
Exercise Sheet 6

Submission: 18.06.2024, 12:15 PM (start of lecture)

Exercise 1. (5 points)

Consider the upper half plane $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric

$$g_p := \frac{1}{y^q} \delta_{ij}$$

where $q \in \mathbb{R}_+$ and $q \neq 2$. Show that (\mathbb{H}, g_p) is not geodesically complete.

Exercise 2. (3 points)

Consider the manifold $U = \{p \in \mathbb{R}^n \mid \|p\| < 1\}$ equipped with the hyperbolic metric

$$g_{ij}|_p := \frac{4}{(1 - \|p\|^2)^2} \delta_{ij}.$$

Determine the Christoffel symbols Γ_{ij}^k and the sectional curvature K of (U, g) explicitly.

Exercise 3. (3 points)

Let (M, g) be a Riemannian manifold and let $\{E_1, \dots, E_{n-1}, X\}$ be an orthonormal basis of $T_p M$ at some point p in M . Denote with $(\mathbb{T}_p)_i$ the plane in $T_p M$ spanned by E_i and X . Show that

$$\text{Ric}(X, X) = \sum_{i=1}^{n-1} K_{(\mathbb{T}_p)_i}$$

where $K_{(\mathbb{T}_p)_i}$ denotes the sectional curvature and Ric the Ricci tensor¹.

¹See also the script dealing with sectional curvature.