CHAPTER 9

Topological dynamics: basic notions and examples

We introduce the notion of a dynamical system, over a given semigroup S. This is a (compact Hausdorff) topological space on which the semigroup S operates in the sense defined in 9.3. Dynamical systems over S are closely connected to the compact right topological semigroup βS for several reasons.
First, βS turns out be helpful in explaining some notions for dynamical system over S. Using the theory of βS developed so far, we are able to prove theorems on dynamical systems.
Moreover, βS can be viewed as a dynamical system over S with a particular universal role in the study of S-systems. And conversely, quite often it is possible to prove theorems on βS or on the combinatorics on S, using results on dynamical systems over S.

In this chapter, we will only define basic notions on dynamical systems, note some simple facts, and give examples, some of which are really simple and intuitive, some of a quite abstract character, to be used in the next chapters.

Many notions and results on dynamical systems work more naturally if S is assumed to be a monoid, not just a semigroup.

9.1. Definition A *monoid* is a triple $(S, \cdot, 1_S)$ in which $(S, \cdot)$ is a semigroup and 1_S is an identity of S, i.e. $1_S x = x 1_S$ holds for all $x \in S$.
An identity of a semigroup S, if it exists, is uniquely detemined. We will therefore notationally abbreviate a monoid $(S, \cdot, 1_S)$ as $(S, \cdot)$ or even as S; instead of 1_S, we also write 1.

Thus, $(\omega, +)$ is a monoid with identity 0; $(\mathbb{N}, +)$ is not a monoid. Every semigroup $(S, \cdot)$ can be extended to a monoid $(S^+, \cdot, i)$ where i is an element not in S and we define $ix = xi = x$ for all $x \in S^+$.

1. Basic notions

We split the definition of a dynamical system into two parts: a purely set resp. monoid theoretic one and a topological one.

9.2. Definition Assume that X is an arbitrary set and S is a monoid. An *action* of S on X is a function μ such that

- μ maps $S \times X$ to X; we use the multiplicative notation sx or $s \cdot x$ for $\mu(s, x)$
- $1_S \cdot x = x$ holds for every $x \in X$

- $s(tx) = (st)x$ holds for all $x \in X$ and $s, t \in S$.

We say that S acts on X, via μ.

As a simple but somewhat abstract example of an S-action, assume that $(T, \cdot, 1_T)$ is a monoid and that S is a submonoid of T, in particular that $1_T = 1_S \in S$. Then S acts on T by left multiplication: we simply put $\mu(s, x) = s \cdot x$ for $s \in S$ and $x \in T$. Similarly, for V a vector space over a field K, the multiplicative semigroup $(K, \cdot)$ acts on V by $\mu(k, v) = kv$, for $k \in K$ and $v \in V$.

9.3. Definition Assume that S is a monoid. A *dynamical system* over S or an S-system is a pair (X, μ) where

- X is a non-empty compact Hausdorff space
- μ is an action of S on X
- for every $s \in S$, the function $m_s : X \to X$ given by $m_s(x) = sx$ is continuous.

I.e. S acts on X in a continuous fashion. The space X is called the *phase space* of the system.

9.4. Remark Every function $\mu : S \times X \to X$ induces a map $m : S \to {}^X X$ by putting $m(s) = m_s$ for $s \in S$; conversely, given any function $m : S \to {}^X X$, we obtain $\mu : S \times X \to X$ by putting $\mu(s, x) = m_s(x)$.
Let us recall from 1.6 the semigroup $({}^X X, \circ)$ of all functions from X to itself, with id_X as its identity. Clearly, μ is an operation of S on X iff the induced map $m : S \to {}^X X$ is a monoid homomorphism, i.e. $m(1_S) = id_X$ and $m_{st} = m_s \circ m_t$ holds for s, t in S.
For a topological space X, write $C(X)$ for the subsemigroup of $({}^X X, \circ)$ consisting of all continuous functions from X to itself. Then (X, μ) is a dynamical system over S iff X is compact Hausdorff and the monoid homomorphism $m : S \to {}^X X$ maps S into $C(X)$.

9.5. Remark In additive notation for S, the property $m_{st} = m_s \circ m_t$ of 9.4 turns to $m_{s+t} = m_s \circ m_t$.
Intuitively, we consider the elements of S as time intervals; given a point $x \in X$, $sx = m_s(x)$ is the point to which x is moved after time s. The equality $m_{st} = m_s \circ m_t$ says that the point $m_t(x)$ is moved, after time s, to the same point to which x is moved after time $s + t$.

Dynamical systems over the monoid $(\omega, +)$ can be described in a less abstract way.

9.6. Remark (a) Dynamical systems over the monoid $(\omega, +)$ are called *discrete.*
(b) Let X be a compact Hausdorff space and $t : X \to X$ a continuous function. For $n \in \omega$, denote by t^n the n'th iteration of t, i.e. $t^n = \underbrace{t \circ \cdots \circ t}_{n \text{ times}}$; in particular, we set $t^0 = id_X$. Putting $n \cdot x = t^n(x)$ gives a discrete dynamical system with phase space X, because $t^{n+m} = t^n \circ t^m$.
(c) Conversely, every discrete dynamical system (X, μ) with phase space X arises in this way. To see this, we define a continuous function $t : X \to X$ by $t(x) = 1 \cdot x$.

(To avoid confusions, note that the identity of $(\omega, +)$ is the natural number 0, not 1!) Then for every $n \in \omega$ and $x \in X$, we have

$$n \cdot x = (1 + \cdots + 1) \cdot x = (t \circ \cdots \circ t)(x) = t^n(x),$$

i.e. the multiplication with n is the n'th iteration of t.
(d) Thus discrete dynamical systems with phase space X are usually identified with pairs (X, t), where $t : X \to X$ is continuous.

A fairly general way of producing dynamical systems is as follows.

9.7. Remark Assume that X is a compact Hausdorff space and S is a submonoid of $C(X)$. Then X becomes a dynamic system over S by putting $s \cdot x = s(x)$, for $s \in S$ and $x \in X$. I.e. the action of S on X is the application of functions in S to points in X.
For example, let T be an arbitrary set of continuous functions from X into itself, i.e. $T \subseteq C(X)$. Let $S = \langle C(X) \rangle$ be the submonoid of $(C(X), \circ)$ generated by T (cf. **??**), i.e. the elements of S are the finite products $t_1 \circ \cdots \circ t_n$ where $n \in \omega$ and $t_i \in T$. In particular for $n = 0$, the empty product $s = id_X$ is the identity of S. Thus every $T \subseteq C(X)$ gives rise to a dynamical system on $S = \langle C(X) \rangle$, with phase space X.

2. Subsystems

We begin to study a natural notion in dynamical systems.

9.8. Definition Let X be a dynamical system over S.
(a) A subset of Y of X is said to be *invariant* if $sy \in Y$ holds for all $y \in Y$ and $s \in S$, i.e. if $\bigcup_{s \in S} m_s[Y] \subseteq Y$. Note that if Y is invariant, then so is $cl_X(Y)$, the topological closure of Y in X, because the maps m_s are continuous.
(b) $Y \subseteq X$ is a *subsystem* of X (with respect to S) or an S-subsystem if it is non-empty, topologically closed, and invariant. We then consider Y as a dynamical system over S, with the continuous functions $m_s \restriction Y$, for $s \in S$.

As an easy example, consider the discrete dynamical system (X, t) where X is the compact subinterval $[0, 1]$ of the reals and t is the function mapping $x \in X$ to $x/2$. Clearly, every open subinterval $[0, a)$, where $0 \leq a \leq 1$, is invariant and every closed subinterval $[0, a]$ is a subsystem.

9.9. Definition Assume that X is a dynamical system over S and x is a point in X.
(a) The (S-)orbit of x in X is the set

$$\text{orb } (x) = \{sx : s \in S\};$$

it is clearly the least invariant subset of X containing x; note that $x = 1_S x \in \text{orb } (x)$.
(b) Consequently, the set

$$\bar{x} = cl_X \text{orb } (x)$$

is the least subsystem of X containing x, the *orbit closure* of x in X.
(c) We call a subsystem Y of X *simply generated* if $Y = \bar{x}$ for some $x \in X$.

E.g. in the example mentioned after 9.8, the orbit of $x \in X$ is $\{t^n(x) : n \in \omega\} = \{x/2^n : n \in \omega\}$, and the orbit closure is $\{x/2^n : n \in \omega\} \cup \{0\}$.

Minimal subsystems of a dynamical system X will be studied quite intensively.

9.10. Definition Assume that X is a dynamical system over S. A subsystem M of X is *minimal* if every subsystem of M coincides with M. i.e. if M has no proper subsystem.

For example, in a discrete dynamical system (X, t), every fixed point x of t (i.e. $t(x) = x$) gives the minimal subsystem $\{x\}$. Slightly more generally, if x is a periodic point with respect to t, i.e. $t^n(x) = x$, $n \geq 1$ and n is minimal with this property, then $\{x, t(x), t^2(x), \ldots, t^{n-1}(x)\}$ is a minimal subsystem.

We state two central facts on minimal subsystems.

9.11. Theorem *Every dynamical system X over S has a minimal subsystem. More generally, for every subsystem Y of X, there is a minimal subsystem of Y.*

Proof. As in 1.21, this is a routine application of Zorn's lemma. We consider the set $(\mathbb{P}, \supseteq)$ of all S-subsystems of Y, a nonempty partial ordering under reverse inclusion. Every chain $\mathcal{C}$ in $\mathbb{P}$ has $Z = \bigcap_{C \in \mathcal{C}} C$ as an upper bound – note that Z is non-empty, $\mathcal{C}$ being a family of closed subsets of X with the finite intersection property. So $\mathbb{P}$ has a maximal element, i.e. a minimal subsystem of Y. □

9.12. Definition For $U \subseteq X$ and $s \in S$, we write $s^{-1}U$ for the set $\{x \in X : sx \in U\}$.

9.13. Proposition *For every dynamical system X over S, the following are equivalent.*
(a) X is a minimal S-system
(b) for every $x \in X$, the orbit closure $\bar{x}$ of x is X
(c) for every $x \in X$, the orbit of x is a dense subset of X
(d) for every non-empty open subset U of X, there is a finite subset e of S such that $X = \bigcup_{s \in e} s^{-1}U$.

Proof. (a) and (b) are equivalent because the orbit closure of any point x is a subsystem of X and conversely, every subsystem of X includes an orbit closure. Equivalence of (b) and (c) is trivial. For the equivalence of (c) and (d), we note that (c) says that for every $x \in X$ and every non-empty open $U \subseteq X$ there is some $s \in S$ satisfying $sx \in U$. This means that $X = \bigcup_{s \in S} s^{-1}U$ holds for every non-empty open U. Since X is compact and the sets $s^{-1}U$ are open (by continuity of left multiplication with s), even $X = \bigcup_{s \in e} s^{-1}U$ holds for some finite $e \subseteq S$. □

It is clear from the remarks in 9.9 and also stated in 9.13 that minimal dynamical systems are simply generated. In many situations, we are mainly interested in minimal systems and therefore often study simply generated systems. These will be described in an abstract but quite elegant way in Section 5.

3. Mathematical examples

We give examples of dynamical systems with an intuitive mathematical content, starting with simple ones and ending with a less trivial one. In particular, we will describe their minimal subsystems. These examples will not be used in the rest of the text; we present them here just because of their fascinating character.

9.14. Example Consider the discrete dynamical system (X, t) from Section 2, where $X = [0, 1]$ and $t(x) = x/2$. Here t has 0 as its unique fixed point, so $\{0\}$ is a minimal subsystem. For every $x \in X$, the orbit closure of x contains the point 0. Thus, $\{0\}$ is the least subsystem and hence the only minimal subsystem of (X, t).

9.15. Example Now consider $X, t)$ where $X = [0, 1]$ and t maps $x \in X$ to x^2. Here t has 0 and 1 as its only fixed points. As in 9.14, $\{0\}$ and $\{1\}$ are the only minimal subsystems of X, because for every $x \neq 1$, the orbit closure of x contains 0.

9.16. Example We consider the unit disk

$$K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

of the real plane. The points of K can be written in polar coordinates

$$x = x_{r\phi} = (r \cos \phi, r \sin \phi)$$

where ϕ, r are real numbers, and $0 \leq r \leq 1$. The set $\mathbb{R}_+$ of non-negative real numbers is a monoid under addition.
We claim that K becomes a dynamical system over $\mathbb{R}_+$ by putting

$$t \cdot x_{r\phi} = x_{r'\phi'}$$

, with $\phi' = \phi + t$ and $t' = re^{-t}$. (So after time t, the distance of x from the origin is reduced by the factor e^{-t}, and its argument is increased by t.) This holds because the identity of $(\mathbb{R}_+, +)$ is the number 0, clearly $0 \cdot x = x$, and for $s, t \in \mathbb{R}_+$, note that $(re^{-t})e^{-s} = re^{-(s+t)}$.
For a point x of K, the orbit of x is a spiral line $L(x)$ starting at x and contracting to 0; the orbit closure of x is $L(x) \cup \{0\}$. Therefore, as in 9.14, $\{0\}$ is the unique minimal subsystem.

9.17. Example Here we work in the complex plane and consider the boundary

$$X = \{x \in \mathbb{C} :\mid x \mid= 1\}$$

of the unit disk, where $\mid x \mid$ is the absolute value of a complex number x. The points of X are written in polar coordinates as

$$x_\phi = \cos \phi + i \sin \phi,$$

for $0 \leq \phi < 2\pi$. We fix an arbitrary angle $\alpha \in [0, 2\pi)$ and obtain a discrete dynamical system by putting

$$t(x) = x \cdot x_\alpha$$

(complex multiplication). So $t(x_\phi) = x_{\phi+\alpha}$; the map t is the rotation of X by the angle α. For simplicity, let us assume that α is positive. The behaviour of the system (X, t) strictly depends on the angle α.
Case 1. $\alpha = q \cdot 2\pi$ where q is a rational number, say $q = m/n$; without loss of generalization, $m \leq n$ and m, n are relatively prime. In this case, $x_\alpha^n = 1$ and $x_\alpha^k \neq 1$ for $k < n$; so t^n is the identity on X. Every $x \in X$ is periodic; its orbit $orb(x) = \{x, x \cdot x_\alpha, x \cdot x_\alpha^2, \ldots, x \cdot x_\alpha^{n-1}\}$ has size n, and X is the union of its minimal subsystems $orb(x)$, $x \in X$.
Case 2. $\alpha = q \cdot 2\pi$ where q is irrational. Then the set $\{x_\alpha^n : n \in \omega\}$ of powers of

x_α is a dense subset of X; see the proof below. It follows that for every $x \in X$, the orbit $orb(x) = \{x \cdot x_\alpha^n : n \in \omega\}$ is dense in X. Hence in this case, X is a minimal system.

PROOF. The topology of X is induced by the metric δ where $\delta(x, y)$ denotes the length of the (shorter) arc from x to y, in X. Moreover, X is a group, under complex multiplication. And the metric δ is invariant under multiplication, i.e. for $x, y, z \in X$, we have $\delta(zx, zy) = \delta(x, y)$, because multiplication with z is a rotation on X.
For $k \in \mathbb{Z}$, we write p_k for x_α^k; the set $G = \{p_k : k \in \mathbb{Z}\}$ is the subgroup of X generated by x_α. We also define $G_+ = \{p_k : k \geq 1\}$.
Step 1. We first note that G is infinite, because $m \neq n$ and $p_m = p_n$ would contradict the irrationality of $\alpha/2\pi$.
Step 2. We show that for every $\varepsilon > 0$, there is some $g \in G_+$ such that $\delta(g, 1) \leq \varepsilon$. For this, consider for $x \in X$ the open arc $U(x)$ with center x and length ε. The arcs $U(x)$ cover X; by compactness, there is a finite subset e of X such that $X = \bigcup_{x \in e} U(x)$. By Step 1, there is an $x \in e$ and two distinct elements of G, say g and h, lying in the same arc $U(x)$. Then $\delta(g, h) < \varepsilon$, $f = g^{-1}h$ is an element of $G \setminus \{1\}$, and $\delta(1, f) = \delta(g, gf) = \delta(g, h) < \varepsilon$. Now $f = p_k$ for some $k \neq 0$; if $k \geq 1$, then f works for the claim. Otherwise, $f^{-1} = p_{-k} \in G_+$ and $\delta(f^{-1}, 1) = \delta(1, f) < \varepsilon$, so f^{-1} works.
Step 3. We now show that G_+ is dense in X: let $x \in X$ and $\varepsilon > 0$; we find some $g \in G_+$ such that $\delta(x, g) < \varepsilon$. For the sake of intuitiveness, let us assume that $\varepsilon \leq \pi/2$. Let ϕ be the real satisfying $0 \leq \phi < 2\pi$ and $x = x_\phi$.
By Step 2, pick $g \in G_+$ such that $a = \delta(g, 1) < \varepsilon$. Let $k \in \omega$ be the natural number with

$$ka < 2\pi < (k+1)a.$$

In the real line, the interval $[0, 2\pi)$ is covered by the intervals $[0, a]$, $[a, 2a]$, ..., $[ka, (k+1)a]$, so there is some $i \in \{0, \dots, k\}$ such that $ia \leq \phi \leq (i+1)a$. This means that, in the unit circle X, the point $x = x_\phi$ belongs in the closed arc from g^i to g^{i+1}, which has length $\delta(g, 1) = a$. It follows that $\delta(x, g^i) \leq a < \varepsilon$.
Step 4. It follows that for arbitrary $x \in X$, the orbit $\{x \cdot x_\alpha^n : n \in \omega\}$ is dense in X, being the image of G_+ under right mutiplication with x. □

4. An abstract example

We present here an abstract example of a dynamical system which plays a prominent role in subsequent results.

9.18. Example (a) We start with a finite set c of colours, say $c = \{1, \dots, r\}$, for $r = 2$, we will also use $c = \{0, 1\}$ instead of $c = \{1, 2\}$. Let $X = {}^\omega c$ be the product space of ω copies of the discrete finite space c, a compact Hausdorff (even Boolean) space. A function $x \in X$ may then be considered as the colouring of ω assigning colour $x(i) \in c$ to $i \in \omega$. We define the *shift* map $t : X \to X$ by

$$t(x)(i) = x(i+1),$$

for $x \in X$ and $i \in \omega$. Writing x_i for $x(i)$, t maps $x = (x_0, x_1, x_2, \dots)$ to $t(x) = (x_1, x_2, x_3, \dots)$. It is continuous since every finite initial segment of $t(x)$ is determined by a finite initial segment of x. This makes (X, t) into a discrete dynamical system, the *shift system*.
(b) The above construction generalizes to arbitrary monoids $(S, 1_S)$, letting X be

the product space ${}^S c$, i.e. X is the space of all colourings of S with colours in c. For $x \in X$ and $s \in S$, we define $sx \in X$ by

$$(sx)(i) = x(is)$$

for $i \in S$. Resp. if we write x as the sequence $(x_i)_{i \in S}$ where $x_i = x(i)$, then $(sx)_i = x_{is}$. Again left multiplication with a fixed $s \in S$ is a continuous map from X into itself and this makes X into a dynamical system over S: we still have to check that $(st)x = s(tx)$ holds for $x \in X$ and $s, t \in S$. In fact, for every $i \in S$, we see that $(st)x$ maps i to $x(ist)$ and $s(tx)$ maps i to $(tx)(is) = x(ist)$.

5. βS as a dynamic system

In this section, we consider an arbitrary monoid $S = (S, \cdot, 1_S)$ and its Stone-Čech compactification βS. Note that βS is a monoid with identity 1_S, too, by continuity of left multiplication with 1_S.

9.19. Definition We make βS into a dynamical system over S in the following way.
We know that βS is a (compact right topological) monoid with S as a submonoid; so S operates on βS by left multiplication, as explained after 9.2. More explicitly, the product $sp \in \beta S$ is defined for every $s \in S$ and every $p \in \beta S$, and we define $\mu : S \times \beta S \to \beta S$ by $\mu(s, p) = sp$. We will henceforth write, as usual, sp for $\mu(s, p)$. In fact, all requirements for a dynamical system over S are satisfied: left multiplication with a fixed $s \in S$ is continuous on βS as mentioned in Chapter 4; $1_S p = p$ holds for all $p \in \beta S$ as noted above, and $s(tp) = (st)p$ is simply a special case of the associativity of the multiplication on βS.

The dynamical properties of the S-system βS can be described, in a straightforward manner, by the algebraic ones of the semigroup $(\beta S, \cdot)$.

9.20. Remark (a) For $p \in \beta S$, the orbit of p is $orb(p) = \{sp : s \in S\} = S \cdot p$; it follows, by continuity of right multiplication with p in βS, that the orbit closure of p is $\bar{p} = \beta S \cdot p$. Thus the simply generated subsystems of βS are the simply generated (closed) left ideals $\beta S \cdot p$ of the semigroup βS, where $p \in \beta S$. More generally, the S- subsystems of βS are exactly the topologically closed left ideals of βS.
(b) Hence the minimal subsystems of βS are exactly the minimal left ideals of the semigroup βS, i.e. the ideals $\beta S \cdot p$ where $p \in K(\beta S)$.
(c) The orbit closure of $1_s \in S \subseteq \beta S$ is $\beta S \cdot 1_S = \beta S$; so the dynamical system βS is simply generated.

The next notion and theorem show that βS is, in a sense made precise below, a universal dynamical system over S.

9.21. Definition Assume that X and Y are dynamical systems over the monoid S. A function $f : X \to Y$ is said to be an S-homomorphism, or simply a homomorphism, from X to Y if it is continuous and the equality

$$f(sx) = sf(x)$$

holds for all $x \in X$ and $s \in S$; here sx is, of course, computed in X and $sf(x)$ in Y. I.e. f commutes with the multiplication by s on X resp. Y, for every $s \in S$.

9.22. Theorem *For every dynamical system X over S and every point x of X, there is a unique S-homomorphism $f = f_x : \beta S \to X$ mapping 1_S to x. The range*

of f is the orbit closure $\bar{x}$ of x.
In particular, every simply generated S-system is a homomorphic image of βS.

Proof. For the uniqueness statement, assume that f and g are S-homomorphisms from βS into X mapping 1_S to x. Then the set $M = \{p \in \beta S : f(p) = g(p)\}$ contains 1_S. Next, every $s \in S$ is in M, because $f(s) = f(s1_S) = sf(1_S) = sx$ and similarly $g(s) = sx$. Finally M is closed, by continuity of f and g. Thus M is the closure of S in βS, i.e. $M = \beta S$ and $f = g$.
For existence of f, we use the universal property of the compactification βS of S: we consider the function $h : S \to X$ given by $h(s) = sx$ and show that its Stone-Čech extension $f : \beta S \to X$, a continuous function, works for the theorem. Now $f(1_S) = h(1_S) = 1_S x = x$; we still have to show that $f(sp) = sf(p)$ holds for all $p \in \beta S$ and $s \in S$.
But this is true if $p = t \in S$, since $f(st) = h(st) = (st)x$ and $sf(t) = sh(t) = s(tx)$, and it carries over to arbitrary $p \in \beta S$ by continuity.
Concerning the range of f, note that the image of S under h is $S \cdot x$, the orbit of x in the S-system X; hence the image of βS under f is the orbit closure of x. □

By its very definition as the Stone-Čech extension $\tilde{h}$ of h, the S-homomorphism $f_x : \beta S \to X$ in 9.22 can be written as

$$f_x(p) = \tilde{h}(p) = p - \lim_{s \in S} h(s) = p - \lim_{s \in S}(sx).$$

We introduce a very convenient notation connected to this fact.

9.23. Definition and Remark Let X be a dynamical system over a monoid S.
(a) For every $p \in \beta S$ and $x \in X$, we define

$$p \cdot x = px = p - \lim_{s \in S}(sx).$$

I.e. $px = f_x(p)$, where f_x is the S-homomorphism from βS to X mapping 1_S to x.
(b) Assigning $px \in X$ to every $p \in \beta S$ (for fixed $x \in X$) is a continuous map, being the S-homomorphism f_x. But the map assigning $px \in X$ to every $x \in X$ (for fixed $p \in \beta S$) is, in general, not continuous; see part (d) of this remark.
(c) The function $\mu : \beta S \times X \to X$ such that $\mu(p, x) = px$ extends the action of S on X. It is, in fact, an action of the semigroup βS on the space X. To verify that $(pq)x = p(qx)$ holds for all $x \in X$ and $p, q \in \beta S$, we note that this holds if $p = s$ and $q = t$ are elements of S. It carries over by continuity to the situation where $q \in \beta S$ and $p = s \in S$, and then to the general situation where p, q are both arbitrary elements of βS.
But this action does not make X a *dynamical system* over βS, because for $p \in \beta S$, the function $x \mapsto px$ is generally not continuous.
(d) In the special case where X is the space βS, we realize that the multiplication $\mu : \beta S \times X \to X$ just defined coincides with the semigroup multiplication on βS introduced in Chapter 4, because for $p, q \in \beta S$, $\mu(p, q) = p - \lim_{s \in S}(sq)$ (cf. part (a)), coinciding with the definition of pq in Chapter 4.
(e) The remark in 9.22 that the range of f_x is the orbit closure $\bar{x}$ of x can now be written in a very suggestive way: $\bar{x} = f_x[\beta S]$ and $f_x(p) = p \cdot x$ for $p \in \beta S$, so

$$\bar{x} = \{p \cdot x : p \in \beta S\} = \beta S \cdot x.$$

Exercises

(1) Let (X, t) be the discrete dynamical system in which X is the unit circle in the complex plain, and $t(x) = x^2$. Try to determine the minimal subsystems. (This is less trivial than it may occur.)

(2) On the compact unit interval $X = [0, 1]$, we define $t_0, t_1 : X \to X$ by $t_0(x) = x/2$ and $t_1(x) = (x + 1)/2$. We let S be the submonoid of $C(X)$ generated by t_0 and t_1 and view X as a dynamical system over S. Prove that X is minimal, as an S-system.

(3) In the shift system $(X =^{\omega} c, t)$ (cf. 9.18), show that there are points $x \in X$ having orbit closure X – i.e. X is simply generated.