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Version: 1

Differential Geometry III – Homework 1

Submission: 1. November 2024, until 8:15 am (start of the exercise class).

This sheet contains 6 bonus points.

1. Exercise (6 points)

The Gauss-Bonnet theorem for a Riemannian surface M with boundary ∂M states that

$$\int_M K \,\mathrm{d}A + \int_{\partial M} k \,\mathrm{d}s = 2\pi\chi(M).$$

The first term is the integral of the Gauss curvature over M, the second term is the integral of the curvature k of the boundary curve ∂M and $\chi(M)$ denotes the Euler characteristic of M. If ∂M is a piece-wise geodesic curve, then one has

$$\int_{\partial M} k \,\mathrm{d}s = \sum_p \alpha_p,$$

where the right hand side is the sum of the angles α_p between adjacent tangents over all points p where the boundary curve ∂M is not differentiable.

Use the Gauss-Bonnet theorem to solve the following problems:

i) Prove, that a hyperbolic triangle T in the upper half-plane model $\mathbb{H}$ satisfies

$$\mathrm{area}_{\mathbb{H}}(T) = \pi - (\alpha + \beta + \gamma), \tag{1}$$

where $\mathrm{area}_{\mathbb{H}}(T)$ is the area of T in $\mathbb{H}$ and α, β, γ are the interior angles of T.

Hint: The Euler characteristic satisfies $\chi(M) = 1$ if M is simply connected.

ii) Derive a generalization of (1) and use it to calculate the area of a single spherical or hyperbolic tile with Schläfli symbol $\{p, q\}$.

iii) Use (1) to calculate the hyperbolic area of the hyperbolic triangle T in $\mathbb{H}$ with vertices i, $4 + i$ and $2 + 2i$.

Hint: Recall, that all geodesics in $\mathbb{H}$ are given by halfes of circles with midpoint on the real axis.

Please turn page!

2. Exercise (6 points)

i) Determine all solutions of a regular tiling $\{p, q, r\}$ according to the theorem provided in the lecture. Which tile $\mathbb{S}^3$, $\mathbb{R}^3$, and $\mathbb{H}^3$ and why? Which of them are dual to each other?

Hint: The dihedral angle θ of a regular solid $\{p, q\}$ fulfills $\sin(\theta/2) = \frac{\cos(\pi/q)}{\sin(\pi/p)}$.

ii) The tilings $\{5, 3, 3\}$, $\{4, 3, 3\}$, and $\{3, 4, 3\}$ are called 120-, 8-, and 24-cell, respectively. Calculate the number of vertices, edges, faces, and cells (regular solids) of each $\mathbb{S}^3$ tiling from i). What are the names of the tilings not mentioned above?

Hint: The tiling $\{3, 3, 3\}$ can be imagined by a complete graph on n vertices for some $n \in \mathbb{N}$. Further, the Euler characteristic of these tilings is $N_v - N_e + N_f - N_c = 0$, with N_v, N_e, N_f, N_c the numbers of vertices, edges, faces, and cells, respectively.

iii) Consider an octahedron $\{3, 4\}$ with edge length 2. Calculate the lenghts and dihedral angles of its characteristic tetrahedron and provide a sketch of it inside the octahedron.

Total: 12