Prof. Dr. Konrad Polthier
Dr. Eric Zimmermann
Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 9

Submission: 09.07.2024, 12:15 PM (start of lecture)

Exercise 1. (3 points)
Show that the fundamental group of a symmetric space is abelian.

Exercise 2. (3 points)
Show that a homogeneous manifold is geodesically complete.

Exercise 3. (8 points)
Let $\mathcal{Q} := \{z = a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ be the set of *quaternions* where

$$i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik.$$

$\mathbb{R}^4$ equipped with this multiplication is called *Hamiltonian quaternions*. The imaginary part of $z \in \mathcal{Q}$ is equal to $bi + jc + kd$ and the conjugate element of z is $\bar{z} = a - ib - jc - kd$.

i) Show that $z\bar{z} = |z|^2$ for $z \in \mathcal{Q}$ and the Euclidean norm.

ii) Conclude that $\mathbb{S}^3 = \{z \in \mathcal{Q} \mid z\bar{z} = 1\}$ equipped with the multiplication mentioned above is a Lie group.

iii) Consider $\mathcal{Q}_8 := \{z \in \mathcal{L} \mid |z| = 1\} \subseteq \mathcal{L} := \{z \in \mathcal{Q} \mid a, b, c, d \in \mathbb{Z}\}$. Describe the object given by the points in $\mathcal{Q}_8$. Can you give it a suitable name?