Freie Universität Berlin WiSe 2024/25
Institut für Mathematik
Prof. Dr. K. Polthier, Dr. T. Kleiner Version: 2

Differential Geometry III – Homework 5

Submission: 29. November 2024, until 8:15 am (start of the exercise class).

---

1. Exercise (3 points)

Let $F = (f_1, f_2, f_3)$ be a conformal parameterization of a surface and $\varphi_i := (f_i)_u - \mathrm{i}(f_i)_v$ for $i = 1, 2, 3$. Complete the lecture from 5th November by proving the following statements:

i) It holds $\langle \Delta F, N \rangle = 2\lambda H$, where H is the mean curvature, N the normal vector and $g = \lambda \cdot \mathrm{id}$ is the metric induced by F.

ii) The surface F is regular if and only if $\varphi_1\overline{\varphi_1} + \varphi_2\overline{\varphi_2} + \varphi_3\overline{\varphi_3} \neq 0$ everywhere.

iii) It holds $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ for $\varphi_1 = \frac{1}{2}f(1 - g^2)$, $\varphi_2 = \frac{\mathrm{i}}{2}f(1 + g^2)$ and $\varphi_3 = f \cdot g$.

2. Exercise (5 points)

Three triply periodic minimal from the lecture, Schwarz P, Schwarz D and the Gyroid, are each nicely approximated by exactly one of the implicit surfaces

$$M_i := \big\{(x, y, z) \in \mathbb{R}^3 \,\big|\, I_i(x, y, z) = 0\big\}, \qquad i = 1, 2, 3,$$

where

$$I_1(x, y, z) := \sin x + \sin y + \sin z,$$
$$I_2(x, y, z) := \sin x \sin y \sin z + \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z,$$
$$I_3(x, y, z) := \sin x \cos y + \sin y \cos z + \sin z \cos x.$$

i) Find all straight lines and all planar geodesics contained in the Schwarz P, Schwarz D and the Gyroid, see Figure 1. Sketch those that you have found.

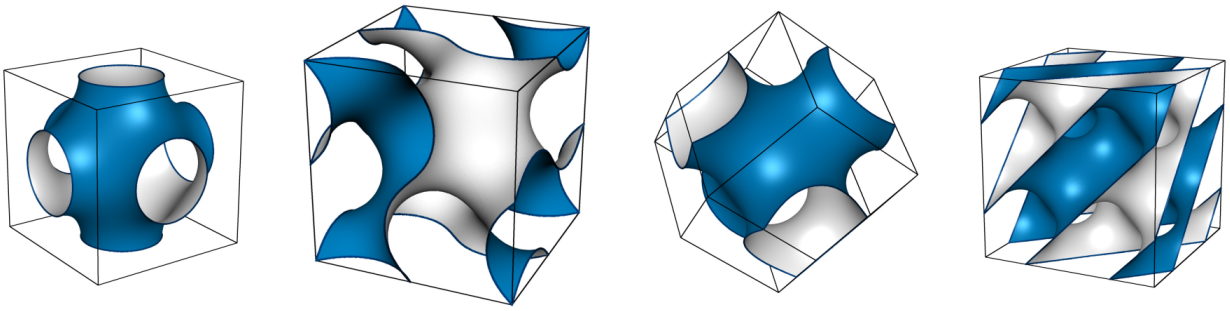
Hint: Use the theorem from the lecture on the 19th November.

ii) Determine which of the surfaces M_i corresponds to Schwarz P, Schwarz D or the Gyroid. Argue by comparing the symmetry properties of the functions I_i to the symmetry properties of the minimal surfaces, shown in Figure 1.

Hint: Figure out where the origin and the unit vectors e_x, e_y, e_z have to be placed in the unit cells in Figure 1, such that the coordinate system fits to the functions I_i. In order to distinguish all three functions and surfaces by their symmetry properties, it suffices to consider permutations of the coordinates and translations.

Please turn the page!

Total: 8



Schwarz P
cube

Gyroid
cube

Schwarz D
rhombic dodecahedron

Schwarz D
cube

Figure 1: Translational units for triply periodic minimal surfaces. The picture is taken from "Discrete Gyroid Surface", U. Reitebuch, M. Skrodzki, K. Polthier (2019).