Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
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Differential Geometry III – Homework 13

Submission: 7. February 2025, until 8:15 am (start of the exercise class).

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1. Exercise (5 points)

A common way to define the Laplace-Beltrami operator on a discrete manifold M with underlying simplicial complex K with underlying abstract set of vertices $\mathscr{V}$ operating on a function $f\colon \mathscr{V} \to \mathbb{R}$ is the cotan formula[I]

$$\Delta f(v) = \sum_{e=\{v,w\}\in K^{(1)}} \frac{\cot \alpha_e + \cot \beta_e}{2}(f(w) - f(v)) \qquad \text{for } v \in \mathscr{V}. \tag{1}$$

Here α_e and β_e are the opposite inner angles of two triangles t and s, respectively, that are adjacent to the edge e. The sum ranges over all edges e that contain the vertex v.

In the following, we consider the octahedron, with vertices $(1, 0, 0)$, $(0, 1, 0)$ and so on, as the discrete manifold M.

i) Calculate the matrix representation of the Laplace-Betrami operator on the octahedron using the cotan formula (1).

ii) Determine the eigenvalues of the Laplace-Beltrami operator on the octahedron and find a basis for each eigenspace.

   Hint: The eigenvectors correspond to oscillations on the octahedron and there are oscillations appearing in three different frequencies (corresponding to the eigenvalues). Taking a look at the geometry of the octahedron and its symmetries can help you guess the eigenvectors.

Please, turn the page!

[I] The cotan formula can be justified from linearly interpolating discrete functions $f\colon \mathscr{V} \to \mathbb{R}$ to continuous functions $f\colon M \to \mathbb{R}$.

2. Exercise (3 points)

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,t) = a_0 + \sum_{k\in\mathbb{N}} a_k \cos(kx)\cos(c_k t + e_k) + \sum_{k\in\mathbb{N}} b_k \sin(kx)\cos(d_k t + f_k) \quad \text{for } x,t \in \mathbb{R},$$

where $a_0, a_k, b_k, c_k, d_k, e_k, f_k \in \mathbb{R}$ and only a finite number of the a_k and b_k are non-zero. Recall, that $\{1, \cos(kx), \sin(kx) : k \in \mathbb{N}\}$ is a linearly independent set of functions in the variable x (and analogous for the variable t).

i) Determine c_k and d_k such that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x,t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(x,t) \qquad \text{for all } x,t \in \mathbb{R}.$$

ii) Prove that $a_k = 0$ must hold for all $k \in \mathbb{N}_0$ if $f(0,t) = 0$ for all $t \in \mathbb{R}$.

iii) Prove that $a_k = 0$ or $e_k \in \pi(\mathbb{Z} + \frac{1}{2})$ must hold for all $k \in \mathbb{N}_0$ if $f(-x,0) = -f(x,0)$ for all $x \in \mathbb{R}$.

Total: 8