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Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 7

Submission: 25.06.2024, 12:15 PM (start of lecture)

Note: This sheet contains 4 bonus points.

Exercise 1. (4 points)
Consider the manifold $U = \{p \in \mathbb{R}^n \mid \|p\| < 1\}$ equipped with the hyperbolic metric

$$g_{ij}|_p := \frac{4}{(1 - \|p\|^2)^2}\delta_{ij}.$$

Determine the Ricci curvature Ric and the scalar curvature[1] S of (U, g) explicitly.

Exercise 2. (6 points)
Let M be a manifold, $p \in M$ a point and denote by $\pi_1(M, p)$ the first fundamental group of M at p.

i) Proof that the group multiplication on $\pi_1(M, p)$ is associative.

ii) Give examples for M and p such that

   a) $\pi_1(M, p)$ is Abelian,

   b) $\pi_1(M, p)$ is not Abelian.

   Justify your solutions.

Exercise 3. (8 points)
Let M and $\tilde{M}$ be two geodesically complete, connected Riemannian manifolds and let $\pi : \tilde{M} \to M$ be a *local isometry*, i.e. for each point $p \in \tilde{M}$, there exists an open neighborhood $U \subseteq \tilde{M}$ of p such that $\pi|_U$ is an isometry.

i) Show that π fulfills the *lifting property for geodesics*: for every geodesic $\gamma : [0, 1] \to M$ and each point $p \in \tilde{M}$ with $\pi(p) = \gamma(0)$ there exists a unique geodesic $\tilde{\gamma} : [0, 1] \to \tilde{M}$ such that $\pi(\tilde{\gamma}) = \gamma$ and $\tilde{\gamma}(0) = p$.

ii) Show that π is surjective.

iii) Conclude that π is a smooth covering map, i.e. is a smooth surjective map with the property that every $p \in M$ has a connected neighborhood V such that each component of $\pi^{-1}(V)$ is mapped diffeomorphically onto V by π.

[1]For Ricci and Scalar curvature see also the script dealing with sectional curvature.