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Version: 1

Differential Geometry II
Summer Semester 2024
Freie Universität Berlin

Exercise Sheet 5

Submission: 04.06.2024, 12:15 PM (start of lecture)

Exercise 1. (3 points)
Let (M, g) be a Riemannian manifold with a unit-speed geodesic $\gamma : I \to M$. A Jacobi field J along γ with $J \perp \dot{\gamma}$ everywhere is called a *normal* Jacobi field along γ. Show that $\ddot{J} + CJ = 0$ for any normal Jacobi field if (M, g) has constant sectional curvature $C \in \mathbb{R}$.

Exercise 2. (4 points)
Consider the upper half plane $\{(x, y) \in \mathbb{R}\times]0, \infty[\}$ equipped with the metric

$$g = \frac{1}{y^2}(\delta_{ij}).$$

Let γ be a parametrization of constant speed of $\{(x_0, y) : y \in]0, \infty[\}$ for fixed x_0.

i) Sketch the situation and show that γ is a geodesic.

ii) Show that $J = \partial_x$ is a Jacobi field along γ.

Exercise 3. (4 points)
Let (M, g) be a Riemannian manifold and $\bar{M} \subset M$ a submanifold of codimension 1 with normal field N. You may use without proof that $\bar{\nabla}_V W = \nabla_V W + b(V, W)N$ where $b(V, W) \coloneqq \langle \nabla_V N, W\rangle = -\langle \nabla_V W, N\rangle$.

i) Show that $\langle \bar{R}(V, W)X, Y\rangle = \langle R(V, W)X, Y\rangle + b(V, Y)b(W, X) - b(V, X)b(W, Y)$.

ii) Deduce that for a plane $\Pi_p \subset T_p\bar{M} \subset T_pM$ it is $\bar{K}(\Pi_p) = K(\Pi_p) + \det b\,|_{\Pi_p}$.