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Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 10 - Bonus

Submission: 16.07.2024 (either in lecture or directly at Tillmann Kleiner)

Note: All points on this sheet are bonus points.

Exercise 1. (9 points)
Consider the *special orthogonal group* SO(3) consisting of all orthogonal matrices whose determinant is equal to 1. Show the following parts to show that SO(3) is a 3-manifold:

i) Consider the *Cayley map* given by

$$\mathrm{Cay} : \mathbb{R}^3 \to \mathrm{SO}(3), \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (I_3 + A) \cdot (I_3 - A)^{-1},$$

where $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ is skew symmetric and I_3 denotes the 3×3 identity matrix.

a) Show that $\mathrm{Cay}(A) \in \mathrm{SO}(3)$ for arbitrary $(a, b, c)^T \in \mathbb{R}^3$.

b) Show that Cay is injective.

ii) Characterize all elements of SO(3) not lying in the image of Cay.

iii) Use exercise 1 from exercise sheet 1 to cover these elements and construct an atlas for SO(3).

Exercise 2. (2 points)
Show that the tangent space $T_p\mathrm{SO}(3)$ of the special orthogonal group

$$\mathrm{SO}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \mid A \cdot A^T = \mathrm{id}, \det A = 1 \right\}$$

at the point $p = \mathrm{id} \in \mathrm{SO}(3)$ can be identified with the set of skew-symmetric 3×3 matrices.

Exercise 3. (4 points)
By considering the covering $\mathbb{S}^2 \to \mathbb{R}P^2$, prove that $\pi_1(\mathbb{R}P^2)$ contains a non-contractible loop. Show further that any non-contractible loop in $\mathbb{R}P^2$ is null-homotopic in $\mathbb{R}P^2$ if it is passed through twice.