{p-1}]$, and we can define a polynomial $g(X_1, \ldots, X_{p-1}) \in \mathbb{Z}[X_1, \ldots, Z_{p-1}]$ by

$$g(X_1, \ldots, X_{p-1}) = \frac{1}{p} g_0(X_1, \ldots, X_{p-1}). \tag{9.3.4}$$

Let $Z \subset \mathbb{P}(H)$ be the hypersurface defined by g. Since g is G-invariant, the action of G on $\mathbb{P}(H)$ induces an action on Z. We denote by $\overline{g}$ the reduction of g in $\mathbb{F}_p[X_1, \ldots, X_{p-1}]$. We first study the singular points of $Z_{\mathbb{F}_p}$. They are solutions of the system of homogenous equations $\partial\overline{g}/\partial X_i = 0$, $1 \leq i \leq p-1$, which can be written as

$$\begin{cases} X_1^{p-1} = (X_2 - X_1)^{p-1} \\ (X_2 - X_1)^{p-1} = (X_3 - X_2)^{p-1} \\ \vdots = \vdots \\ (X_{p-1} - X_{p-2})^{p-1} = (-X_{p-1})^{p-1} \end{cases}. \tag{9.3.5}$$

Lemma 9.4. *Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$.*

(i) *The solutions of (9.3.5) in $\mathbb{P}^n(\overline{\mathbb{F}}_p)$ belong to $\mathbb{P}^n(\mathbb{F}_p)$, and they correspond bijectively to the families $(u_1, \ldots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$ such that*

$$1 + u_1 + \cdots + u_{p-1} = 0. \tag{9.4.1}$$

(ii) *For* $u \in \mathbb{F}_p^\times$, *let* $\tilde{u} = [u] \in \mu_{p-1}(\mathbb{Z}_p)$ *be its Teichmüller representative. Then a point* $x \in \mathbb{P}^n(\mathbb{F}_p)$ *which is a solution of* (9.3.5) *belongs to* $Z_{\mathbb{F}_p}$ *if and only if*

$$1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-1} \in p^2\mathbb{Z}_p, \tag{9.4.2}$$

where $(u_1, \ldots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$ *corresponds to* x *by* (i).

Proof. Given $(u_1, \ldots, u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$ satisfying (9.4.1), the corresponding solution $x = (\xi_1 : \ldots : \xi_{p-1}) \in \mathbb{P}^n(\mathbb{F}_p)$ of the system (9.3.5) is obtained by choosing $\xi_1 \in \mathbb{F}_p^\times$, setting

$$\xi_i - \xi_{i-1} = u_{i-1}\xi_1 \quad \text{for } 2 \leq i \leq p-1, \tag{9.4.3}$$

and observing that (9.4.1) implies that $-\xi_{p-1} = u_{p-1}\xi_1$. Assertion (i) is then straightforward.

Let $\eta_1 \in \mathbb{Z}_p$ be a lifting of ξ_1, and let η_i be defined inductively for $2 \leq i \leq p-1$ by

$$\eta_i - \eta_{i-1} = \tilde{u}_{i-1}\eta_1. \tag{9.4.4}$$

Define $\alpha \in \mathbb{Z}_p$ by

$$1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-1} = p\alpha. \tag{9.4.5}$$

Then we get by adding the equations in (9.4.4)

$$\eta_{p-1} = (1 + \cdots + \tilde{u}_{p-2})\eta_1 = (p\alpha - \tilde{u}_{p-1})\eta_1. \tag{9.4.6}$$

We can now substitute (9.4.4) and (9.4.6) in g_0, and we obtain the relation

$$\begin{aligned} g_0(\eta_1, \ldots, \eta_{p-1}) &= \eta_1^p(1 + \tilde{u}_1^p + \ldots + \tilde{u}_{p-2}^p + (\tilde{u}_{p-1} - p\alpha)^p) \\ &= \eta_1^p(1 + \tilde{u}_1 + \ldots + \tilde{u}_{p-2} + \tilde{u}_{p-1} + \sum_{j=1}^{p} \binom{p}{j} \tilde{u}_{p-1}^{p-j}(-p\alpha)^j) \\ &\equiv p\alpha\eta_1^p \mod p^2\mathbb{Z}_p. \end{aligned} \tag{9.4.7}$$

Hence we get

$$g(\eta_1, \ldots, \eta_{p-1}) \equiv \alpha\eta_1^p \mod p\mathbb{Z}_p, \tag{9.4.8}$$

and assertion (ii) follows. □

Lemma 9.5. (i) *The action of* G *on* $Z_{\mathbb{F}_p}$ *is free.*

(ii) *If* p *is not a Fermat number, then* $Z_{\mathbb{F}_p}$ *is singular, and is not the special fibre of a semi-stable scheme.*

Let us recall that the Fermat numbers are the integers of the form $2^{2^n} + 1$ with $n \geq 0$, that any prime number of the form $2^n + 1$ with $n > 0$ is a Fermat number, and that the only known prime Fermat numbers are 3, 5, 17, 257 and 65537.

Proof. A linear algebra computation shows that the only fixed point of σ in $\mathbb{P}^n(\overline{\mathbb{F}}_p)$ is the point $x_0 = (1 : 2 : \ldots : p-1)$. This point is the solution of (9.3.5) corresponding to $u_1 = \ldots = u_{p-1} = 1$. Lemma 9.4 (ii) implies that it does not belong to $Z_{\mathbb{F}_p}$, which proves assertion (i).

As the system (9.3.5) has only a finite number of solutions, the singular points of $Z_{\mathbb{F}_p}$ are isolated. In particular, since $\dim Z_{\mathbb{F}_p} \geq 4$, $Z_{\mathbb{F}_p}$ cannot be the special fibre of a semi-stable scheme if it has a singular point. To find a singular point on $Z_{\mathbb{F}_p}$, Lemma 9.4 shows that it suffices to construct a family $(\tilde{u}_i)_{1\le i\le p-1}$ of $(p-1)$-th roots of unity in $\mathbb{Z}_p$ such that $1+\sum_i \tilde{u}_i \in p^2\mathbb{Z}_p$. Since p is not a Fermat number, $p-1$ has an odd prime factor q. We can choose a primitive q-th root of unity ζ, and set $\tilde{u}_i = \zeta^i$ for $1 \le i \le q-1$, $\tilde{u}_i = 1$ for $q \le i \le q+(p-q)/2-1$, $\tilde{u}_i = -1$ for $q+(p-q)/2 \le i \le p-1$. So $Z_{\mathbb{F}_p}$ is singular. □

9.6. We now address the regularity condition in 9.1 (1). We replace Z by another equivariant lifting of $Z_{\mathbb{F}_p}$ defined as follows. Let R be the ring of integers of a finite extension K of $\mathbb{Q}_p$, of degree > 1, with residue field k. If $K/\mathbb{Q}_p$ is unramified, we set $\pi = p$, otherwise we choose a uniformizer π of R. Let $\lambda \in R$ be an element satisfying the following condition:

a) If $K/\mathbb{Q}_p$ is unramified, then the reduction of λ mod p does not belong to $\mathbb{F}_p$;

b) If $K/\mathbb{Q}_p$ is ramified, then $\lambda \in R^\times$.

Let $h \in \mathbb{Z}[X_1,\dots,X_{p-1}]$ be the product of the elements of the orbit of X_1, i.e.,

$$h(X_1,\dots,X_{p-1}) = X_1(-X_{p-1})\prod_{i=2}^{p-1}(X_i - X_{i-1}), \tag{9.6.1}$$

and let $f \in R[X_1,\dots,X_{p-1}]$ be defined by

$$f = g + \pi\lambda h. \tag{9.6.2}$$

We define $Y \subset \mathbb{P}^n_R$ to be the hypersurface with equation f. Since f is invariant under G, the action of G on $\mathbb{P}^n_R$ induces an action on Y. Its special fibre Y_k is equal to Z_k, on which G acts freely by Lemma 9.5. Then the fixed locus of σ is a closed subscheme of Y, and its projection on $\operatorname{Spec} R$ is a closed subset which does not contain the closed point. Therefore it is empty, and the action of G on Y is free. We define X to be the quotient scheme $X = Y/G$.

Proposition 9.7. *Assume that p is an odd prime which is not a Fermat number. Then the scheme X defined above satisfies conditions* (1) - (6) *of* 9.1.

Proof. As observed in 9.2 (iii), it suffices to check that X satisfies conditions (1), (3") and (4), and condition (3") holds by construction.

The hypersurface Y is projective and flat over R, since g is not divisible by π. So X is also projective and flat. As $Y_k = Z_k$, Lemma 9.5 (ii) implies that Y is not semi-stable. Since $Y \to X$ is étale and semi-stability is a local property for the étale topology, X is not semi-stable either. So we only have to prove that X is regular. This is again a local property for the étale topology, hence it suffices to prove that Y is regular. Because Y is excellent, its singular locus is closed, and the same holds for its projection to $\operatorname{Spec} R$. So it is enough to check the regularity of Y at the points of its special fibre. The regularity is clear at the smooth points of Y_k, and we need to prove it at the singular points.

Let $x = (\xi_1 : \dots : \xi_{p-1}) \in \mathbb{P}^n(k)$ be a singular point of Y_k. As $Y_k = Z_k$, x corresponds by Lemma 9.4 to a family $(u_1,\dots,u_{p-1}) \in (\mathbb{F}_p^\times)^{p-1}$ such that

$$1 + \tilde{u}_1 + \cdots + \tilde{u}_{p-1} = p^2\beta \tag{9.7.1}$$

for some $\beta \in \mathbb{Z}_p$. We have seen in the proof of Lemma 9.4 that $\xi_1 \in \mathbb{F}_p^\times$, so we may assume that $\xi_1 = 1$. We set $\eta_1 = 1$, and we define inductively η_i for $2 \leq i \leq p-1$ by (9.4.4). This allows to work on the affine space $\mathbb{A}_R^n = D_+(X_1) \subset \mathbb{P}_R^n$, and we will denote

$$a_*(X_2, \ldots, X_{p-1}) := a(1, X_2, \ldots, X_{p-1})$$

for any homogenous polynomial $a(X_1, \ldots, X_{p-1}) \in R[X_1, \ldots, X_{p-1}]$. For $2 \leq i \leq p-1$, we set

$$X_i = \eta_i + Y_i,$$

so that $(\pi, Y_2, \ldots, Y_{p-1})$ is a regular sequence of generators of the maximal ideal $\mathfrak{m}_x$ of the regular local ring $\mathcal{O}_{\mathbb{A}_R^n,x}$.

We want to prove that $\mathcal{O}_{\mathbb{A}_R^n,x}/(f_*)$ is regular, i.e., that $f_* \notin \mathfrak{m}_x^2$. We first claim that

$$g_* \equiv p\beta \mod \mathfrak{m}_x^2. \tag{9.7.2}$$

Indeed, applying (9.4.7) with $\alpha = p\beta$, we obtain the congruence

$$g_{0\,*}(\eta_2, \ldots, \eta_{p-1}) \equiv p^2\beta \mod p^3\mathbb{Z}_p,$$

hence

$$g_*(\eta_2, \ldots, \eta_{p-1}) \equiv p\beta \mod p^2\mathbb{Z}_p \subset \mathfrak{m}_x^2. \tag{9.7.3}$$

On the other hand, equations (9.4.4) show that, for $2 \leq i \leq p-2$,

$$\frac{\partial g_*}{\partial X_i}(\eta_2, \ldots, \eta_{p-1}) = 0. \tag{9.7.4}$$

Finally, equations (9.4.4) and (9.4.6) show that

$$\begin{aligned} \frac{\partial g_*}{\partial X_{p-1}}(\eta_2, \ldots, \eta_{p-1}) &= (\eta_{p-1} - \eta_{p-2})^{p-1} - \eta_{p-1}^{p-1} \\ &= 1 - (p^2\beta - \tilde{u}_{p-1})^{p-1} \\ &\equiv 0 \mod p^2\mathbb{Z}_p \subset \mathfrak{m}_x^2. \end{aligned} \tag{9.7.5}$$

Applying (9.7.3), (9.7.4) and (9.7.5) to the Taylor development of g_* proves (9.7.2).

From the definition of h, we obtain

$$h_*(\eta_2, \ldots, \eta_{p-1}) = -(p^2\beta - \tilde{u}_{p-1}) \prod_{i=1}^{p-2} \tilde{u}_i \equiv \prod_{i=1}^{p-1} \tilde{u}_i \mod \mathfrak{m}_x. \tag{9.7.6}$$

As $h_* \equiv h_*(\eta_2, \ldots, \eta_{p-1}) \mod \mathfrak{m}_x$, f_* satisfies the congruence

$$f_* = g_* + \pi\lambda h_* \equiv \pi(\frac{p}{\pi}\beta + \lambda \prod_{i=1}^{p-1} \tilde{u}_i) \mod \mathfrak{m}_x^2. \tag{9.7.7}$$

Let $w = \frac{p}{\pi}\beta + \lambda \prod_i \tilde{u}_i$. If $K/\mathbb{Q}_p$ is ramified, then condition 9.6 b) implies that w is a unit. If $K/\mathbb{Q}_p$ is unramified, then $\pi = p$, and condition 9.6 a) implies that the reduction mod p of w is non-zero, hence w is again a unit. In each case, $f_* \notin \mathfrak{m}_x^2$, and $\mathcal{O}_{Y,x}$ is regular. $\square$

Appendix: Complete intersection morphisms of virtual relative dimension 0

As mentioned in the introduction, we explain here the construction of the morphism $\tau_f : \mathbb{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$ for a proper complete intersection morphism $f : Y \to X$ of virtual dimension 0, and we give a proof of Theorem 3.1.

The Appendix consists of two sections. In section A, we recall the construction of the invertible sheaf $\omega_{Y/X}$ associated to a complete intersection morphism $f : Y \to X$, and we prove some of its properties. We do not use duality theory here, even if we keep for convenience the terminology "relative dualizing sheaf". Instead, we use the complete intersection assumption to deduce our constructions from the elementary properties of smooth morphisms and regular immersions, thanks to the canonical isomorphisms defined by Conrad [Co00, 2.2]. It is then easy to define the canonical section δ_f of $\omega_{Y/X}$ when f has virtual relative dimension 0, and to prove its basic properties.

In section B, we assume that X is noetherian and has a dualizing complex. We then use duality theory and the identification $\omega_{Y/X} \xrightarrow{\sim} f^!\mathcal{O}_X$ to deduce τ_f from the canonical section δ_f. To translate the properties of δ_f into the properties of τ_f listed in Theorem 3.1, we need to use the fundamental identifications of duality theory, as well as the various compatibilities between these identifications. Our proofs rely in an essential way on Conrad's exposition [Co00].

It may be worth pointing out that we need in this article the compatibility of τ_f with base change in a context which is not covered by the base change results of [Co00]. Indeed, we consider morphisms f which are not flat in general (such as in Theorem 1.5), and base change morphisms which are not flat either (such as reduction mod p^n in the proof of Proposition 8.6). The key property we use here, which is familiar to the experts, but not so well documented in the literature, is the Tor-independence of f and the base change morphism.

A. The canonical section of the relative dualizing sheaf

We recall now the construction of the invertible sheaf $\omega_{Y/X}$ for a complete intersection morphism, and we explain some of its properties. As usual, the main work is to prove that the constructions are well-defined, and in particular to check the sign conventions. As the details are easy but tedious, we leave most of them as exercises, and only sketch the main steps of the verifications.

We first recall a standard base change result for complete intersection morphisms.

Proposition A.1. *Let $f : Y \to X$ be a complete intersection morphism of virtual relative dimension m, and let*

(A.1.1)
$$\begin{array}{ccc} Y' & \xrightarrow{v} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{u} & X \end{array}$$

be a cartesian square such that X' and Y are Tor-independent over X.

(i) *The morphism f' is a complete intersection morphism of virtual relative dimension m.*

(ii) *Assume that X is quasi-compact, and that f is separated of finite type. If $\mathcal{E}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{O}_Y)$ is of finite Tor-dimension over $\mathcal{O}_Y$, then $\mathbb{R}f_*\mathcal{E}^\bullet$ is of finite Tor-dimension over $\mathcal{O}_X$, and the base change morphism*

$$\mathbb{L}u^*\mathbb{R}f_*\mathcal{E}^\bullet \longrightarrow \mathbb{R}f'_*\mathbb{L}v^*\mathcal{E}^\bullet \tag{A.1.2}$$

is an isomorphism.

Proof. The first claim is local on Y', so we may assume that there exists a factorization $f = \pi \circ i$ such that $\pi : P \to X$ is a smooth morphism of relative dimension n, and $i : Y \hookrightarrow P$ is a closed immersion of codimension $d = n - m$. Then i is a regular immersion, defined by an ideal $\mathcal{I} \subset \mathcal{O}_P$, and, since the claim is local, we may assume that $\mathcal{I}$ is generated by a regular sequence $t_1, \dots, t_d$ of sections of $\mathcal{O}_P$. Then the Koszul complex $K_\bullet(t_1, \dots, t_d)$ is a resolution of $\mathcal{O}_Y$ by $\mathcal{O}_P$-modules which are flat relatively to X. Let $P' = X' \times_X P$, and let $t'_1, \dots, t'_d$ be the images of $t_1, \dots, t_d$ in $\mathcal{O}_{P'}$. Since X' and Y are Tor-independent over X, the Koszul complex $K(t'_1, \dots, t'_d)$ is a resolution of $\mathcal{O}_{Y'}$ over $\mathcal{O}_{P'}$, which shows that f' is a complete intersection morphism of virtual relative dimension m.

Assume now that the hypotheses of (ii) are satisfied. Since X is quasi-compact, it suffices to check that $\mathbb{R}f_*\mathcal{E}^\bullet$ is of finite Tor-dimension when X is affine. We can then choose a finite covering $\mathfrak{U}$ of Y by affine open subsets U_α, and we may assume that the U_α are small enough so that the restriction f_α of f to U_α can be factorized as $f_\alpha = \pi_\alpha \circ i_\alpha$, where $\pi_\alpha : P_\alpha \to X$ is smooth and $i_\alpha : U_\alpha \hookrightarrow P_\alpha$ is a closed immersion defined by a regular sequence of sections of $\mathcal{O}_{P_\alpha}$. For each sequence $\alpha_0 < \cdots < \alpha_r$, denote $U_{\underline{\alpha}} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}$, $j_{\underline{\alpha}} : U_{\underline{\alpha}} \hookrightarrow Y$, and let $f_{\underline{\alpha}}$ be the restriction of f to $U_{\underline{\alpha}}$. If $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{E}^\bullet$, then the alternating Čech complex $\check{\mathrm{C}}^\bullet(\mathfrak{U}, \mathcal{I}^\bullet)$ is a resolution of $\mathcal{E}^\bullet$. Since $j_{\underline{\alpha}}$ is an affine open immersion, the complex $j_{\underline{\alpha}*}j^*_{\underline{\alpha}}\mathcal{I}^\bullet = \mathbb{R}j_{\underline{\alpha}*}j^*_{\underline{\alpha}}\mathcal{E}^\bullet$ belongs to $D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_Y)$ for each $\underline{\alpha}$. Therefore it suffices to prove that $\mathbb{R}f_*\mathcal{E}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_X)$ for complexes $\mathcal{E}^\bullet$ of the form $\mathbb{R}j_*\mathcal{F}^\bullet$, where j is the inclusion of an affine open subscheme U, and $\mathcal{F}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_U)$. This reduces the proof to the case where Y is affine. Then there exists a bounded complex of $\mathcal{O}_Y$-modules $\mathcal{P}^\bullet$ with flat quasi-coherent terms, and a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{E}^\bullet$. Since $\mathcal{O}_Y$ has finite Tor-dimension over $\mathcal{O}_X$, so does any flat $\mathcal{O}_Y$-module, and the first assertion of (ii) follows.

The complex $\mathbb{L}v^*\mathcal{E}^\bullet$ belongs to $D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_{Y'})$, and the base change morphism (A.1.2) can be defined by adjunction as usual. Arguing as before, it suffices to prove that it is an isomorphism when X is affine and $\mathcal{E}^\bullet$ is of the form $\mathbb{R}j_*\mathcal{F}^\bullet$, where j is the inclusion of an affine open subscheme $U \subset Y$, and $\mathcal{F}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_U)$. Let $U' = X' \times_X U$, and let $w : U' \to U$ be the projection, $j' : U' \hookrightarrow Y'$ the pullback of j. Since j is an affine morphism and $\mathcal{F}^\bullet \in D^{\mathrm{b}}_{\mathrm{qc,fTd}}(\mathcal{O}_U)$, the base change morphism $\mathbb{L}v^*\mathbb{R}j_*\mathcal{F}^\bullet \to \mathbb{R}j'_*\mathbb{L}w^*\mathcal{F}^\bullet$ is an isomorphism. This implies that the base change morphism for f and $\mathcal{E}^\bullet$ is an isomorphism if and only if the base change morphism for $f \circ j$ and $\mathcal{F}^\bullet$ is an isomorphism. If one chooses a bounded, flat, quasi-coherent resolution $\mathcal{P}^\bullet$ of $\mathcal{F}^\bullet$, the Tor-independence assumption implies that, for each n, $(f \circ j)_*\mathcal{P}^n$ is u^*-acyclic. It follows easily that the base change morphism for $\mathcal{P}^\bullet$ is an isomorphism, which ends the proof. □

A.2. Let $f : Y \to X$ be a complete intersection morphism of relative dimension m. Recall that, if $f = \pi \circ i$ is a factorization of f where $\pi : P \to X$ is a smooth morphism of relative dimension n and $i : Y \hookrightarrow P$ a closed immersion of codimension $d = n - m$, defined by a regular ideal $\mathcal{I} \subset \mathcal{O}_P$, one defines an invertible $\mathcal{O}_Y$-module $\omega_{Y/X}$, called the *relative dualizing sheaf* of Y/X (or f), by setting

$$\text{(A.2.1)} \qquad \begin{aligned} \omega_{Y/X} &= \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^*\omega_{P/X} \\ &= \wedge^d((\mathcal{I}/\mathcal{I}^2)^\vee) \otimes_{\mathcal{O}_Y} i^*\Omega^n_{P/X}. \end{aligned}$$

We also recall how this construction is made independent of the choice of the factorization, up to canonical isomorphism. Let $f = \pi' \circ i'$ be another factorization of f through a smooth morphism $\pi' : P' \to X$, and let $\omega^P_{Y/X}$ and $\omega^{P'}_{Y/X}$ be the invertible $\mathcal{O}_Y$-modules defined by (A.2.1) using the two factorizations. Assume first that there exists an X-morphism $u : P' \to P$ such that $u \circ i' = i$, and which is either a smooth morphism or a regular immersion. Then, one defines an isomorphism $\varepsilon^{P',P}(u) : \omega^P_{Y/X} \xrightarrow{\sim} \omega^{P'}_{Y/X}$ by the commutative diagram

$$\text{(A.2.2)} \qquad \begin{array}{ccc} \omega^P_{Y/X} = \omega_{Y/P} \otimes i^*\omega_{P/X} & \xrightarrow[\sim]{\zeta'_{i',u} \otimes \mathrm{Id}} & \omega_{Y/P'} \otimes i'^*\omega_{P'/P} \otimes i'^*u^*\omega_{P/X} \\ & \overset{\sim}{\underset{\varepsilon^{P',P}(u)}{\searrow}} & \sim \uparrow \mathrm{Id} \otimes i'^*(\zeta'_{u,\pi}) \\ & & \omega_{Y/P'} \otimes i'^*\omega_{P'/X} = \omega^{P'}_{Y/X}. \end{array}$$

The definitions of $\zeta'_{i',u}$ and $\zeta'_{u,\pi}$ depend upon whether u is a smooth morphism or a regular immersion (the two definitions agree when u is an open and closed immersion):

a) If u is smooth, then $\zeta'_{i',u}$ is defined by [Co00, p. 29, (d)], and $\zeta'_{u,\pi}$ is defined by [Co00, p. 29, (a)].

b) If u is a regular immersion, then $\zeta'_{i',u}$ is defined by [Co00, p. 29, (b)], and $\zeta'_{u,\pi}$ is defined by [Co00, p. 29, (c)].

Let $f = \pi'' \circ i''$ be a third factorization of f through a smooth morphism $\pi'' : P'' \to X$, let $\omega^{P''}_{Y/X}$ be defined by (A.2.1) using this factorization, and assume that there exists an X-morphism $v : P'' \to P'$ such that $v \circ i'' = i'$ and such that each of the morphisms v and $u \circ v$ is either a smooth morphism or a regular immersion. Then it follows readily from Conrad's general transitivity relation for compositions of smooth morphisms and regular immersions [Co00, (2.2.4)] that

$$\text{(A.2.3)} \qquad \varepsilon^{P'',P'}(v) \circ \varepsilon^{P',P}(u) = \varepsilon^{P'',P}(u \circ v).$$

If $f = \pi \circ i = \pi' \circ i'$ are any factorizations as above, let now $P'' = P' \times_X P$, and let $i'' : Y \hookrightarrow P''$ be the diagonal immersion, and $q : P'' \to P, q' : P'' \to P'$ the two projections. One defines the canonical isomorphism $\varepsilon^{P',P} : \omega^P_{Y/X} \xrightarrow{\sim} \omega^{P'}_{Y/X}$ by setting

$$\text{(A.2.4)} \qquad \varepsilon^{P',P} := \varepsilon^{P'',P'}(q')^{-1} \circ \varepsilon^{P'',P}(q).$$

Whenever there exists a smooth morphism or a regular immersion $u : P' \to P$ as above, it follows from (A.2.3) that $\varepsilon^{P',P}(u) = \varepsilon^{P',P}$. One checks similarly that the isomorphisms $\varepsilon_{P',P}$ satisfy the usual cocycle condition for three factorizations.

Note that, thanks to this cocycle condition, one can define the invertible sheaf $\omega_{Y/X}$ even when there does not exist a global factorization $f = \pi \circ i$ as above, by choosing local factorizations and glueing the invertible sheaves obtained locally by the previous construction. By construction, the sheaf $\omega_{Y/X}$ commutes with Zariski localization, and is equipped with a canonical isomorphism for which we keep the notation ζ':

$$\zeta'_{\pi,i} : \omega_{Y/X} \xrightarrow{\sim} \omega_{Y/P} \otimes_{\mathcal{O}_Y} i^*\omega_{P/X}, \tag{A.2.5}$$

for any factorization $f = \pi \circ i$ where π is a smooth morphism and i is a regular immersion.

If m is the virtual relative dimension of Y over X, we will need to work with the complex $\omega_{Y/X}[m]$ which is the single $\mathcal{O}_Y$-module $\omega_{Y/X}$ sitting in degree $-m$. If $f = \pi \circ i$ as above, we define in $D^{\mathrm{b}}(\mathcal{O}_Y)$ the isomorphism of complexes

$$\zeta'_{i,\pi} : \omega_{Y/X}[m] \xrightarrow{\sim} \omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}[n]) \tag{A.2.6}$$

by (A.2.5) in degree $-m$, without any sign modification. If f is a smooth morphism or a regular immersion, this definition is consistent with [Co00, (2.2.6)]. By [Co00, (1.3.6)], the isomorphism (A.2.6) is equal to the composed isomorphism

$$\begin{aligned}\omega_{Y/X}[m] \xrightarrow[\text{(A.2.5)}[m]]{\sim} (\omega_{Y/P} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}))[m] &\xrightarrow{\sim} (\omega_{Y/P} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}[n]))[-d] \\ &\xrightarrow{\sim} \omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}[n])\end{aligned}$$

and differs from the composed isomorphism

$$\begin{aligned}\omega_{Y/X}[m] \xrightarrow[\text{(A.2.5)}[m]]{\sim} (\omega_{Y/P} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}))[m] &\xrightarrow{\sim} (\omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}))[n] \\ &\xrightarrow{\sim} \omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}[n])\end{aligned}$$

by multiplication by $(-1)^{dn}$.

Lemma A.3. *Under the assumptions of Proposition A.1, there exists a canonical isomorphism*

$$\mathbb{L}v^*(\omega_{Y/X}) \cong v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}. \tag{A.3.1}$$

Moreover, if the assumptions of Proposition A.1 (ii) are satisfied, the canonical base change morphism

$$\mathbb{L}u^*\mathbb{R}f_*(\omega_{Y/X}) \to \mathbb{R}f'_*(\omega_{Y'/X'}). \tag{A.3.2}$$

is an isomorphism.

Proof. Since $\omega_{Y/X}$ is invertible, $\mathbb{L}v^*(\omega_{Y/X}) \xrightarrow{\sim} v^*(\omega_{Y/X})$. To prove the isomorphism (A.3.1), assume first that there exists a factorization $f = \pi \circ i$ where π is smooth and i is a regular immersion. Let $f' = \pi' \circ i'$ be the factorization deduced from $f = \pi \circ i$ by base change. Then, if $\mathcal{I}$ and $\mathcal{I}'$ are the ideals defining i

and i', the Tor-independence assumption implies that the canonical homomorphism $u^*(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}'/\mathcal{I}'^2$ is an isomorphism, which defines (A.3.1). It is easy to check that, for two factorizations of f, the corresponding isomorphisms are compatible with the identifications (A.2.4). This provides the isomorphism (A.3.1) in the general case.

When the assumptions of A.1 (ii) are satisfied, the isomorphism (A.3.2) follows from (A.3.1) and (A.1.2). □

A.4. Let

$$\begin{array}{ccc} Y' & \xrightarrow{v} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{u} & X \end{array}$$

be a cartesian square, and assume that:

a) f and u are complete intersection morphisms of relative dimensions m and n;

b) X' and Y are Tor-independent over X.

Then Lemma A.3 provides canonical isomorphisms

$$v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}, \qquad f'^*(\omega_{X'/X}) \xrightarrow{\sim} \omega_{Y'/Y}.$$

One defines the canonical isomorphism

$$\chi_{f,u} : \omega_{Y'/Y} \otimes_{\mathcal{O}_{Y'}} v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X}) \tag{A.4.1}$$

as being the product by $(-1)^{mn}$ of the composite

$$\omega_{Y'/Y} \otimes_{\mathcal{O}_{Y'}} v^*(\omega_{Y/X}) \xrightarrow{\sim} f'^*(\omega_{X'/X}) \otimes_{\mathcal{O}_{Y'}} \omega_{Y'/X'} \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X}),$$

where the first isomorphism is the product of the previous base change isomorphisms, and the second one is the usual commutativity isomorphism of the tensor product (see [De83, Appendix, (a)] and [Co00, p. 215-216]).

The following relations follow easily from the local description of the isomorphisms $\zeta'_{f,g}$ given in [Co00, p. 30, (a) - (d)]:

(i) In the above cartesian square, assume that each of the three morphisms u, f and $u \circ f' = f \circ v$ is either a smooth morphism or a regular immersion. Then the following isomorphisms $\omega_{Y'/X} \xrightarrow{\sim} \omega_{Y'/X'} \otimes_{\mathcal{O}_{Y'}} f'^*(\omega_{X'/X})$ are equal:

$$\zeta'_{f',u} = \chi_{f,u} \circ \zeta'_{v,f}. \tag{A.4.2}$$

(ii) Let

$$\begin{array}{ccccc} & & Y' & \overset{v}{\hookrightarrow} & Y \\ & {\scriptstyle j}\nearrow & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X'' & \overset{i}{\hookrightarrow} & X' & \overset{u}{\hookrightarrow} & X \end{array}$$

be a commutative diagram in which the square is cartesian, f is smooth, i and u are regular immersions. Then the following isomorphisms

$$\omega_{X''/X} \xrightarrow{\sim} \omega_{X''/Y'} \otimes_{\mathcal{O}_{X''}} j^*(\omega_{Y'/X'}) \otimes_{\mathcal{O}_{X''}} i^*(\omega_{X'/X})$$

are equal:

$$(\zeta'_{j,f'} \otimes \mathrm{Id}) \circ \zeta'_{i,u} = (\mathrm{Id} \otimes j^*(\chi_{f,u})) \circ (\zeta'_{j,v} \otimes \mathrm{Id}) \circ \zeta'_{vj,f}. \tag{A.4.3}$$

(iii) Let

$$\begin{array}{ccccc} Y'' & \xrightarrow{v'} & Y' & \xrightarrow{v} & Y \\ {\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X'' & \xrightarrow{u'} & X' & \xrightarrow{u} & X \end{array}$$

be a commutative diagram in which both squares are cartesian, each of the morphisms f, u, u' and $u \circ u'$ is either a smooth morphism or a regular immersion, X' and Y are Tor-independent over X, and X'' and Y' are Tor-independent over X' (so that X'' and Y are Tor-independent over X, and all immersions are regular). Then the following isomorphisms

$$\omega_{Y''/Y} \otimes_{\mathcal{O}_{Y''}} (vv')^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y''/X''} \otimes_{\mathcal{O}_{Y''}} f''^*(\omega_{X''/X'} \otimes_{\mathcal{O}_{X''}} u'^*(\omega_{X'/X}))$$

are equal:

$$\text{(A.4.4)} \qquad (\mathrm{Id} \otimes f''^*(\zeta'_{u',u})) \circ \chi_{f,uu'} = (\chi_{f',u'} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes v'^*(\chi_{f,u})) \circ (\zeta'_{v',v} \otimes \mathrm{Id}).$$

We will also need to extend the isomorphism $\chi_{f,u}$ to the derived category. We define

$$\text{(A.4.5)} \qquad \chi_{f,u} : \omega_{Y'/Y}[n] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y'}} \mathbb{L}v^*(\omega_{Y/X}[m]) \xrightarrow{\sim} \omega_{Y'/X'}[m] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y'}} \mathbb{L}f'^*(\omega_{X'/X}[n])$$

by (A.4.1) in degree $-(m+n)$, without any further sign modification. Because of the sign convention in the commutativity isomorphism for the derived tensor product [Co00, p. 11], $\chi_{f,u}$ can also be described as the composite

$$\begin{aligned} \omega_{Y'/Y}[n] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y'}} \mathbb{L}v^*(\omega_{Y/X}[m]) &\xrightarrow{\sim} \mathbb{L}f'^*(\omega_{X'/X}[n]) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y'}} \omega_{Y'/X'}[m] \\ &\xrightarrow{\sim} \omega_{Y'/X'}[m] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y'}} \mathbb{L}f'^*(\omega_{X'/X}[n]), \end{aligned}$$

where the first isomorphism is the tensor product of the base change isomorphisms, and the second one is the commutativity isomorphism. With this definition, the previous relations (A.4.2) to (A.4.4) remain valid in $D^{\mathrm{b}}(\mathcal{O}_{Y'})$.

A.5. We now consider the composition of two complete intersection morphisms $f : Y \to X$, $g : Z \to Y$, and define a canonical isomorphism

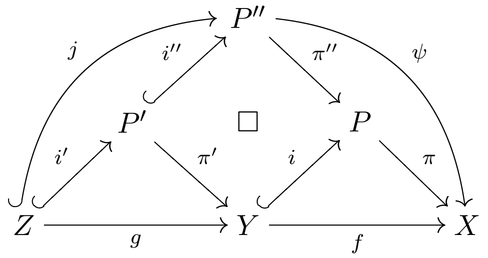
$$\text{(A.5.1)} \qquad \zeta'_{g,f} : \omega_{Z/X} \xrightarrow{\sim} \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^*(\omega_{Y/X})$$

extending the isomorphism (A.2.5).

Assume first that there exists a factorization $f = \pi \circ i$, where $\pi : P \to X$ is a smooth morphism, and a factorization $i \circ g = \pi'' \circ j$, where $\pi'' : P'' \to P$ is a smooth morphism (such factorizations always exist when X, Y and Z are affine).

Let $\pi' : P' \to Y$ be the pull-back of π'', so that we get a commutative diagram

(A.5.2)



where the middle square is cartesian. Using (A.2.5) for (j, ψ), and the isomorphisms $\zeta'_{i',i''} \otimes j^*(\zeta'_{\pi'',\pi})$, we obtain an isomorphism

$$\begin{aligned}\omega_{Z/X} \;\cong\; \omega_{Z/P''} \otimes j^*(\omega_{P''/X}) &\xrightarrow{\sim} (\omega_{Z/P'} \otimes i'^*(\omega_{P'/P''})) \otimes j^*(\omega_{P''/P} \otimes \pi''^*(\omega_{P/X}))\\ &\xrightarrow{\sim} \omega_{Z/P'} \otimes i'^*(\omega_{P'/P''} \otimes i''^*(\omega_{P''/P})) \otimes g^* i^*(\omega_{P/X}).\end{aligned}$$

Using the isomorphism

$$\chi_{\pi'',i} : \omega_{P'/P''} \otimes i''^*(\omega_{P''/P}) \xrightarrow{\sim} \omega_{P'/Y} \otimes \pi'^*(\omega_{Y/P})$$

defined in A.4, and $(\zeta'_{i',\pi'} \otimes g^*(\zeta'_{i,\pi}))^{-1}$, we then obtain the composed isomorphism

$$\begin{aligned}\omega_{Z/X} &\xrightarrow{\sim} \omega_{Z/P'} \otimes i'^*(\omega_{P'/Y} \otimes \pi'^*(\omega_{Y/P})) \otimes g^* i^*(\omega_{P/X})\\ &\xrightarrow{\sim} (\omega_{Z/P'} \otimes i'^*(\omega_{P'/Y})) \otimes g^*(\omega_{Y/P} \otimes i^*(\omega_{P/X}))\\ &\xrightarrow{\sim} \omega_{Z/Y} \otimes g^*(\omega_{Y/X}),\end{aligned}$$

which defines (A.5.1).

To prove that this isomorphism is well defined, and to glue the local constructions to obtain a global one when a diagram (A.5.2) does not exist globally, we must check that it does not depend on the chosen factorizations. If we have two diagrams (A.5.2), with factorizations $f = \pi_k \circ i_k$, $i_k \circ g = \pi''_k \circ j_k$, for $k = 1, 2$, we can embed Y diagonally in $P_1 \times_X P_2$, and Z in $P''_1 \times_X P''_2$. This allows to reduce the verification to the case where there exists a smooth X-morphism $u : P_2 \to P_1$ such that $u \circ i_2 = i_1$, and a smooth morphism $u'' : P''_2 \to P''_1$ such that $\pi''_1 \circ u'' = u \circ \pi''_2$, and $j_1 = u'' \circ j_2$. Morever, the same argument shows that we may assume that the morphism $P''_2 \to P''_1 \times_{P_1} P_2$ is smooth. The verification can then be reduced to the following two cases:

(i) The morphism $P''_2 \to P''_1 \times_{P_1} P_2$ is an isomorphism;

(ii) The morphism $P_2 \to P_1$ is an isomorphism.

In each of these cases, the equality of the two definitions of (A.5.1) breaks down to a succession of elementary commutative diagrams involving isomorphisms of the form $\zeta'_{f,g}$ and $\chi_{f,u}$. We omit details here, and only point out that, in addition to [Co00, (2.2.4)], the first case uses relation (A.4.2), and the second one uses relation (A.4.3). In particular, the sign convention introduced in the definition of $\chi_{f,u}$ in A.4 is necessary for this independence result.

If m and m' are the virtual relative dimensions of f and g, we define as in A.2 the derived category variant of (A.5.1) as being the morphism

$$\zeta'_{g,f} : \omega_{Z/X}[m+m'] \xrightarrow{\sim} \omega_{Z/Y}[m'] \otimes^{\mathbb{L}}_{\mathcal{O}_Z} \mathbb{L}f^*(\omega_{Y/X}[m]) \tag{A.5.3}$$

defined by applying (A.5.1) to the underlying modules (sitting in degree $-m-m'$), without any sign modification.

With the definition of $\zeta'_{g,f}$ provided by (A.5.1) (resp. (A.5.3)), we now extend to complete intersection morphisms Conrad's transitivity relation [Co00, (2.2.4)].

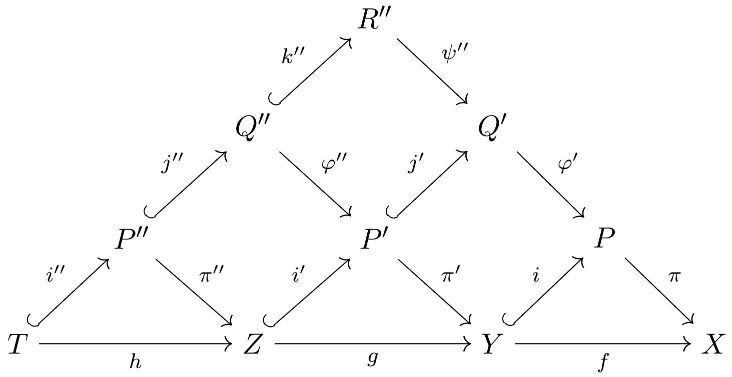
Proposition A.6. *Let*

$$T \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$$

be three complete intersection morphisms. Then

$$(\mathrm{Id} \otimes h^*(\zeta'_{g,f})) \circ \zeta'_{h,fg} = (\zeta'_{h,g} \otimes \mathrm{Id}) \circ \zeta'_{gh,f}. \tag{A.6.1}$$

Proof. As the verification is local on T, we may assume that there exists a commutative diagram



in which the three squares are cartesian, the morphisms π, φ', ψ'' are smooth, and the immersions i, i', i'' are regular. Using [Co00, (2.2.4)] and the relation (A.4.4), the proof of (A.6.1) again breaks into a succession of elementary commutative diagrams, which we do not detail here. □

A.7. We now assume that $f : Y \to X$ is a complete intersection morphism of (virtual) relative dimension 0, and, under this hypothesis, we define a section $\delta_f \in \Gamma(Y, \omega_{Y/X})$, which we call the *canonical section.*

We first assume that there is a factorization $f = \pi \circ i$ such that $\pi : P \to X$ is a smooth morphism of relative dimension n, and $i : Y \hookrightarrow P$ is a regular closed immersion, necessarily of codimension n since f has relative dimension 0. Let $\mathcal{I} \subset \mathcal{O}_P$ be the ideal defining i. The canonical derivation $d : \mathcal{O}_P \to \Omega^1_{P/X}$ induces an $\mathcal{O}_Y$-linear homomorphism $\bar{d} : \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{P/X}$. Taking its n-th exterior power, we obtain a linear homomorphism

$$\wedge^n \bar{d} : \wedge^n(\mathcal{I}/\mathcal{I}^2) \longrightarrow i^*\Omega^n_{P/X}. \tag{A.7.1}$$

Through the canonical isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_Y}(\wedge^n(\mathcal{I}/\mathcal{I}^2), i^*\Omega^n_{P/X}) &\cong (\wedge^n(\mathcal{I}/\mathcal{I}^2))^\vee \otimes_{\mathcal{O}_Y} i^*\Omega^n_{P/X}\\ &\cong \wedge^n((\mathcal{I}/\mathcal{I}^2)^\vee) \otimes_{\mathcal{O}_Y} i^*\Omega^n_{P/X}\\ &= \omega_{Y/X},\end{aligned}$$

it can be seen as a section of $\omega_{Y/X}$, which is the section δ_f. If $(t_1, \dots, t_n)$ is a regular sequence of generators of $\mathcal{I}$ on a neighbourhood U of some point $y \in Y$, then

(A.7.2) $$\delta_f = (\bar{t}_1^\vee \wedge \dots \wedge \bar{t}_n^\vee) \otimes i^*(dt_n \wedge \dots \wedge dt_1) \ \in \Gamma(U, \omega_{Y/X}),$$

since the canonical isomorphism $(\wedge^n(\mathcal{I}/\mathcal{I}^2))^\vee \cong \wedge^n((\mathcal{I}/\mathcal{I}^2)^\vee)$ maps $(\bar{t}_n \wedge \dots \wedge \bar{t}_1)^\vee$ to $\bar{t}_1^\vee \wedge \dots \wedge \bar{t}_n^\vee$.

To end the construction of δ_f, it suffices to check that the section obtained in this way does not depend on the chosen factorization. Using the diagonal embedding, it suffices as usual to compare the sections δ_f and δ'_f defined by two factorizations $f = \pi \circ i = \pi' \circ i'$ when there exists a smooth X-morphism $u : P' \to P$ such that $u \circ i' = i$. Let $\mathcal{I}'$ be the ideal of Y in P', and

$$\omega'_{Y/X} = \wedge^{n'}((\mathcal{I}'/\mathcal{I}'^2)^\vee) \otimes_{\mathcal{O}_Y} i'^*\Omega^{n'}_{P'/X},$$

where n' is the codimension of Y in P'. Then the canonical identification $\omega_{Y/X} \cong \omega'_{Y/X}$ is given by (A.2.2), case a), and, thanks to (A.7.2), the equality $\delta_f = \delta'_f$ follows from [Co00, p. 30, (a) and (d)].

Proposition A.8. *Let $f : Y \to X$ be a complete intersection morphism of virtual relative dimension* 0.

(i) *Let $g : Z \to Y$ be a second complete intersection morphism of virtual relative dimension* 0*. The image of δ_{fg} under the isomorphism $\zeta'_{g,f}$ defined in* (A.5.1) *is given by*

(A.8.1) $$\zeta'_{g,f}(\delta_{fg}) = \delta_g \otimes g^*(\delta_f).$$

(ii) *For any cartesian square* (A.1.1)*, the isomorphism* (A.3.1)

$$v^*(\omega_{Y/X}) \xrightarrow{\sim} \omega_{Y'/X'}$$

maps $v^*(\delta_f)$ to $\delta_{f'}$.

Proof. As the first claim is local on Z, we may assume that there exists a diagram (A.5.2) in which the immersion i is defined by a regular sequence $(t_1, \dots, t_n)$, and the immersion $j = i'' \circ i'$ by a regular sequence $(t'_1, \dots, t'_{n'}, t''_1, \dots, t''_n)$, with $t''_i = \pi''^*(t_i)$. If we set $\bar{t}'_i = i''^*(t'_i)$, then i' is defined by the regular sequence $(\bar{t}'_1, \dots, \bar{t}'_{n'})$. By construction, δ_{fg} corresponds by $\zeta'_{j,\psi}$ to the section

$$(t''^\vee_n \wedge \dots \wedge t''^\vee_1 \wedge t'^\vee_{n'} \wedge \dots \wedge t'^\vee_1) \otimes j^*(dt'_1 \wedge \dots \wedge dt'_{n'} \wedge dt''_1 \wedge \dots \wedge dt''_n)$$

of $\omega_{Z/P''} \otimes j^*(\omega_{P''/X})$, which is mapped by $\zeta'_{i',i''} \otimes j^*(\zeta'_{\pi'',\pi})$ to the section

$$((-1)^{nn'}(\bar{t}'^\vee_{n'} \wedge \dots \wedge \bar{t}'^\vee_1) \otimes i'^*(t''^\vee_n \wedge \dots \wedge t''^\vee_1)) \otimes j^*((dt'_1 \wedge \dots \wedge dt'_{n'}) \otimes \pi''^*(dt_1 \wedge \dots \wedge dt_n))$$

of $(\omega_{Z/P'} \otimes i'^*(\omega_{P'/P''})) \otimes j^*(\omega_{P''/P} \otimes \pi''^*(\omega_{P/X}))$. We then get via $\chi_{\pi'',i}$ the section

$$(\bar{t}'^\vee_{n'} \wedge \dots \wedge \bar{t}'^\vee_1) \otimes i'^*(d\bar{t}'_1 \wedge \dots \wedge d\bar{t}'_{n'}) \otimes i'^*\pi'^*(t_n^\vee \wedge \dots \wedge t_1^\vee) \otimes j^*\pi''^*(dt_1 \wedge \dots \wedge dt_n)),$$

of $\omega_{Z/P'} \otimes i'^*(\omega_{P'/Y}) \otimes i'^*\pi'^*(\omega_{Y/P}) \otimes j^*\pi''^*(\omega_{P/X})$, which, by construction, corresponds by $(\zeta'_{i',\pi'} \otimes g^*(\zeta'_{i,\pi}))^{-1}$ to the section $\delta_g \otimes g^*(\delta_f)$ of $\omega_{Z/Y} \otimes g^*(\omega_{Y/X})$.

The second claim follows from (A.7.2). □

B. The trace morphism τ_f on $\mathbb{R}f_*(\mathcal{O}_Y)$

Let $f : Y \to X$ be a proper complete intersection morphism of virtual relative dimension 0. This section is devoted to the construction of the "trace morphism" $\tau_f : \mathbb{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$, derived from the canonical section of $\omega_{Y/X}$ defined in A.7. The key step is to define an identification λ_f between $\omega_{Y/X}$ as defined in A.2, and $f^!\mathcal{O}_X$. The construction is then a straightforward application of the relative duality theorem, and the properties of τ_f listed in Theorem 3.1 follow from corresponding properties of δ_f and λ_f.

B.1. For the whole section, we assume that X is a noetherian scheme with a dualizing complex. Let $f : Y \to X$ be a complete intersection morphism of virtual relative dimension m. We first explain the relation between the relative dualizing module $\omega_{Y/X}$ defined in the previous section, and the extraordinary inverse image functor $f^!$ defined in [Ha66, VII 3.4] and [Co00, 3.3].

Let $r \in \mathbb{Z}$ be an integer, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and $\mathcal{E} = \mathcal{L}[r] \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$. We define

$$f^\sharp(\mathcal{E}) := \omega_{Y/X}[m] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*(\mathcal{E}), \tag{B.1.1}$$

and we observe that $f^\sharp(\mathcal{E})$ is another complex concentrated in a single degree, with an invertible cohomology sheaf. We can then construct a canonical isomorphism

$$\lambda_{f,\mathcal{E}} : f^\sharp(\mathcal{E}) \xrightarrow{\sim} f^!(\mathcal{E}) \tag{B.1.2}$$

as follows.

a) If f is smooth, then the functor $f^\sharp$ defined above is the usual one, and we set

$$\lambda_{f,\mathcal{E}} = e_f : f^\sharp(\mathcal{E}) \xrightarrow{\sim} f^!(\mathcal{E}), \tag{B.1.3}$$

where e_f is the isomorphism defined by [Co00, (3.3.21)].

b) If f is a regular immersion, then we define $\lambda_{f,\mathcal{E}}$ to be the composition

$$\lambda_{f,\mathcal{E}} : f^\sharp(\mathcal{E}) \underset{\sim}{\xrightarrow{\eta_f^{-1}}} f^\flat(\mathcal{E}) \underset{\sim}{\xrightarrow{d_f}} f^!(\mathcal{E}), \tag{B.1.4}$$

where η_f is defined by [Co00, (2.5.3)] and d_f by [Co00, (3.3.19)].

c) In the general case, let us assume first that there exists a factorization $f = \pi \circ i$, where $\pi : P \to X$ is a smooth morphism of relative dimension n, and i is a regular immersion of codimension $d = n - m$. Then we define $\lambda_{f,\mathcal{E}}$ by the commutative diagram

$$\begin{array}{ccc} \omega_{Y/X}[m] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E} & \xrightarrow[\sim]{\quad\lambda_{f,\mathcal{E}}\quad} & f^!\mathcal{E} \\ {\scriptstyle \zeta'_{i,\pi}\otimes\mathrm{Id}}\downarrow\wr & & \wr\downarrow{\scriptstyle c_{i,\pi}} \\ \omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*\pi^\sharp\mathcal{E} \xrightarrow[\sim]{\lambda_{i,\pi^\sharp\mathcal{E}}} i^!\pi^\sharp\mathcal{E} & \xrightarrow[\sim]{i^!(\lambda_{\pi,\mathcal{E}})} & i^!\pi^!\mathcal{E} \end{array} \tag{B.1.5}$$

where $c_{i,\pi}$ is the transitivity isomorphism [Co00, (3.3.14)].

This isomorphism is actually independent of the chosen factorization. To check it, one can argue as in A.2 to reduce the comparison between the isomorphisms (B.1.2) defined by two factorizations $f = \pi \circ i = \pi' \circ i'$ to the case where there is a smooth X-morphism $u : P' \to P$ such that $u \circ i' = i$. It is then a long but straightforward verification, using various functorialities, the compatibility between $\zeta'_{i',u}$ and the isomorphism $i^\flat \simeq i'^\flat u^\sharp$ [Co00, (2.7.4)], the compatibility between $\zeta'_{u,\pi}$ and the isomorphism $\pi'^\sharp \simeq u^\sharp \pi^\sharp$ [Co00, (2.2.7)], and the properties (VAR1), (VAR3) and (VAR5) of the functor $f^!$ (see [Ha66, III, Th. 8.7] and [Co00, p. 139]).

Since $f^!\mathcal{O}_X$ is acyclic outside degree $-m$, a morphism $\omega_{Y/X}[m] \to f^!\mathcal{O}_X$ in $D(\mathcal{O}_Y)$ is simply a module homomorphism $\omega_{Y/X} \to \mathcal{H}^{-m}(f^!\mathcal{O}_X)$. Therefore, the previous construction provides in the general case local isomorphisms which can be glued to define a global isomorphism even if there does not exist a global factorization $f = \pi \circ i$ as above.

When $\mathcal{E} = \mathcal{O}_X$, the isomorphism (B.1.2) will simply be denoted

$$\lambda_f : \omega_{Y/X}[m] \xrightarrow{\sim} f^!\mathcal{O}_X. \tag{B.1.6}$$

If f is flat, hence is a CM map, it provides the identification between the construction of $\omega_{Y/X}$ used in this article, and the construction of Conrad for CM maps [Co00, 3.5, p. 157].

We now give for the isomorphisms $\lambda_{f,\mathcal{E}}$ a transitivity property which generalizes (B.1.5).

Proposition B.2. *Let $g : Z \to Y$ be a second complete intersection morphism, with virtual relative dimension m'. Then the diagram*

(B.2.1)

$$\begin{array}{ccccc}
\omega_{Z/X}[m+m'] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} \mathbb{L}(fg)^*\mathcal{E} & & \xrightarrow[\sim]{\lambda_{fg,\mathcal{E}}} & & (fg)^!\mathcal{E} \\
\zeta'_{g,f}\otimes \mathrm{Id} \downarrow \wr & & & & \wr \downarrow c_{g,f} \\
\omega_{Z/Y}[m'] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} \mathbb{L}g^* f^\sharp\mathcal{E} & \xrightarrow[\sim]{\lambda_{g,f^\sharp\mathcal{E}}} & g^! f^\sharp \mathcal{E} & \xrightarrow[\sim]{g^!(\lambda_{f,\mathcal{E}})} & g^! f^! \mathcal{E} \\
\Vert & & & & \Vert \\
\omega_{Z/Y}[m'] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} \mathbb{L}g^* f^\sharp\mathcal{E} & \xrightarrow[\sim]{g^\sharp(\lambda_{f,\mathcal{E}})} & \omega_{Z/Y}[m'] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} \mathbb{L}g^* f^!\mathcal{E} & \xrightarrow[\sim]{\lambda_{g,f^!\mathcal{E}}} & g^! f^! \mathcal{E}
\end{array}$$

commutes.

Proof. The commutativity of the lower part of the diagram is due to the functoriality of the isomorphism λ_g with respect to morphisms between two complexes concentrated in the same degree.

We first observe that the commutativity of (B.2.1) is clear in the following cases:

a) If f is smooth and g is a closed immersion, the diagram is (B.1.5), which commutes by construction.

b) If f and g are smooth, the isomorphism $(fg)^\sharp \cong g^\sharp f^\sharp$ is defined by $\zeta'_{g,f}$, hence the commutativity of (B.2.1) is the compatibility of the isomorphisms e_f with composition, i.e., property (VAR3) of the functor $f^!$ [Co00, p. 139].

c) If f and g are regular immersions, then isomorphisms such as η_{fg} commute with $\zeta'_{g,f}$ and $c_{g,f}$ [Co00, Th. 2.5.1], and the commutativity of (B.2.1) follows from the compatibility of the isomorphisms d_f with composition, i.e., property (VAR2) of the functor $f^!$ [Co00, p. 139].

We will also use the following remark. Let $h : T \to Z$ be a third complete intersection morphism, yielding the four couples of composable complete intersection morphisms (h, g), (g, f), (gh, f) and (h, fg). Then, if the diagrams (B.2.1) for the couples (h, g) and (g, f) are commutative, the commutativity of (B.2.1) for (gh, f) is equivalent to the commutativity of (B.2.1) for (h, fg): this is a consequence of (A.6.1) and of the compatibility of the isomorphisms $c_{g,f}$ with triple composites (i.e., property (VAR1) of the functor $f^!$ [Co00, p. 139]).

In the general case, the complexes entering in (B.2.1) are concentrated in the same degree, hence its commutativity can be checked locally. So we may assume that there exists a diagram (A.5.2). Thanks to the three particular cases listed above, one can then deduce the commutativity of (B.2.1) for (f, g) from the commutativity of (B.2.1) for (π', i), by applying the previous remark successively to the triples (i', i'', π''), $(i''i', \pi'', \pi)$, (i', π', i) and (g, i, π).

To prove the commutativity of (B.2.1) for (π', i), we use the factorization $i \circ \pi' = \pi'' \circ i''$ to define $\lambda_{i\pi',\mathcal{E}}$. Let d be the codimension of Y in P, and n' the relative dimension of P'' over P. Then, if $\mathcal{E}$ is a complex on P as in B.1, (B.2.1) for (π', i) is made of the exterior composites in the diagram

(B.2.2)

$$
\begin{array}{ccc}
 & \omega_{P'/P}[n'-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}(i\pi')^*\mathcal{E} & \\
 \swarrow {\scriptstyle \zeta'_{\pi',i}\otimes \mathrm{Id}},\ \sim & & \sim,\ {\scriptstyle \zeta'_{i'',\pi''}\otimes \mathrm{Id}} \searrow \\
\omega_{P'/Y}[n'] \overset{\mathbb{L}}{\otimes} \pi'^*(\omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}i^*\mathcal{E}) & \xrightarrow[\sim]{\chi_{\pi'',i}\otimes \mathrm{Id}} & \omega_{P'/P''}[-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}i''^*(\omega_{P''/P}[n'] \overset{\mathbb{L}}{\otimes} \pi''^*\mathcal{E}) \\
\sim \big\downarrow {\scriptstyle \pi'^\sharp(\eta_i^{-1})} & & {\scriptstyle \eta_{i''}^{-1}} \big\downarrow \sim \\
\pi'^\sharp i^\flat \mathcal{E} & \xrightarrow{\sim} & i''^\flat \pi''^\sharp \mathcal{E} \\
\sim \big\downarrow {\scriptstyle \pi'^\sharp(d_i)} & & {\scriptstyle d_{i''}} \big\downarrow \sim \\
\pi'^\sharp i^! \mathcal{E} & & i''^! \pi''^\sharp \mathcal{E} \\
\sim \big\downarrow {\scriptstyle e_{\pi'} = \lambda_{\pi', i^!\mathcal{E}}} & & {\scriptstyle i''^!(\lambda_{\pi'',\mathcal{E}}) = i''^!(e_{\pi''})} \big\downarrow \sim \\
\pi'^! i^! \mathcal{E} \xleftarrow[c_{\pi',i}]{\sim} & (i\pi')^!\mathcal{E} = (\pi''i'')^!\mathcal{E} & \xrightarrow[c_{i'',\pi''}]{\sim} i''^! \pi''^! \mathcal{E}.
\end{array}
$$

$$\pi'^\sharp(\lambda_{i,\mathcal{E}}) : \omega_{P'/Y}[n'] \overset{\mathbb{L}}{\otimes} \pi'^*(\omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}i^*\mathcal{E}) \to \pi'^\sharp i^! \mathcal{E}, \qquad \lambda_{i'',\pi''^\sharp\mathcal{E}} : \omega_{P'/P''}[-d] \overset{\mathbb{L}}{\otimes} \mathbb{L}i''^*(\omega_{P''/P}[n'] \overset{\mathbb{L}}{\otimes} \pi''^*\mathcal{E}) \to i''^! \pi''^\sharp \mathcal{E}$$

Here, the middle horizontal arrow is the standard isomorphism [Co00, Lemma 2.7.3], and the lower rectangle commutes thanks to property (VAR4) of the functor $f^!$ [Co00, Theorem 3.3.1]. The upper triangle commutes thanks to (A.4.2). To check the comutativity of the middle rectangle, one observes on the one hand that η_i commutes with the flat base change π'' and that $\eta_{i''}$ commutes with tensorisation by the invertible sheaf $\omega_{P''/P}$ (see [Co00, last paragraph of p. 54]). On the other hand, $\eta_{i''}$ commutes also with the translation by n', provided that the convention [Co00, (1.3.6)] is used for the commutation of the tensor product with translations applied to the second argument (see the discussion on [Co00, p. 53]). This requires here multiplication by $(-1)^{dn'}$ on $\omega_{P'/P''} \otimes i''^{*}\omega_{P''/P}$, since $\omega_{P'/P''}$ sits in degree d. As this is the sign entering in the definition of $\chi_{\pi'',i}$, this ends the proof. $\square$

B.3. Assume now that f is proper. As in B.1, let $\mathcal{E} = \mathcal{L}[r] \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$, $\mathcal{L}$ being an invertible $\mathcal{O}_X$-module and r an integer. Using (B.1.2), we can define the trace morphism $\mathrm{Tr}^{\sharp}_{f,\mathcal{E}}$ on $\mathbb{R}f_*f^{\sharp}\mathcal{E} = \mathbb{R}f_*(\omega_{Y/X}[m] \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E})$ as the composite

$$\mathrm{Tr}^{\sharp}_{f,\mathcal{E}} : \mathbb{R}f_*f^{\sharp}\mathcal{E} \xrightarrow[\sim]{\mathbb{R}f_*(\lambda_{f,\mathcal{E}})} \mathbb{R}f_*f^{!}\mathcal{E} \xrightarrow{\mathrm{Tr}_f} \mathcal{E}, \tag{B.3.1}$$

where Tr_f denotes the classical trace morphism defined in [Ha66, VII, Cor. 3.4] and [Co00, 3.4]. When $\mathcal{E} = \mathcal{O}_X$, we will use the shorter notation

$$\mathrm{Tr}^{\sharp}_{f} : \mathbb{R}f_*(\omega_{Y/X}[m]) \to \mathcal{O}_X. \tag{B.3.2}$$

We first give some basic properties of the morphism $\mathrm{Tr}^{\sharp}_{f}$.

Lemma B.4. *With the previous hypotheses, let*

$$\mu_{f,\mathcal{E}} : \mathbb{R}f_*(\omega_{Y/X}[m]) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathbb{R}f_*(\omega_{Y/X}[m] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E}) = \mathbb{R}f_*f^{\sharp}\mathcal{E} \tag{B.4.1}$$

be the isomorphism given by the projection formula [Ha66, II, Prop. 5.6]. *Then the diagram*

$$\begin{array}{ccc} \mathbb{R}f_*(\omega_{Y/X}[m]) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\sim]{\mu_{f,\mathcal{E}}} & \mathbb{R}f_*f^{\sharp}\mathcal{E} \\ & {\scriptstyle \mathrm{Tr}^{\sharp}_f\otimes\mathrm{Id}}\searrow\sim \quad \sim\swarrow{\scriptstyle \mathrm{Tr}^{\sharp}_{f,\mathcal{E}}} & \\ & \mathcal{E} & \end{array} \tag{B.4.2}$$

commutes.

Proof. When f is flat, it suffices to invoke [Co00, Th. 4.4.1]. Since we make no such assumption on f, we give a direct argument, which is made a lot simpler by the very special nature of the complex $\mathcal{E}$.

Let K be a residual complex on X, and let $f^{\Delta}K$ be its inverse image on Y in the sense of residual complexes, which is a residual complex on Y. Then K and $f^{\Delta}K$ define respectively duality δ-functors D_X on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ and D_Y on $D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$. Recall that, by definition, $f^! = D_Y \circ \mathbb{L}f^* \circ D_X$. Using the fact that $\mathcal{E} = \mathcal{L}[r]$, with $\mathcal{L}$ invertible, one easily sees that there is a canonical isomorphism which commutes with translations acting on $\mathcal{E}$

$$f^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E} \xrightarrow{\sim} f^!\mathcal{E}. \tag{B.4.3}$$

On the other hand, we have by definition a canonical isomorphism

$$f^{\sharp}\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E} \xrightarrow{\sim} f^{\sharp}\mathcal{E}, \tag{B.4.4}$$

which also commutes with translations. A first observation is that the diagram

$$\begin{array}{ccc} f^{\sharp}\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E} & \xrightarrow[\sim]{\text{(B.4.4)}} & f^{\sharp}\mathcal{E} \\ {\scriptstyle \lambda_f\otimes\mathrm{Id}}\downarrow\wr & & \wr\downarrow{\scriptstyle \lambda_{f,\mathcal{E}}} \\ f^{!}\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E} & \xrightarrow[\sim]{\text{(B.4.3)}} & f^{!}\mathcal{E} \end{array} \tag{B.4.5}$$

commutes. Indeed, all complexes are concentrated in the same degree $m - r$, hence the verification can be done locally. This allows to assume that $\mathcal{L} = \mathcal{O}_X$, which reduces the verification to the commutation of the vertical arrows with translations acting on $\mathcal{E}$. This now follows from the fact that the isomorphisms e_f, η_f and d_f used in the construction of λ_f commute with translations.

Applying $\mathbb{R}f_*$ to this diagram, and using the functoriality of the projection formula isomorphism, the proof is reduced to proving the commutativity of the diagram

(B.4.6)
$$\begin{array}{ccccc} \mathbb{R}f_*f^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{E} & \xrightarrow[\sim]{\nu_f} & \mathbb{R}f_*(f^!\mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{E}) & \xrightarrow[\sim]{(B.4.3)} & \mathbb{R}f_*f^!\mathcal{E} \\ & \searrow^{\sim}_{\mathrm{Tr}_f\otimes\mathrm{Id}} & & \swarrow^{\sim}_{\mathrm{Tr}_f} & \\ & & \mathcal{E} & & \end{array},$$

where ν_f is the projection formula isomorphism. As all morphisms of the diagram commute with translations, we may assume that $r = 0$. We recall that Tr_f is defined as the morphism of functors defined by the composite

$$\mathbb{R}f_*f^!(\cdot) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(D_X(\cdot), f_*f^{\Delta}K) \xrightarrow{\mathrm{Tr}_{f,K}} \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(D_X(\cdot), K) \xleftarrow{\sim} \mathrm{Id},$$

where the first isomorphism follows from the definition of $f^!$ and the adjunction between $\mathbb{L}f^*$ and $\mathbb{R}f_*$, the second morphism is defined by the trace morphism for residual complexes $\mathrm{Tr}_{f,K}$ and the last isomorphism is the local biduality isomorphism (see [Co00, p. 146]). Each of these morphisms has a natural compatibility with respect to the tensor product of the argument by an invertible sheaf. Putting together these compatibilities yields the commutativity of (B.4.6). □

Proposition B.5. *Let $g : Z \to Y$ be a second proper complete intersection morphism, with virtual relative dimension m'. Then the diagram*
(B.5.1)
$$\begin{array}{ccc} \mathbb{R}f_*\mathbb{R}g_*(\omega_{Z/X}[m'+m]) & \xrightarrow[\sim]{\mathbb{R}f_*\mathbb{R}g_*(\zeta'_{g,f})} & \mathbb{R}f_*\mathbb{R}g_*(\omega_{Z/Y}[m'] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Z} \mathbb{L}g^*(\omega_{Y/X}[m])) \\ & & \downarrow{\scriptstyle\sim} \\ \downarrow{\scriptstyle \mathrm{Tr}^\sharp_{fg}} & & \mathbb{R}f_*(\mathbb{R}g_*(\omega_{Z/Y}[m']) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \omega_{Y/X}[m]) \\ & & \downarrow{\scriptstyle \mathbb{R}f_*(\mathrm{Tr}^\sharp_g\otimes\mathrm{Id})} \\ \mathcal{O}_X & \xleftarrow{\mathrm{Tr}^\sharp_f} & \mathbb{R}f_*(\omega_{Y/X}[m]) \end{array}$$

(*where the second isomorphism is given by the projection formula*) *is commutative.*

Proof. It follows from Lemma B.4 that the right vertical arrow is equal to the morphism

$$\mathbb{R}f_*\mathbb{R}g_*(\omega_{Z/Y}[m'] \otimes g^*(\omega_{Y/X}[m])) \xrightarrow{\mathrm{Tr}^\sharp_{g,\omega_{Y/X}[m]}} \mathbb{R}f_*(\omega_{Y/X}[m]).$$

Then, using adjunction between $\mathbb{R}f_*$ and $f^!$, and adjunction between $\mathbb{R}g_*$ and $g^!$, one sees that the commutativity of (B.5.1) is equivalent to the commutativity of (B.2.1). □

Proposition B.6. *With the hypotheses of Proposition A.1, assume in addition that X and X' are noetherian schemes with dualizing complexes, and that one of the following conditions is satisfied:*

a) *f is projective;*
b) *f is proper and u is residually stable* [Co00, p. 132]*;*
c) *f is proper and flat.*

Then the triangle

(B.6.1)
$$\begin{array}{ccc} \mathbb{L}u^*\mathbb{R}f_*(\omega_{Y/X}[m]) & \xrightarrow{\mathbb{L}u^*(\mathrm{Tr}^\sharp_f)} & \\ {\scriptstyle (A.3.2)}\ \downarrow \wr & & \mathcal{O}_{X'} \\ \mathbb{R}f'_*(\omega_{Y'/X'}[m]) & \xrightarrow{\mathrm{Tr}^\sharp_{f'}} & \end{array}$$

is commutative.

Proof of Case a). We can choose a factorization $f = \pi \circ i$, where $\pi : P \to X$ is the structural morphism of some projective space $P = \mathbb{P}^n_X$ over X, and i is a regular immersion of codimension $d = n - m$. Let $f' = \pi' \circ i'$ be the factorisation of f' defined by base change, with $\pi' : P' = \mathbb{P}^n_{X'} \to X'$, and let $w : P' \to P$ be the projection.

The isomorphisms $\zeta'_{i,\pi}$ and $\zeta'_{i',\pi'}$ are clearly compatible with the base change isomorphisms (A.3.1) relative to f and u, and the same holds for the projection formula isomorphisms $\mu_{i,\omega_{P/X}[n]}$ and $\mu_{i',\omega_{P'/X'}[n]}$, and the base change isomorphisms (A.3.1) relative to i and w. Then, using Proposition B.5, one sees that it suffices to prove the proposition for $f = i$ and for $f = \pi$.

When $f = \pi : \mathbb{P}^n_X \to X$, let $X_0, \dots, X_n$ be the canonical coordinates on $\mathbb{P}^n_X$, and $x_i = X_i/X_0$, $1 \le i \le n$. If $\mathfrak{U}$ is the relatively affine covering of $\mathbb{P}^n_X$ defined by $X_0, \dots, X_n$, the corresponding alternating Čech resolution provides a canonical isomorphism

$$f_*(\check{\mathrm{C}}^\bullet(\mathfrak{U}, \omega_{P/X})[n]) \xrightarrow{\sim} \mathbb{R}f_*(\omega_{P/X}[n]). \tag{B.6.2}$$

Recall that $e_\pi : \pi^\sharp \cong \pi^!$ identifies the trace morphism for projective spaces Trp_π with the general trace morphism Tr_π [Co00, Lemma 3.4.3, (TRA3)]. Then the commutativity of (B.6.1) for π follows from the fact that Trp_π can be characterized as the only morphism which, via (B.6.2), induces on $\mathcal{H}^0$ the map sending the cohomology class $dx_1 \wedge \ldots \wedge dx_n/x_1 \cdots x_n$ to $(-1)^{n(n+1)/2}$ [Co00, (2.3.1)-(2.3.3)].

When $f = i : Y \hookrightarrow P$, recall that $d_i : i^\flat \cong i^!$ identifies the trace morphism for finite morphisms Trf_i with the general trace morphism Tr_i [Co00, Lemma 3.4.3, (TRA2)], and that $\mathrm{Trf}_i : \mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_P) \to \mathcal{O}_P$ is the canonical morphism induced by $\mathcal{O}_P \twoheadrightarrow \mathcal{O}_Y$. Using local cohomology with supports in Y, it can be factorized as

$$\mathrm{Trf}_i : \mathbb{R}\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_P) \to \mathbb{R}\underline{\Gamma}_Y(\mathcal{O}_P) \to \mathcal{O}_P. \tag{B.6.3}$$

On the other hand, there exists a canonical morphism

$$\mathbb{L}w^*\mathbb{R}\underline{\Gamma}_Y(\mathcal{O}_P) \longrightarrow \mathbb{R}\underline{\Gamma}_{Y'}(\mathcal{O}_{P'}), \tag{B.6.4}$$

which is an isomorphism: to check this, it suffices to choose a finite affine covering $\mathfrak{V}$ of $V = P \setminus Y$, and to identify $\mathbb{R}\underline{\Gamma}_Y(\mathcal{O}_P)$ with its flat resolution provided by the total complex

$$\mathcal{O}_P \to j_*\check{\mathrm{C}}(\mathfrak{V}, \mathcal{O}_V),$$

where j denotes the inclusion of V in P and $\mathcal{O}_P$ sits in degree 0. Moreover, this shows that the diagram

$$\begin{array}{ccc} \mathbb{L}w^*\mathbb{R}\underline{\Gamma}_Y(\mathcal{O}_P) & \longrightarrow & \mathbb{L}w^*(\mathcal{O}_P) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{R}\underline{\Gamma}_{Y'}(\mathcal{O}_{P'}) & \longrightarrow & \mathcal{O}_{P'} \end{array}$$

commutes. Therefore, it suffices to prove the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{L}w^*i_*(\omega_{Y/P}[-d]) & \longrightarrow & \mathbb{L}w^*\mathbb{R}\underline{\Gamma}_Y(\mathcal{O}_P) \\ \sim\downarrow & & \downarrow\sim \\ \omega_{Y'/P'}[-d] & \longrightarrow & \mathbb{R}\underline{\Gamma}_{Y'}(\mathcal{O}_{P'}) \end{array} . \tag{B.6.5}$$

Since $Y' \hookrightarrow P'$ is a regular immersion of codimension d, all complexes in this diagram are acyclic except in degree d, so that, up to translation by $-d$, the diagram is actually a diagram of morphisms of $\mathcal{O}_{P'}$-modules. It follows that its commutativity can be checked locally on P'. Thus we may assume that P is affine, and that the ideal $\mathcal{I}$ of Y in P is generated by a regular sequence $t_1, \ldots, t_d$. Then the ideal $\mathcal{I}'$ of Y' in P' is generated by the images $t'_1, \ldots, t'_d$ of $t_1, \ldots, t_d$ in $\mathcal{O}_{P'}$, which form a regular sequence. Let $\mathfrak{V} = (V_1, \ldots, V_d)$ be the open covering of $P \setminus Y$ defined by the sequence $(t_1, \ldots, t_d)$. For any section $a \in \Gamma(P, \mathcal{O}_P)$, let us still denote by $a/t_1 \cdots t_d$ the image of $a/t_1 \cdots t_d \in \Gamma(V_1 \cap \ldots \cap V_d, \mathcal{O}_P)$ under the canonical homomorphisms

$$\Gamma(V_1 \cap \ldots \cap V_d, \mathcal{O}_P) \to H^{d-1}(P \setminus Y, \mathcal{O}_P) \to H^d_Y(P, \mathcal{O}_P) = \Gamma(P, \mathcal{H}^d_Y(\mathcal{O}_P)).$$

Then the canonical morphism

$$\omega_{Y/P} \xrightarrow{\sim} \mathcal{E}xt^d_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_P) \to \mathcal{H}^d_Y(\mathcal{O}_P)$$

maps $(\bar{t}_1^\vee \wedge \ldots \wedge \bar{t}_d^\vee) \otimes a$ to $\varepsilon(d)a/t_1 \cdots t_d$, where $\varepsilon(d) \in \{\pm 1\}$ only depends upon d (see [Co00, p. 252-254]). The commutativity of (B.6.5) follows. □

Proof of Case b). When u is residually stable, the diagram analogous to (B.6.1) commutes, thanks to [Co00, 3.4.3, (TRA4)]. Moreover, the isomorphisms e_π and d_i entering in the local definition of λ_f in B.1.2 c) also commute with base change by u, thanks to [Co00, p. 139, (VAR6)]. Then it suffice to observe that η_i commutes with flat base change, which is clear. □

Proof of Case c). When f is flat, f is a CM map, and the results of [Co00, 3.5 - 3.6] can be applied. Then the commutativity of (B.6.1) follows from [Co00, Theorem 3.6.5], provided one checks that λ_f identifies the base change isomorphism (A.3.1) for $\omega_{Y/X}$ with the more subtle base change isomorphism $\beta_{f,u}$ for ω_f defined in [Co00, Theorem 3.6.1]. As we will not use Case c) in this article, we leave the details to the reader. □

B.7. Let X be a noetherian scheme with a dualizing complex, and $f : Y \to X$ a proper complete intersection morphism of virtual relative dimension 0. One can define in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ a "trace morphism"

$$\tau_f : \mathbb{R}f_*(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \tag{B.7.1}$$

as follows. Thanks to the relative duality theorem (see [Ha66, VII, 3.4] or [Co00, Th. 3.4.4]), defining τ_f is equivalent to defining a morphism $\mathcal{O}_Y \to f^!\mathcal{O}_X$. Using the isomorphism λ_f, this is also equivalent to defining a morphism

$$\varphi_f : \mathcal{O}_Y \longrightarrow \omega_{Y/X}, \tag{B.7.2}$$

i.e., a section of the invertible sheaf $\omega_{Y/X}$. We define φ_f as being the morphism which maps 1 to the canonical section δ_f of $\omega_{Y/X}$, defined in A.7.

From this construction, it follows that the morphism τ_f can be described equivalently either as the composition

$$\tau_f : \mathbb{R}f_*(\mathcal{O}_Y) \xrightarrow{\mathbb{R}f_*(\lambda_f \circ \varphi_f)} \mathbb{R}f_*(f^!\mathcal{O}_X) \xrightarrow{\mathrm{Tr}_f} \mathcal{O}_X, \tag{B.7.3}$$

or as the composition

$$\tau_f : \mathbb{R}f_*(\mathcal{O}_Y) \xrightarrow{\mathbb{R}f_*(\varphi_f)} \mathbb{R}f_*(\omega_{Y/X}) \xrightarrow{\mathrm{Tr}^\sharp_f} \mathcal{O}_X, \tag{B.7.4}$$

where $\mathrm{Tr}^\sharp_f$ is the trace map defined in (B.3.1).

Before proving Theorem 3.1, we relate τ_f to the residue symbol defined in [Co00, (A.1.4)] (which differs by a sign from Hartshorne's definition in [Ha66]).

Proposition B.8. *With the hypotheses of B.7, assume in addition that f is finite and flat, and that $f = \pi \circ i$, where π is smooth of relative dimension d, and i is a closed immersion, globally defined by a regular sequence $(t_1, \dots, t_d)$ of sections of $\mathcal{O}_P$. Then, for any section a of $\mathcal{O}_P$, with reduction $\bar{a}$ on Y, we have*

$$\tau_f(\bar{a}) = \mathrm{Res}_{P/X}\begin{bmatrix} a\,dt_1 \wedge \dots \wedge dt_d \\ t_1, \dots, t_d \end{bmatrix}. \tag{B.8.1}$$

Proof. Let $\omega = a\,dt_1 \wedge \dots\dots \wedge dt_d$. By construction, the residue symbol is given by

$$\mathrm{Res}_{P/X}\begin{bmatrix} \omega \\ t_1, \dots, t_d \end{bmatrix} = (-1)^{d(d-1)/2}\,\varphi_\omega(1), \tag{B.8.2}$$

where $\varphi_\omega : f_*\mathcal{O}_Y \to \mathcal{O}_X$ is the image of $(t_1^\vee \wedge \dots \wedge t_d^\vee) \otimes i^*(\omega)$ by the isomorphism of complexes concentrated in degree 0 [Co00, (A.1.3)]

$$\omega_{Y/P}[-d] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mathbb{L}i^*(\omega_{P/X}[d]) \underset{\sim}{\xrightarrow{\eta_i^{-1}}} i^\flat\pi^\sharp\mathcal{O}_X \underset{\sim}{\xrightarrow{\psi_{i,\pi}^{-1}}} f^\flat\mathcal{O}_X; \tag{B.8.3}$$

here $f^\flat\mathcal{O}_X = \mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ viewed as a $\mathcal{O}_Y$-module, and $\psi_{i,\pi}$ is the canonical isomorphism of functors $f^\flat \xrightarrow{\sim} i^\flat\pi^\sharp$. Since Trf_f is the morphism

$f_*\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X) \to \mathcal{O}_X$ given by evaluation at 1, we can use the isomorphism $d_f : f^\flat \xrightarrow{\sim} f^!$ and the equality $\mathrm{Tr}_f \circ f_*(d_f) = \mathrm{Trf}_f$ [Co00, 3.4.3, (TRA2)] to write (B.8.4)

$$\mathrm{Res}_{P/X}\left[\begin{array}{c}\omega\\ t_1,\dots,t_d\end{array}\right] = (-1)^{d(d-1)/2}\,\mathrm{Tr}_f(f_*(d_f \circ \psi_{i,\pi}^{-1} \circ \eta_i^{-1})(t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(\omega))).$$

On the other hand, we have by definition

$$\zeta'_{i,\pi}(\delta_f) = (-1)^{d(d-1)/2}\, t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(dt_1 \wedge \dots\dots \wedge dt_d),$$

so we deduce from (B.7.3) the equality

$$\begin{aligned}\tau_f(\bar{a}) &= \mathrm{Tr}_f(f_*(\lambda_f \circ \varphi_f)(\bar{a}))\\ &= (-1)^{d(d-1)/2}\,\mathrm{Tr}_f(f_*(c_{i,\pi}^{-1} \circ i^!(e_\pi) \circ d_i \circ \eta_i^{-1})(t_1^\vee \wedge \dots \wedge t_d^\vee \otimes i^*(\omega))).\end{aligned}$$

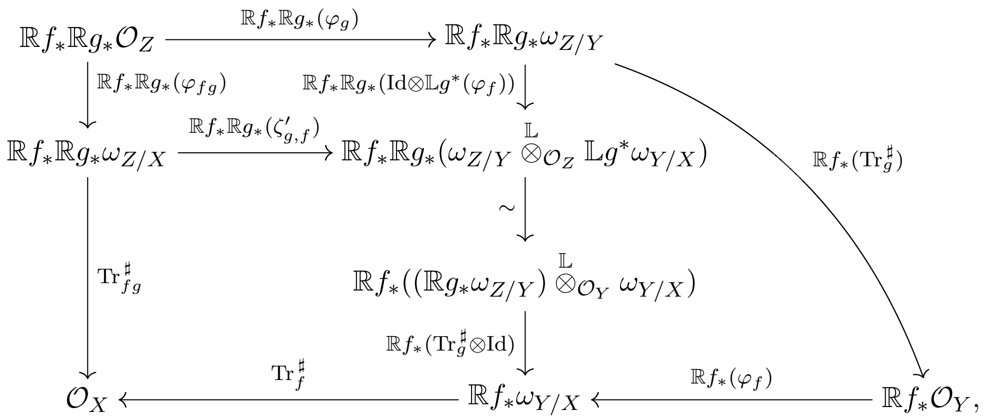
Therefore, it suffices to check that

$$d_f \circ \psi_{i,\pi}^{-1} = c_{i,\pi}^{-1} \circ i^!(e_\pi) \circ d_i,$$

and this results from (VAR5) [Co00, (3.3.26)]. □

B.9. *Proof of Theorem 3.1.*

(i) The transitivity formula (3.1.1) is the equality of the exterior composites in the diagram



where the upper left square commutes thanks to (A.8.1), the lower left square is the commutative square (B.5.1), and the right triangle commutes by functoriality.

(ii) Thanks to Proposition B.6 and to the description (B.7.4) of τ_f, the assertion follows from the compatibility of the canonical section δ_f with Tor-independent pull-backs (proved in Proposition A.8 (ii)) and the functoriality of the base change morphism.

(iii) To prove (3.1.3), it suffices to prove that the equality holds in the henselization $\mathcal{O}_{X,x}^{\mathrm{h}}$ of the local ring of X at each point x. As the morphism $\mathrm{Spec}\,\mathcal{O}_{X,x}^{\mathrm{h}} \to X$ is residually stable [Co00, p. 132], Proposition B.6 and the commutation with base change of the classical trace map for the finite locally free algebra $f_*\mathcal{O}_Y$ allow to assume that $X = \mathrm{Spec}\,A$, where A is a henselian noetherian local ring. Then Y is a disjoint union of open subschemes $Y_i = \mathrm{Spec}\,B_i$, where B_i is a finite local algebra over A. Each of the morphisms $Y_i \to X$ is a complete intersection morphism of virtual relative dimension 0 (since this is a local condition on Y), and the additivity of the trace (valid both for Tr_f, hence for τ_f, and for $\mathrm{trace}_{f_*\mathcal{O}_Y/\mathcal{O}_X}$) shows that it suffices to prove (3.1.3) for each morphism $Y_i \to X$. So we may assume that B is local. We can choose a presentation $B \cong C/I$, where C is a smooth A-algebra, and I is an ideal in C. Let $P = \operatorname{Spec} C$, $\mathcal{I} = I\mathcal{O}_P$, and let $y \in Y \subset P$ be the closed point. Then $\mathcal{I}_y$ is generated by a regular sequence $(t_1, \dots, t_d)$. Shrinking P if necessary, we may assume that $t_1, \dots, t_d$ generate $\mathcal{I}$ globally on P, so that the hypotheses of B.8 are satisfied. Then (3.1.3) follows from (B.8.1) and from property (R6) of the residue symbol [Co00, p. 240]. □

References

[Ax64] J. Ax, *Zeroes of polynomials over finite fields*, Amer. Journal of Math. **86** (1964), 255–261.

[Be74] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique p*, Lecture Notes in Math. **407**, Springer-Verlag (1974).

[BBE07] P. Berthelot, S. Bloch, H. Esnault, *On Witt vector cohomology for singular varieties*, Compositio Math. **143** (2007), 363–392.

[BO78] P. Berthelot, A. Ogus, *Notes on crystalline cohomology*, Mathematical Notes **21**, Princeton University Press (1978).

[Bo70] N. Bourbaki, *Algèbre*, Chapitres 1–3, Springer-Verlag (1970).

[Co00] B. Conrad, *Grothendieck Duality and Base Change*, Lecture Notes in Math. **1750**, Springer-Verlag (2000).

[CT03] B. Chiarellotto, N. Tsuzuki, *Cohomological descent of rigid cohomology for étale coverings*, Rend. Sem. Mat. Univ. Padova **109** (2003), 63–215.

[De71] P. Deligne, *Théorie de Hodge : II*, Publ. Math. Inst. Hautes Études Sci. **40** (1971), 5–57.

[De74] P. Deligne, *Théorie de Hodge : III*, Publ. Math. Inst. Hautes Études Sci. **44** (1974), 5–77.

[De83] P. Deligne, *Intégration sur un cycle évanescent*, Inv. Math. **76** (1983), 129–143.

[dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

[Dw64] B. Dwork, *On the Zeta Function of a Hypersurface: II*, Annals of Math. **80** (1964), 227–299.

[Es06] H. Esnault, *Deligne's integrality theorem in unequal characteristic and rational points over finite fields*, Annals of Math. **164** (2006), 715–730.

[Ek84] T. Ekedahl, *On the multiplicative properties of the de Rham-Witt complex I*, Arkiv För Matematik **22** (1984), 185–239.

[El78] F. El Zein, *Complexe dualisant et applications à la classe fondamentale d'un cycle*, Mém. Soc. Math. France **58** (1978), 5–66.

[Fo94] J.-M. Fontaine, *Représentations p-adiques semi-stables*, in *Périodes p-adiques*, Séminaire de Bures 1988, Astérisque **223** (1994), 113–184.

[Gr69] A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Topology **8** (1969), 299–303.

[Ha66] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. **20**, Springer-Verlag (1966).

[Hy91] O. Hyodo, *On the de Rham-Witt complex attached to a semi-stable family*, Compositio Math. **78** (1991), 241–260.

[HK94] O. Hyodo, K. Kato, *Semi-stable reduction and crystalline cohomology with logarithmic poles*, in *Périodes p-adiques*, Séminaire de Bures 1988, Astérisque **223** (1994), 221–268.

[Il79] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Éc. Norm. Sup. **12** (1979), 501–661.

[Il90] L. Illusie, *Ordinarité des intersections complètes générales*, in *The Grothendieck Festschrift*, vol. II, Birkhäuser (1990), 375–405.

[IR83] L. Illusie, M. Raynaud, *Les suites spectrales associées au complexe de de Rham-Witt.* Publ. Math. Inst. Hautes Études Sci. **57** (1983), 73–212.

[Ka89] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in *Algebraic analysis, geometry and number theory*, John Hopkins Univ. Press (1989), 191–224.

[Kz71] N. Katz, *On a theorem of Ax*, Amer. Journ. Math. **93** (1971), 485–499.

[LZ04] A. Langer, T. Zink, *De Rham-Witt cohomology for a proper and smooth morphism*, J. Inst. Math. Jussieu **3** (2004), 231–314.

[Ma72] B. Mazur, *Frobenius and the Hodge filtration*, Bull. Amer. Math. Soc. **78** (1972), 653–667.

[Ma73] B. Mazur, *Frobenius and the Hodge filtration (estimates)*, Annals of Math. **98** (1973), 58–95.

[Na09] Y. Nakkajima, *Weight filtration and slope filtration on the rigid cohomology of a variety*, preprint (2009).

[Ra70] M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. Inst. Hautes Études Sci. **38** (1970), 27–76.

[Se58] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, Symp. Int. Top. Alg. Mexico (1958), 24–53.

[SGA 6] *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), dirigé par P. Berthelot, A. Grothendieck et L. Illusie, avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Math. **225**, Springer-Verlag (1971).

[SGA 7 II] *Groupes de monodromie en géométrie algébrique. II*, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7), dirigé par P. Deligne et N. Katz. Lecture Notes in Math. **340**, Springer-Verlag (1973).

[Ts98] T. Tsuji, *p-adic Hodge Theory in the Semi-Stable Reduction Case*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 207–216.

[Ts99] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Inventiones Math. **137** (1999), 233–411.

[Tz04] N. Tsuzuki, *Cohomological descent in rigid cohomology*, in *Geometric aspects of Dwork theory*, edited by A. Adolphson et alii, vol. 2, De Gruyter (2004), 931–981.

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