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Version: 2

Differential Geometry III – Homework 14

Submission: 14. February 2025, until 8:15 am (start of the exercise class).

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1. Exercise (3 points)

Let T be a non-degenerate triangle in the euclidean plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ with vertices $v_1 = (0, 0)$, $v_2 = (v_{2,x}, 0)$ and $v_3 = (v_{3,x}, v_{3,y})$, where $v_{2,x} \neq 0$ and $v_{3,y} \neq 0$. Further, let f and g be affine linear functions $\mathbb{R}^2 \to \mathbb{R}$.[I]

i) Express the partial derivatives $\partial_x f$ and $\partial_y f$ in terms of the function values $f(v_i)$, $i = 1, 2, 3$ at the vertices of the triangle and the coordinates $v_{2,x}$, $v_{3,x}$, $v_{3,y}$.

ii) Show that the scalar product of $f|_T$ and $g|_T$ can be written as

$$\langle \nabla f, \nabla g \rangle = \sum_{i,j=1,2,3} B_{ij} f(v_i) g(v_j)$$

with real constants $B_{ij} = B_{ji}$ for $i, j = 1, 2, 3$ that depend only on $v_{2,x}$, $v_{3,x}$ and $v_{3,y}$.

Please, turn the page!

[I]That is, functions of the form $f(x, y) = ax + by + c$ with constants $a, b, c \in \mathbb{R}$.

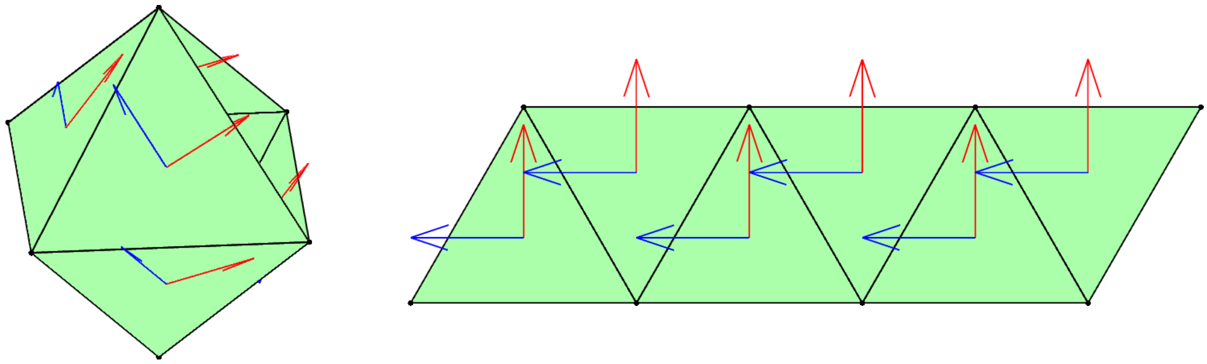

Figure 1: Left: Octahedron with two opposing triangles removed and two tangential vector fields X and Y drawn in red and blue, respectively. Right: The same vectors fields are drawn in an unfolding of the octahedron in the plane.

2. Exercise (5 points)

Let M be the discrete manifold that is obtained by removing two opposing triangles from an octahedron with edge lengths $\sqrt{2}$, resulting in a strip, as shown on the left in Figure 1. The unfolding of the strip is shown on the right. We define two triangle-wise constant tangential vector fields X and Y on the octahedron, by defining them as constant vector fields on the unfolding as indicated in Figure 1 in red and blue, respectively.

i) Calculate the dimensions of the spaces $\mathscr{X}_h$, $\mathscr{H}_h$, $\mathscr{H}_{h,D}$ and $\mathscr{H}_{h,N}$.[II]

ii) Prove that the space $\mathscr{H}_{h,D}$ respectively $\mathscr{H}_{h,N}$ of harmonic vector fields on M that are perpendicular respectively "almost tangential" to the boundary is generated by X respectively Y.

Hint: Use the formula for $\partial_x f$ from Exercise 1,i).

iii) Give an example of a vector field $Z \in \mathscr{H}_h \setminus (\mathscr{H}_{h,D} \cup \mathscr{H}_{h,N})$.

iv) Give an example of a vector field $Z \in \mathscr{X}_h \setminus \mathscr{H}_h$.

Total: 8

[II]Recall the notations

$$
\begin{aligned}
\mathscr{H}_h &:= \{X \in \mathscr{X}_h : \forall \varphi \in \mathscr{L}_0, \psi \in \mathscr{F}_0 : \langle X, \nabla\varphi\rangle = \langle X, \mathcal{J}\nabla\psi\rangle = 0\},\\
\mathscr{H}_{h,D} &:= \{X \in \mathscr{H}_h : \forall \psi \in \mathscr{F} : \langle X, \mathcal{J}\nabla\psi\rangle = 0\},\\
\mathscr{H}_{h,N} &:= \{X \in \mathscr{H}_h : \forall \varphi \in \mathscr{L} : \langle X, \nabla\varphi\rangle = 0\},
\end{aligned}
$$

where $\mathscr{X}_h$ is the set of piecewise constant vector fields on M (also denoted as $\Lambda_h(M)$) and

$$
\mathscr{L} := \left\{\varphi : \begin{smallmatrix}\varphi|_T \text{ linear on each triangle } T \text{ and}\\ \varphi \text{ globally continuous}\end{smallmatrix}\right\}, \qquad \mathscr{L}_0 := \left\{\varphi \in \mathscr{L} : \begin{smallmatrix}\varphi(v_b) = 0 \text{ at all}\\ \text{boundary vertices } v_b\end{smallmatrix}\right\},
$$

$$
\mathscr{F} := \left\{\psi : \begin{smallmatrix}\psi|_T \text{ linear on each triangle } T \text{ and}\\ \psi \text{ continuous at each midpoints}\end{smallmatrix}\right\}, \qquad \mathscr{F}_0 := \left\{\psi \in \mathscr{L} : \begin{smallmatrix}\psi(m_{e_b}) = 0 \text{ at all}\\ \text{boundary edge midpoints } m_{e_b}\end{smallmatrix}\right\}.
$$