Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
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Differential Geometry III – Homework 10

Submission: 17. January 2025, until 8:15 am (start of the exercise class).

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1. Exercise (4 points)

Consider the surface of a tetrahedron with side lengths 2 as a simplicial surface M and let $v \in \Lambda_h(M)$ be the tangential vector field on M that is depicted in red in Figure 1.

i) Determine the Hodge-Helmholtz decomposition of the vector field v into the vector fields $v_1 \in \operatorname{grad} S_h(M)$, $v_2 \in \mathcal{J} \operatorname{grad} S_h^*(M)$ and $v_3 \in H(M)$. You can do this by providing a suitable function $f \in S_h(M)$ resp. $g \in S_h^*(M)$ for the representation $v_1 = \operatorname{grad} f$ resp. $v_2 = \mathcal{J} \operatorname{grad} g$. It is sufficient to describe the function f resp. g by its values on the vertices resp. edges of the tetrahedron.

ii) Sketch the vector fields v_1, v_2 and v_3, analogous to Figure 1.

iii) What is the dimension of $H(M)$?

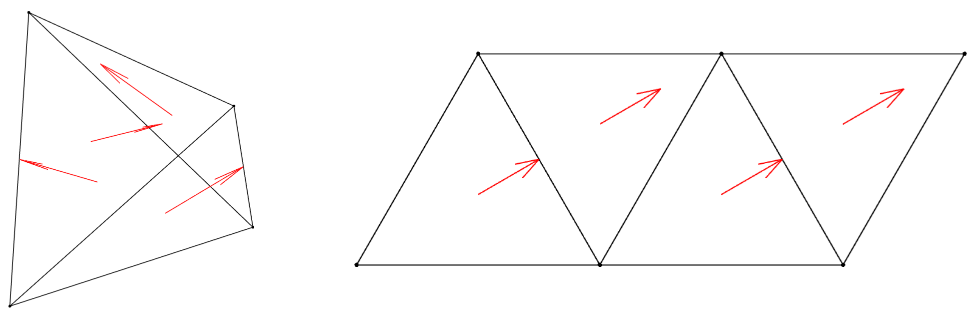

Figure 1: Left: Tetrahedron surface mesh with the tangential vector field $v \in \Lambda(M)$ drawn in red. Right: The same drawn for an unfolding of the tetrahedron.

2. Exercise (1 point)

Let $P \subset \mathbb{R}^3$ be a non-empty finite set of points in euclidean space and $k \in \mathbb{N}$. The k-neighborhood of a point $p \in P$ is defined as the set[I] of those k points from P that are closest to the point p (including p itself). Give a mathematical formula for U_k using the operator "argmin".

Please turn the page!

[I] In general, this definition is not unique, as there might be two distinct points having the same distance to p. To simply this exercise, we assume that the latter does not happen for the given set P.

3. Exercise (3 points)

Write down the simplicial complexes $\mathscr{V}_1$, $\mathscr{V}_2$ and $\mathscr{V}_3$ of the following geometric objects:

1: the solid bipyramid with an equilateral triangle as base,

2: the boundary of the latter (= the hollow bipyramid),

3: the graph of the latter, which is depicited in Figure 2.

Further, calculate the Link and the Star for the top vertex and for one of the vertices of the base in each of the three simplicial complexes $\mathscr{V}_1$, $\mathscr{V}_2$ and $\mathscr{V}_3$.

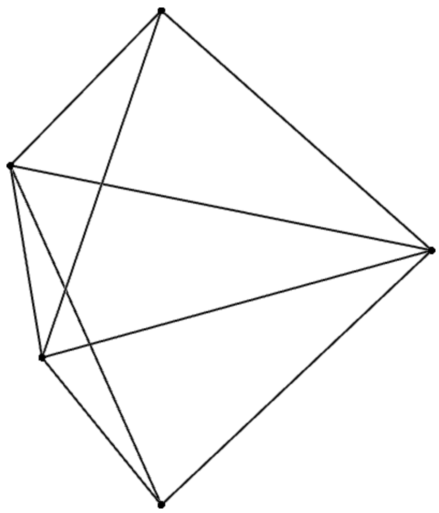

Figure 2: Graph of a bipyramid with an equilateral triangle as base.

Total: 8