Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
Prof. Dr. K. Polthier, Dr. T. Kleiner Version: 2

Differential Geometry III – Homework 6

Submission: 6. Dezember 2024, until 8:15 am (start of the exercise class).

All points on this sheet are bonus points.

1. Exercise (2 points)

Consider all regular tilings $\{p, q\}$ of S^2, which correspond to the five platonic solids.

i) Using the Gauss-Bonnet theorem, calculate the number of tiles required to cover the whole S^2 for each tiling.

   Hint: Recall the solution for Exercise 1,ii).

ii) Calculate the number of vertices and edges for each tiling as well.

2. Exercise (2 points)

Consider a tetrahedron with vertices at $V_0 = (1, 1, 1)$, $V_1 = (1, -1, -1)$, $V_2 = (-1, 1, -1)$ and $V_3 = (-1, -1, 1)$. Draw the tetrahedron and its characteristic tetrahedron. Calculate its edge lengths and the dihedral angles, as for Exercise 2,iii) from Sheet 1. What is the volume of the characteristic tetrahedron in comparison to the initially given tetrahedron?

Please turn the page!

3. Exercise (1 point)

Show that the stereographic projection $\hat{st}_n \colon S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ from Sheet 2 is conformal.

4. Exercise (3 points)

Consider the 3-dimensional sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

and, as a subset, the Clifford torus

$$C = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = \tfrac{1}{2}, x_3^2 + x_4^2 = \tfrac{1}{2}\}.$$

i) Prove that the Clifford torus maps to a torus of revolution $T_{R,r}$ under the stereographic projection $\hat{st}_3 \colon S^3 \setminus e_4 \to \mathbb{R}^3$ and determine its toroidal and poloidal radii, R and r.

   Hint: The implicit description of the torus of revolution in $\mathbb{R}^3$ is

   $$T_{R,r} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid (y_1^2 + y_2^2 + y_3^2 + R^2 - r^2)^2 - 4R^2(y_1^2 + y_2^2) = 0\}.$$

   It just remains to find R and r.

ii) The Clifford torus is known to separate two domains of the same shape in S^3. Sketch the full torus enclosed by the surface $T_{R,r} \subset \mathbb{R}^3$ (the inner domain) and explain how the outer domain of $T_{R,r}$ can be interpreted as a full torus as well.

Please turn the page!

5. Exercise (2 points)

Let Φ be the Weierstrass function

$$\Phi = \begin{pmatrix} \frac{1}{2} f(1-g^2) \\ \frac{\mathrm{i}}{2} f(1+g^2) \\ fg \end{pmatrix}. \tag{1}$$

Provide a transform $(f, g) \mapsto (\tilde{f}, \tilde{g})$ such that $\tilde{\Phi}$ is ...

i) ... the reflection of Φ at the xy-plane $(x, y, z) \mapsto (x, y, -z)$,

ii) ... the inversion of Φ at the origin $(x, y, z) \mapsto (-x, -y, -z)$,

iii) ... the reflection of Φ at the xy-diagonal plane $(x, y, z) \mapsto (y, x, z)$.

6. Exercise (2 points)

Let F be the Enneper surface as given by the Weierstrass-Enneper representation

$$F(z) = \int_0^z \Phi(w)\,\mathrm{d}w$$

with Φ as in Equation (1), and with the functions $f(z) = 2$ and $g(z) = z$. Calculate the associated family F^θ. What do you observe geometrically?

Hint: A reparameterization of the form $\tilde{F}^\theta(z) = F^\theta(z \cdot a^\theta)$, with $a^\theta \in \mathbb{C}$ chosen appropriately, can make the geometrical relation between the surfaces F and F^θ more clear.

Please turn the page!

7. Exercise (4 points)

Consider the triply periodic minimal surfaces (TPMS) Neovius and Schwarz I-WP. Nodal (= non minimal) approximations for these are given by

$$I(x, y, z) := 3(\cos x + \cos y + \cos z) + 4 \cos x \cos y \cos z,$$
$$J(x, y, z) := \cos x \cos y + \cos x \cos z + \cos y \cos z + \tfrac{1}{4},$$

and depicted in Figure 1. (Here $\{I = 0\} \approx$ Neovius and $\{J = 0\} \approx$ Schwarz I-WP.) These approximants are known to have the same symmetries as their minimal counter parts.

i) Find all straight lines and all planar geodesics contained in the Neovius and the Schwarz I-WP surface, using your geometric intuition.

ii) Formulate the corresponding symmetry operation for one of each type of straight line or planar geodesic. Then verify your findings from i) algebraically by explicitly applying the corresponding symmetry operations to the functions I and J.

   Example: The mirror symmetry of the Neovius surface at the xy-plane follows from the equation $I(x, y, -z) = I(x, y, z)$ for all $x, y, z \in \mathbb{R}$.

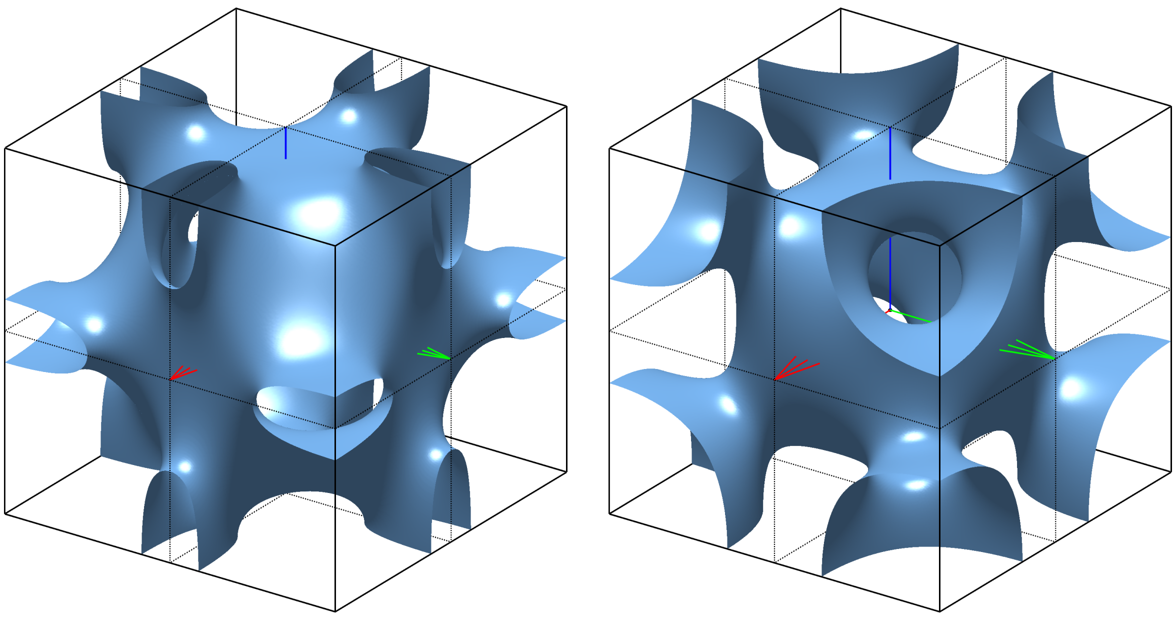

Figure 1: Left: Nodal approximation of the Neovius surface. Right: Nodal approximation of the Schwarz I-WP surface. The ranges are $-\pi \leq x, y, z \leq \pi$ in both plots, the red, green and blue arrows are the vectors $(\pi, 0, 0)$, $(0, \pi, 0)$ and $(0, 0, \pi)$, respectively.

Total: 16